# Quantum mechanics of the vacuum state in two-dimensional QCD with adjoint fermions

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A study of two-dimensional QCD on a spatial circle with Majorana fermions in the adjoint representation of the gauge groups SU(2) and SU(3) is performed. The main emphasis is put on the symmetry properties related to the homotopically nontrivial gauge transformations and the discrete axial symmetry of this model. Within a gauge-fixed canonical framework, the delicate interplay of topology on the one hand and Jacobians and boundary conditions arising in the course of resolving the Gauss law on the other hand is exhibited. As a result, a consistent description of the residual $Z_N$ gauge symmetry [for SU($N$)] and the "axial anomaly" emerges. For illustrative purposes, the vacuum of the model is determined analytically in the limit of a small circle. There, the Born-Oppenheimer approximation is justified and reduces the vacuum problem to simple quantum mechanics. The issue of fermion condensates is addressed and residual discrepancies with other approaches are pointed out.

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## I. INTRODUCTION

After the discovery of instantons [1] it became clear that quantum mechanics of the vacuum state in non-Abelian gauge theories is nontrivial. Thus, in QCD it was shown that a noncontractible path in the space of fields exists; i.e., one of the directions in the (infinitely dimensional) functional space is a closed circle. Correspondingly, quantum mechanics of the vacuum state is analogous to that of a particle living on a circle. This fact results in a wave function of Bloch-type and in the occurrence of the vacuum angle $\theta$ [2, 3].

At the same time evidence has been mounting that the simple picture suggested in Refs. [2, 3] is not quite complete, and a variety of distinct situations can take place in different theories; in particular, the presence of fermions may affect the underlying quantum mechanics. Thus, chiral Ward identities in QCD [4] imply that correlation functions in QCD with massless quarks are periodic in $\theta$ with a period, which seemingly depends on the number of massless quarks considered and is not equal to $2\pi$, as one would expect in the problem of a particle on the circle. Another indication of a more complicated vacuum structure comes from the four-dimensional supersymmetric Yang-Mills (SUSYM) theories. Indeed, in such theories the gaugino condensate $\langle\lambda\lambda\rangle$ is nonvanishing and, moreover, exactly calculable [5]. The calculation is based on a theorem establishing the holomorphic dependence of $\langle\lambda\lambda\rangle$ on certain parameters in the SUSYM Lagrangian [6] (for a brief review see the reprint volume cited in [1]). While the direct calculation [5] proves that $\langle\lambda\lambda\rangle \neq 0$ the standard pattern [2,3] with one circle of unit length in the $K$ direction ($K$ is the Chern-Simons charge; see Fig. 1) gives no hint whatsoever on the condensation of $\lambda\lambda$, since the number of the gaugino zero modes in the tunneling transition with $|\Delta K| = 1$ is larger than two for any gauge group.

If for the unitary groups one can at least hope that torons [7] resolve the paradox by reducing the length of the noncontractible contour from 1 to $1/N$ for SU($N$) [7, 8], for the orthogonal and exceptional groups such a way out is absent, since the torons do not exist in this case. Moreover, the tentative toron solution of the problem for the unitary groups does not seem to be appealing, since the phenomenon of condensation of $\lambda\lambda$ is quite universal and the condensate $\langle\lambda\lambda\rangle \neq 0$ is present in the SUSYM theories with arbitrary gauge group.

Thus, it is clear that the existing ideas of quantum mechanics of the vacuum state in the non-Abelian gauge theories are incomplete. The possibility of more sophisticated dynamical patterns remains open. Quite recently it was shown [9], for instance, that in a "twisted" two-dimensional Schwinger model with two flavors the standard picture of Fig. 1(a) valid in the one-flavor model

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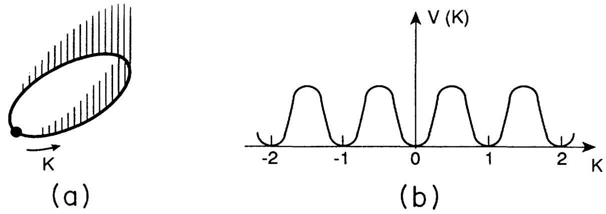


FIG. 1. Topology in the space of gauge fields in QCD. (a) A circle in the space of the gauge fields in the $K$ direction. The length of the circle is 1. The vertical lines indicate the strength of a potential acting on the effective degree of freedom living on the circle. If we unwind the circle onto a line we get the picture of (b).

(for a review see, e.g., [10]) gives place to that of Fig. 2. For each given value of $\theta$ there are two vacuum states. The twofold degeneracy and the possibility for the system to live in the second well, "half way around the circle," results in the occurrence of a bilinear fermion condensate, which is not generated in the two-flavor model otherwise. In a sense, the standard gauge principle can be extended in the two-flavor model at a price of incorporating an additional symmetry, which reduces the length of the noncontractible circle in the space of fields by 1/2.[1]

In summary, it appears that the standard topological considerations may not be sufficient to account for all relevant symmetry aspects of gauge theories; as a consequence the theoretical analysis has to be performed on a more detailed level for achieving a complete overview of the underlying symmetries. Such detailed analyses are possible for a few model theories only. Two-dimensional QCD with fermions in the adjoint representation of the SU($N$) group ($\mathrm{QCD}_2^{\mathrm{adj}}$) represents such a model. It is sufficiently simple—transverse degrees of freedom of the gauge fields are absent, and we are left with a nontrivial topological structure in its pure form, not overshadowed by dynamics of "perturbative gluons." Moreover, if the model is considered on a spatial circle of a small size (as we shall do in most parts of this paper) all interesting phenomena take place in the weak-coupling regime, so that quasiclassical methods are fully applicable. This

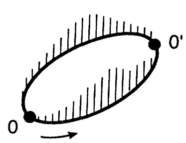


FIG. 2. An effective potential energy for the degree of freedom living on a circle in the twisted two-flavor Schwinger model. The point $O'$ is *not* the gauge image of $O$. The true vacuum state is the symmetric or antisymmetric linear combination of the states concentrated near $O$ and $O'$.

[1]Such an interpretation of the results referring to the twisted Schwinger model was emphasized by A. Vainshtein.

makes the problem solvable. On the other hand, the model is sufficiently rich and some of the dynamical phenomena to be discussed are hopefully relevant also for four-dimensional QCD. Most importantly, with all fields in the adjoint representation, this model exhibits nontrivial topological properties [11,37]; the gauge group is $SU(N)/Z_N$ and not SU($N$) as is the case for fermions in the fundamental representation. Subtle differences in topology of the configurational space can be the source of qualitative differences in the quantum mechanics of systems that are similar otherwise. This is well known from other branches of physics; an example is provided, for instance, by the so-called nematic systems. The difference in topological properties of the magnetic and nematic systems leads to distinct quantum mechanics. The configurational spaces in these two cases are $S^2$ and $S^2/Z_2 \sim RP^2$, respectively; this "small" distinction gives rise to macroscopically different properties such as the stability of certain singularities (line singularities) in the nematic but not in the magnetic substances [12].

Related to the nontrivial topological properties ($\pi_1[SU(N)/Z_N] = Z_N$) instantons and bilinear fermion condensates appear in $\mathrm{QCD}_2^{\mathrm{adj}}$ [11] much in the same way as in $\mathrm{QCD}_4$. The fact that $\mathrm{QCD}_2$ with adjoint fermions has a kind of $\theta$ vacuum was noted in Ref. [13], where it was also found that the SU($N$) theory has $N$ vacuum states. Some other general features of four-dimensional QCD also find parallels in this two-dimensional gauge theory, as follows from the recent analysis of the large $N$ limit of the model [14].

In this work we deal mainly with the gauge groups SU(2) and SU(3). It will be shown that the quantum-mechanical structure of the vacuum state in these two cases differs significantly.

We shall investigate $\mathrm{QCD}_2^{\mathrm{adj}}$ within the canonical "gauge-fixed" formalism, which is easily derived from the axial-gauge representation of (3+1)-dimensional QCD [15]. Within this formalism, the dynamics is described exclusively in terms of unconstrained degrees of freedom. As a consequence of the elimination of redundant gauge degrees of freedom the topological structure in the space of gauge fields is implemented in a peculiar although quite explicit form. This is generally the case whenever gauge theories are formulated in terms of physical variables only. One actually finds in such unconstrained formulations residual symmetries, i.e., symmetries that are not associated with small gauge transformations. The generators of these residual symmetries in both Abelian and non-Abelian gauge theories have been completely specified [15–17] and play a key role in our analysis of quantum mechanics emerging in $\mathrm{QCD}_2^{\mathrm{adj}}$.

The questions of gauge fixing and quantization in two-dimensional QCD were discussed in the literature more than once [18–25]. As we will see below, some of the conclusions and results obtained previously are incorporated in our analysis, while we disagree with others.

In this work we take up the issue of residual symmetries (as formulated in [15]) in the specific context of $\mathrm{QCD}_2^{\mathrm{adj}}$. Consideration of this model helps us understand, in the language of quantum mechanics, implications of the topological properties. Furthermore, our analysis demonstrates a possibility of revealing symmetries in the process of elimination of redundant variables—symmetries which are seen in the framework of unconstrained degrees of freedom but are implicit in the original formulation.

We work within the Hamiltonian formalism, which provides a particularly convenient framework for eliminating the redundant gauge degrees of freedom along the lines suggested in Ref. [15] for $\mathrm{QCD}_4$; in turn, the results of the present detailed investigation of $\mathrm{QCD}_2^{\mathrm{adj}}$ will shed light on general properties of such a gauge-fixed formalism. Technically, the fermions are assumed to be described by real Majorana fields. Then $\mathrm{QCD}_2^{\mathrm{adj}}$ is formulated on a finite-size interval (spatial circle), with the requirement

$$gL \ll 1, \tag{1}$$

where $g$ is the gauge coupling constant and $L$ is the size of the interval. This additional condition ensures that the quantum-mechanical reduction of the problem emerging in this way belongs to the weak-coupling regime, and the structure of the vacuum state can be treated quasiclassically. The general strategy is very close in spirit to that accepted previously [10] in the Hamiltonian analysis of the Schwinger model on a circle (cf. also [26]). Among other more technical issues, the spectral flow of the fermion levels is considered, and the topological reasons for the occurrence of the fermion zero modes [11] are explained.

As we will see, the residual symmetries of the theory can manifest themselves in fermion condensates forming in the corresponding ground state. The fermion condensates serve as a convenient indicator of a vacuum structure. The question of the fermion condensates in $\mathrm{QCD}_2^{\mathrm{adj}}$ and a related interpretation of the structure of the vacuum state were discussed recently in Ref. [11] within the Euclidean (path integral) formulation. If $\mathrm{QCD}_2^{\mathrm{adj}}$ is treated on a cylinder $S^1 \otimes R$, it has instantons—trajectories interpolating between gauge equivalent points in the space of fields corresponding to minima of the effective potential energy. These points are connected by gauge transformations that are not continuously deformable to unity (although in this case the situation is more complicated than just a simple circle of $\mathrm{QCD}_4$). The instantons generate fermion zero modes. If the gauge group is SU(2) the number of zero modes is two, exactly what is needed to produce the bilinear fermion condensate. For higher gauge groups, however, the number of the fermion zero modes in the instanton transition is larger than two, so that the bilinear condensate does not appear within the standard instanton calculus. On the other hand, an independent solution of the model based on bosonization seems to show that the bilinear fermion condensate develops irrespectively of what particular gauge group is considered. This paradox, mentioned in Ref. [11], is obviously perfectly identical to the one we face in the four-dimensional SUSYM theory.

In our explicit construction of the vacuum wave function, the symmetry properties and related tunneling phenomena are realized explicitly, in the familiar context of the quantum mechanics of few dynamical degrees of freedom. Thereby an intuitive picture of dynamics emerges, which may be useful in future attempts to resolve the "condensate paradox."

The paper is organized as follows. In Sec. II we review the Hamiltonian approach to $\mathrm{QCD}_2^{\mathrm{adj}}$. The canonical quantization is carried out, and the remaining (unconstrained) gauge variables are specified. The topology of the corresponding fundamental domains is discussed in detail. Section III is devoted to the structure of the vacuum state. The vacuum wave function is explicitly built in the limit $gL \ll 1$. The analysis is conducted separately for SU(2) and SU(3) theories. The vacuum wave function is then used for calculating the fermion condensates. We also derive anomaly relations and index theorems relevant to $\mathrm{QCD}_2^{\mathrm{adj}}$. Finally we extend our symmetry considerations beyond the weak-coupling approximation. Section IV contains our conclusions and an outlook.

## II. HAMILTONIAN APPROACH TO $\mathrm{QCD}_2^{\mathrm{adj}}$

### A. Formulation of the problem

The Lagrangian of (1+1)-dimensional QCD coupled to Majorana fermions $\psi$ in the adjoint representation is

$$\mathcal{L} = \mathrm{tr}\left\{ -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}D_\mu\gamma^\mu\psi \right\} \tag{2}$$

with the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu, A_\nu]$. The standard matrix notation is used so that

$$F_{\mu\nu} \equiv F^a_{\mu\nu}t^a \text{ and } \psi \equiv \psi^a t^a,$$

where $t^a$ are the generators of the group, and $[t^a, t^b] = if^{abc}t^c$ [$t^a = (1/2)\sigma^a$ for SU(2) and $t^a = (1/2)\lambda^a$ for SU(3)]. Moreover, the covariant derivative acts as[2]

$$D_\mu = \partial_\mu + ig[A_\mu,\ ]\ .$$

With the following choice of the $\gamma$ matrices,

$$\gamma^0 = \sigma_2\,, \quad \gamma^1 = i\sigma_1\,, \quad \gamma^5 = \sigma_3, \tag{3}$$

the Majorana spinors can be chosen real. In this "chiral" representation the right-handed fermion field is the upper component of $\psi$, while the left-handed fermion field is the lower component.

The theory is considered on a finite-size spatial interval,

$$0 \le x \le L\ ,$$

compactified to a circle by imposing periodic boundary conditions on the gauge fields,

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[2] If $T^a$ is the generator of the gauge group in the *adjoint* representation, $(T^a)_{mn} = if^{man}$, then the covariant derivative can be written as $iD_\mu = i\partial_\mu - gA^a_\mu T^a$. Correspondingly, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$. The definitions of the left- and right-handed fields and $\gamma_5$ below follow Bjorken and Drell [38].

$$A_\mu(x=0) = A_\mu(x=L)\,.$$

As for the fermion fields, it is slightly more convenient to impose antiperiodic boundary conditions:

$$\psi(x=0) = -\psi(x=L)\,.$$

Before we proceed to quantization let us discuss some general topological properties [11, 13] of the theory defined by the Lagrangian (2). First of all, since all fields considered are in the adjoint representation of the gauge group, the elements of the center of the group act on them trivially. In other words, the gauge group of the model is not SU($N$), but, rather, SU($N$)/$Z_N$. This fact is very important, since it leads to a nontrivial topology in the space of the gauge fields. Indeed, $\pi_1[\mathrm{SU}(N)]$ is trivial, while $\pi_1[\mathrm{SU}(N)/Z_N] = Z_N$. The latter aspect reminds us of four-dimensional QCD. The parallel is not perfect, though. Indeed in QCD$_4$ it is $\pi_3[\mathrm{SU}(N)] = Z$ that counts, and we arrive at one noncontractible contour with the topology of a circle (Fig. 1). In other words, if $U(\mathbf{x})$ is a gauge matrix corresponding to the winding number $K=1$ (one full rotation over the circle), $U^2$ will represent two rotations, $U^3$ three, etc. In QCD$_2^{\mathrm{adj}}$, say with the SU(2) gauge group, the analogous matrix $U(x)$ is nontrivial, but $U^2$ is already continuously deformable to the unit matrix, so that two rotations have the same effect as no rotation at all. Correspondingly, the two-instanton configuration [the $N$ instanton configuration in SU($N$)] is topologically trivial.

Graphically the topology of the manifold of an analogue quantum-mechanical system might be depicted as in Fig. 3 [for SU(2)]. The two circles are actually equivalent to each other, but to make both of them visible they are split in two and slightly distorted. The analogue particle moves, starting from the point $O$, along the larger circle in the direction indicated by the arrow, approaches $O$ and then moves along the smaller circle in the opposite direction, "unwinding" the path. At the point $O$ we, clearly, deal with a singularity whose presence seems to invalidate the whole picture. The general topological arguments do not indicate how these properties can actually be realized in the quantum mechanical context of the relevant gauge degree of freedom. Our detailed calculation will unravel the solution to this problem.

From the general properties of the canonical formalism the wave function will be shown to vanish at this point independently of any details of dynamics. Therefore, the particle motion can be considered only on one of the two circles of Fig. 3; the configurational space of the problem is a manifold *with boundaries*. This manifold will be referred to as the fundamental domain and will be discussed in more detail in Sec. II C. Here we only note that dynamics on the second circle of Fig. 3 is

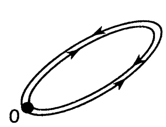


FIG. 3. Topology in the space of fields in QCD$_2^{\mathrm{adj}}$ with the gauge group SU(2).

not independent and is just a gauge replica of that of the fundamental domain.

If a single Majorana field in the adjoint representation is considered, as we shall do here, such a theory has no continuous fermion symmetries whatsoever. Indeed the real fermion field does not allow any phase rotation, neither for the left-handed nor for the right-handed components. There exist no conserved gauge-invariant (colorless) fermion currents, since

$$\mathrm{tr}(\bar\psi\gamma_\mu\psi) \equiv 0 \ \text{ and } \ \mathrm{tr}(\bar\psi\gamma_\mu\gamma_5\psi) \equiv 0\ . \tag{4}$$

We shall see later, however, that one can define a colored vector fermion current which, in a *certain gauge*, is conserved in the normal sense—its regular, not covariant, divergence vanishes. The divergence of its $\gamma_\mu\gamma_5$ partner has an anomaly looking similar to the axial anomaly in the Schwinger model (in the limit $gL \ll 1$).

With $n$ Majorana fields the Lagrangian of the model would possess a global $O(n)\times O(n)$ symmetry. Any bilinear fermion condensate would necessarily break a part of this symmetry spontaneously. Since in two dimensions spontaneous breaking of continuous symmetries cannot take place, such a model would be protected from the generation of bilinear condensates and, although essential features of the dynamics remain intact, the analysis would become more involved. This is the reason why for the time being we focus on the one-flavor case. Our model is engineered to be a prototype of QCD$_4$ with one massless quark. Multiflavor generalizations would be closer to QCD$_4$ with two or more massless quarks.

### B. Canonical quantization

We now proceed to the quantization of the system described above. Since, as in any gauge theory, we deal here with a large number of redundant (gauge) degrees of freedom, two alternative aproaches might be applied. One possibility is eliminating the gauge degrees of freedom at the classical level and then quantizing the remaining physical degrees of freedom. This is the most commonly used procedure, which, however, seems to be fraught with certain ambiguities concerning the kinetic energy of the gauge-fixed degrees of freedom (cf. Refs. [19, 27] versus Ref. [15]). The other possibility is to quantize the system in the Weyl gauge,

$$A_0 = 0, \tag{5}$$

with part of the redundant variables (the space components of $A_\mu$) subject to constraints, and to resolve the constraints at the quantum level. This second alternative has the attractive feature that the quantization is completely straightforward. We shall explain in detail how this works in the present case and how we deal with the constraints. Our final results will shed light on the consequences associated with the ambiguities of the "classical" gauge-fixing procedure and thereby help to resolve these issues.

In the Weyl gauge, the Gauss law is not an equation of motion but has to be imposed "by hand." Canonical quantization is straightforward, yielding $N^2-1$ gauge field variables $A_1^a$ and their conjugate variables, the electric fields $E^a$. The Hamiltonian for $\mathrm{QCD}_2^{\mathrm{adj}}$ is

$$H = \int_0^L dx \,\mathrm{tr}\left\{\frac{1}{2}E^2 - i\psi\sigma_3 D_1\psi\right\} . \tag{6}$$

The Gauss law is implemented as a constraint on the physical states:

$$\left\{\partial_1 E(x) - ig\left[A^1(x), E(x)\right] - g\rho(x)\right\}|\Phi\rangle = \left[D_1 E(x) - g\rho(x)\right]|\Phi\rangle = 0 . \tag{7}$$

Here $\rho$ is the fermion color charge density in the matrix representation:

$$\rho^{ij} \equiv \rho^a (t^a)_{ij} = \frac{1}{2}\left(\psi^{ik}\psi^{kj} - \psi^{kj}\psi^{ik}\right), \tag{8}$$

where the summation over the Lorentz indices of $\psi$ is implied. From now on, we always have in mind Eq. (7) when talking about physical states or the physical sector of Hilbert space.

In this framework, the technically demanding part is not the quantization but the resolution of the Gauss law. In two-dimensional QCD, this issue is particularly crucial, since the dynamics of the gauge field is almost exclusively governed by the Gauss law constraint. Various techniques to resolve the Gauss law have been advocated in the literature. Recently, a method based on unitary transformations in the Hilbert space (rather than on a change of variables in the Schrödinger representation [28]) has been developed [16, 17] and successfully applied to four-dimensional QCD on a torus, in a modified axial gauge [15]. Since the axial gauge singles out one space direction, the results of Ref. [15] can easily be adapted for $\mathrm{QCD}_2$. Without repeating all details, we briefly sketch this gauge-fixing procedure and collect the pertinent results.

In the first step, one tries to resolve the Gauss law constraint (7) for $E(x)$ and to substitute the corresponding expression for $E(x)$ in the Hamiltonian (6), restricting oneself from the outset to the physical sector. This requires inversion of the covariant derivative $D_1$ viewed as operator in color and configuration space, which is most conveniently done after diagonalization. One finds that $D_1$ has generically (i.e., for arbitrary $A^1$) $N-1$ zero modes. The projections of $E$ onto these zero modes, denoted by $\frac{1}{L}e^p, p = 1, \ldots, N-1$, are obviously not constrained by the Gauss law and remain as physical variables in the Hamiltonian; however, because of the use of a dynamical basis—the eigenfunctions of $D_1$ depend on the gauge field $A^1$—they are non-Hermitian quantum-mechanical operators. The projections of $E$ onto the nonzero modes of $D_1$ on the other hand can be eliminated in the space of physical states in favor of the (matter) color charge density, $\rho$, and $A^1$ (through the inverse of the operator $D_1'$, where the prime means that zero modes are omitted); schematically:

$$\langle\Phi|E^a E^a|\Phi\rangle = \left\langle\Phi\left|\left(\frac{1}{L^2}e^{p\dagger}e^p + 2\mathrm{tr}\, g\rho\frac{1}{D_1'}g\rho\right)\right|\Phi\right\rangle . \tag{9}$$

Note that here and throughout this paper, we shall use the letters $a$, $b$, etc., belonging to the beginning of the alphabet for the $N^2-1$ generators of the algebra, while those of the Cartan subalgebra will be denoted by $p$, $q, \ldots$. The corresponding summation conventions are implied in equations such as Eq. (9).

At this stage, the Hamiltonian still contains the full gauge field $A^1$. However, one can construct a unitary transformation, which eliminates all modes of $A^1$ except for $N-1$ spatially constant quantum-mechanical variables $a^p$, related to the eigenvalues of the path ordered exponential winding around the circle:

$$\mathcal{P}\exp\left(ig\int_0^L dx A^1(x)\right) = V e^{igaL}V^\dagger \quad (a = a^p t^p) . \tag{10}$$

This procedure is satisfactory because one can show that the constant electric fields $e^p$ (projection of $E$ onto the zero modes of $D_1$), and the zero mode "gluons" $a^q$ (closely related to the eigenvalues of $D_1$) are conjugate variables satisfying standard commutation relations

$$[e^p, a^q] = \frac{1}{i}\delta_{pq} \quad p, q = 1, \ldots, N-1 . \tag{11}$$

The resulting theory still has global constraints: The projection of the Gauss law onto the zero modes of $D_1$ yields $N-1$ residual, $x$-independent constraints, which, after the gauge-fixing unitary transformation, are reduced to the condition that the neutral fermion charges vanish:

$$Q^p|\Phi\rangle = \int_0^L dx\rho^p(x)|\Phi\rangle = 0 \qquad (p = 1, \ldots, N-1) . \tag{12}$$

One characteristic feature of this method is the fact that $e^p$ is not Hermitian, although the Hamiltonian is. This is completely analogous to the transition from Cartesian to polar coordinates in quantum mechanics, where $e^p$ would correspond to the non-Hermitian radial momentum operator $\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}/i$. In both cases, a Hermiticity defect arises when projecting the momentum operator on a dynamical, i.e., coordinate-dependent ($A^1$ and $\mathbf{r}$, respectively), basis. As in the familiar case of the polar coordinates, the kinetic energy can be rewritten with the help of an appropriate Jacobian in the form

$$K_a = \frac{1}{2L}e^{p\dagger}e^p = -\frac{1}{2L}\frac{1}{J[a]}\frac{\partial}{\partial a^p}J[a]\frac{\partial}{\partial a^p} . \tag{13}$$

The Jacobian $J[a]$, in turn, can be readily deduced from the non-Hermitian part of $e^p$. In the SU($N$) case one finds that $J[a]$ is just the Haar measure of this group,

$$J[a] = \prod_{i>j}\sin^2\left(\frac{1}{2}gL(a_i - a_j)\right) , \tag{14}$$
$$a_i = a^p t^p_{ii} \quad \text{(no summation over } i) , \quad \sum_i a_i = 0 ,$$

reflecting the transition from elements of the algebra (vector potentials) to elements of the group (path ordered exponentials) in the gauge-fixing process.

Eventually, we arrive at the following Hamiltonian describing the dynamics of the gauge variables $a^p$ and the fermion degrees of freedom in the space of physical states:

$$H = K_a + H_C + H_{\rm ferm}\ . \tag{15}$$

$K_a$ is the kinetic energy of the gauge degrees of freedom, Eq. (13). $H_C$ is the color electrostatic energy ("Coulomb energy") appearing in the elimination of the electric field by means of the Gauss law; cf. the second term on the right-hand side of Eq. (9):

$$H_C = \frac{g^2}{L}\sum_{n=-\infty}^{\infty}\sum_{i,j}\int_0^L dy\int_0^L dz\,(1-\delta_{ij}\delta_{n0})\,\frac{\rho^{ij}(y)\,\rho^{ji}(z)}{\left(\frac{2\pi n}{L}+g\,(a_i-a_j)\right)^2}e^{2i\pi n(y-z)/L}. \tag{16}$$

The fermionic part of the Hamiltonian has the form

$$H_{\rm ferm} = -i\int_0^L dx\ {\rm tr}\ \{\psi\sigma_3\,(\partial_1\psi - ig[a,\psi])\}\ . \tag{17}$$

The fermion degrees of freedom are quantized in a standard way via

$$\{\psi_\alpha^{ij}(x),\psi_\beta^{kl}(y)\} = \frac{1}{2}\delta_{\alpha\beta}\delta^{(a)}(x-y)\,[\delta_{il}\delta_{kj} - (1/N)\delta_{ij}\delta_{kl}]\ , \tag{18}$$

where the $\delta$ function $\delta^{(a)}(x-y)$ is the one appropriate for an interval of length $L$ and antiperiodic boundary conditions.

It is convenient to classify all fermion fields with respect to color degrees of freedom in the following way. First, we single out the "neutral" components of the fermion field, $\psi^p$, $p=1,...,N-1$. It is obvious that they are not coupled to the gauge degrees of freedom $a^p$. They only take part in the Coulomb interaction, which is suppressed in the small interval limit.

The off-diagonal components of the fermions are charged with respect to the neutral zero-mode gluons $a^p$. We introduce $\frac{1}{2}N(N-1)$ charged fields

$$\varphi^{ij} = \sqrt{2}\psi^{ij}\ ,\quad \varphi^{ij\dagger} = \sqrt{2}\psi^{ji}\quad (\text{for } i<j)\ , \tag{19}$$

satisfying the standard canonical anticommutation relations [cf. Eq. (18)]

$$\left\{\varphi_\alpha^{ij}(x)\,,\varphi_\beta^{kl\dagger}(y)\right\} = \delta_{\alpha\beta}\delta_{ik}\delta_{jl}\delta^{(a)}(x-y)\ . \tag{20}$$

With these definitions the fermion part of the Hamiltonian takes the form

$$H_{\rm ferm} = H_\psi + H_\varphi(a), \tag{21}$$

where

$$H_\psi = \frac{1}{2i}\int_0^L dx\psi^p\sigma_3\partial_1\psi^p, \tag{22}$$

and

$$H_\varphi(a) = \sum_{i<j}\int_0^L dx\varphi^{ij\dagger}\sigma_3\left(\frac{1}{i}\partial_1 - g(a_i-a_j)\right)\varphi^{ij}. \tag{23}$$

The theory is supplemented by the neutrality conditions (12), which in our present notation read

$$Q_i|\Phi\rangle = 0\quad (i=1,\ldots,N-1), \tag{24}$$

where [cf. Eqs. (8) and (19)]

$$Q_i = \int_0^L dx\rho^{ii} = \frac{1}{2}\int_0^L dx\left(\sum_{k>i}\varphi^{ik\dagger}\varphi^{ik} - \sum_{k<i}\varphi^{ki\dagger}\varphi^{ki}\right) \tag{25}$$

(no summation over $i$).

The appearance of the Jacobian [cf. Eqs. (13) and (14)] in the kinetic energy of the quantum-mechanical gauge degrees of freedom will turn out to be vital for the self-consistency of our quantum-mechanical solution of the problem. The eigenvalue problem with the Hamiltonian $H$ defined in Eqs. (13), (15), (16), (22), and (23) has the form

$$H|\Phi\rangle = E|\Phi\rangle,$$

where $|\Phi\rangle$ denotes a state vector in the physical space. We can perform a similarity transformation and proceed to a different wave function, which, for brevity, will be referred to as "radial":

$$\tilde\Phi[a] = \sqrt{J[a]}\Phi[a]. \tag{26}$$

In the space of the radial wave functions, the kinetic energy operator acts as

$$-\frac{1}{2L}\frac{1}{\sqrt{J}}\partial_pJ\partial_p\frac{1}{\sqrt{J}} = -\frac{1}{2L}\partial_p\partial_p + \frac{1}{2L}\frac{1}{\sqrt{J}}\left(\partial_p\partial_p\sqrt{J}\right) \tag{27}$$

(with the notation $\partial_p = \partial/\partial a^p$), i.e., as the standard Laplace operator supplemented by an "effective potential." In the present case, this effective potential turns out to be a constant

$$\frac{1}{2L}\frac{1}{\sqrt{J}}\left(\partial_p\partial_p\sqrt{J}\right) = -\frac{(gL)^2}{48}N(N^2-1), \tag{28}$$

which can simply be dropped, since it merely shifts all energy eigenvalues by the same amount. The only reminiscence of the Jacobian will then be the boundary condition for the radial wave function,

$$\tilde\Phi[a] = 0\quad \text{if}\quad J[a] = 0\ , \tag{29}$$

analogous to the condition that the radial wave function vanishes at $r=0$ when working with polar coordinates.

We also note the correspondence between the electrostatic field energy $H_C$ and the centrifugal barrier in the conventional radial Schrödinger equation. Both are singular at the points where the respective Jacobians have zeros. Indeed, the Coulomb interaction part $H_C$, Eq. (16), is formally ill defined whenever $a_i - a_j = 2\pi n/gL$, exactly where the Jacobian (14) vanishes.

Before discussing the symmetries of this theory, let us briefly comment on the alternative quantization scheme mentioned at the beginning of this section ("complete" classical gauge fixing followed by quantization). Here one starts from the observation that all nonconstant modes of $A^1$ can be "gauged away." The remaining constant modes of $A^1$, by global rotations in the color space, can be reduced to a diagonal matrix. Thus, the physical degrees of freedom in $A^1$ are the constant modes of $A^{1,p}$ [where the index $p$ runs again over $N-1$ values corresponding to the Cartan subalgebra of SU($N$)]. Moreover, $A_0$, the time component of the gauge potential, enters the Lagrangian without time derivatives. This means that it is not dynamical and can be eliminated altogether by means of the equations of motion. In this process, the Coulomb interaction is generated.

This scheme has been followed, for example, in Ref. [27] (in light-cone coordinates, but this is immaterial for the present discussion of gauge fixing). For the Schwinger model, the resulting formulation is indistinguishable from the one derived via the Weyl gauge. For $\mathrm{QCD}_2$, however, the Jacobian was missed, so that the Hamiltonian looks exactly like the one above (after going to the radial wave function and the standard form of the kinetic energy), but without the boundary condition (29) and the effective potential. We conclude that this framework is not yet completely defined, unless one is able to recover the Jacobian from some other source.

The origin of this discrepancy can easily be singled out: In $\mathrm{QED}_2$, the physical (gauge-invariant) variable on the circle is the zero mode of $A^1(x)$:

$$a = \frac{1}{L}\int_0^L dx A^1(x) \ . \tag{30}$$

Therefore, it is sufficient to simply discard the nonzero modes of $A^1$ as one would do in naively fixing the gauge to $\partial A^1/\partial x = 0$. In $\mathrm{QCD}_2$, the correct gauge-invariant variables are not the standard zero modes of $A^1$, but rather the elements of the diagonal matrix $a$ defined as [cf. Eq. (10)]

$$a = \frac{1}{igL}\ln\left[V^\dagger \mathcal{P}\exp\left(ig\int_0^L dx A^1(x)\right)V\right] \ . \tag{31}$$

One cannot keep track of the difference between Eqs. (30) and (31), which is responsible for the nontrivial Jacobian in the QCD case, if one simply drops certain color and Fourier components of the $A^1$ field at the classical level. In order to do it correctly, one presumably would have to implement the gauge condition classically as a change of variables, in a Hamiltonian framework, and then quantize the theory in curvilinear coordinates. The fact that at least in the present example one only misses a boundary condition suggests that there might be some shortcut, but we are not aware of it (some discussion relevant for this issue can be found in Refs. [22–25]). In any case, the conceptual advantage of the Weyl gauge is so tangible that it seems the most natural method within the canonical framework. Moreover, the technique of resolving the Gauss law via unitary transformations has already proven to be helpful in clarifying the physical meaning of the residual gauge symmetries in the case of Abelian theories [16, 17].

It is in order to note that the Hamiltonian emerging in this way in SU(2) $\mathrm{QCD}_2^{\mathrm{adj}}$ after the transformation (25) is *locally* indistinguishable from the Hamiltonian one gets in the Schwinger model after the gauge-fixing procedure (provided that we neglect, for the time being, the term $H_C$, which is inessential in the limit $gL \ll 1$). The "analog" Schwinger model we end up with is the model of one Dirac fermion with unit charge coupled to a neutral "photon" $a$. Similarly, the SU(3) case looks locally like a generalization of the Schwinger model to the gauge group U(1)×U(1); here, two neutral "photons" $a^3$ and $a^8$ are coupled to three Dirac fermions:

$$\varphi^{12}, \quad \varphi^{13}, \quad \text{and } \varphi^{23}.$$

The $a^3$ charges of these fermions are 1, 1/2, and −1/2, respectively, while the $a^8$ charges are 0, $\sqrt{3}/2$, and $\sqrt{3}/2$. The distinction between $\mathrm{QCD}_2^{\mathrm{adj}}$ and the Schwinger model, which is absolutely crucial, shows up only in the global properties, namely, the domain where the variables $a$ live, how different points on the corresponding manifolds are glued, etc. We proceed to a discussion of this issue.

### C. Fundamental domain, residual symmetries, and large gauge transformations

In the preceding section the dynamics of the gauge fields was reduced to that of the residual gauge variables $a^p$ by resolving the Gauss law. Let us first discuss the range of values these variables can take on. In the process of gauge fixing, the $a^p$ are only defined via group elements, namely, the eigenvalues of the path ordered integral around the circle; cf. Eq. (10). Therefore, $gLa_i$ are angular variables, which are only defined modulo $2\pi$. If we impose the two conditions that the parametrization is one to one and permutations of the eigenvalues do not lead out of the domain, then the definition of such a domain for the variables $a^p$ is unique. We will call this domain *the elementary cell.* For SU(2), it is the interval

$$-\frac{2\pi}{gL} \le a^3 \le \frac{2\pi}{gL} \tag{32}$$

with the end points identified [Fig. 4(a)]. The points where two eigenvalues of the matrix Eq. (10) cross are 0 and $\pm\frac{2\pi}{gL}$; therefore, exchanging $a_1$ with $a_2$ maps the two half intervals onto each other: $a^3 \to -a^3$ (recall that $a_1 = a^3/2, \ a_2 = -a^3/2$).

For SU(3), the corresponding construction can easily be found with the help of the standard Gell-Mann matrices $\lambda^3$ and $\lambda^8$, which yield

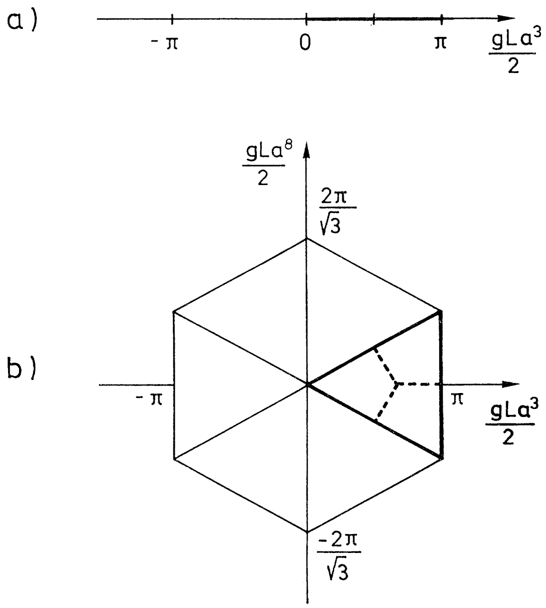


FIG. 4. Elementary cell and fundamental domain of the gauge field variables $a^p$ as defined in the text. (a) SU(2) and (b) SU(3) gauge group. The thick lines delimit the fundamental domains. The dashed lines in (b) and the midpoint of the interval $[0,\pi]$ in (a) separate regions within the fundamental domain, which are gauge copies under large gauge transformations.

$$a_1 = \frac{1}{2}\left(a^3 + \frac{1}{\sqrt{3}}a^8\right),$$
$$a_2 = \frac{1}{2}\left(-a^3 + \frac{1}{\sqrt{3}}a^8\right),$$
$$a_3 = -\frac{1}{\sqrt{3}}a^8 . \tag{33}$$

Here, two out of the three eigenvalues of the matrix equation (10) cross along the lines

$$\frac{gLa^3}{2} = n\pi \ , \quad \pm\frac{gLa^3}{2} + \sqrt{3}\frac{gLa^8}{2} = 2n\pi \quad (n \in Z) . \tag{34}$$

At the points of intersection of these lines, all three eigenvalues are equal. The resulting elementary cell is the hexagon shown in Fig. 4(b) in the $\{a^3, a^8\}$ plane. If we tile the plane with such hexagons, then shifting any of the three angles $gLa_i$ by $2\pi$ maps this domain onto another such hexagon. Our choice is singled out by the condition that permutations of the eigenvalues just map the different subtriangles onto each other, so that the full domain is invariant. Note that opposing sides of the hexagon should be identified, since they correspond to angles differing by $2\pi$.

Construction of the elementary cell is not the end of the story, however. We could have factorized it with respect to the Weyl group. In other words, we still have the freedom of performing (topologically trivial) gauge transformations, which will order the eigenvalues. For instance, in the SU(2) case, if $a^3$ is negative, we can perform a global rotation in the color space by $\pi$ around the first axis. This rotation maps the interval $-2\pi/(gL) < a^3 < 0$ onto the interval $0 < a^3 < 2\pi/(gL)$ (simultaneously, $\varphi^{12} \leftrightarrow \varphi^{21}$). In this way we arrive at the notion of fundamental domain, a domain of variation of $a$'s emerging after we use all gauge freedom residing in the topologically trivial gauge transformations. Further reduction of this domain (identification of points that are gauge images of one another) is possible only by invoking "large" (topologically nontrivial) gauge transformations.

Generically, if we specify the gauge by prescribing the ordering of the eigenvalues, we restrict ourselves to the fundamental domain smaller than the elementary cell by a factor of $(1/N!)$, namely, half the interval in SU(2) and one equilateral triangle in SU(3). In fact, these fundamental domains are separated by points [in SU(2)] or lines [in SU(3)] of the vanishing Jacobian; hence, they are dynamically decoupled anyway. This singles out these smaller, *fundamental* domains as the relevant ones, and we choose them as (see Fig. 4)

$$0 \le a^3 \le \frac{2\pi}{gL} \qquad \text{in SU(2) and SU(3)},$$
$$-\frac{1}{\sqrt{3}}a^3 \le a^8 \le +\frac{1}{\sqrt{3}}a^3 \qquad \text{in SU(3)}. \tag{35}$$

The fact that these fundamental domains are a closed interval or triangle guarantees the existence of discrete spectra and normalizable wave functions, in contrast with the case of polar coordinates.

We now turn to the formal symmetries of our Hamiltonian. For this discussion it is actually advantageous not to restrict the range of variables as described above but rather work with variables of the covering space. As shown in Ref. [15], these symmetries are the leftovers of the original local gauge symmetry in a gauge-fixed formulation. Since these residual symmetries can easily be verified in the present case, we only summarize the results.

Displacements $T_D(\mathbf{k})$:

$$a_i \to a_i + \frac{2\pi}{gL}k_i \qquad \left(\sum_i k_i = 0, k_i \in Z\right),$$
$$\varphi^{ij} \to e^{2\pi i(k_i - k_j)x/L}\varphi^{ij} \ , \quad \psi^p \to \psi^p. \tag{36}$$

Central conjugations $T_C(n)$:

$$a_i \to a_i + \frac{2\pi}{gL}\nu_i \quad [\nu_i = n(1/N - \delta_{iN}),\ n = 1, \ldots, N-1],$$
$$\varphi^{ij} \to e^{2\pi i(\nu_i - \nu_j)x/L}\varphi^{ij} \ , \quad \psi^p \to \psi^p. \tag{37}$$

Permutations of the basis of color $[P(1)\cdots P(N)]$:

$$a_i \to a_{P(i)} \; , \quad \psi^{ij} \to \psi^{P(i)P(j)}. \tag{38}$$

Global, color diagonal transformations:

$$a_i \to a_i \; , \quad \varphi^{ij} \to e^{i(\beta_i-\beta_j)}\varphi^{ij} \; , \quad \psi^p \to \psi^p, \tag{39}$$

where $\beta_i$ are real numbers. The permutations (38) are awkward to write down in the basis of $\psi^p, \varphi^{ij}$ for general $N$, so that we have used here the original fermion fields $\psi^{ij}$ instead. For SU(2) and SU(3), the conversion can easily be made if needed. The global symmetry (39) reflects the fact that the residual neutrality condition (24), the remnant of the original Gauss law constraint, has not yet been implemented, and trivially reduces to **1** in the space of physical states.

It is instructive to confront these formal symmetries of the system when using variables in the covering space with the choice of the fundamental domain. The displacements (36) shift the fundamental domain to some other possible domain, since an angle $gLa_i$ gets shifted by $2\pi$. The permutations have already been discussed. The central conjugations amount to a shift of $a^3$ by $2\pi/(gL)$ (half the length of the elementary cell) for SU(2), and to a shift of $a^8$ by $4\pi/(\sqrt{3}gL)$ (half the "diameter" of the hexagon) for SU(3). Clearly, restriction of the variables to the fundamental domain renders most of these symmetries not very useful at this stage. How this is resolved can best be explained with the help of the concrete examples below.

As a last item, we discuss the large gauge transformations and their relation to the symmetries of our Hamiltonian. First consider SU(2). As discussed in Sec. II A, the elements of the center of the group act trivially if the fermions are in the adjoint representation of the group. The gauge group is actually not SU(2) but, rather, $SU(2)/Z_2$ [or SO(3)]. Thus, we need to discuss mappings of the (spatial) circle onto $SU(2)/Z_2$. As is well known, $\pi_1[SU(2)/Z_2] = Z_2$; i.e., there exist matrices $U(x)$ defined on the circle and not continuously deformable to the unit matrix. It is easy to construct an example of such a matrix. Indeed, the group space of $SU(2)/Z_2$ is the three-sphere $S^3$ with the end points of any diameter identified. A mapping of the circle starting from the south pole of the sphere, going along a meridian and ending up at the north pole is, therefore, allowed, and it obviously cannot be continuously deformed to the trivial mapping: for example,

$$U(x) = e^{i\frac{\pi x}{L}\sigma^3} \; . \tag{40}$$

More generally, all $U(x)$ can be classified according to the relation

$$U_\pm(L) = \pm U_\pm(0) \; . \tag{41}$$

All transformations $U_-$ are topologically nontrivial, but can be deformed into each other by small gauge transformations $U_+$. The square of any $U_-$ clearly belongs to the topologically trivial class.

It is obvious how to generalize this to SU($N$): The gauge group is $SU(N)/Z_N$. Topologically nontrivial transformations are just the gauge transformations differing by a nontrivial center element if one goes around the circle,

$$U_z(L) = zU_z(0) \; , \tag{42}$$

where $z$ is the $n$th root of unity ($\neq 1$):

$$z = -1 \;\; \text{for SU(2)} \; , \quad z = e^{i2\pi/3}\; , e^{i4\pi/3} \;\; \text{for SU(3)} \; . \tag{43}$$

How does $a^3$ behave under these gauge transformations? If we go back to the defining relation (10) and perform a local gauge transformation $U(x)$, we get

$$\begin{aligned} \mathcal{P}\exp\left(ig\int_0^L dx A^1(x)\right) & \\ &\to U(L)\mathcal{P}\exp\left(ig\int_0^L dx A^1(x)\right)U(0) \\ &= U(L)Ve^{igaL}V^\dagger U(0) \; . \end{aligned} \tag{44}$$

Together with Eq. (41), this yields

$$Ve^{igaL}V^\dagger \to \pm U_\pm(0)Ve^{igaL}V^\dagger U_\pm^\dagger(0) \; , \tag{45}$$

where the plus sign holds for small and the minus sign for large gauge transformations. It is tempting to conclude from this that $a^3$ gets shifted by $\frac{4\pi}{gL}$ or $\frac{2\pi}{gL}$. The detailed transformation property depends on the precise definition of the fundamental domain. Since one only knows how the exponential transforms, one cannot infer from that how the exponent transforms, unless one fully specifies how to take the logarithm. Furthermore, the ordering of the eigenvalues is involved. Fortunately, it will turn out that the present formalism projects out the relevant symmetry without the necessity of submerging into these subtle issues. Moreover, the gauge transformations gluing the points at the boundaries of the fundamental domains are easily identifiable, both for SU(2) and SU(3).

Repeating a similar analysis for SU($N$) yields the transformation property

$$Ve^{igaL}V^\dagger \to zU_z(0)Ve^{igaL}V^\dagger U_z^\dagger(0) \tag{46}$$

under gauge transformations belonging to the homotopy class labeled by the center element $z$. Because of the restriction to the fundamental domain, it is again difficult to directly read off the transformation properties of the $a^p$. The presence of the Jacobian allows us to proceed without detailed knowledge of this.

The preceding discussion refers to the Weyl gauge, where we still have the freedom of local, time-independent gauge transformations. The question arises what happens to the gauge transformations after resolution of the Gauss law, and how they are connected to the formal symmetries of the gauge-fixed Hamiltonian.

On general grounds, one would expect the following: Topologically trivial gauge transformations can be generated by the Gauss law operator and should be reduced to **1** in the physical space. All nontrivial transformations $U_z(x)$ belonging to a given $z$ are deformable into each other and therefore equivalent in the physical space. Thus, after gauge fixing, we expect a global residual symmetry with $Z_N$ group structure as the only remnant of the original local gauge symmetry, directly exhibiting the topology of the large gauge transformations.

How do the local gauge transformations $U(x)$ behave under gauge fixing? In $QED_4$ or the Schwinger model, it is indeed easy to follow this explicitly [16, 17]. It was found that the residual symmetry [here only a U(1) displacement symmetry] corresponds directly to the homotopically nontrivial gauge transformations. In QCD, the situation is less clear [15]. One can show that any gauge transformation is reduced in the process of gauge fixing to a residual symmetry transformation, but the connection between the original gauge function and the particular residual symmetry transformation could not be established in all details due to technical difficulties. In addition, the large number of formal symmetries of the type (36)–(39) does not seem to match the expected group structure ($Z$ for the case of $QCD_4$, $Z_N$ for the case of $QCD_2^{adj}$). The present model will enable us to clarify this issue. Rather than trying to do it formally, we proceed to concrete applications to SU(2) and SU(3), where the general theoretical expectations will indeed be confirmed.

## III. THE STRUCTURE OF THE VACUUM STATE

Once the quantum-mechanical reduction of $QCD_2^{adj}$ is completed we can proceed to determine the wave function of the vacuum state. The construction can be carried out in the Born-Oppenheimer approximation: We first freeze the variables $a^p$ and calculate the effective potential energy as a function of $a^p$ by "integrating out the fermion degrees of freedom." Then we study quantum mechanics of the variables $a^p$ living on the manifolds specified above. At this point the boundary conditions on $\tilde{\Phi}$, the vanishing of the "radial" wave function at zeros of the Jacobian, become important. The use of the Born-Oppenheimer approximation is justified *a posteriori*, since the typical frequencies of the quantum-mechanical variables $a^p$ are of order $g$, while those characteristic of the fermion degrees of freedom are of order $1/L$, (almost) everywhere inside the fundamental domain. If $gL \ll 1$, as we shall assume, the fermion frequencies are much larger than those referring to $a^p$. The Born-Oppenheimer approximation and the effective quantum-mechanical description in terms of $a^p$ may fail only near exceptional points where the fermion levels cross the zero-energy mark. These points exist and play their role in our analysis. They will be discussed in due course.

The general strategy in calculating the effective $a^p$-dependent potential is the same as the one usually applied in the Hamiltonian approach to the Schwinger model; see, e.g., Ref. [10]. For each given value of $a^p$ we fill the Dirac sea appropriately, so that all negative energy levels are occupied and all positive energy levels are free, and then we find the energy of the Dirac sea thus constructed. Since some of the fermion levels cross the zero-energy mark at certain values of $a^p$ inside the fundamental domain, the Dirac sea must be restructured correspondingly. Hence, in different sectors of the fundamental domain the fermion component of the wave function is different, and the effective potential energy for the $a^p$'s is also different. The full vacuum wave function is a linear combination of the wave functions in different sectors. We shall explicitly construct the fermion component in each of the sectors, find the effective potential, and solve the Schrödinger equation in the variables $a^p$. Two specific cases, SU(2) and SU(3), will be considered separately.

### A. SU(2): Vacuum, symmetries, and fermion condensate

The physics content of the preceding formal developments is first exhibited by application to the technically simple case of SU(2) $QCD_2^{adj}$ on a small interval ($gL \ll 1$). In this limit, the relevant part of the Hamiltonian is that of the charged fermion, neutral gluon system of Eqs. (13) and (23):

$$\tilde{H}_{\varphi,a} = \int_0^L dx \varphi^\dagger \sigma_3 \left(\frac{1}{i}\partial_1 - ga\right)\varphi - \frac{1}{2L}\frac{\partial^2}{\partial a^2}, \quad (47)$$

where we have denoted the fermion and gauge-field degrees of freedom by

$$\varphi = \varphi^{12} \quad , \quad a = a^3 = a_1 - a_2 \quad (48)$$

to ease the notation. This Hamiltonian acts on the space of radial wave functions satisfying the constraint

$$\tilde{\Phi}\,(a = 2n\pi/gL) = 0\ . \quad (49)$$

The neutrality condition [cf. Eq. (24)] reads

$$Q^3|\Phi\rangle = (Q_1 - Q_2)\,|\Phi\rangle = \int_0^L dx \varphi^\dagger\varphi|\Phi\rangle = 0\ . \quad (50)$$

The formal symmetry transformations leaving the Hamiltonian invariant [cf. Eqs. (36)–(39)] can be reduced here to shifts of the gauge degrees of freedom accompanied by phase changes of the charged fermions (central conjugations),

$$T_C(n): \qquad a \to a + \frac{2\pi n}{gL} \quad , \quad \varphi \to e^{2i\pi n x/L}\varphi, \quad (51)$$

and reflections accompanied by charge conjugations (interchange of color labels 1,2):

$$R: \qquad a \to -a \quad , \quad \varphi \to \varphi^\dagger \quad (52)$$

[we ignore the global gauge transformations (33), which play no role in the physical sector; displacements $T_D(n,-n)$, on the other hand, are the same as $T_C(2n)$ for SU(2)]. We now proceed to determine in the small interval limit the ground state of this system. The normal mode expansion

$$\varphi(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} e^{ip_k x} \quad , \quad p_k = \frac{2\pi}{L}\left(k + \frac{1}{2}\right) \tag{53}$$

yields, for the Hamiltonian $\tilde{H}_{\varphi,a}$ [Eq. (47)] and the charge $Q^3$ [Eq. (50)],

$$\tilde{H}_{\varphi,a} = \sum_{k=-\infty}^{\infty} \left(a_k^\dagger a_k - b_k^\dagger b_k\right)(p_k - ga) - \frac{1}{2L}\frac{\partial^2}{\partial a^2}, \tag{54}$$

$$Q^3 = \sum_{k=-\infty}^{\infty} \left(a_k^\dagger a_k + b_k^\dagger b_k\right) . \tag{55}$$

In the adiabatic approximation, the ground state of the fast fermion degrees of freedom for a fixed value of the slow gauge degree of freedom is obtained by filling the fermion negative energy states. Assuming the $a$ modes to be filled for $k < M$ and the $b$ modes for $k \geq M'$ the (heat kernel regularized) charge of such a state is given by

$$Q^3|M,M'\rangle = \lim_{\lambda\to 0}\left(\sum_{k=-\infty}^{M-1} e^{\lambda(p_k - ga)} + \sum_{k=M'}^{\infty} e^{-\lambda(p_k - ga)}\right)|M,M'\rangle = (M - M')\,|M,M'\rangle . \tag{56}$$

Neutrality (50) requires the Fermi levels to satisfy

$$M' = M . \tag{57}$$

The regularized energy of such a neutral state is

$$\lim_{\lambda\to 0}\sum_{k=-\infty}^{\infty}\left(a_k^\dagger a_k e^{\lambda(p_k - ga)} - b_k^\dagger b_k e^{-\lambda(p_k - ga)}\right)(p_k - ga)\,|M,M\rangle = U_M(a)\,|M,M\rangle ,$$

$$U_M(a) = \frac{2\pi}{L}\left(M - \frac{gLa}{2\pi}\right)^2 . \tag{58}$$

As is well known from similar treatments of the Schwinger model or of QCD with fermions in the fundamental representation [27] the adiabatic ground state of the system is obtained by adjusting the occupation of the fermionic neutral states to the variations in the gauge degree of freedom. Obviously $U_M(a)$ is minimal with the following choice of $M$:

$$\left|M(a) - \frac{gLa}{2\pi}\right| \leq \frac{1}{2} . \tag{59}$$

The occupation changes whenever a pair of fermion levels crosses zero energy:

$$gLa = 2\pi\left(n + \frac{1}{2}\right) . \tag{60}$$

In this way, a periodic potential energy of the gauge degrees of freedom,

$$U(a) := U_{M(a)}(a) , \tag{61}$$

[where $M(a)$ is defined implicitly via Eq. (59)] is obtained (Fig. 5). The analogy with the Schwinger model with its discrete translational symmetry generated by large gauge transformations (topology of a circle) is only superficial. Physics consequences are very different due to the presence of the constraint (49) on the radial wave function of QCD. This constraint requires solution of the Schrödinger equation in one fundamental interval defined by two consecutive zeros of the wave function. Because of the translational symmetry of $U$, we therefore can restrict the calculations to the fundamental domain discussed in Sec. II C:

$$0 \leq gLa \leq 2\pi . \tag{62}$$

Then, the $k = 0$ fermion $a$ and $b$ modes cross zero energy in the middle, at $gLa = \pi$.

As the characteristic property of $QCD_2$ with fermions

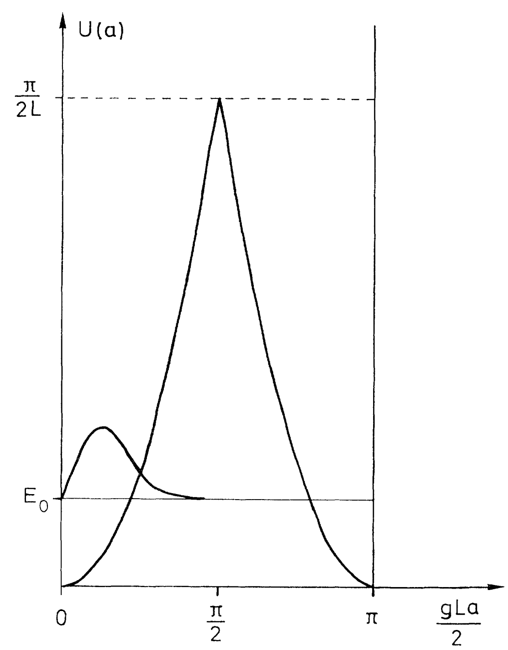


FIG. 5. Adiabatic potential for the gauge degree of freedom in the SU(2) case; cf. Eq. (61). Also shown is the ground-state wave function $\tilde{\Phi}_I(a)$, Eq. (63), in one of the classical minima, for the value $gL = 1/3$ (arbitrary units).

in the adjoint representation, we note the reflection symmetry of U($a$) around $gLa = \pi$. The double-well structure yields a twofold degenerate ground state in the semiclassical limit with ground-state wave functions given by

$$\tilde{\Phi}_{\rm I}\,(a) = \frac{2(gL)^{3/4}}{\pi^{5/8}}\, a \,\exp\left\{-\frac{gL}{2\sqrt{\pi}}a^2\right\} \ ,$$
$$\tilde{\Phi}_{\rm II}\,(a) = \tilde{\Phi}_{\rm I}\left(\frac{2\pi}{gL} - a\right) \ , \tag{63}$$

corresponding to localization close to one of the two minima of the potential energy (see Fig. 5). As a further consequence of the Jacobian in the kinetic energy of $a$, we note the vanishing of the wave function at the minima of the potential energy. The ground-state energy is

$$E_0 = \frac{3}{2}\frac{g}{\sqrt{\pi}} \ . \tag{64}$$

It is instructive to compare this result with the one obtained for fermions in the fundamental representation. The result for the potential energy in the corresponding adiabatic approximation is

$$U^f\,(a) = \frac{4\pi}{L}\left(M - \frac{gLa}{4\pi}\right)^2 , \tag{65}$$

while the constraint (49) remains unchanged. The effective reduction in the coupling constant by a factor of 2 follows directly from the difference in the minimal substitution, which couples fermions and gauge degrees of freedom via $a_1$ and $a_2 = -a_1$ for fundamental and via $a_2 - a_1$ for adjoint fermions. As a consequence of this reduced coupling $U^f\,(a)$ depends on $a$ monotonically in the fundamental interval and no degeneracy in the ground state occurs.

At this point the origin of the symmetry of the adiabatic potential energy in one fundamental interval remains to be understood. The discrete translational symmetry of the potential energy $U\,(a)$ of Eq. (61),

$$U\left(a + \frac{2n\pi}{gL}\right) = U(a), \tag{66}$$

and the reflection symmetry

$$U(-a) = U(a) \tag{67}$$

around $a = 0$ are of no direct relevance, since they relate wave functions in dynamically decoupled intervals. These symmetries taken together, however, entail

$$U\left(\frac{\pi}{gL} + a\right) = U\left(\frac{\pi}{gL} - a\right), \tag{68}$$

i.e., a reflection symmetry around the center of the fundamental interval. Although derived in the adiabatic approximation, it is straightforward to generalize this argument and to identify an exact and relevant symmetry for the system under consideration. We start with the formal and, in the above sense, irrelevant symmetries of Eqs. (51) and (52). The particular combination of a central conjugation and a reflection,

$$S = T_C(1)R: \quad SaS^\dagger = -a + \frac{2\pi}{gL} \ ,$$
$$S\varphi S^\dagger = e^{2i\pi x/L}\varphi^\dagger, \tag{69}$$

connects states with the gauge degrees of freedom restricted to the fundamental interval and therefore represents a relevant symmetry of the system. This operator can be chosen to satisfy

$$S^2 = 1 \quad \text{and} \quad [S, H] = 0, \tag{70}$$

and the stationary states can be classified generally as symmetric and antisymmetric ones. $S$ transforms the wave functions $\tilde{\Phi}_{\rm I}(a)$ and $\tilde{\Phi}_{\rm II}(a)$ into each other; cf. Eq. (63). The action of $S$ on the fermionic states $|M,M\rangle$ [cf. Eq. (57)] can easily be calculated:

$$S|M,M\rangle = e^{i\alpha_M}|-M+1,-M+1\rangle \ . \tag{71}$$

Here, the phase $e^{i\alpha_M}$ is not determined by the above properties of $S$, except for the condition

$$e^{i\alpha_M} = e^{-i\alpha_{-M+1}}, \tag{72}$$

which follows from $S^2 = 1$. Using only the fermionic part of $S$, this implies the reflection symmetry of the adiabatic potential:

$$\langle M,M|H_\varphi\,(a)\,|M,M\rangle = \langle -M+1,-M+1|H_\varphi\left(-a + \frac{2\pi}{gL}\right)|-M+1,-M+1\rangle \ . \tag{73}$$

For fermions in the fundamental representation, also both types of formal symmetries are present but the displacements in the gauge variable $a$ occur in twice as large steps $a \to a + \frac{4\pi}{gL}$. As a consequence no relevant symmetry can be constructed by combining displacements and reflections.

Finally we are able to write down the full vacuum state vector, which we take to be an eigenstate of the symmetry operator $S$ with eigenvalue $\pm 1$:

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\left\{|0,0\rangle|\tilde{\Phi}_{\rm I}\rangle \pm e^{i\alpha_0}|1,1\rangle|\tilde{\Phi}_{\rm II}\rangle\right\} \ , \tag{74}$$

where $\alpha_0$ is the phase introduced in Eq. (71).

Thus, we arrive at two vacua: one is described by $|\Psi_+\rangle$, the other by $|\Psi_-\rangle$. The two linear conbinations in Eq. (74) are the $Z_2$ analog of the $\theta$ vacua of $\rm QCD_4$, where the relevant symmetry is $Z$. The necessity of superimposing the states I and II in the vacuum wave function, Eq. (74), is due to the fact that only under this choice the property of the cluster decomposition will be satisfied. Moreover, if a fermion mass term $m\bar{\psi}\psi$ is introduced as a small perturbation, physics will be smooth in $m$ under the choice (74). On the contrary, had we chosen the states I and II as the vacuum states, introducing a small mass term would result in a drastic restructuring.

With these considerations we have succeeded in clarifying the symmetry properties of [SU(2)] $\mathrm{QCD}_2^{\mathrm{adj}}$ in the canonical formulation with the unconstrained degrees of freedom. The crucial element in the construction, valid beyond this particular model, is the emergence of relevant symmetry transformations by combining a sequence of irrelevant ones.

Using the above expression for the vacuum it is straightforward to calculate the fermion condensate $\langle\bar{\psi}^a\psi^a\rangle$. The contribution from the charged fermions to the condensate operator is

$$\mathcal{C} = \frac{1}{L}\int_0^L dx\left(\varphi^\dagger\gamma^0\varphi + \varphi\gamma^0\varphi^\dagger\right) = \frac{2i}{L}\sum_k\left(b_k^\dagger a_k - a_k^\dagger b_k\right) \tag{75}$$

(the neutral fermions $\psi^3$ do not contribute in this approximation). This yields the condensate

$$\langle\Psi_\pm|\bar{\psi}^a\psi^a|\Psi_\pm\rangle = \pm\frac{2}{L}\sin\alpha_0\langle\tilde{\Phi}_{\mathrm{I}}|\tilde{\Phi}_{\mathrm{II}}\rangle \tag{76}$$

with the overlap of the two localized wave functions of the gauge degree of freedom given by

$$\int_0^{2\pi/gL} da\tilde{\Phi}_{\mathrm{I}}(a)\,\tilde{\Phi}_{\mathrm{II}}(a) = \frac{4\pi^{3/2}}{gL}\exp\left(-\frac{\pi^{3/2}}{gL}\right). \tag{77}$$

In addition to the expected nonperturbative exponential suppression ($gL \ll 1$) the overlap (77) is actually enhanced by an inverse power of $gL$ in the prefactor. This is a direct consequence of the fact that $\tilde{\Phi}$ has nodes at the boundaries of the fundamental domain. As a consequence of the constraint (49) on the radial wave function, the system is pushed away from the minima of the potential energy into the classically forbidden region of the potential barrier. The result in Eq. (77) agrees with the result obtained previously in Ref. [11] by functional integration in the Euclidean formulation of $\mathrm{QCD}_2^{\mathrm{adj}}$ as far as the exponent and the $gL$ dependence of the prefactor are concerned. Without the Jacobian, we would have missed the factor $(gL)^{-1}$.

The value of the condensate depends explicitly on the choice of phases. To clarify this dependence we observe that in addition to the scalar condensate one might also use the pseudoscalar condensate to characterize the ground state. The value of this condensate is

$$\langle\Psi_\pm|\bar{\psi}^a\gamma^5\psi^a|\Psi_\pm\rangle = \pm\frac{2i}{L}\cos\alpha_0\langle\tilde{\Phi}_{\mathrm{I}}|\tilde{\Phi}_{\mathrm{II}}\rangle\,, \tag{78}$$

which therefore can be combined with the scalar condensate to form a complex quantity with the modulus being independent of the phase $\alpha_0$. The arbitrariness in the phase can be removed by adding a weak disturbance to the Hamiltonian, which will lift the twofold degeneracy [Eq. (74)]. Choosing this disturbance to be proportional to a scalar (mass) term ($\propto\bar{\psi}^a\psi^a$) or a pseudoscalar term ($\propto\bar{\psi}^a\gamma^5\psi^a$) fixes $\alpha_0$ to be $\pm\pi/2$ and $\pi$, respectively.

### B. Corrections to the adiabatic approximation

We discuss here the corrections to the small interval limit. The Coulomb energy [Eq. (16)] acts as a centrifugal barrier on the gauge degrees of freedom, which, in general, are therefore prevented, beyond the effect of the Jacobian, to approach the points $gLa = 2n\pi$. The relevant part of the Coulomb energy is determined by the off-diagonal color charge densities, which in turn contain both charged and neutral fermion fields. In the adiabatic, small interval limit the ground state of the neutral fermions is that of the noninteracting system. The normal mode expansion of the Hermitian neutral field is

$$\psi^3(x) = \frac{1}{\sqrt{L}}\sum_{k\geq 0}\left[\left(c_k e^{ip_k x} + c_k^\dagger e^{-ip_k x}\right)\begin{pmatrix}1\\0\end{pmatrix} + \left(d_k e^{ip_k x} + d_k^\dagger e^{-ip_k x}\right)\begin{pmatrix}0\\1\end{pmatrix}\right]. \tag{79}$$

The free field Hamiltonian (22) reads, in terms of these creation and annihilation operators,

$$H_\psi = \sum_{k\geq 0} p_k\left(c_k^\dagger c_k - d_k^\dagger d_k\right), \tag{80}$$

and therefore the ground state satisfies

$$c_k|0_\psi\rangle = 0, \quad d_k^\dagger|0_\psi\rangle = 0, \quad k\geq 0\,. \tag{81}$$

With the choice (62) of the fundamental interval, the Coulomb energy becomes singular if, for $n = 0$, 1, or $-1$, certain elements of the matrix

$$\tilde{\rho}^{ij}(n) = \int_0^L dx e^{2i\pi nx/L}\rho^{ij}(x) \tag{82}$$

are nonvanishing when acting on the ground state. For the case of SU(2) in particular, the relevant part of the Coulomb energy is

$$\frac{g^2}{L}\sum_n\frac{\tilde{\rho}^{21}(n)\tilde{\rho}^{12}(-n) + \tilde{\rho}^{12}(-n)\tilde{\rho}^{21}(n)}{(2\pi n/L - ga)^2}, \tag{83}$$

so that the potentially dangerous terms are $\tilde{\rho}^{12}(0)$, $\tilde{\rho}^{21}(0)$, $\tilde{\rho}^{12}(-1)$, and $\tilde{\rho}^{21}(1)$.

Using the normal mode expansions [Eqs. (53) and (79)], the off-diagonal charge is given by

$$\tilde{\rho}^{12}(0) = \sum_{k\geq 0}\left(a_{-k-1}c_k + a_k c_k^\dagger + b_{-k-1}d_k + b_k d_k^\dagger\right), \tag{84}$$

with $\tilde{\rho}^{21}(0) = \left[\tilde{\rho}^{12}(0)\right]^\dagger$. Explicit calculation shows the direct product of the neutral states, $|M, M\rangle$ with $M = 0$ [cf. Eqs. (56) and (57)] and $|0_\psi\rangle$ of Eq. (81) to be an eigenstate of the off-diagonal charge with a vanishing eigenvalue:

$$\tilde{\rho}^{12}(0)\,|0,0;0_\psi\rangle = \tilde{\rho}^{21}(0)\,|0,0;0_\psi\rangle = 0\,; \tag{85}$$

i.e., the adiabatic ground state is a singlet state with respect to the fermionic color charge. Under reflections generated by $S$, the off-diagonal charges transform as

$$S\tilde{\rho}^{12}(0)\,S^{\dagger} = -\tilde{\rho}^{21\dagger}(1)\,,$$
$$S\tilde{\rho}^{21}(0)\,S^{\dagger} = -\tilde{\rho}^{12\dagger}(-1)\,, \tag{86}$$

which, together with the transformation properties of the states [cf. Eq. (71)], yield

$$\tilde{\rho}^{21}(1)\,|1,1;0_{\psi}\rangle = \tilde{\rho}^{12}(-1)\,|1,1;0_{\psi}\rangle = 0\,. \tag{87}$$

Thus in the evaluation of the adiabatic ground-state expectation value of the color electrostatic energy (16),

$$\theta\,(\pi - gLa)\,\langle 0,0;0_{\psi}|H_C|0,0;0_{\psi}\rangle + \theta\,(gLa - \pi)\,\langle 1,1;0_{\psi}|H_C|1,1;0_{\psi}\rangle, \tag{88}$$

the residues of the singularities in the Laurent expansion in $gLa$ are zero and the contribution to the energy is $\propto g^2L$ and therefore negligible in the small interval limit. For finite interval length, vanishing fermionic color charges are no longer energetically favored, and, as a result of the electrostatic centrifugal barrier, (radial) wave function components will be admixed, which vanish faster than required by the Jacobian (this effect has been demonstrated to actually occur for static color charges in the fundamental representation [29]).

### C. The vacuum in the SU(3) case

The calculation of the vacuum structure in the adiabatic approximation described above for the SU(2) case can be generalized to SU($N$). We start from the $\frac{1}{2}N(N-1)$ charged fermion fields $\varphi^{ij}(i<j)$ introduced in Eq. (19) and the normal mode expansion analogous to Eq. (53):

$$\varphi^{ij}(x) = \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty}\begin{pmatrix} a_k^{ij} \\ b_k^{ij}\end{pmatrix} e^{ip_kx} \quad , \quad p_k = \frac{2\pi}{L}\left(k+\frac{1}{2}\right)\,. \tag{89}$$

The adiabatic fermion ground state is assumed to have the "$a^{ij}$ modes" filled for $k < M_{ij}$ and the "$b^{ij}$ modes" filled for $k \geq M'_{ij}$. These states are eigenstates of $H_\varphi(a)$, Eq. (23), with regularized eigenvalues

$$H_{\varphi}(a)\,|\mathbf{M},\mathbf{M}'\rangle = \frac{\pi}{L}\sum_{i<j}\left[\left(M_{ij} - \frac{gL}{2\pi}(a_i - a_j)\right)^2 + \left(M'_{ij} - \frac{gL}{2\pi}(a_i - a_j)\right)^2\right]|\mathbf{M},\mathbf{M}'\rangle \tag{90}$$

($\mathbf{M},\mathbf{M}'$ denote $[N(N-1)/2]$-dimensional vectors with components $M_{ij}, M'_{ij}, i<j$). For given values of the gauge variables $a_i$, the choice

$$M_{ij} = M'_{ij}\ , \quad \left|M_{ij}(a) - \frac{gL}{2\pi}(a_i - a_j)\right| \leq \frac{1}{2}\ , \quad i<j \tag{91}$$

yields the lowest eigenvalue, i.e., the effective potential. Of course, we have to verify that these states satisfy the neutrality condition (24). Exactly as in the SU(2) case, one first shows that (after regularization)

$$\int_0^L dx\varphi^{ij\dagger}\varphi^{ij}|\mathbf{M},\mathbf{M}'\rangle = \left(M_{ij} - M'_{ij}\right)|\mathbf{M},\mathbf{M}'\rangle\ , \tag{92}$$

where the indices $(i,j)$ are kept fixed. For the state of lowest energy, the right-hand side vanishes. Hence, in the expression (25) for the neutral charge operators $Q_i$, each term in the sum vanishes separately when acting on the adiabatic ground state, so that this latter is indeed a physical state.

Although this part of the calculation can be done trivially for arbitrary $N$, the following step, namely, solving the Schrödinger equation in the gauge variables and discussing in detail the symmetries, condensates, spectral flow, etc., quickly becomes complicated due to the increasing number of variables and the geometrical intricacies associated with the fundamental domain. For our purpose it is more useful to exhibit in detail the results for SU(3). Together with the SU(2) case, this provides enough understanding to be able to foresee what will happen for larger $N$, at least qualitatively.

For the explicit calculation in SU(3), it is preferable to change from the diagonal matrix elements $a_i$ of the gauge degrees of freedom to the neutral amplitudes $a^3, a^8$ [cf. Eq. (33)], which are the natural variables for the kinetic energy term of the gauge field and, more importantly, the independent ones. The Schrödinger equation in the radial form now corresponds to a quantum-mechanical problem on a plane,

$$\left[-\frac{1}{2L}\left(\frac{\partial^2}{\partial(a^3)^2} + \frac{\partial^2}{\partial(a^8)^2}\right) + U_{\text{eff}}(a^3,a^8)\right]\tilde{\Phi}(a^3,a^8) = E\tilde{\Phi}(a^3,a^8), \tag{93}$$

with the condition that $\tilde{\Phi}$ vanishes along the boundary of the fundamental triangle [cf. Fig. 4 and Eq. (35)]. The effective potential is given by

$$U_{\text{eff}}\left(a^3,a^8\right) = U\left(a^3\right) + U\left(\frac{1}{2}(a^3+\sqrt{3}a^8)\right) + U\left(\frac{1}{2}(-a^3+\sqrt{3}a^8)\right) \tag{94}$$

with $U(a)$ defined in Eqs. (59)–(61). As indicated in Fig. 6, we can further subdivide the fundamental domain into four congruent triangles, in each of which the adiabatic potential is that of an isotropic, two-dimensional harmonic oscillator:

$$U^{\text{I}}\left(a^3,a^8\right) = \frac{3g^2L}{4\pi}\left[(a^3)^2+(a^8)^2\right], \tag{95}$$

$$U^{\text{II}}\left(a^3,a^8\right) = U^{\text{I}}\left(a^3-\frac{2\pi}{gL}, a^8-\frac{2\pi}{\sqrt{3}gL}\right),$$
$$U^{\text{III}}\left(a^3,a^8\right) = U^{\text{I}}\left(a^3-\frac{2\pi}{gL}, a^8+\frac{2\pi}{\sqrt{3}gL}\right),$$
$$U^{\text{IV}}\left(a^3,a^8\right) = U^{\text{I}}\left(a^3-\frac{4\pi}{3gL}, a^8\right)+\frac{2\pi}{3L}. \tag{96}$$

Generalizing the reflection symmetry of the SU(2) case, this potential also exhibits discrete symmetries: It is invariant under rotations around the center of the fundamental triangle by 120° or 240°, as well as under reflections with respect to three lines joining its center with each corner (see Figs. 7 and 8).

As in the SU(2) case, the discrete rotational symmetry is a property of the exact theory, valid beyond the adiabatic approximation. We can follow the analysis of the SU(2) case. As emphasized above, the formal residual

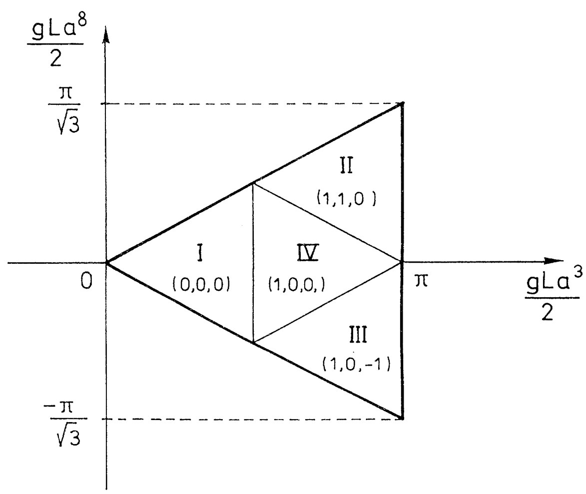


FIG. 6. Fundamental domain of the gauge field variables in SU(3). The four subtriangles I-IV are the regions in which the Dirac sea has a particular filling. The numbers in parentheses stand for $(M_{12}, M_{13}, M_{23})$, the corresponding Fermi levels.

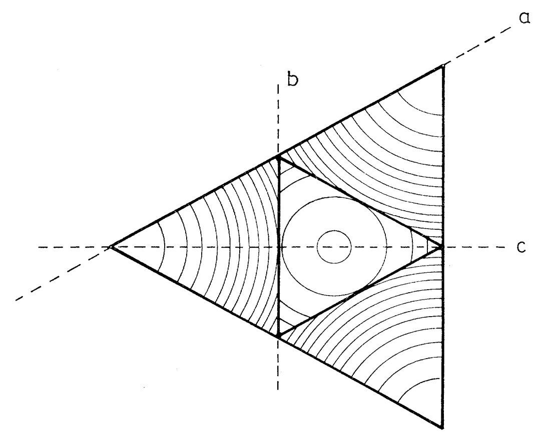


FIG. 7. Contour plot of the adiabatic potential $U(a^3,a^8)$ for SU(3), Eqs. (94)–(96). The axes are the same as in Fig. 6. This figure is complemented by Fig. 8 for the sake of clarity.

symmetries are separately irrelevant, since they do not respect the fundamental domain. Guided by the observation that we are dealing with a discrete subgroup of the two-dimensional Euclidean group $E_2$ in the $a^3, a^8$ plane, it is easy to identify the particular combination of the symmetries that does not lead out of the fundamental domain: The cyclic permutation

$$C: \quad \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \tag{97}$$

of the color labels rotates the triangle by 120° around the origin of the $(a^3,a^8)$ plane into another fundamental domain (see Fig. 4). It can be shifted back into the original domain by combining a displacement ($k_1=0, k_2=-1, k_3=1$) with a central conjugation ($n=1$):

$$S = T_D(0,-1,1)T_C(1)C\ . \tag{98}$$

The operator $S$ acts as follows on our variables:

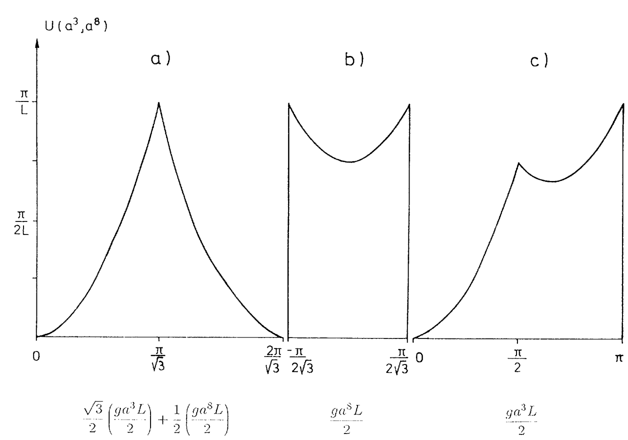


FIG. 8. Adiabatic potential $U(a^3,a^8)$ for SU(3), Eqs. (94)–(96). Shown are cuts through the potential surface along the dashed lines denoted by $a$, $b$, and $c$ in Fig. 7.

$$
\begin{aligned}
Sa^3S^\dagger &= a^{3\,\prime} = -\frac{1}{2}\left(a^3+\sqrt{3}a^8\right)+\frac{2\pi}{gL},\\
Sa^8S^\dagger &= a^{8\,\prime} = \frac{1}{2}\left(\sqrt{3}a^3-a^8\right)-\frac{2\pi}{\sqrt{3}gL},\\
S\varphi^{12}S^\dagger &= e^{i2\pi x/L}\varphi^{13\dagger}\ ,\quad S\varphi^{23}S^\dagger = e^{-i2\pi x/L}\varphi^{12}\ ,\\
S\varphi^{13}S^\dagger &= \varphi^{23\dagger}\ .
\end{aligned}
\tag{99}
$$

$S^2$ is the inverse operator of $S$, so that $S^3=1$. For later convenience we also note the inverse relations

$$
\begin{aligned}
S^\dagger a^3 S &= a^{3\,\prime\prime} = -\frac{1}{2}\left(a^3-\sqrt{3}a^8\right)+\frac{2\pi}{gL}\ ,\\
S^\dagger a^8 S &= a^{8\,\prime\prime} = -\frac{1}{2}\left(\sqrt{3}a^3+a^8\right)+\frac{2\pi}{\sqrt{3}gL}\ .
\end{aligned}
\tag{100}
$$

This is the quantum-mechanical formulation of the $Z_3$ symmetry of rotations of the fundamental triangle. Since any residual symmetry transformation can be written as a permutation followed by a shift, it is easy to see that this exhausts the exact symmetries of the model. The additional reflection symmetries exhibited by the effective potential are not symmetries of the full Hamiltonian. Geometrically, the reason is the following: These reflections involve noncyclic permutations of the color indices. Such permutations change the orientation of the basic triangle in the plane by $\pm\,60°$, so that it cannot be mapped back onto the fundamental domain by a subsequent translation.

Let us now again build the vacuum state vector. First consider the fermions. The occupation of the Dirac sea changes across the boundary of the internal triangle (IV) in Fig. 6, where pairs of fermion levels cross zero energy, one pair at each boundary. We denote the adiabatic fermionic states by $|M_{12},M_{13},M_{23}\rangle$ (the $M'_{ij}$ are redundant owing to the neutrality condition). In Fig. 6, we also display the "Fermi levels" $M_{ij}$ in the four different sectors of the fundamental domain. The action of $S$ on these states in general is

$$
S|M_{12},M_{13},M_{23}\rangle = |M_{23}+1,-M_{12}+1,-M_{13}\rangle\ ,
\tag{101}
$$

where we have made a definite choice of the phases. Specifically, for the case at hand,

$$
\begin{aligned}
S|0,0,0\rangle &= |1,1,0\rangle\ ,\quad S|1,1,0\rangle = S|1,0,-1\rangle\ ,\\
S|1,0,-1\rangle &= |0,0,0\rangle\ ,\quad S|1,0,0\rangle = |1,0,0\rangle\ ,
\end{aligned}
\tag{102}
$$

in accordance with the discrete rotational symmetry. The Schrödinger equation for $\tilde\Phi$ clearly has a threefold degeneracy with wave functions localized near the three corners related by actions of $S$:

$$
\begin{aligned}
\tilde\Phi_{\rm II}(a^3,a^8) &= S\tilde\Phi_{\rm I}(a^3,a^8) = \tilde\Phi_{\rm I}\left(a^{3\,\prime\prime},a^{8\,\prime\prime}\right)\ ,\\
\tilde\Phi_{\rm III}(a^3,a^8) &= S\tilde\Phi_{\rm II}(a^3,a^8) = \tilde\Phi_{\rm I}\left(a^{3\,\prime},a^{8\,\prime}\right)
\end{aligned}
\tag{103}
$$

with the primed and double-primed arguments defined in Eqs. (99) and (100). The ground state wave function in particular corresponds to an $n=3$, $f$-wave excited state of the two-dimensional isotropic harmonic oscillator, due to the boundary condition that it has two nodal lines subtending an angle of 60°,

$$
\tilde\Phi_{\rm I}(a^3,a^8) = \frac{2}{gL}\left(\frac{32\nu^4}{\pi}\right)^{1/2}\rho^3\cos(3\phi)e^{-\nu\rho^2}
\tag{104}
$$

with plane polar coordinates defined as

$$
\frac{gLa^3}{2} = \rho\cos\phi\ ,\quad \frac{gLa^8}{2} = \rho\sin\phi
\tag{105}
$$

and

$$
E_0 = 4\omega\ ,\quad \omega^2 = \frac{3g^2}{2\pi}\ ,\quad \nu = \frac{1}{gL}\sqrt{\frac{6}{\pi}}\ .
\tag{106}
$$

It is then easy to construct, with the above choice of phases, simultaneous eigenstates of $H$ and $S$:

$$
|\Psi_z\rangle = \frac{1}{\sqrt{3}}\left\{|0,0,0\rangle|\tilde\Phi_{\rm I}\rangle + z^2|1,1,0\rangle|\tilde\Phi_{\rm II}\rangle + z|1,0,-1\rangle|\tilde\Phi_{\rm III}\rangle\right\}
\tag{107}
$$

with

$$
S|\Psi_z\rangle = z|\Psi_z\rangle\ ,\quad z\in\left\{1,e^{2\pi i/3},e^{4\pi i/3}\right\}\ .
\tag{108}
$$

Since the fermionic components differ in the occupation of two pairs of levels, the bilinear condensate vanishes in this approximation. On the other hand, one could construct an operator of fourth order in the fermion fields with nonzero matrix elements between the adiabatic fermion states in sectors I, II, and III but vanishing contribution within one sector. In this case, a quartic condensate

$$
i\,{\rm tr}\,\{\bar\psi(1-\gamma_5)\psi\bar\psi(1-\gamma_5)\psi\}
$$

or

$$
i\,{\rm tr}\,\{\bar\psi(1+\gamma_5)\psi\bar\psi(1+\gamma_5)\psi\}
$$

appears, being determined by the overlap of the bosonic wave functions. The relevant (bosonic) matrix element is again calculable (for $gL\ll 1$) and yields

$$
\langle\tilde\Phi_{\rm I}|\tilde\Phi_{\rm II}\rangle = 2\pi^4\nu^2 e^{-2\nu\pi^2/3}.
\tag{109}
$$

The effect discussed above for SU(2), enhancement of the overlap due to the Jacobian, is even more pronounced in SU(3), where it forces us into higher harmonic oscillator states; it yields a prefactor $\sim (gL)^{-2}$.

### D. Anomalies and spectral flow of fermion levels

Up to this point, the charge operators relevant for constructing the vacuum state vectors were those that enter the neutrality condition; cf. Eqs. (24) and (25). Let us now consider the corresponding *axial* charges

$$
Q_{5i} = \sum_{k>i}(Q_5)_{ik} - \sum_{k<i}(Q_5)_{ki}
\tag{110}
$$

with

$$(Q_5)_{ik} = \frac{1}{2}\int_0^L dx\varphi^{ik\dagger}\gamma^5\varphi^{ik}. \tag{111}$$

They provide the most direct access to the issue of axial anomalies in our framework (see also Ref. [21]). Exactly like the ordinary charges $Q_i$, the axial charges $Q_{5i}$ will be regularized in a way that is invariant under the displacements and central conjugations [cf. Eqs. (36) and (37)]. Heat kernel regularization yields

$$(Q_5)_{ij}|\mathbf{M},\mathbf{M}\rangle = \left(M_{ij} - \frac{gL}{2\pi}(a_i - a_j)\right)|\mathbf{M},\mathbf{M}\rangle\ . \tag{112}$$

The presence of $\gamma^5$ $(= \sigma_3)$ introduces a minus sign between the contributions from right- and left-handed fermions, so that two terms, which cancel in the ordinary charge [cf. Eqs. (50) and (92)], add up in the axial charge, leading to a dependence of $Q_{5i}$ on the gauge variables $a_i$. It is convenient to define purely fermionic operators by

$$(\tilde{Q}_5)_{ij} = (Q_5)_{ij} + \frac{gL}{2\pi}(a_i - a_j) \tag{113}$$

and the corresonding combinations [$\tilde{Q}_{5i}, \tilde{Q}_5^p$, cf. Eq. (110)], which by construction are not invariant under displacements or central conjugations. However, the charges $\tilde{Q}_5^p$ are conserved in the small interval limit. On the other hand, the invariant axial charges $Q_5^p$ do not commute with the small interval Hamiltonian [cf. Eq. (15)], but yield the "anomalous" result

$$[K_a + H_{\text{ferm}}, Q_5^p] = i\frac{Ng}{4\pi}\left(e^p + e^{p\dagger}\right)\ . \tag{114}$$

We note in passing the explicit expressions for the relevant axial charges in SU(2) and SU(3):

$$\text{SU}(2):\quad Q_5^3 = \int_0^L dx\varphi^{12\dagger}\gamma^5\varphi^{12}\ , \tag{115}$$

$$\text{SU}(3):\quad Q_5^3 = \int_0^L dx\left(\varphi^{12\dagger}\gamma^5\varphi^{12} + \frac{1}{2}\varphi^{13\dagger}\gamma^5\varphi^{13} - \frac{1}{2}\varphi^{23\dagger}\gamma^5\varphi^{23}\right),$$
$$Q_5^8 = \int_0^L dx\frac{\sqrt{3}}{2}\left(\varphi^{13\dagger}\gamma^5\varphi^{13} + \varphi^{23\dagger}\gamma^5\varphi^{23}\right). \tag{116}$$

At this point a comparison with the Schwinger model reveals the essential structures underlying these definitions. In the Schwinger model, the residual gauge symmetry is that of the displacements [similar to (36)]. Therefore, the above regularization renders both the corresponding vector charge $Q$ and axial charge $Q_5$ gauge invariant, i.e., invariant under the residual gauge symmetry transformations. In SU(2) QCD, the residual symmetry transformation $S$ of Eq. (69) consists of a combined reflection and central conjugation. Neither of the two axial charges $Q_5^3, \tilde{Q}_5^3$ [cf. Eqs. (112) and (113)] are invariant [nor is the charge $Q^3$, cf. Eqs. (55) and (56)]. The above regularization, however, guarantees that $Q_5^3$ transforms covariantly under the residual gauge transformation

$$SQ_5^3S^\dagger = -Q_5^3. \tag{117}$$

The axial charge $\tilde{Q}_5^3$, on the other hand, transforms inhomogeneously:

$$S\tilde{Q}_5^3S^\dagger = -\tilde{Q}_5^3 + 2\,. \tag{118}$$

For SU(3) similar arguments apply and corresponding relations can be derived easily on the basis of the transfomation properties

$$\begin{aligned} S(Q_5)_{12}S^\dagger &= -(Q_5)_{13}\ , & S(\tilde{Q}_5)_{12}S^\dagger &= -(\tilde{Q}_5)_{13} + 1, \\ S(Q_5)_{13}S^\dagger &= -(Q_5)_{23}\ , & S(\tilde{Q}_5)_{13}S^\dagger &= -(\tilde{Q}_5)_{23}, \\ S(Q_5)_{23}S^\dagger &= (Q_5)_{12}\ , & S(\tilde{Q}_5)_{23}S^\dagger &= (\tilde{Q}_5)_{12} - 1. \end{aligned} \tag{119}$$

The gauge-variant charges $\tilde{Q}_5^p$ commute with the small interval Hamiltonian $(K_a + H_{\text{ferm}})$ of the system. Therefore, in the small interval limit, all eigenstates can be classified with respect to the values of the $\tilde{Q}_5^p$ charges. Moreover, in the SU(3) theory $\tilde{Q}_5^3$ and $\tilde{Q}_5^8$ commute with each other. The states corresponding to different eigenvalues of $\tilde{Q}_5^p$ are necessarily orthogonal to each other.

It is also possible to formulate in the small interval limit the axial anomaly in a "local" form,

$$\partial^\mu J_{\mu 5}^p = \frac{Ng}{4\pi L}\left(e^p + e^{p\,\dagger}\right)\ , \tag{120}$$

with the neutral components of the axial vector current:

$$J_{\mu 5}^p = \frac{1}{2}\psi\gamma_0\gamma_\mu\gamma^5T^p\psi\ . \tag{121}$$

This should be compared to the nonanomalous vector current $J_\mu^p = \frac{1}{2}\psi\gamma_0\gamma_\mu T^p\psi$, the neutral components of which are conserved in the ordinary sense,

$$\partial^\mu J_\mu^p = 0\ , \tag{122}$$

since in our specific gauge the only remnants of the gauge fields are the neutral variables $a^p$.

In the canonical framework, Eq. (120) is obtained by evaluating the commutator of the axial charge density with the small interval Hamiltonian. The relevant Hamiltonian $H_\varphi(a)$ of Eq. (23) can be equally well interpreted as the weak-coupling Hamiltonian of $N(N-1)/2$ charged particles coupled to $N-1$ different photons. Therefore, in the gauge-fixed formulation and the weak-coupling limit, the non-Abelian anomaly of SU($N$) $\text{QCD}_2$ is identical to the anomaly of an Abelian $\text{U}(1)^{N-1}$ gauge theory.

By and large, the analysis of the anomaly of the Schwinger model can be repeated with minor modifications to establish in the weak-coupling-limit index theorems, to classify the wave functions in different sectors, and to address the issue of the spectral flow.

Let us first discuss this last point. As usual the problem is formulated as follows. One adiabatically varies the gauge field along a certain trajectory starting at $\{a^p\}_i$

at $t = -T$ and arriving at $\{a^p\}_f$ at $t = T$ (assuming $T \to \infty$). At each given value of $t$, i.e., for a given gauge field, the energy eigenvalues of relevant fermion degrees of freedom in the first-quantized eigenvalue problem are found. In our case this is a particularly simple excercise, since the Coulomb interaction has been neglected, the background gauge field configuration reduces to spatial constants $a^p$, and then each fermion mode is the eigenfunction of the one-particle equation

$$\sigma_3\{(1/i)\partial_1 - g(a_i - a_j)\}\varphi_k^{ij} = E_k\varphi_k^{ij} ,$$

where $k$ numbers the fermion modes.

At the next stage one studies the evolution of $E_k$ vs $t$. If some of the levels cross zero, this phenomenon is in one-to-one correspondence with the anomalous nonconservation of some of the fermion charges, see, e.g., [10]. Moreover, using arguments similar to those of Ref. [30] one can relate the fact of the crossover to the occurrence of the zero modes in the two-dimensional Dirac operator in background fields interpolating between $\{a^p\}_i$ and $\{a^p\}_f$.

Since the SU(2) and SU(3) theories differ in important technical details, it is convenient to consider them separately. In the SU(2) theory we have only one gauge-field variable $a^3$ and hence one topological charge

$$q = \frac{2g}{8\pi}\int \epsilon^{\mu\nu}F^3_{\mu\nu}d^2x = \frac{gL}{2\pi}\left\{a^3(+T) - a^3(-T)\right\} . \tag{123}$$

The interval of variation of the variable $a^3$ to be considered is the fundamental interval $[0, 2\pi/(gL)]$. Two end points of this interval are gauge equivalent. The gauge transformation gluing the end points is a large gauge transformation, see Eq. (40). Any trajectory connecting them is characterized by $|q| = 1$. Equation (120) shows then that, in this transition,

$$\Delta Q_5^3 = 2 ;$$

i.e., two fermion levels, one right handed and one left handed, cross the zero in the opposite directions when we adiabatically proceed from the origin to $a^3 = \frac{2\pi}{gL}$. In agreement with our general remarks above, it does not come as a surprise that in this weak-coupling limit the spectral flow of SU(2) QCD is identical to that in the Schwinger model with one fermion field, see Fig. 1 in [10], restricted to the interval $0 < a < \frac{2\pi}{gL}$.

Proceeding to the SU(3) case, we observe that the fundamental domain where the spectral flow is to be analyzed is given by the triangle of Fig. 6. The three vertices of the triangle are gauge equivalent. The corresponding large gauge transformations gluing them are $U = \exp\left\{\pm(2\pi ix/L\sqrt{3})t^8 + (2\pi ix/L)t^3\right\}$ [remember that $t^a = \lambda^a/2$ in the SU(3) theory]. Although one can consider any trajectory running inside the fundamental domain and the associated spectral flow, of special interest are the trajectories starting at one vertex at $t = -T$ and ending up at another at $t = +T$.

Now we can introduce three topological charges,

$$q_\pm = \frac{3g}{16\pi}\int \varepsilon^{\mu\nu}\left(F^3_{\mu\nu} \pm \frac{1}{\sqrt{3}}F^8_{\mu\nu}\right)d^2x ,$$

$$q = \frac{\sqrt{3}g}{8\pi}\int \varepsilon^{\mu\nu}F^8_{\mu\nu}\,d^2x , \tag{124}$$

subject to a constraint

$$q_+ - q_- = q .$$

On each of the three trajectories connecting different vertices, the absolute value of one of the topological charges is 1, while for the two others we have 1/2. The one with the maximal absolute value is relevant for the given trajectory.

For instance, if the trajectory runs along the upper side of the triangle $a^8 = a^3/\sqrt{3}$, and the effective color current coupled to the "gluon field" is proportional to $J^3_\mu + J^8_\mu/\sqrt{3}$, from Eq. (116) we see that the corresponding axial charge is that of the two-flavor Schwinger model ($\varphi^{12}$ and $\varphi^{13}$). The spectral flow along this trajectory coincides with that of the two-flavor Schwinger model, with two pairs of levels crossing the zero-energy mark. The trajectories running along two other sides of the fundamental triangle have the same properties modulo cyclic permutations of $\varphi^{12}$, $\varphi^{13}$, and $\varphi^{23}$. Generically, with the antiperiodic boundary conditions on the fermion fields, the lines where a pair of the fermion levels crosses zero form an equilateral triangle inside the fundamental domain; see Fig. 6.

In Ref. [11] the instanton solutions were found in $\text{QCD}_2^{\text{adj}}$—trajectories interpolating between the vertices of the fundamental triangle. It was shown that these solutions are accompanied by $2(N-1)$ fermion zero modes for SU($N$). Usually the fermion zero modes are associated with index theorems of the Atiyah-Singer type. The latter relate the difference in the numbers of the left-handed and right-handed zero modes to the topological charge of the given background field. There is obviously a close kinship between the index theorems, anomaly relations, and the issue of the topological charge. In $\text{QCD}_4$ the index theorem was established in Ref. [31]; see also [32] for a thorough discussion. We understand now that analogous index theorems exist in $\text{QCD}_2^{\text{adj}}$, but they necessarily involve a topological charge whose definition is given in a particular gauge. For instance, in the SU(2) theory

$$n_L - n_R = q , \quad n_R^\dagger - n_L^\dagger = q , \tag{125}$$

where $n_{L,R}$ and $n^\dagger_{L,R}$ are the numbers of the corresponding zero modes for $\psi$ and $\psi^\dagger$, respectively. Equation (125) explains why for any field trajectory interpolating between $a^3 = 0$ and $a^3 = \frac{2\pi}{gL}$, not necessarily the instanton solution, the zero modes will persist.

### E. Symmetries beyond weak coupling

So far our analysis has been performed in the small interval or weak-coupling limit. For the case of the

Schwinger model or the U(1) anomaly in $QCD_4$, corrections to the weak-coupling limit do not affect the anomaly. This is different in the case of $QCD_2^{adj}$. The symmetry of the weak-coupling limit is significantly higher than that of the full theory. The full theory does not exhibit any continuous axial symmetry. Before gauge fixing, the only continuous symmetries are gauge symmetries. After gauge fixing no continuous symmetry survives; the axial symmetries of the weak-coupling limit are manifestly broken by the Coulomb interaction $H_C$ [cf. Eqs. (15) and (16)]. Beyond the weak-coupling limit, the theory still exhibits a discrete axial symmetry. The full Hamiltonian (15) remains invariant under changes of sign of either all the right- (upper components) or all the left-handed (lower components) fermion fields. The unitary transformation, generating sign changes of right-handed fields,

$$\tilde{R} = \exp\left( i\pi \left[ \sum_{i,k(i<k)} [(\tilde{Q}_5)_{ik} + Q_{ik}] + \sum_{p=1}^{N-1} \sum_{k\geq 0} c_k^{p\,\dagger} c_k^p \right] \right) , \tag{126}$$

is a discrete symmetry transformation of $QCD_2^{adj}$ beyond the weak-coupling limit [the vector charges $Q_{ik}$ are defined in analogy to Eq. (111)]:

$$\tilde{R} H \tilde{R}^\dagger = H . \tag{127}$$

We can define similarly left-handed transformations. No new structures, however, are encountered, since the transformation reversing the sign of all fermion fields commutes with all other symmetry transformations. Here it is not convenient to define a corresponding transformation generated by the axial charge only. This would involve a redefinition of the phase of all fermion fields, which have been assumed to be real. We nevertheless will refer to $\tilde{R}$ as a discrete axial charge transformation.

The operators $c_k^{p\dagger}, d_k^{p\,\dagger}$ create neutral fermions and are defined in analogy with the SU(2) operators of Eq. (79). To characterize how this symmetry is realized, we introduce the condensate operators

$$\Gamma_{0,(5)} = \sum_{i,k(i<k)} \varphi^{ik\dagger} \gamma^0 \left(\gamma^5\right) \varphi^{ik} + \sum_{p=1}^{N-1} \psi^p \gamma^0 \left(\gamma^5\right) \psi^p , \tag{128}$$

and it is easy to verify that these condensate operators are odd under the discrete axial transformations

$$\tilde{R} \Gamma_{0,(5)} \tilde{R}^\dagger = -\Gamma_{0,(5)} . \tag{129}$$

A nonvanishing scalar or pseudoscalar condensate associated with these bilinear fermion operators can therefore develop only if the system is not in an eigenstate of $\tilde{R}$:

$$\langle \Phi | \Gamma_{0,(5)} | \Phi \rangle \neq 0 \quad \text{only if} \quad \tilde{R} |\Phi\rangle \neq \pm |\Phi\rangle. \tag{130}$$

The appearance of quartic condensates on the other hand, as considered above in connection with SU(3), is not constrained by these symmetry properties. We obviously have

$$\tilde{R} i \,\mathrm{tr} \left\{ \bar{\psi}(1 \pm \gamma_5) \psi \bar{\psi}(1 \pm \gamma_5) \psi \right\} \tilde{R}^\dagger = i \,\mathrm{tr} \left\{ \bar{\psi}(1 \pm \gamma_5) \psi \bar{\psi}(1 \pm \gamma_5) \psi \right\} , \tag{131}$$

and therefore nonvanishing vacuum expectation values of these quartic fermion operators may develop irrespective of the realization of the discrete axial symmetry $\tilde{R}$. In turn, such condensates cannot be used as order parameters to characterize the different phases of the discrete axial symmetry.

In addition to this discrete axial symmetry, the full system exhibits the discrete residual gauge symmetry $S$, which has been explicitly constructed for SU(2) [Eq. (69)] and SU(3) [Eq. (98)]. Using the results (118) and (119) as well as the invariance of the neutral fermions under displacements [Eq. (36)] and central conjugations [Eq. (37)] and the transformation properties under permutations of the color basis [Eq. (38)] we obtain

$$\begin{aligned} \mathrm{SU}(2){:}\quad & S\tilde{R}S^\dagger = -\tilde{R} , \\ \mathrm{SU}(3){:}\quad & S\tilde{R}S^\dagger = \tilde{R} . \end{aligned} \tag{132}$$

In SU(2), the discrete axial symmetry $\tilde{R}$ is "anomalous"; i.e., this symmetry cannot be realized simultaneously with the discrete residual gauge symmetry (this is an example of the *global* anomaly). "Gauge invariant," stationary states are therefore twofold degenerate with $\tilde{R}$ connecting these states. Furthermore, the condensate operators develop in general nonvanishing expectation values. By contrast, in SU(3), the discrete residual gauge symmetry and axial symmetry can be realized simultaneously; i.e., stationary states can be labeled by two quantum numbers characteristic for these two symmetries. In general, the system does not exhibit degeneracies, and condensates (of operators quadratic in the fermion fields) are not present. As in the case of anomalous continuous symmetries, the anomaly of the discrete $\tilde{R}$ transformation can be cured by supplementing the fermionic operators by appropriate gauge field operators. This is achieved easily by replacing the axial charges $(\tilde{Q}_5)_{ik}$ in the definition of $\tilde{R}$ by $(Q_5)_{ik}$ [cf. Eqs. (112) and (113)]

$$R = \exp\left( i\pi \left[ \sum_{i,k(i<k)} [(Q_5)_{ik} + Q_{ik}] + \sum_{p=1}^{N-1} \sum_{k\geq 0} c_k^{p\,\dagger} c_k^p \right] \right) = \delta \tilde{R} . \tag{133}$$

By construction, $R$ is invariant under displacements (36) and central conjugations (37)

$$T_D(\mathbf{k}) R T_D^\dagger(\mathbf{k}) = R , \quad T_C(n) R T_C^\dagger(n) = R . \tag{134}$$

As in the case of anomalous continuous symmetries, inclusion of gauge degrees of freedom via $\delta$ in the definition of $R$ results in a nonvanishing commutator

$$R H R^\dagger \neq H , \tag{135}$$

and therefore $R$ is not a symmetry transformation of SU($N$) $\mathrm{QCD}_2^{\mathrm{adj}}$. This operator is nevertheless useful in clarifying the general symmetry properties for SU($N$). An elementary calculation shows that the operator $\delta$ acting on the gauge degrees of freedom transforms as

$$T_D(\mathbf{k})\delta T_D^\dagger(\mathbf{k}) = \delta \ ,$$
$$T_C(n)\delta T_C^\dagger(n) = (-1)^{n(N-1)}\delta \ . \tag{136}$$

Given the invariance of $R$ [Eq. (134)] and the invariance of $\tilde{R}$ under permutations of the color basis (38), the appropriate relevant SU($N$) symmetry transformation $S$ constructed with these building blocks transforms $\tilde{R}$ as

$$S\tilde{R}S^\dagger = (-1)^{n(N-1)}\tilde{R} \ . \tag{137}$$

Thus, our result derived in detail for SU(3) is generally valid for SU($N$) with $N$ odd. In these systems residual gauge symmetries and the discrete axial symmetry are simultaneously realized by the stationary states. No degeneracy occurs as a result of a conflict between discrete residual gauge and axial symmetries and, in general, no condensates are formed. For even $N$, in general, the axial symmetry is anomalous unless the construction of the "relevant" symmetry $S$ would not involve the "elementary" choice $n = 1$ (or more generally $n$ odd) of the parameter specifying the central conjugations. (We cannot rule out this possibility at present, since we have not yet constructed the symmetry operator $S$ for $N > 3$.) Assuming for the moment that $S$ does involve an odd value of $n$, we conclude that the stationary states are degenerate. The transformation property

$$S\tilde{R}S^\dagger = -\tilde{R} \tag{138}$$

implies that the stationary state $|E,z\rangle$ with

$$S|E,z\rangle = z|E,z\rangle \tag{139}$$

is degenerate with $|E,-z\rangle$. Thus, in SU($N$) with $N$ even, all the stationary states might be twofold degenerate and correspondingly develop condensates. This result also shows that the relation (138) cannot be true for odd $N$, where $z$ and $-z$ cannot both belong to the spectrum of $S$.

Obviously these general results encompass those of the weak coupling limit. In particular, we confirm the degeneracy of the SU(2) ground state with the concomitant formation of a condensate beyond weak coupling, while the threefold degeneracy of the ground state in SU(3) [cf. Eqs. (107) and (108)] is revealed as being due to symmetries that do not persist beyond weak coupling. The residual gauge symmetry $S$ is still present and realized and allows one to characterize stationary states by the corresponding eigenvalue $z$. However, as a result of the Coulomb interaction these states must be expected to split energetically. Concerning the issue of a possibility of an SU(3) condensate, our discussion only rules out an anomalous symmetry as the origin for a condensate associated with the fermionic bilinear operators (128). Such condensates nevertheless may appear "dynamically," as does the chiral condensate in $\mathrm{QCD}_4$. As emphasized above, the appearance of the quartic condensate [cf. Eq. (131)] in the weak-coupling limit of SU(3) is not associated with a particular realization of the discrete axial symmetry. For reasons of continuity, we can conclude that this "quartic" condensate persists beyond the weak-coupling limit.

We finally comment on extensions of the local symmetry considerations beyond the weak-coupling limit. Strictly speaking, there is no continuous axial symmetry, which would be broken by effects of regularization of associated currents. Unlike the axial anomaly of QCD or of the Schwinger model, the *local* axial anomaly of $\mathrm{QCD}_2^{\mathrm{adj}}$ is a valid concept only in the weak-coupling limit. It is of interest to repeat the calculation leading to Eq. (120) when including the Coulomb interaction. The structure of the additional term, which arises when commuting the axial charge with $H_C$, suggests introducing for SU(2) the "time component" of the vector potential:

$$\alpha_0^{ij}(x) = -\frac{g}{L}\sum_{n=-\infty}^{\infty}\int_0^L dy\,(1-\delta_{ij}\delta_{n0})\,\frac{\rho^{ij}(y)}{\left(\frac{2\pi n}{L}+(i-j)\,a^3\right)^2}e^{2i\pi n(x-y)/L}. \tag{140}$$

With the help of $\alpha_0$, the time derivative in the continuity equation is replaced by a covariant derivative

$$\partial^0 J_{05}^3(x) - g\epsilon^{ab3}\frac{1}{2}\left\{J_{05}^a(x), \alpha_0^b(x)\right\} + \partial^1 J_{15}^3(x) = \frac{g}{2\pi L}\left(e^3 + e^{3\dagger}\right). \tag{141}$$

Here we have extended the definition (121) to currents that have off-diagonal color components. In accordance with our above discussion, this result displays the two different sources for the nonconservation of the axial vector current. In addition to the nonvanishing divergence of the axial vector current arising from the anomaly (the electric field term $e^3$), the current is manifestly not conserved by the appearence of a "covariant" time derivative ($\propto g$).

## IV. SUMMARY

Our investigation of $\mathrm{QCD}_2^{\mathrm{adj}}$ has focused on symmetry properties of this model field theory and their implications for the vacuum structure of these models. General topological arguments exhibit the center symmetries, i.e., $Z_N$ symmetries in SU($N$) to be the only relevant gauge-related symmetries. These residual gauge symmetries are present only if the Yang-Mills field is coupled to fermions in the adjoint representation. The difference in homotopy properties of SU($N$)/$Z_N$, the manifold relevant if fermions are in the adjoint representation, and SU($N$) in the case of fundamental fermions reveals the presence or absence of these residual symmetries. The nature of the realization of the symmetries and consequences for the vacuum structure such as the possible formation of condensates remain unspecified by such general topological reasoning. In gauge theories, the connection between general symmetry properties and their dynamical implementation is a remote one. Topological properties are most easily discussed in a formalism involving redundant gauge fields; the dynamics on the other hand may be more conveniently described in terms of unconstrained, physical degrees of freedom. It has been the purpose of this work to investigate this rather intricate connection between general topological properties and specific dynamical realizations in the context of $\mathrm{QCD}_2^{\mathrm{adj}}$.

We have chosen to perform the detailed dynamical study in the canonical framework of the Weyl gauge. We have imposed, furthermore, periodic boundary conditions, i.e., gauge and matter fields live on a spatial circle. In this way, the singular infrared properties are kept under control. In the canonical formalism, the redundant gauge fields are eliminated by explicitly implementing the Gauss law constraint. The unconstrained, physical degrees of freedom are the fermion fields and, for SU($N$), $N-1$ gauge degrees of freedom, which can be interpreted as zero momentum neutral gluons. Most important with regard to the topological properties is the peculiar form of the electric field energy of these leftover gluons. In the course of eliminating the gauge fields the kinetic energy of these gluons acquires a nontrivial Jacobian in much the same way as the kinetic energy of a quantum-mechanical particle moving on the surface of a sphere does. By appropriate definition of a "radial" wave function the kinetic energy can be transformed to the standard Cartesian form supplemented, however, by the constraint on the wave function to vanish wherever the Jacobian vanishes. In this way, the configuration space of the gauge degrees of freedom becomes a manifold with boundaries. For SU(2), this manifold is a compact interval, and an equilateral triangle for SU(3). Along the boundaries, the radial wave function and Jacobian vanish. Much of the characteristics of the dynamics of $\mathrm{QCD}_2$ is due to these topological properties. For instance, the different dynamics of SU(2) $\mathrm{QCD}_2$ and the Schwinger model, with their common one-dimensional configuration space of the residual gauge degree of freedom, can be traced back to a large extent to the topological difference between a compact interval and a circle. (In electrodynamics, the electric field energy appears without further redefinition of the wave function in Cartesian form.) The physics of a quantum mechanical particle moving on a circle or in a periodic potential is significantly different from that of a particle enclosed, e.g., in an infinite square well. This difference is indicative of what happens in two-dimensional electrodynamics and SU(2) chromodynamics, respectively.

The definition of the configuration space of the gauge degrees of freedom of $\mathrm{QCD}_2$ is independent of the characteristics of the matter fields. The color structure of the matter fields is, however, relevant as far as the existence of symmetries acting on the combined configuration space of fermion fields and gauge degrees of freedom is concerned. We have analyzed in detail these symmetries. As one of the main results of our studies we have explicitly constructed the residual gauge symmetries for $\mathrm{QCD}_2^{\mathrm{adj}}$. For SU(2), these symmetry transformations are reflections of the gauge variable at the midpoint of the interval defined by the zeros of the Jacobian accompanied by a charge conjugation of the fermion fields. In SU(3), the symmetry transformations consist of a rotation of the fundamental equilateral triangle by $2\pi/3$ with concomitant color rotations of the fermions. The mere presence of these residual gauge symmetries would have very few consequences as far as the spectrum or the structure of the vacuum is concerned were it not for the presence of yet another symmetry in this class of theories. $\mathrm{QCD}_2^{\mathrm{adj}}$ exhibits, in addition to the gauge symmetries, a discrete axial symmetry; i.e., the Hamiltonian is invariant under a separate change of sign of the totality of right- or of left-"handed" fermions. The interplay of these two discrete symmetries is reflected in spectrum and ground-state properties. In particular, these systems provide an example of an anomaly in discrete symmetries. For SU($2N$), there is the possibility that the reflection symmetry is anomalous, in which case the stationary states cannot be simultaneously "gauge invariant" and of definite chirality. As a consequence, the ground state is expected in general to be degenerate and to develop a condensate associated with the standard scalar or pseudoscalar density. For SU($2N+1$) no conflict between the two symmetries arises, and, in general, neither do degeneracies occur nor scalar or pseudoscalar densities develop expectation values.

A more specific characterization of the dynamics is possible in the small interval or equivalently the weak-coupling limit. Most of our detailed investigations, which finally have led to the general results discussed above, have been performed in this limit. For weak coupling an adiabatic treatment with the gauge degrees of freedom representing the slow and the fermions representing the fast degrees of freedom is possible. This adiabatic approximation has allowed us to study very explicitly the symmetry properties incorporated in the adiabatic potentials. The reflection symmetry in the SU(2) case or the threefold discrete rotational symmetry of SU(3) $\mathrm{QCD}_2^{\mathrm{adj}}$ is manifestly exhibited by the corresponding adiabatic potential. This explicit construction reflects, in particular, in a very intuitive way, the difference in symmetry arising if the gauge degrees of freedom are coupled to fermions in either the fundamental or the adjoint representation.

In the adiabatic picture the formation of condensates is connected with the overlap of wave functions of the gauge degrees of freedom associated with different fermionic vacua. The study of this overlap has revealed another interesting consequence of the presence of the Jacobian in the electric field energy. The vanishing of the Jacobian and the radial wave function occurs when the values of gauge degrees describe pure gauges and therefore coincide with the minima of the potential energy. As a consequence, the modified kinetic energy forces the system off the classical equilibrium position and thereby enhances the "tunneling" probability to other configurations. In general, the calculations in the adiabatic, weak-coupling limit are in full agreement with the exact results and illustrate the rather surprising odd-even effect of condensate formation in SU($N$). Finally, we have shown that in the weak-coupling limit, the axial symmetry is extended to a continuous symmetry. The presence of continuous symmetries opens the possibility of studying properties associated with the anomaly using the classical tools. In particular, we have formulated index theorems associated with the spectral flow of fermion levels and could thereby provide further qualitative insights into the symmetry properties of $QCD_2^{adj}$.

The analysis of the quantum mechanics of the vacuum state of the model at hand carried out above gives us a new understanding of the "condensate" problem. However, the discrepancy between fermionic approaches like the present one and bosonization techniques remains to be clarified. Our results yield no hint on the bilinear condensate in the SU(3) case, while the bosonization arguments seemingly indicate that it develops. If this is indeed true, such a condensate must have a dynamical origin not directly related to the gauge symmetry or the discrete chiral symmetry of the theory and should not be present in the limit $gL \ll 1$; only then would there be no obvious contradiction with our investigation.

We conclude by emphasizing those issues, which beyond the particular structure of $QCD_2^{adj}$ might be of relevance for gauge theories in higher dimensions. The appearence of Jacobians and centrifugal barriers, which has been of such crucial importance for the structure of the gauge fixed theory, is clearly not limited to gauge theories in one spatial dimension. Physical and unphysical degrees of freedom in non-Abelian theories cannot be expected to simply factorize as in QED, and this complication does not depend on the number of space dimensions. Indeed the corresponding modifications of the electric field energy have been found in a variety of gauges [28, 33–35]. In most cases, an explicit evaluation of the corresponding Jacobian is missing. However, irrespective of the detailed structure of the Jacobians, their zeros effectively introduce boundaries into the infinite dimensional configuration space of the corresponding theory. Therefore, the issue of realization of symmetries in the presence of constraints on the wave functional must come up also in gauge theories in higher dimensions. Furthermore, in the axial gauge representation of $QCD_4$, in which the Jacobian can be evaluated explicitly, the pure gauges, i.e., gauge field configurations corresponding to vanishing magnetic fields, are seen to be located on the "hyperplanes" of vanishing Jacobians. Thus, an appropriately defined radial wave functional is forced to vanish at the classical equilibrium points and the mechanism of enhanced tunneling processes, which we found in the one-dimensional case, will be effective in higher dimensions too. Finally, our discussion of anomalies and associated index theorems should be relevant for gauge theories in higher dimensions. In the context of $QCD_2^{adj}$ our discussion was necessarily restricted to the weak-coupling limit. In $QCD_4$ on the other hand, the corresponding continuous axial symmetries and therefore the phenomenon of "non-Abelian" anomalies (cf. [36]) persist beyond the small volume limit. In particular it appears promising to extend to higher dimensions our method of expressing, within a gauge-fixed formulation, these non-Abelian anomalies as Abelian anomalies of the color neutral axial vector currents, i.e., axial vector currents associated with the SU($N$) Cartan subalgebra.

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