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More on scattering of Chern-Simons vortices

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I derive a general formalism for finding kinetic terms of the effective Lagrangian for slowly moving Chern-Simons vortices. Deformations of fields linear in velocities are taken into account. From the equations they must satisfy I extract the kinetic term in the limit of coincident vortices. For vortices passing one over the other there is locally right-angle scattering. The method is based on the analysis of field equations instead of the action functional so it may be useful also for nonvariational equations in nonrelativistic models of condensed matter physics.

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I. INTRODUCTION

There is quite a large family of field-theoretical models both relativistic [1–8] and nonrelativistic [9–12] with solitons which possess a Bogomol'nyi [13] limit. In this limit static field equations in a given topological sector can be reduced for a minimal energy configuration to first-order differential equations. These equations generically admit static multisoliton solutions characterized by a finite set of parameters such as positions of solitons and their internal orientations. The configurations are static; if we think about solitons as we do about particles, there are no net static forces between them.

It was an idea of Manton [14] that low-energy scattering of monopoles in the Bogomol'nyi-Prasad-Sommerfield model [13,6] can be modeled by reduction of the dynamics to a finite-dimensional manifold of parameters of static multisoliton solutions. The kinetic part of the Lagrangian of the original theory after integrating out the spatial dependence of the fields with by now time-dependent parameters yields the kinetic part of the effective Lagrangian quadratic in time derivatives of parameters. A metric on moduli space can be read out of it. A fundamental idea in this approach is that configurations satisfying the Bogomol'nyi lower bound are at the bottom of a potential well. A slow motion of solitons can lead only to small deformations of fields with respect to static configurations.

The idea was succesfully applied to scattering of monopoles, vortices in the Abelian Higgs model [4,15–17] and CP^1 solitons [18,5]. Recently also extensions of the method to the case of Chern-Simons vortices both relativistic and nonrelativistic were done [19–24]. However, as was first pointed out in [20] in these cases a new problem arises. Lagrangians of these models contain terms linear in time derivatives such as the Chern-Simons term and/or Schrödinger action. By just promoting parameters to the role of collective coordinates one can reliably calculate only terms in the effective Lagrangian linear in velocities. To compute the kinetic part one has to take into account small deformations of the fields with respect to static configurations. It is enough to consider only deformations linear in velocities. Such deformations can be in principle calculated from full field equations linearized in deformations and terms linear in velocities. However, also terms linear in accelerations and third time derivatives arise and as I have discussed in [22] there is no apparent reason why they should be negligible as compared to velocities. Such an "approximation" can lead to serious inconsistencies.

In this paper I put the problem on a slightly different footing. The acceleration terms are not neglected. There are no net static forces between vortices and so their accelerations must be zero for vanishing velocities. Thus we can assume that the acceleration vector is at least linear in velocities. We can expect this linear term to be nonzero because there are charge-flux interactions by Lorenz-like forces. Thus acceleration is not neglected but replaced by a position-dependent matrix ω times velocity. The same procedure can be applied iteratively to the third time derivative. Finally in a special limit of coinciding vortices such a unique form of the ω matrix is extracted which admits regular deformations of fields. Knowledge of ω is enough to establish the form of moduli space metrics. A brief comment on how this approach works in the Abelian Higgs model is added.

II. MODEL, ZERO MODES, AND USEFUL NOTATIONS

We take the Lagrangian of the self-dual Chern-Simons-Higgs model in the form

$$L = \frac{\kappa}{2}\varepsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha + D_\mu\phi^\star D^\mu\phi - V(\mid\phi\mid) \ , \qquad (1)$$

where $V(\mid\phi\mid) = \frac{1}{\kappa^2}\mid\phi\mid^2(\mid\phi\mid^2 - 1)^2$, $D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$, the signature of the flat (2+1)-dimensional metrics is $(+,-,-)$, and the Levi-Civita symbol is chosen so that $\varepsilon^{012} = -1$. A variation of the action with respect to A_0 leads to Gauss' law constraint

$$2\phi^\star\phi(\partial_t\chi - A_0) = \kappa B \ , \qquad (2)$$

where we have introduced $\phi = \mid\phi\mid e^{i\chi}$ and the magnetic field is $B = -F_{12}$. When one takes into account Gauss' law one can effectively rewrite the original Lagrangian in

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a new form:

$$L = \dot{f}^2 + \kappa B\partial_t\chi + \frac{\kappa}{2}\varepsilon^{ij}A_i\partial_t A_j - \left[\partial_i f\partial_i f + f^2(\partial_i\chi - A_i)^2 + \frac{\kappa^2 B^2}{4f^2} + \frac{1}{\kappa^2}f^2(f^2-1)^2\right] . \quad (3)$$

We have just introduced $f = |\phi|$ and ε^{ij} such that $\varepsilon^{12} = 1$. The energy density for a static configuration with a positive topological index is

$$\varepsilon = [\partial_i f - f\varepsilon_{ik}(\partial_k\chi - A_k)]^2 + \frac{1}{f^2}\left[\frac{\kappa B}{2} + \frac{f^2(1-f^2)}{\kappa}\right]^2 - B . \quad (4)$$

The magnetic flux is quantized as $2\pi n$ with n being the winding number. Thus energy is bounded from below by this value of the magnetic flux. If $\kappa > 0$, this Bogomol'nyi lower bound is saturated by static configurations being solutions to first-order differential equations:

$$\partial_i f = f\varepsilon_{ik}(\partial_k\chi - A_k) , \quad (5)$$

$$\varepsilon_{ij}\partial_i A_j = \frac{2}{\kappa^2}f^2(1-f^2) , \quad (6)$$

$$A_0 = \frac{1-f^2}{\kappa} . \quad (7)$$

These equations admit static multivortex solutions parametrized by a set of $2n$ real parameters. In the Coulomb gauge one can take $\chi = \sum_{v=1}^{n} Arg(z - z_v)$, where the sum runs over vortices labeled by v's and complex coordinates on the plane are used. In this gauge there is only one second-order differential equation to solve:

$$\nabla^2 \ln f = \frac{2}{\kappa^2}f^2(f^2-1) + 2\pi\sum_{v=1}^{n}\delta^{(2)}(z - z_v) . \quad (8)$$

Once f is known, other fields can be obtained from Eqs. (2),(5). The singular sources on the right-hand side (RHS) of the above equation enforce the modulus f in a close vicinity of the p-fold zero z_0 to behave like $|z-z_0|^p$ (it is a leading term of the expansion). This equation in particular admits a cylindrically symmetric vortex solution with winding number n: $\phi = f(r)\exp(in\theta)$. We will parametrize multivortex solutions in the equivalent forms

$$\phi = (z - z_1)\cdots(z - z_n)W(z,\bar{z},z_v) \equiv \left(z^n - \sum_{k=0}^{n-1}\lambda_k z^k\right)W(z,\bar{z},z_v) \quad (9)$$

and

$$\chi = \frac{1}{2i}\ln\frac{\prod_{v=1}^{n}(z - z_v)}{\prod_{w=1}^{n}(\bar{z} - \bar{z}_w)} \equiv \frac{1}{2i}\ln\frac{z^n - \sum_{k=0}^{n-1}\lambda_k z^k}{\bar{z}^n - \sum_{l=0}^{n-1}\bar{\lambda}_l\bar{z}^l} . \quad (10)$$

λ's are complex coefficients of the n-th degree polynomial with roots z_v and W is a positive real function. Although the parametrizations are equivalent, it will later appear that λ's are more efficient in the description of vortices passing one over another.

Each of these multivortex solutions possesses $2n$ zero modes:

$$\delta f(z,\lambda) = \frac{\partial f}{\partial\lambda_p^A}(z,\lambda)\delta\lambda_p^A \equiv f(z,\lambda)h_p^A(z,\lambda)\delta\lambda_p^A ,$$

$$\delta\chi(z,\lambda) = \frac{\partial\chi}{\partial\lambda_p^A}\delta\lambda_p^A = -\mathrm{Im}\left(\frac{l^A z^p}{z^n - \lambda_s z^s}\right)\delta\lambda_p^A ,$$

$$\delta A_k = \partial_k\delta\chi + \varepsilon_{kl}\partial_l\frac{\delta f}{f} = \left(\partial_k\frac{\partial\chi}{\partial\lambda_p^A} + \varepsilon_{kl}\partial_l h_p^A\right)\delta\lambda_p^A ,$$

$$\delta A_0 = -\frac{2}{\kappa^2}f^2 h_p^A\delta\lambda_p^A , \quad (11)$$

where $\lambda_p = \lambda_p^1 + i\lambda_p^2$ and $l^A = (1,i)$. Equation (8) linearized in fluctuations becomes

$$\nabla^2 h_p^A + \frac{4}{\kappa^2}\rho(1-2\rho)h_p^A = 0 . \quad (12)$$

ρ denotes moduli squared f^2. From the asymptotics of f close to its zeros we can extract the leading term in the fluctuations:

$$h_p^A \approx -\mathrm{Re}\left(\frac{l^A z^p}{z^n - \lambda_s z^s}\right) \quad (13)$$

as $(z^n - \lambda_s z^s) \approx 0$. This is the only singular term in the expansion around the actual zero of the Higgs field. This singularity is fine-tuned by the singularity in $\delta\chi$ to yield regular δA_k [see (11)].

Now because of future applications let us take a closer look at the coincident n-vortex solution $\phi = f(r)\exp(in\theta)$ with $f(r)$ satisfying

$$ff'' + \frac{ff'}{r} - (f')^2 = \frac{2}{\kappa^2}f^4(f^2-1) , \quad (14)$$

with boundary conditions $f(0) = 0$ and $f(\infty) = 1$. Close to zero it behaves like $f \sim f_0 r^n - \frac{f_0^3}{2\kappa^2(n+1)^2}r^{3n+2} + \cdots$. Fluctuations for small λ's can be written as

$$h(r,\theta) = h_p^A(r,\theta)\lambda_p^A = -H_{n-p}[\lambda_{n-p}^1\cos(n-p)\theta + \lambda_{n-p}^2\sin(n-p)\theta] . \quad (15)$$

H's satisfy the equations

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(n-p)^2}{r^2}\right)H_{n-p} + \frac{4}{\kappa^2}\rho(1-2\rho)H_{n-p} = 0 , \quad (16)$$

with a normalization that, close to zero, $H_m \sim \frac{1}{r^m}$ and it asymptotically vanishes at infinity.

III. SLOWLY MOVING VORTICES

For more clarity in this paragraph and in what follows we will rescale the gauge fields $A_\mu \to \kappa^{-1}A_\mu$ and coordinates $x^\mu \to \kappa x^\mu$. On the level of field equations it amounts to fixing $\kappa = +1$.

The aim of this paper is to investigate the slow motion of vortices in the adiabatic approximation. In the case of Chern-Simons solitons a new difficulty arises because there are terms linear in time derivatives in the Lagrangian of the theory. Only terms linear in velocities can be correctly calculated by direct application of former methods. As pointed out in [20], to obtain the kinetic term one has not only to make the static fields time dependent by time variation of their $2n$ parameters, but one also has to take into account deformations of the "static" fields up to terms linear in velocities. It is something like a generalized Lorenz transformation. In this paragraph we will develop a general formalism for calculating such corrections. It will appear that to have regular solutions one cannot neglect terms proportional to accelerations as was anticipated in [22].

Let us make the fields time dependent by variation of the parameters and add to them deformations linear in velocities, say, $f(z,\lambda) \to f[z,\lambda(t)] + \triangle f[z,\lambda(t),\dot{\lambda}(t)]$, etc. Evaluation of the effective Lagrangian is accomplished by substitution of such fields into Lagrangian (3) and integrating out their planar dependence. The only terms which can contribute to the part of the effective Lagrangian linear in velocities are

$$L^{(1)}_{\text{eff}} = \int d^2x\, [B\dot{\chi} + \tfrac{1}{2}\varepsilon^{ij} A_i \dot{A}_j] \ , \tag{17}$$

and one needs to take into account only effects of promoting parameters to the role of collective coordinates. All the other terms are quadratic in velocities. The contribution of the second term in Eq. (17) vanishes in the Coulomb gauge. For fairly separated vortices one can approximate under the integral $B = -2\pi \sum_v \delta^{(2)}(z - z_v)$ to obtain

$$L^{(1)}_{\text{eff}} \approx -2\pi \frac{d}{dt} \sum_{v>w} Arg(z_v - z_w) \ . \tag{18}$$

With the explicit form of the phase (10) and some integration by parts one can rewrite the linear Lagrangian as

$$L^{(1)}_{\text{eff}} = -2\pi \sum_v \dot{x}^i_v A_i(z_v) \ , \tag{19}$$

where $z_v = x^1_v + i x^2_v$.

The second-order term of the effective Lagrangian is (in the gauge $\Delta X = 0$)

$$\begin{aligned} L^{(2)}_{\text{eff}} = {} & \dot{f}^2 + 2\dot{f}(\triangle f)^{\cdot} + (\triangle f)^{\cdot}(\triangle f)^{\cdot} - (\partial_i \triangle f)^2 - \frac{1}{2}V''(f)(\triangle f)^2 + f^2(\dot{\chi} - \triangle A_0)^2 - f^2(\triangle A_i)^2 + (\triangle f)^2 A_0^2 \\ & -(\triangle f)^2(\partial_i\chi - A_i)^2 + \triangle A_0 \triangle B + \frac{1}{2}\varepsilon_{ij}(2\dot{A}_i \triangle A_j + (\triangle Ai)^{\cdot} \triangle A_j) \\ & -4f\triangle f A_0(\dot{\chi} - \triangle A_0) + 4f\triangle f(\partial_i\chi - A_i)\triangle A_i \ . \end{aligned} \tag{20}$$

The small corrections to the fields have to be calculated from following equations obtained by linearization of the full field equations:

$$\begin{aligned} \nabla^2\left(\frac{\triangle f}{f}\right) + 4\rho(2-3\rho)\left(\frac{\triangle f}{f}\right) + (\partial_k \ln\rho)\left[\partial_k\left(\frac{\triangle f}{f}\right) + \varepsilon_{kl}(\triangle A_l - \partial_l \triangle\chi)\right] - 2(\rho-1)(\triangle A_0 - \dot{\chi}) & \\ = \frac{\ddot{f}}{f} + 2(\rho-1)\dot{\triangle}\chi + \frac{\ddot{\triangle} f}{f} & \ , \\ \nabla^2\triangle\chi - \partial_k\triangle A_k - 2(2\rho-1)\frac{\dot{f}}{f} + (\partial_k\ln\rho)\left(\varepsilon_{kl}\partial_l\frac{\triangle f}{f} + \partial_k\triangle\chi - \triangle A_k\right) = \ddot{\chi} - \dot{\triangle} A_0 + \ddot{\triangle}\chi - 2(1-\rho)\frac{\dot{\triangle} f}{f} & \ , \\ \varepsilon_{kl}\partial_k\triangle A_l - 4f(1-\rho)\triangle f + 2\rho(\dot{\chi} - \triangle A_0) = -2\rho\dot{\triangle}\chi & \ , \\ \partial_n\triangle A_0 + 2\partial_n\rho\frac{\triangle f}{f} + 2\rho\varepsilon_{nk}(\partial_k\triangle\chi - \triangle A_k) - \partial_n\dot{\chi} - \varepsilon_{nk}\partial_k\left(\frac{\dot{f}}{f}\right) = \dot{\triangle} A_n & \ . \end{aligned} \tag{21}$$

The equations were simplified with a use of static field equations satisfied by background fields. Now the crucial observation is that in the Bogomol'nyi limit any forces exerted on vortices must be zero for vanishing velocities, and so accelerations are at least linear in velocities,

$$\ddot{\lambda}^A_p = \omega^{AB}_{pq}(\lambda)\dot{\lambda}^B_q \ . \tag{22}$$

ω for a given λ is a matrix in the pair of indices (A,p) and (B,q). This relation can be iterated to give

$$\frac{d^3}{dt^3}\lambda^A_p = \omega^{AB}_{pq}(\lambda)\ddot{\lambda}^B_q + \frac{\partial\omega^{AB}_{pq}}{\partial\lambda^C_r}\dot{\lambda}^B_q\dot{\lambda}^C_r \approx \omega^{AB}_{pq}\omega^{BC}_{qr}\dot{\lambda}^C_r \ , \tag{23}$$

where once again we have preserved only terms linear in velocities. We can also make the following definitions and approximations

$$\dot{f} = f h_p^A \dot{\lambda}_p^A \ , \quad \ddot{f} \approx f h_p^A \omega_{pq}^{AB} \dot{\lambda}_q^B \ ,$$
$$\triangle f \equiv f s_p^A \dot{\lambda}_p^A \ , \quad \dot{\triangle f} \approx f s_p^A \omega_{pq}^{AB} \dot{\lambda}_q^B \ , \quad \ddot{\triangle f} \approx f s_p^A \omega_{pr}^{AC} \omega_{rq}^{CB} \dot{\lambda}_q^B \ ,$$
$$\ddot{\chi} \approx \frac{\partial \chi}{\partial \lambda_p^A} \omega_{pq}^{AB} \dot{\lambda}_q^B \ , \quad \triangle A_k \equiv A_k^{Ap} \dot{\lambda}_p^A; \ , \quad \triangle A_0 \equiv a_p^A \dot{\lambda}_p^A \ . \tag{24}$$

With these formulas and with the gauge $\triangle\chi = 0$, Eqs. (21) can be rewritten as

$$\dot{\lambda}_p^A \left[\nabla^2 s_p^A + 4\rho(2-3\rho) s_p^A + (\partial_k \ln\rho)(\partial_k s_p^A + \varepsilon_{kl} A_l^{Ap}) - 2(\rho-1)\left(a_p^A - \frac{\partial\chi}{\partial\lambda_p^A}\right)\right] = h_p^A \omega_{pq}^{AB} \dot{\lambda}_q^B + s_p^A \omega_{pr}^{AC} \omega_{rq}^{CB} \dot{\lambda}_q^B \ ,$$
$$\dot{\lambda}_p^A [-\partial_k A_k^{Ap} - 2(2\rho-1) h_p^A + (\partial_k \ln\rho)(\varepsilon_{kl}\partial_l s_p^A - A_k^{Ap})] = \left[\frac{\partial\chi}{\partial\lambda_p^A} - a_p^A - 2(1-\rho) s_p^A\right] \omega_{pq}^{AB} \dot{\lambda}_q^B \ ,$$
$$\dot{\lambda}_p^A \left[\varepsilon_{kl}\partial_k A_l^{Ap} - 4\rho(1-\rho) s_p^A + 2\rho\left(\frac{\partial\chi}{\partial\lambda_p^A} - a_p^A\right)\right] = 0 \ ,$$
$$\dot{\lambda}_p^A \left[\partial_n a_p^A + 2\partial_n \rho s_p^A - 2\rho\varepsilon_{nk} A_k^{Ap}) - \left(\partial_n \frac{\partial\chi}{\partial\lambda_p^A} + \varepsilon_{nk}\partial_k h_p^A\right)\right] = A_n^{Ap} \omega_{pq}^{AB} \dot{\lambda}_q^B \ . \tag{25}$$

They should yield field deformations in approximations up to terms linear in velocities. In the limit of very small velocities field deformations become small as compared to background fields, and so our linearizations of field equations with respect to deformations are justified. Because full field deformations are regular functions, in the limit of slow motion also Eqs. (25) must have regular solutions, being good approximations to full deformations. With this in mind we can take for granted the existence of regular solutions and derive the necessary conditions for the regularity. The unknown parameters in Eqs. (25) are elements of the matrices $\omega_{pq}^{AB}(\lambda)$. Our strategy from now on is to adjust such unique values of these parameters which allow solutions to be regular. Once we know the parameters, we will also know the equations of motion for λ's linearized in velocities:

$$\ddot{\lambda}_p^A = \omega_{pq}^{AB} \dot{\lambda}_q^B \ . \tag{26}$$

Since we know the linear part of the effective Lagrangian (18), knowledge of these equations enables us to restore also its quadratic part up to total derivatives. The general form of the Lagrangian is

$$L_{\text{eff}} = g_{pq}^{AB}(\lambda) \dot{\lambda}_p^A \dot{\lambda}_q^B + b_{pq}^{AB}(\lambda) \lambda_p^A \dot{\lambda}_q^B \ . \tag{27}$$

The metric tensor on the moduli space, g, must be symmetric under exchange of pairs of indices (A,p) and (B,q) and invertible:

$$(g^{-1})_{pq}^{AB} g_{qr}^{BC} = \delta^{AC}\delta_{pr} \ . \tag{28}$$

The equations of motion linearized in velocities are

$$2g_{pq}^{AB} \ddot{\lambda}_p^A + (b_{pq}^{AB} - b_{qp}^{BA}) \dot{\lambda}_p^A + \left(\frac{\partial b_{pq}^{AB}}{\partial \lambda_s^D} - \frac{\partial b_{ps}^{AD}}{\partial \lambda_q^B}\right) \lambda_p^A \dot{\lambda}_s^D \ . \tag{29}$$

The components of the metric tensor must solve the equations

$$g_{qr}^{BC} \omega_{rp}^{CA} + \frac{1}{2}(b_{pq}^{AB} - b_{qp}^{BA}) + \frac{1}{2}\left(\frac{\partial b_{rq}^{CB}}{\partial \lambda_p^A} - \frac{\partial b_{rp}^{CA}}{\partial \lambda_q^B}\right) \lambda_r^C = 0. \tag{30}$$

This is a system of $2n \times 2n$ linear inhomogenous equations. The basic condition for the system to have a unique solution is that the matrix ω be invertible in pairs of indices (A,p) and (C,r), $\det\omega \neq 0$. This means [see Eq. (26)] that, whatever the small velocity is, it is always a source of acceleration already in linear terms. It should be a quite generic case except for some anomalous sets of measure zero in a model with magnetic interaction. The metrics can be extended to these exceptional points by continuity. One such point is certainly the limit of infinitely separated vortices where magnetic interaction degenerates to a purely topological term [Eq. (18)]. But in this limit the effective Lagrangian can be accurately calculated with the help of the product ansatz of independently Lorenz-boosted vortices. The quadratic term reads

$$L_{\text{eff}}^{(2)} \approx \pi \sum_v \dot{z}_v \dot{z}_v^\star \ . \tag{31}$$

The second and third terms in Eq. (30) are explicitly antisymmetric under exchange of pairs (A,p) and (B,q), and so the first term also has to be antisymmetric, $\omega^T g = -g\omega$. This condition means that to leading order acceleration is orthogonal to velocity with respect to the metric g. Once again it should be so for forces due to magnetic interactions and they are the only forces linear in velocities. If this condition is satisfied, we are left with $2n^2 - n$ independent equations necessary to establish the same number of metric tensor components.

IV. TWO VORTICES IN THE CENTER-OF-MASS FRAME

We will consider the by now classic example of two vortices in the c.m. frame. The system is well described by two parameters λ_1 and λ_2 which can be identified with the former notation:

$$\lambda_1 = \lambda_0^1 \quad , \quad \lambda_2 = \lambda_0^2 \ . \tag{32}$$

The Higgs field is $\phi = (z^2 - \lambda)W(z,\bar{z},\lambda)$. Equations (25) are on both sides linear in velocities, and so they can be cast in a form of a velocity-independent matrix multiplying a vector of velocities. For the product to be always zero any velocity vector must belong to kernels of the matrices. A matrix is zero if and only if it annihilates any vector from the basis spanning the space of velocities. If we choose $\dot{\lambda}_1 \neq 0$, and $\dot{\lambda}_2 = 0$, Eqs. (25) reduce to

$$\begin{aligned}
\nabla^2 s^{(1)} + 4\rho(2-3\rho)s^{(1)} + (\partial_k \ln\rho)(\partial_k s^{(1)} + \varepsilon_{kl}A_l^{(1)}) - 2(\rho-1)\left(a^{(1)} - \frac{\partial\chi}{\partial\lambda^{(1)}}\right) &= h^{(A)}\omega^{A1} + s^{(B)}\omega^{BA}\omega^{A1} \ , \\
-\partial_k A_k^{(1)} - 2(2\rho-1)h^{(1)} + (\partial_k \ln\rho)(\varepsilon_{kl}\partial_l s^{(1)} - A_k^{(1)}) &= \left[\frac{\partial\chi}{\partial\lambda^{(A)}} - a^{(A)} - 2(1-\rho)s^{(A)}\right]\omega^{A1} \ , \\
\varepsilon_{kl}\partial_k A_l^{(1)} - 4\rho(1-\rho)s^{(1)} + 2\rho\left(\frac{\partial\chi}{\partial\lambda_1} - a^{(1)}\right) &= 0 \ , \\
\partial_n a^{(1)} + 2\partial_n\rho s^{(1)} - 2\rho\varepsilon_{nk}A_k^{(1)} - \left(\partial_n \frac{\partial\chi}{\partial\lambda^{(1)}} + \varepsilon_{nk}\partial_k h^{(1)}\right) &= A_n^{(A)}\omega^{A1} \ .
\end{aligned} \tag{33}$$

On the other hand, for the choice $\dot{\lambda}_1 = 0$ and $\dot{\lambda}_2 \neq 0$ we obtain

$$\begin{aligned}
\nabla^2 s^{(2)} + 4\rho(2-3\rho)s^{(2)} + (\partial_k \ln\rho)(\partial_k s^{(2)} + \varepsilon_{kl}A_l^{(2)}) - 2(\rho-1)\left(a^{(2)} - \frac{\partial\chi}{\partial\lambda^{(2)}}\right) &= h^{(A)}\omega^{A2} + s^{(B)}\omega^{BA}\omega^{A2} \ , \\
-\partial_k A_k^{(2)} - 2(2\rho-1)h^{(2)} + (\partial_k \ln\rho)(\varepsilon_{kl}\partial_l s^{(2)} - A_k^{(2)}) &= \left[\frac{\partial\chi}{\partial\lambda^{(A)}} - a^{(A)} - 2(1-\rho)s^{(A)}\right]\omega^{A2} \ , \\
\varepsilon_{kl}\partial_k A_l^{(2)} - 4\rho(1-\rho)s^{(2)} + 2\rho\left(\frac{\partial\chi}{\partial\lambda_2} - a^{(2)}\right) &= 0 \ , \\
\partial_n a^{(2)} + 2\partial_n\rho s^{(2)} - 2\rho\varepsilon_{nk}A_k^{(2)} - \left(\partial_n \frac{\partial\chi}{\partial\lambda^{(2)}} + \varepsilon_{nk}\partial_k h^{(2)}\right) &= A_n^{(A)}\omega^{A2} \ .
\end{aligned} \tag{34}$$

These two sets of equations have to be satisfied simultaneously.

Now let us consider the limit of coincident vortices. Such a configuration is rotationally symmetric and the direction in which it is split should make no difference. This motivates the limiting form of the effective Lagrangian:

$$L_{\text{eff}}^{\lambda\to 0} \sim h_0 \frac{1}{2} \mid \lambda \mid^{\xi} \delta^{AB}\dot{\lambda}_A\dot{\lambda}_B - g_0\varepsilon^{AB}\lambda_A\dot{\lambda}_B \ , \tag{35}$$

with the coefficient h_0 and the power ξ to be determined. g_0 can be extracted with the help of Eq. (17) and fluctuations (11),(15):

$$g_0 = 4\pi \int_0^\infty dr \ \frac{\rho(1-2\rho)H_2(r)}{r} \ . \tag{36}$$

The value of g_0 was estimated numerically to be $g_0 \approx 0.0194$.

Motivated by null powers of λ's in Eqs. (33),(34) and by

$$\frac{\partial\chi}{\partial\lambda_1} = \frac{\sin 2\theta}{r^2} \quad , \quad \frac{\partial\chi}{\partial\lambda_2} = -\frac{\cos 2\theta}{r^2} \quad , \tag{37}$$

in the limit of vanishing λ's we can restrict the field deformations to the following forms in the polar coordinates r,θ:

$$\begin{aligned}
s^{(1)} &= s(r)\sin 2\theta \ , \quad s^{(2)} = -s(r)\cos 2\theta \ , \\
a^{(1)} &= a(r)\sin 2\theta \ , \quad a^{(2)} = -a(r)\cos 2\theta \ , \\
A_r^{(1)} &= b(r)\cos 2\theta \ , \quad A_r^{(2)} = b(r)\sin 2\theta \ , \\
A_\theta^{(1)} &= c(r)\sin 2\theta \ , \quad A_\theta^{(2)} = -c(r)\cos 2\theta \ ,
\end{aligned} \tag{38}$$

together with a form of the matrix $\omega^{AB} = \omega\varepsilon^{AB}$. Equations (33) and (34) reduce to

$$
\begin{aligned}
s'' + \frac{s'}{r} - \frac{4s}{r^2} + \omega^2 s + 4\rho(2-3\rho)s + \frac{\rho'}{\rho}(s'+c) - 2(\rho-1)\left(a - \frac{1}{r^2}\right) &= \omega H_2 \ , \\
b' + \frac{b}{r} + \frac{2c}{r} - 2(2\rho-1)H_2 + \frac{\rho'}{\rho}\left(b - \frac{2s}{r}\right) &= -\frac{\omega}{r^2} + \omega a - 2\omega(\rho-1)s \ , \\
c' + \frac{c}{r} + \frac{2b}{r} - 4\rho(1-\rho)s + 2\rho\left(\frac{1}{r^2} - a\right) &= 0 \ , \\
a' + 2\rho' s - 2\rho c + \omega b &= -\frac{2}{r^3} + \frac{2H_2}{r} \ , \\
\frac{2a}{r} + 2\rho b - \omega c &= \frac{2}{r^3} + H_2' \ .
\end{aligned}
\tag{39}
$$

ω is an adjustable parameter we have to choose in such a way that solutions are regular. A short inspection shows that the fourth equation in the above set of equations can be derived from the fifth, second, and third equations. Regularity of $\triangle f$ means that s cannot be more divergent than $O(r^{-2})$. Thus we can expand the regular solution around $r = 0$ as

$$
s(r) = \sum_{k=-2}^{\infty} s_k r^k \ , \ a(r) = \sum_{k=0}^{\infty} a_k r^k \ , \ b(r) = \sum_{k=1}^{\infty} b_k r^k \ , \ c(r) = \sum_{k=1}^{\infty} c_k r^k \ . \tag{40}
$$

Substitution of the first few terms in the expansions to Eqs. (39) shows that they are in contradiction unless $\omega = -2$. If we adopt this value of ω the following leading terms will be obtained:

$$
\begin{aligned}
s &= (-v-\xi)r^2 + \cdots \ , \\
a &= (-v-2\xi)r^2 + \cdots \ , \\
b &= (-2v-2\xi)r + \cdots \ , \\
c &= (2v+2\xi)r \ ,
\end{aligned}
\tag{41}
$$

where $v \approx 1\,387\,000$ is a coefficient in the expansion $H_2 = r^{-2} + vr^2 + \cdots$ and ξ is a free parameter coming from the "homogenous" part of the solution. For the solution regular at infinity numerical analysis has given $\xi \approx -1\,387\,017.5$.

With the value of $\omega = -2$ we can conclude that the limiting form of the effective Lagrangian for $\lambda \to 0$ is

$$
L_{\text{eff}}^{\lambda\to 0} = g_0\left(\tfrac{1}{2}\delta^{AB}\dot\lambda_A\dot\lambda_B - \varepsilon^{AB}\lambda_A\dot\lambda_B\right) \ . \tag{42}
$$

In the generic case of two vortices in the c.m. frame at (R,Θ) and $(R,\Theta+\pi)$ the effective Lagrangian must take the form

$$
L_{\text{eff}} = F(R)\dot R^2 + 2G(R)\dot R R\dot\Theta + H(R)R^2\dot\Theta^2 + B(R)R\dot\Theta. \tag{43}
$$

The functions F, G, H are to be determined. It is a general form of the metric tensor invariant with respect to rotations. The function G is in general nonzero since the Chern-Simons term breaks parity invariance. Equation (20) also contains terms which break parity and there does not seem to be any reason why these terms should vanish. In polar coordinates the ω matrix reads

$$
\omega^{AB} = \frac{\frac{d}{dR}[RB(R)]}{2R}(g^{-1})^{AC}\varepsilon^{CB} = \frac{J'(R)}{2R}(g^{-1})^{AC}\varepsilon^{CB} \ , \tag{44}
$$

with $J(R)$ being the total spin of two vortices separated by a distance $2R$. It is an invertible matrix and acceleration is indeed to leading order orthogonal to velocity. I have attempted to calculate ω, but because there is less symmetry (less constraints) in the problem for generic R, then for $R = 0$ it cannot be extracted just from the asymptotices close to zeros of the Higgs field. Matching with asymptotics at infinity would be necessary.

Finally a comment on analogous scattering of vortices in the Abelian Higgs model (AHM) is in order. In this model $G(R) = 0$ and also $B(R) = 0$. Similar considerations as above lead to a conclusion that $\omega^{AB} = 0$. Thus in the AHM accelerations are at least quadratic in velocities. In equations analogous to (21) terms linear in acceleration and its time derivative can be neglected as compared to those linear in velocities. These terms were indeed neglected in Appendix B of Ref. [20] and it was shown that rearrangement of the effective Lagrangian analogous to (20) with the help of equations fulfilled by deformation leads to the same expression as that derived by Samols [15]. Here we have shown the justification of these steps when performed in the AHM.

Let us consider a general Bogomol'nyi theory and try to decide what are the conditions for the effective Lagrangian to be purely quadratic in time derivatives. A Lagrangian of the theory can be written as

$$
L = G_{ab}[\psi]\dot\psi_a\dot\psi_b + K_a[\psi]\dot\psi_a - \varepsilon[\psi] \ , \tag{45}
$$

where ψ's are a set of fields, $\varepsilon[\psi]$ is a static energy density functional, and $G_{ab}[\psi]$ is an invertible, symmetric, and positively definite tensor. The only contribution to the linear part of the effective Lagrangian is

$$
L_{\text{eff}}^{(1)} = K_a[\psi]\dot\psi_a \ , \tag{46}
$$

where, as in the rest of this paper ψ's are the fields of the static self-dual background and the time derivative means a total derivative with respect to time-dependent parameters. This term certainly vanishes if $K_a[\psi] = 0$ for the given background. A more subtle possibility is that the whole expression (46) can yield a zero result when its spatial dependence is integrated out. Relativistic gauge theory contains linear terms as in Eq. (45). In the Abelian Higgs model such a term is equal to $-\partial_i A_0 \dot{A}_i + e\psi^\star \psi A_0 \dot{\chi}$ but the self-dual background has the property that $A_0 = 0$ and that is why there is no linear term in the effective Lagrangian. An interesting example of the Maxwell-Higgs self-dual model with relativistic kinetics can be found in [7]. A uniform background charge density forces a nonzero A_0. Vortices in this model feel both Magnus force and mutual magnetic interactions.

The quadratic term of the general effective Lagrangian is

$$L^{(2)}_{\text{eff}} = G_{ab}[\psi](\dot{\psi}_a\dot{\psi}_b + 2\dot{\psi}_a \triangle\dot{\psi}_b + \triangle\dot{\psi}_a\triangle\dot{\psi}_b) + \frac{\delta K_a}{\delta\psi_b}(\dot{\psi}_a\triangle\psi_b - \triangle\psi_a\dot{\psi}_b - \triangle\psi_a\triangle\dot{\psi}_b) - \frac{\delta^2\varepsilon}{\delta\psi_a\delta\psi_b}[\psi]\triangle\psi_a\triangle\psi_b \ , \tag{47}$$

where we have introduced time derivatives of background fields and deformations of fields. Now we restrict ourselves to the case of $L^{(1)}_{\text{eff}} = 0$. This means that accelerations are at least quadratic in velocities and the above formula can be reduced to

$$L^{(2)}_{\text{eff}} = G_{ab}[\psi]\dot{\psi}_a\dot{\psi}_b + \frac{\delta K_a}{\delta\psi_b}(\dot{\psi}_a\triangle\psi_b - \triangle\psi_a\dot{\psi}_b) - \frac{\delta^2\varepsilon}{\delta\psi_a\delta\psi_b}[\psi]\triangle\psi_a\triangle\psi_b \ , \tag{48}$$

We can linearize the field equations with respect to velocities. The field deformations must satisfy a simple equation

$$-\frac{\delta^2\varepsilon}{\delta\psi_a\delta\psi_b}\triangle\psi_b = \left(\frac{\delta K_a}{\delta\psi_b} - \frac{\delta K_b}{\delta\psi_a}\right)\dot{\psi}_b \ . \tag{49}$$

This relation enables us to simplify Eq. (48) to the compact form

$$L_{\text{eff}} = G_{ab}[\psi]\dot{\psi}_a\dot{\psi}_b \ , \tag{50}$$

plus higher-order terms negligible in the adiabatic approximation.

Thus whenever there is no linear term in the effective Lagrangian the quadratic term can be correctly calculated with the help of background fields with their parameters promoted to the role of collective coordinates.

V. CONCLUSIONS

The limiting form of the term quadratic in velocities of the effective Lagrangian for two Chern-Simons vortices was extracted from equations satisfied by deformations of the fields with respect to a static background. This form shows that as we trace, locally, trajectories of vortices passing one over another there is the celebrated right-angle scattering. Globally, if two vortices were pushed from a large distance one against the other with a zero impact parameter, they would avoid direct collision, their trajectories being curved by charge-flux interactions. The difference between the total spin of the pair of vortices when they are infinitely separated and when they sit on top of one another is -2π (for $\kappa = 1$). Thus the necessary condition for the zeros to meet and the right-angle scattering to occur is that an impact parameter d with respect to the center of mass and an initial velocity v satisfy $dv = \frac{1}{2}$. This condition can become sufficient only for d small enough because vortex-vortex magnetic interactions are falling exponentially with a distance.

The local right-angle scattering is a hint that the moduli space manifold is similar to a smoothed cone. The missing volume can show itself in a thermodynamics of a vortex gas by an excluded volume in a van der Waals state equation similarly as for the vortex gas in the Abelian Higgs model [25].

Finally let me stress that problems partially overcome in this paper are not at all special to relativistic Chern-Simons models. They can also appear in nonrelativistic and nonvariational models so celebrated in condensed matter physics because Schrödinger or diffusion terms are linear in time derivatives on the one hand and on the other hand the effective action method may not be able to describe the whole variety of dynamical phenomena. I would like to address such problems in the near future.

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