PHYSICAL REVIEW D VOLUME 51, NUMBER 12 15 JUNE 1995

Dynamics of vortex and monopole production by quench-induced phase separation

A.J. Gill and R.J. Rivers
Blackett Laboratory, Imperial College, South Kensington, London SW7 2BZ, United Kingdom
and Isaac Newton Institute For Mathematical Sciences, 20 Clarkson Road, Cambridge CB3 0EH, United Kingdom
(Received 19 January 1995)

We examine the nonequilibrium dynamics of defect formation for weakly coupled global O(N) theories possessing vortices (strings) and monopoles. It is seen that, as domains form and grow, defects are trapped on their boundaries at a density of about one defect per coherence area in the case of strings, or one per coherence volume in the case of monopoles.

PACS number(s): 11.27.+d, 05.70.Fh, 11.10.Wx, 67.40.Vs

I. INTRODUCTION

The formation of topological defects during symmetry-breaking phase transitions is generic to many physical systems. In particular we cite the vortices and monopoles of superfluid ^{4}He and ^{3}He and the vortices (flux tubes) of high- and low-T_c superconductors. Similar defects, cosmic strings, or monopoles most likely appeared in the early Universe at the grand-unified-theory- (GUT) scale phase transition. All of these systems are described by some form of quantum field theory and, due to the phase transition, their dynamics is intrinsically nonequilibrium. They therefore provide a good means to test nonequilibrium field theory experimentally over a wide range of energies.

Roughly, the dynamics of defect formation proceeds as follows [1]. From some initial state, usually not too far from thermal equilibrium, some change in the bulk properties of the system, such as pressure or volume, induces a phase transition. During this transition, the scalar fields which describe the order parameter fall from the false vacuum into the true vacuum, choosing a point on the vacuum manifold at each point in space, subject to the constraint that they must be continuous and single valued. We shall limit ourselves to continuous or weakly first order transitions, for which this collapse to the true vacuum occurs by spinodal decomposition or phase separation. The resulting field configuration is one of domains within each of which the scalar fields have relaxed to a constant vacuum value.

For a theory permitting defects, it will sometimes happen that the requirements of continuity and single valuedness force the fields to remain in the false vacuum between some of the domains. For example, in the case of a single complex scalar field producing vortices, the phase of the field may change by an integer multiple of 2π on going round a loop in space. This requires at least one zero of the field within the loop, which signifies the presence of a region of unbroken phase. Each zero has topological stability and characterizes a vortex passing through the loop. When the phase transition is complete and there is no longer sufficient thermal energy available for the field to fluctuate into the false vacuum, the topological defects are frozen into the field. From then on, the defect density alters almost entirely by interactions of defects amongst themselves, rather than by fluctuations in the fields. See for example [2].

The major question then, is what fixes the initial defect density and the defect correlations? Only then can the subsequent evolution of defect networks be determined with any accuracy. It was first argued [1] that topological defects should be frozen in at the Ginzburg temperature T_G [3], the temperature above which there is sufficient thermal energy available for the field to fluctuate into the false vacuum without cost [4]. If so, the defect number would be strongly fluctuating above the Ginzburg temperature but frozen in below it. In this case, the relevant scale for the initial defect density would be the coherence length $\xi(T_G)$ of the Higgs field, or fields, at the Ginzburg temperature. For example, in vortex production for the $U(1)$ theory mentioned above, the initial vortex density, that is the number of vortices passing through unit area, would be $\kappa/\xi^2(T_G)$, where κ is a constant of order unity. Similarly, in monopole production the monopole density would be expected to be $\kappa/\xi^3(T_G)$, for similar κ. Thereafter, defect forces are assumed to take over.

Recently, however, more compelling pictures of the way in which the initial density of topological defects is fixed have been proposed. While the mechanism outlined initially is almost certainly correct, in general it is unlikely that the Ginzburg temperature is relevant to anything other than a thermally produced population of defects. This is most obviously so in the experiments using liquid ^{4}He where the final temperature after the quench is greater than the Ginzburg temperature. For the cases of greater interest to us, for example, an expanding Universe, we expect that, as the system is driven from some initial thermal state towards the phase transition, there will come a point when the rate at which the transition is driven is too fast for the evolution of the field to keep up [5]. The transition may now be viewed as a quench and it is no longer clear that either temperature or free energy means anything at all. The current view [5] is that the initial density of topological defects is fixed at this stage since the density of zeros of the field, which will eventually become defects, evolves so slowly in comparison to the rate of the quench. Thus, the initial density of defects is held to be approximately the density of zeros

of the scalar field when it first goes out of equilibrium.

Until the time when the scalar field is actually falling down the potential hill, one can always quench sufficiently fast that this is a good approximation. It is not clear, however, that the approximation will always be valid during the fall from the false to the true vacuum. In the following, we study the evolution of the defect density during the early stages of spinodal decomposition, for a weakly coupled theory. The result is that, for sufficiently weak coupling, during the early stages of the fall into the true vacuum the defect network scales and the defect density decreases. In this case, the approximation is not valid, although it may be so for more strongly coupled theories.

II. THE MODEL

In the following we consider a quench from an initially thermal state in a class of theories where the broken and unbroken symmetries are *global*, thereby guaranteeing that they pass through a *second order* transition. (Had they passed through a strongly first order transition, the mechanism for the transition, bubble nucleation, would lead to different consequences from those outlined below, although it might then be a good approximation to say that the defect density is frozen in when the field first goes out of thermal equilibrium.) We assume that the change of symmetry is sufficiently rapid that the fields are unable to respond immediately, but evolve by means of phase separation or spinodal decomposition and domain formation.

We shall consider the simplest theory, one of N massive relativistic scalar fields ϕ_a, where $a = 1, \ldots, N$, in D spatial dimensions, transforming as the fundamental representation of a globally O(N) invariant theory. Changes in the environment cause the symmetry to be broken to O($N-1$) (i.e., as given by the generalised "wine-bottle" potential) leading to a theory of one massive Higgs boson and $N-1$ massless Goldstone bosons, with the vacuum manifold S^{N-1}. Since the nth homotopy group Π_n of the n sphere is $\Pi_n(S^n) = Z$, the group of integers, the theory possesses global monopoles if $N = D$ and global strings if $N = D-1$. Otherwise the theory does not permit stable defects. We are primarily interested in $D = 3$ dimensions, for which the O(3) theory possesses monopoles, and the O(2) theory possesses strings.[1]

The transition is realized by the changing environment inducing an explicit time dependence in the field parameters. Although we have the early Universe in mind, we remain as simple as possible, in flat space-time with the ϕ-field action:

$$S[\phi] = \int d^{D+1}x \left(\frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m^2(t)\phi_a^2 - \frac{1}{4}\lambda(t)(\phi_a^2)^2 \right).$$

The t dependence of $m^2(t)$ and $\lambda(t)$ is assumed given and O(N) indices are summed over.

We wish to calculate the evolution of the defect density during the fall from the false vacuum to the true vacuum after a rapid quench from an initial thermal state. The simplest assumption, made here, is that the symmetry breaking occurs at time $t = t_0$, with the sign of $m^2(t)$ changing from positive to negative at t_0. Further, after some short period $\Delta t = t - t_0 > 0$, $m^2(t)$ and $\lambda(t)$ have relaxed to their final values, denoted by $m^2 = -\mu^2$ and λ, respectively. The field begins to respond to the symmetry breaking at $t = t_0$ but we assume that its response time is greater than Δt, again ignoring any mode dependence.

To follow the evolution of the defect density during the fall off the hill involves two problems. The first is to follow the evolution of the quantum field and the second is how to count the defects. We take these in turn.

III. EVOLUTION OF THE QUANTUM FIELD

During a symmetry-breaking phase transition, the dynamics of the quantum field is intrinsically nonequilibrium. The normal techniques of equilibrium thermal field theory are therefore inapplicable. Out of equilibrium, one typically proceeds using a functional Schrödinger equation or using the closed time path formalism of Mahanthappa, Schwinger, and Keldysh [6–8]. Here, we employ the latter, following closely the work of Boyanovsky, de Vega and co-authors [9,10].

Take $t = t_0$ as our starting time. Suppose that, at this time, the system is in a pure state, in which the measurement of ϕ would give Φ_0. That is,

$$\hat{\phi}(t_0, \mathbf{x})|\Phi_0, t_0\rangle = \Phi_0(\mathbf{x})|\Phi_0, t_0\rangle.$$

The probability $p_{t_f}[\Phi_f]$ that, at time $t_f > t_0$, the measurement of ϕ will give the value Φ_f is $p_{t_f}[\Phi_f] = |c_{f0}|^2$, where

$$c_{f0} = \int_{\phi(t_0)=\Phi_0}^{\phi(t_f)=\Phi_f} \mathcal{D}\phi \, \exp\left\{ iS[\phi] \right\},$$

in which $\mathcal{D}\phi = \prod_{a=1}^{N} \mathcal{D}\phi_a$ and spatial labels have been suppressed. It follows that $p_{t_f}[\Phi_f]$ can be written in the closed-time-path form

$$p_{t_f}[\Phi_f] = \int_{\phi_\pm(t_0)=\Phi_0}^{\phi_\pm(t_f)=\Phi_f} \mathcal{D}\phi_+ \mathcal{D}\phi_- \times \exp\left\{ i\Big(S[\phi_+] - S[\phi_-] \Big) \right\}.$$

Instead of separately integrating $\phi_\pm$ along the time paths $t_0 \le t \le t_f$, the integral can be interpreted as time ordering of a single field ϕ along the closed path $C_+ \oplus C_-$ where $\phi = \phi_+$ on C_+ and $\phi = \phi_-$ on C_-. The two-field notation is misleading in that it suggests that the ϕ_+ and ϕ_- fields are decoupled. That this is not so follows immediately from the fact that $\phi_+(t_f) = \phi_-(t_f)$. See Fig. 1.

It is necessary to keep this in mind when we extend the contour from t_f to $t = \infty$. Either ϕ_+ or ϕ_- is an equally

[1] However, the vortex production in the $D = 2$ Kosterlitz-Thouless transition has some interest, although we shall not pursue it here.

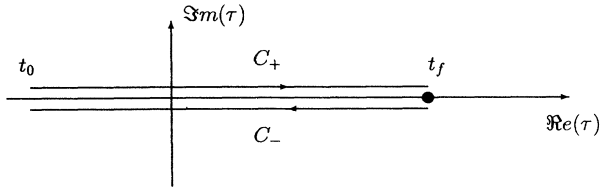

FIG. 1. The closed time path contour $C_+ \oplus C_-$.

good candidate for the physical field, but we choose ϕ_+. See Fig. 2.

With this choice and suitable normalization, p_{t_f} becomes

$$p_{t_f}[\Phi_f] = \int_{\phi_\pm(t_0)=\Phi_0} \mathcal{D}\phi_+\mathcal{D}\phi_-\,\delta[\phi_+(t_f)-\Phi_f] \times \exp\left\{i\left(S[\phi_+]-S[\phi_-]\right)\right\},$$

where $\delta[\phi_+(t)-\Phi_f]$ is a delta functional, imposing the constraint $\phi_+(t,\mathbf{x})=\Phi_f(\mathbf{x})$ for each $\mathbf{x}$.

The choice of a pure state at time t_0 is too simple to be of any use. The one fixed condition is that we begin in a symmetric state with $\langle\phi\rangle = 0$ at time $t = t_0$. Otherwise, our ignorance is parametrized in the probability distribution that at time t_0, $\phi(t_0,\mathbf{x}) = \Phi(\mathbf{x})$. If we allow for an initial probability distribution $P_{t_0}[\Phi]$ then $p_{t_f}[\Phi_f]$ is generalized to

$$p_{t_f}[\Phi_f] = \int \mathcal{D}\Phi P_{t_0}[\Phi] \int_{\phi_\pm(t_0)=\Phi} \mathcal{D}\phi_+\mathcal{D}\phi_-\,\delta[\phi_+(t_f) -\Phi_f]\,\exp\left\{i\left(S[\phi_+]-S[\phi_-]\right)\right\}.$$

At this stage, we have to begin to make approximations. We shall find that analytic calculations can be performed only if $p_{t_f}[\Phi_f]$ is Gaussian. Although there is no compelling reason why this should be exactly so, it provides a natural first guess. So that $p_{t_f}[\Phi_f]$ shall be Gaussian, it is necessary to take $P_{t_0}[\Phi]$ to be Gaussian also, with zero mean. All the cases that we might wish to consider are encompassed in the assumption that Φ is Boltzmann distributed at time t_0 at an effective temperature of $T_0 = \beta_0^{-1}$ according to a *quadratic* Hamiltonian $H_0[\Phi]$. That is,

$$P_{t_0}[\Phi] = \langle\Phi,t_0|e^{-\beta H_0}|\Phi,t_0\rangle = \int_{\phi_3(t_0)=\Phi=\phi_3(t_0-i\beta_0)} \mathcal{D}\phi_3 \exp\left\{iS_0[\phi_3]\right\},$$

for a corresponding action $S_0[\phi_3]$, in which ϕ_3 is taken to be periodic in imaginary time with period β_0. We take $S_0[\phi_3]$ to be quadratic in the O(N) vector ϕ_3 as

$$S_0[\phi_3] = \int d^{D+1}x\left[\frac{1}{2}(\partial_\mu\phi_{3\,a})(\partial^\mu\phi_{3\,a}) - \frac{1}{2}m_0^2\phi_{3\,a}^2\right].$$

We stress that m_0 and β_0 parametrize our uncertainty in the initial conditions. The choice $\beta_0 \to \infty$ corresponds to choosing the $p_t[\Phi]$ to be determined by the ground state functional of H_0, for example. For the sake of argument we take $T_0 = \beta_0^{-1}$ to be a temperature higher than the transition temperature T_c. Whatever the effect is to give an action $S_3[\phi]$ in which we are in thermal equilibrium for $t < t_0$ during which period the mass $m(t)$ takes the constant value m_0 and, by virtue of choosing a Gaussian initial distribution, $\lambda(t) = 0$ for $t < t_0$.

We now have the explicit form for $p_{t_f}[\Phi_f]$

$$p_{t_f}[\Phi_f] = \int \mathcal{D}\Phi \int_{\phi_3(t_0)=\Phi=\phi_3(t_0-i\beta_0)} \mathcal{D}\phi_3\, e^{iS_0[\phi_3]} \int_{\phi_\pm(t_0)=\Phi} \mathcal{D}\phi_+\Phi\mathcal{D}\phi_-\, e^{i(S[\phi_+]-S[\phi_-])}\delta[\phi_+(t_f)-\Phi_f]$$
$$= \int_B \mathcal{D}\phi_3\mathcal{D}\phi_+\mathcal{D}\phi_-\,\exp\left\{iS_0[\phi_3]+i(S[\phi_+]-S[\phi_-])\right\}\delta[\phi_+(t_f)-\Phi_f],$$

where the boundary condition B is $\phi_\pm(t_0) = \phi_3(t_0) = \phi_3(t_0 - i\beta_0)$. This can be written as the time ordering of a single field:

$$p_{t_f}[\Phi_f] = \int_B \mathcal{D}\phi\, e^{iS_C[\phi]}\,\delta[\phi_+(t_f)-\Phi_f],$$

along the contour $C = C_+ \oplus C_- \oplus C_3$, extended to include a third imaginary leg, where ϕ takes the values ϕ_+, ϕ_-, and ϕ_3 on C_+, C_-, and C_3, respectively, for which S_C is $S[\phi_+]$, $S[\phi_-]$, and $S_0[\phi_3]$. See Fig. 3.

We stress again that although $S_0[\phi]$ may look like the

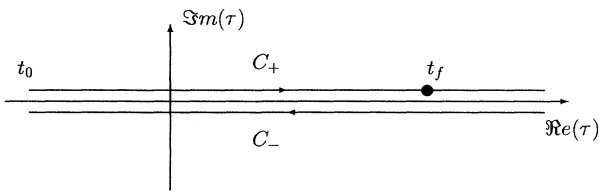

FIG. 2. Extending the integration contour.

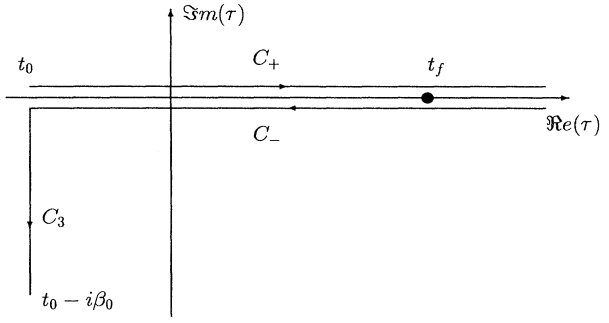

FIG. 3. A third imaginary leg.

quadratic part of $S[\phi]$, its role is solely to encode the initial distribution of configurations Φ and need have nothing to do with the physical action. Henceforth we drop the suffix f on Φ_f and take the origin in time from which the evolution begins as $t_0 = 0$.

We perform one final maneuver with $p_t[\Phi]$ before resorting to further approximation. This will enable us to avoid a nominally ill-defined inversion of a two-point function later on, a consequence of the seeming independence of ϕ_+ and ϕ_- mentioned earlier. Consider the generating functional

$$Z[j_+, j_-, j_3] = \int_B \mathcal{D}\phi \, \exp\left\{ iS_C[\phi] + i \int j\phi \right\},$$

where $\int j\phi$ is shorthand notation for

$$\int j\phi \equiv \int_0^\infty dt \, [\, j_+(t)\phi_+(t) - j_-\phi_-(t) \,] + \int_0^{-i\beta} j_3(t)\phi_3(t)\, dt,$$

omitting spatial arguments. Then introducing $\alpha_a(\mathbf{x})$ where $a = 1, \ldots, N$, we find

$$p_{t_f}[\Phi] = \int \mathcal{D}\alpha \int_B \mathcal{D}\phi \, \exp\left\{ iS_C[\phi] \right\} \exp\left\{ i \int d^D x \alpha_a(\mathbf{x})[\phi_+(t_f, \mathbf{x}) - \Phi(\mathbf{x})]_a \right\}$$
$$= \int \mathcal{D}\alpha \, \exp\left\{ -i \int \alpha_a \Phi_a \right\} Z[\overline{\alpha}, 0, 0],$$

where $\overline{\alpha}$ is the source $\overline{\alpha}(t, \mathbf{x}) = \alpha(\mathbf{x})\delta(t - t_f)$. As with $\mathcal{D}\phi$, $\mathcal{D}\alpha$ denotes $\prod_1^N \mathcal{D}\alpha_a$.

We have said that analytic progress can only be made insofar as $p_t[\Phi]$ is itself Gaussian, requiring in turn that $Z[\overline{\alpha}, 0, 0]$ be Gaussian in the source $\overline{\alpha}$. In order to treat the fall from the false into the true vacuum, at best this means adopting a self-consistent or variational approach i.e., a Hartree approximation or a large N expansion.[2] However, if we limit ourselves to small times t then $p_t[\Phi]$ is genuinely Gaussian since the field has not yet felt the upturn of the potential. That is, we may treat the potential as an inverted parabola until the field begins to probe beyond the spinodal point. The length of time for which it is a good approximation to ignore the upturn of the potential is greatest for weakly coupled theories which, for the sake of calculation we assume, but physically, we expect that if the defect counting approximation is going to fail, then it will do so in the early part of the fall down the hill.

IV. INITIAL CONDITIONS

The onset of the phase transition at time $t = 0$ is characterized by the instabilities of long wavelength fluctuations permitting the growth of correlations. Although the initial value of $\langle\phi\rangle$ over any volume is zero, the resulting phase separation or spinodal decomposition will lead to domains of constant $\langle\phi\rangle$ whose boundaries will evolve so that ultimately, the average value of ϕ in some finite volume will be nonzero. That is, the relativistic system has a nonconserved order parameter. In this sense, the model considered here is similar to those describing the λ transition in liquid helium or transitions in superconductors.

Consider small amplitude fluctuations of ϕ_a at the top of the parabolic potential hill described by $V(\phi) = \frac{1}{2}m^2(t)\phi_a^2$. At $t < 0$, $m^2(t) > 0$ and, for $t > 0$, $m^2(t) < 0$. However, by $t \approx \Delta t$, $m^2(t)$ and λ have achieved their final values, namely $-\mu^2$ and λ. Long wavelength fluctuations, for which $|\mathbf{k}|^2 < -m^2(t)$, begin to grow exponentially. If their growth rate $\Gamma_k \approx \sqrt{-m^2(t) - |\mathbf{k}|^2}$ is much slower than the rate of change of the environment which is causing the quench, then those long wavelength modes are unable to track the quench. For the case in point, this requires $m\Delta t \ll 1$. We take this to be so. To exemplify the growth of domains and the attendant dispersal of defects, it is sufficient to take the idealized case, $\Delta t = 0$, in which the change of parameters at $t = 0$ is instantaneous. That is, $m^2(t)$ satisfies

$$m^2(t) = \begin{cases} m_0^2 > 0 & \text{if } t < 0, \\ -\mu^2 < 0 & \text{if } t > 0, \end{cases}$$

where, for $t < 0$, the field is in thermal equilibrium at inverse temperature β_0. μ is a physical parameter, but m_0 is not. A concrete choice of m_0 is to take it as the effective scalar field mass at temperature $T_0 = \beta_0^{-1}$. That is, in the mean-field approximation we take

$$m_0^2 = -\mu^2 \left(1 - \frac{T_0^2}{T_c^2} \right),$$

where T_0 is greater than the transition temperature T_c, given by $T_c^2 = 12\mu^2/(N+2)\lambda$ in the same approximation. As will be seen, results are insensitive to m_0, provided it is comparable to μ. With the parametrization above, this would be true unless we quenched from very close to the transition, and we do not consider this possibility here. As for $\lambda(t)$, for $t < 0$, it has already been set to zero, so that $p_{t_0}[\Phi]$ be Gaussian. However, for small t, when the amplitude of the field fluctuations is small, the field has yet to experience the upturn of the potential and we can effectively take $\lambda(t) = 0$ then as well when determining the field evolution. At best, this can be valid until the exponential growth $|\phi| \approx \mu e^{\mu t}$ in the amplitude reaches the point of inflection $|\phi_{in}| = \mu/\sqrt{3\lambda}$, that is

[2] In the latter case, given the relationship between N and spatial dimension $D = N$, this corresponds to a large-dimension expansion.

$\mu t \sim \ln(1/\lambda)$. The smaller the coupling, the longer this approximation is valid. As noted earlier, it should be possible to perform more sophisticated calculations with the aim of evolving the defect density right through the transition. For our present purposes, however, the small time or Gaussian approximation is adequate.

There is one final concern. A necessary condition for this rolling down of the field to be valid is that the initial field distribution at $t \leq 0$ should not overhang the point of inflection. That is, the initial thermal field fluctuations about $\phi_a = 0$ should be small enough that the field does not initially feel the non-Gaussian nature of the potential and there is no significant probability that the field is already in the true vacuum.

This is not difficult to check. In equilibrium, the probability $p_{t_f}[\Phi_f]$, now independent of t, is calculable as

$$p[\Phi] = N\, e^{-\beta H[\Phi]}.$$

N is a normalization factor and H permits the expansion in $\beta_0 m_0$ [4]:

$$H[\Phi] = S_0[\Phi] - \frac{1}{24}\beta_0^2 \int d^D x \left(\frac{\delta S_0}{\delta \Phi}\right)^2 + O(\beta_0^4 m_0^4).$$

With calculational simplicity in mind we restrict ourselves to high initial temperatures ($\beta_0 m_0 \ll 1$) for which it is sufficient to retain only the first term.

As always, fluctuations are defined with respect to some length scale. At time $t \leq 0$ we see from the definition of S_0 that the fields Φ_a are correlated over a distance $\xi_0 = m_0^{-1}$. Let v be any correlation volume $v = \xi_0^3$. The fields Φ_a coarse grained to this volume are defined by

$$\Phi_{a,v} = \frac{1}{v}\int_{x \in v} d^D x\, \Phi_a(\mathbf{x})$$

(where v denotes both the position of the correlation volume and its magnitude).

The probability $p_v(\bar{\Phi})$ that $\Phi_v = \bar{\Phi}$ can be written as

$$p_v(\bar{\Phi}) = N \exp[-\bar{\Phi}^2/2(\Delta_v \Phi)^2]\ ,$$

where $(\Delta_v \Phi)^2$ is the coarse-grained, two-point function

$$\langle \Phi_{a,v} \Phi_{b,v} \rangle = \delta_{ab} (\Delta_v \Phi)^2\ ,$$

defined with respect to $S_0[\Phi]$. For the quadratic $S_0[\Phi]$ chosen here it is straightforward to show [4] that, for $v = \xi^3$,

$$(\Delta_v \Phi)^2 \simeq A m_0 T_0\ ,$$

where $A \simeq 10^{-1}$ (depending slightly on the geometry of v).

The condition that there be no overhang is thus

$$(\Delta_v \Phi)^2 \ll \phi_{in}^2\ .$$

There is no difficulty in satisfying this for small coupling. For example, with the parametrization of m_0 given earlier, let us suppose that $T_0^2 = 2T_c^2$, so that $m_0 = \mu$. Then the condition for no overhang for correlation-volume fluctuations is

$$\sqrt{\lambda} \ll \frac{\sqrt{(N+2)/6}}{6A}\ ,$$

automatically satisfied by the small λ that we need for rolling to be relevant.

We note in passing that a similar concern would have arisen had the transition been *first* order, where the condition

$$(\Delta_v \Phi)^2) \ll \langle \Phi \rangle^2$$

is necessary for bubble nucleation to dominate over thermal fluctuations as the mechanism for the transition dynamics. This has been discussed in detail for the electroweak transition in the standard model [11]. The fact that A is such a small number makes thermal fluctuations irrelevant in this case.

Finally, although we are happy to remain with the Gaussian approximation in this preliminary analysis, it is possible to be more general. Using Chebycheff inequalities, even if $S_0[\Phi]$ is not quadratic, it is not difficult to show [4] that

$$p(\Phi_v > \bar{\Phi}) \leq N \exp[-\beta_0 v V_v(\bar{\Phi})]\ ,$$

where $V_v(\bar{\Phi})$ is the effective potential coarse grained to volume v, and we have taken $\bar{\Phi} > 0$ for convenience. This is not the usual effective potential $V_{eff}(\bar{\Phi})$, which is recovered from $V_v(\bar{\Phi})$ at high T_0 only in the $v \to \infty$ limit. For the case of S_0 quadratic the inequality becomes an equality and we recover the previous result, which can be rewritten as

$$V_v(\bar{\Phi}) = \frac{1}{2A} m_0^2 \Phi^2$$

for correlation volumes, ten times steeper than the classical potential.

V. COUNTING THE DEFECT DENSITY

Given a Gaussian $p_t[\Phi]$, the probability that the measurement of the field $\phi(t, \mathbf{x}) = (\phi_1, \phi_2, \ldots, \phi_N)$ would yield the result $\Phi(\mathbf{x}) = (\Phi_1, \Phi_2, \ldots, \Phi_N)$, the defect density can be calculated quite easily. Let us write the Gaussian distribution in the field in the form

$$p_t[\Phi] = \mathcal{N} \exp\Big(- - \frac{1}{2}\int d^D x\, d^D y \times \Phi_a(\mathbf{x}) K_{ab}(\mathbf{x} - \mathbf{y}; t) \Phi_b(\mathbf{y})\Big),$$

with $K_{ab} = \delta_{ab} K$ and $\mathcal{N}$ a normalization. In this we follow Halperin [12], whose work on defect counting in thermal equilibrium will be adapted to our circumstances.

For weakly coupled theories we have argued that, for short times after t_0 at least, a Gaussian $p_t[\Phi]$ will occur. If this is taken for granted it is relatively straightforward to calculate the number density of defects. Postponing the calculation of K until then, we quote those of Halperin's results that are relevant to us.

Suppose that the O(N) vector field $\phi(t, \mathbf{x})$ takes the

particular value $\Phi(\mathbf{x})$. We count the vortices by identifying them with its zeros. The only way for a zero to occur with significant probability is at the center of a topological defect so, but for a set of measure zero, all zeros are topological defects. This will be clarified later.

Consider the O(D) theory in D spatial dimensions, with global *monopoles*. Although less relevant than strings for the early Universe they are slightly easier to perform calculations for. Almost everywhere, monopoles occur at the zeros of $\Phi(\mathbf{x})$, labeled $\mathbf{x_i}$, $i = 1, 2, \ldots$, at which the orientation $\Phi(\mathbf{x})/|\Phi(\mathbf{x})|$ is ill defined. A topological winding number $n_i = \pm 1$ can be associated with each zero $\mathbf{x_i}$ by the rule

$$n_i = \operatorname{sgn} \det(\partial_a \Phi_b)\Big|_{\mathbf{x}=\mathbf{x_i}} .$$

Monopoles with higher winding number are understood as multiple zeros of $\Phi(\mathbf{x})$ at which the n_i are additive. The *net* monopole density is then given by

$$\rho_{\rm net}(\mathbf{x}) = \sum_i n_i \, \delta(\mathbf{x} - \mathbf{x_i}).$$

The volume integral of this gives the number of monopoles minus the number of antimonopoles. The correlations of $\rho_{\rm net}$ give us information on monopole-(anti)monopole correlations but, in the first instance, we are interested in the cruder grand totals. The quantity of greater relevance to us is the *total* monopole density

$$\rho(\mathbf{x}) = \sum_i |n_i| \delta(\mathbf{x} - \mathbf{x_i}) = \sum_i \delta(\mathbf{x} - \mathbf{x_i}) \ ,$$

whose volume integral gives the total number of monopoles *plus* antimonopoles in the volume of integration.

Now consider an ensemble of systems in which the fields Φ are distributed according to $p_t[\Phi]$ at time t. Then, on average, the total monopole density (with suffix m for *monopoles*) is

$$\rho_m(t) = \langle \rho(\mathbf{x}) \rangle_t = \left\langle \sum_i \delta(\mathbf{x} - \mathbf{x_i}) \right\rangle_t ,$$

where the triangular brackets denote averaging with respect to $p_t[\Phi]$. That is,

$$\langle F[\Phi] \rangle_t = \int \mathcal{D}\Phi \, F[\Phi] \, p_t[\Phi],$$

with $p_t[\Phi]$ normalized so that $\int \mathcal{D}\Phi \, p_t[\Phi] = 1$. The translational invariance of the Gaussian kernel of the probabililty density ensures that $\rho(t)$ is independent of $\mathbf{x}$.

In terms of the fields Φ_a, vanishing at $\mathbf{x_i}$, $\rho_m(t)$ can be reexpressed as

$$\rho_m(t) = \langle \, \delta^D[\Phi_c(\mathbf{x})] \, | \det[\partial_a \Phi_b(\mathbf{x})] | \, \rangle_t .$$

The second term in the angular brackets is just the Jacobian of the transformation from $\mathbf{x}$ to $\Phi(\mathbf{x})$, explicable from the form of $\rho(\mathbf{x})$ above. Similarly, although we have less need for it, the *net* monopole density at any time is

$$\langle \rho_{\rm net}(t) \rangle = \langle \, \delta^D[\Phi(\mathbf{x})] \det[\partial_a \Phi_b(\mathbf{x})] \rangle_t .$$

Since field zeros that do not correspond to defects have $\det[\partial_a \Phi_b(\mathbf{x})] = 0$, since otherwise $n_i = \pm 1$ from above, it is apparent that only those zeros corresponding to defects are being counted.

It follows from the Gaussian form of the probability density that the Φ_a are individually and independently Gaussian distributed with zero mean, as

$$\langle \, \Phi_a(\mathbf{x}) \, \Phi_b(\mathbf{x}) \, \rangle_t = \delta_{ab} W(|\mathbf{x} - \mathbf{y}|; t),$$

where $W(|\mathbf{x}-\mathbf{y}|;t) = K^{-1}(\mathbf{x}-\mathbf{y};t)$. So also are the first derivatives of the field $\partial_a \Phi_b$, which are independent of the Φ field,

$$\langle \Phi_c(\mathbf{x}) \partial_a \Phi_b(\mathbf{y}) \rangle_t = 0 \ ,$$

due to the fact that W is dependent only on the magnitude of $\mathbf{x} - \mathbf{y}$.

Thus, the total defect density may be separated into two independent parts:-

$$\rho_m(t) = \langle \, \delta^D[\Phi(\mathbf{x})] \rangle_t \, \langle \, | \det(\partial_a \Phi_b) | \, \rangle_t .$$

The first factor is easy to calculate, the second less so.

Consider first the δ-distribution factor

$$\begin{aligned} \langle \delta[\Phi(\mathbf{x_0})] \rangle_t &= \int \mathcal{D}\Phi \, \delta[\Phi(\mathbf{x_0})] \, \exp\left\{ -\frac{1}{2} \int d^D\mathbf{x} d^D\mathbf{y} \Phi(\mathbf{x}) K(\mathbf{x}-\mathbf{y};t) \Phi(\mathbf{y}) \right\} \\ &= \int \not{d}\alpha \int \mathcal{D}\Phi \, \exp\left\{ i\alpha\Phi(\mathbf{x_0}) - \frac{1}{2} \int d^D\mathbf{x} d^D\mathbf{y} \Phi(\mathbf{x}) K(\mathbf{x}-\mathbf{y};t) \Phi(\mathbf{y}) \right\}, \end{aligned}$$

where O(N) indices and integrals over spatial variables have been suppressed and $\not{d}\alpha = d\alpha/2\pi$. On defining $\overline{\alpha} = \delta(\mathbf{x} - \mathbf{x_0})\alpha$, we find

$$\begin{aligned} \langle \delta[\Phi(\mathbf{x_0})] \rangle_t &= \int \not{d}\alpha \int \mathcal{D}\Phi \, \exp\left\{ \int d^D x \left(i\overline{\alpha}(\mathbf{x})\Phi(\mathbf{x}) - \frac{1}{2} \int d^D x d^D y \Phi(\mathbf{x}) K(\mathbf{x}-\mathbf{y};t) \Phi(\mathbf{y}) \right) \right\} \\ &= \int \not{d}\alpha \exp\left\{ -\frac{1}{2} \int d^D x d^D y \overline{\alpha}(\mathbf{x}) W(|\mathbf{x}-\mathbf{y}|;t) \overline{\alpha}(\mathbf{y}) \right\} \\ &= \int \not{d}\alpha \exp\left\{ -\frac{1}{2} \int d^D x d^D y \delta(\mathbf{x}-\mathbf{x_0}) \alpha W(|\mathbf{x}-\mathbf{x}'|;t) \alpha \delta(\mathbf{x}'-\mathbf{x_0}) \right\} \\ &= \frac{1}{2\pi} \left(\frac{1}{\sqrt{K^{-1}}} \right)^2 = \frac{1}{2\pi \langle \Phi\Phi \rangle} = \frac{1}{2\pi W(0;t)} \ . \end{aligned}$$

Consider now the second factor. Writing out the determinant explicitly for $N = D = 2$, and exploiting the fact that the field is Gaussian, we have

$$\langle\,|\det[\partial_a\phi_b(\mathbf{x})]|\,\rangle_t^2 = \left\langle \left[\det[\partial_a\phi_b(\mathbf{x})]\right]^2 \right\rangle_t$$
$$= \left\langle (\partial_1\phi_1\partial_2\phi_2)^2 + (\partial_1\phi_2\partial_2\phi_1)^2 - 2\partial_1\phi_2\partial_2\phi_1\partial_1\phi_1\partial_2\phi_2 \right\rangle_t .$$

The first term may be factorized into a product of two Gaussian variables and calculated as follows:

$$\langle\,(\partial_1\phi_1\partial_2\phi_2)^2\,\rangle_t = \langle\,(\partial_1\phi_1)^2\,\rangle_t\,\langle\,(\partial_2\phi_2)^2\,\rangle_t$$
$$= [-\delta_{11}\partial_1\partial_1 W(|\mathbf{x}|;t)]\,[-\delta_{22}\partial_2\partial_2 W(|\mathbf{x}|;t)] = [\partial_1\partial_1 W(|\mathbf{x}|;t)]^2,$$

where $W(|\mathbf{x}|;t)$ is as before. Fourier transforming the two-point function, we find:

$$\langle\,(\partial_1\phi_1\partial_2\phi_2)^2\,\rangle_t = \left(\partial_1\partial_1 \int e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{W}(\mathbf{k};t)\,\not{d}^3k\right)\left(\partial_2\partial_2 \int e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{W}(\mathbf{k};t)\,\not{d}^3k\right)$$
$$= \left(\int k^2\cos^2(\theta)W(\mathbf{k};t)\,\not{d}^3k\right)^2$$
$$= [\nabla^2 W(|\mathbf{x}|;t)]^2\Big|_{\mathbf{x}=\mathbf{0}} .$$

A similar result applies for the second term whereas the third term vanishes. The final result is

$$\rho_m(t) = C_N\left|\frac{W''(0;t)}{W(0;t)}\right|^{N/2},$$

where the second derivative in the numerator is with respect to $x = |\mathbf{x}|$. C_N is $1/\pi^2$ for $N = D = 3$ and $1/2\pi$ for $N = D = 2$ had we performed the calculation in $D = 2$ dimensions, the difference coming entirely from the determinant factor.

Let us now consider the case of global strings in $D = 3$ dimensions that arise from the O(2) theory. Strings are identified with lines of zeros of $\phi(t,\mathbf{x}) = \Phi(\mathbf{x})$ and the net vortex density, the density of vortices minus that of antivortices, in a plane perpendicular to the i direction is

$$\rho_{\mathrm{net},i}(\mathbf{x}) = \delta^2[\Phi(\mathbf{x})]\,\epsilon_{ijk}(\partial_j\Phi_1)(\partial_k\Phi_2),$$

with obvious generalizations to $N = D - 1$, for all N in terms of the Levi-Cevita symbol $\epsilon_{i_1,i_2,\ldots,i_D}$. As before, the *total* vortex density is of more immediate use. On a surface perpendicular to the i direction this is

$$\rho_i(\mathbf{x}) = \delta^2[\Phi(\mathbf{x})]\,|\epsilon_{ijk}(\partial_j\Phi_1)(\partial_k\Phi_2)|\ ,$$

in analogy with the monopole case. The expectation value of this total density, when calculated as before reproduces the same expression (where s now denotes *strings*):

$$\rho_{s,i}(t) = \langle\rho_i(\mathbf{x})\rangle = C_N\left|\frac{W''(0;t)}{W(0;t)}\right|^{N/2},$$

independent of i, but for $N = D - 1$. For the case of interest, $N = 2$ and $C_2 = 1/2\pi$.

VI. EVOLUTION OF THE DEFECT DENSITY

We are now in a position to evaluate $p_t[\Phi]$ for $t > 0$, identify K, and calculate the defect density accordingly. $S_C[\phi]$ becomes $S_0[\phi_3]$ on segment C_3. On setting the boundary condition $\phi_+(0,\mathbf{x}) = \phi_3(0,\mathbf{x}) = \phi_3(-i\beta_0,\mathbf{x})$ and we have:-

$$S[\phi_+] = \int d^{D+1}x\left[\frac{1}{2}(\partial_\mu\phi_a)(\partial^\mu\phi_a) + \frac{1}{2}\mu^2\phi_a^2\right],$$

on C_+. The Gaussian integrations can now be performed to give

$$p_t[\Phi] = \int \mathcal{D}\alpha$$
$$\times \exp\left\{-i\int d^Dx\,\alpha_a\Phi_a\right\}$$
$$\times \exp\left\{\frac{i}{2}\int d^Dx\,d^Dy\,\alpha_a(\mathbf{x})G(\mathbf{x}-\mathbf{y};t,t)\alpha_b(\mathbf{y})\right\},$$

where $G(\mathbf{x}-\mathbf{y};t,t)$ is the equal time correlation, or Wightman, function with thermal boundary conditions. Because of the time evolution there is no time translation invariance in the double time label. As this is not simply invertible, we leave the α integration unperformed. The form is then a mnemonic reminding us that $K^{-1} = G$.

In fact, there is no need to integrate the αs since from the previous equation it follows that the characteristic functional $\langle\exp\{i\int J_a\Phi_a\}\rangle_t$ is directly calculable as

$$\left\langle \exp\left\{ i \int j_a \Phi_a \right\} \right\rangle_t = \int \mathcal{D}\Phi \, p_t[\Phi] \exp\left\{ i \int j_a \Phi_a \right\}$$
$$= \exp\left\{ \frac{1}{2} \int d^D x \, d^D y \, j_a(\mathbf{x}) G(\mathbf{x}-\mathbf{y}; t, t) j_a(\mathbf{y}) \right\}.$$

Thus for example, the first factor in the monopole density $\rho_m(t)$ is

$$\langle \delta^D[\Phi(\mathbf{x})] \rangle_t = \left\langle \int dj \, \exp\left[i \Phi_a(\mathbf{x}) j_a \right] \right\rangle_t$$
$$= \int dj \, \exp\left\{ \frac{1}{2} j_a^2 G(\mathbf{0}; t, t) \right\} = [-iG(\mathbf{0}; t, t)]^{-D/2},$$

with suitable normalization, without having to invert $G(\mathbf{0}; t)$. Thus, on identifying $-iG(\mathbf{x}; t, t)$ with $W(\mathbf{x}, t)$ as defined earlier, $\rho_m(t)$ becomes

$$\rho_m(t) = C_N \left| \frac{-iG''(\mathbf{0}; t, t)}{-iG(\mathbf{0}; t, t)} \right|^{N/2}.$$

$-iG(\mathbf{x}; t, t)$ has to be built from the modes $\mathcal{U}^{\pm}_{a,k}$, satisfying the equations of motion

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t) \right] \mathcal{U}^{\pm}_{a,k} = 0,$$

for $m^2(t)$ above, subject to the initial condition of a thermal distribution.

Details are given by Boyanovsky *et al.*, [10] and we quote their results, which give $-iG(\mathbf{x}; t, t)$ as the real, positive quantity:

$$-iG(\mathbf{x}; t, t) = \int \frac{d^D k}{2\omega_<(k)} e^{i\mathbf{k}\cdot\mathbf{x}} \coth[\beta_0 \omega_<(k)/2] \left\{ \left[1 + A_k \{ \cosh[2W(k)t] - 1 \} \right] \theta(\mu^2 - |\mathbf{k}|^2) \right.$$
$$\left. + \left[1 + \alpha_k \{ \cos[2\omega_>(k)t] - 1 \} \right] \theta(|\mathbf{k}|^2 - \mu^2) \right\},$$

with

$$\omega^2_<(k) = |\mathbf{k}|^2 + m_0^2,$$
$$\omega^2_>(k) = |\mathbf{k}|^2 - \mu^2,$$
$$W^2_<(k) = \mu^2 - |\mathbf{k}|^2,$$
$$A_k = \frac{1}{2}\left(1 + \frac{\omega^2_<(k)}{W^2(k)} \right),$$
$$\alpha_k = \frac{1}{2}\left(1 - \frac{\omega^2_<(k)}{\omega^2_>(k)} \right).$$

The first term is the contribution of the unstable long wavelength modes; the second is that of the short wavelength stable modes which provide the noise. The first term will dominate for large times and even though the approximation is only valid for small times, there is a regime, for small couplings, in which t is large enough for $\cosh(2\mu t) \approx \frac{1}{2}\exp(2\mu t)$ and yet μt is still smaller than the time $O(\ln 1/\lambda)$ at which the fluctuations sample the deviation from a parabolic hill. In these circumstances the integral at time t is dominated by a peak in the integrand $k^{D-1} e^{2W(k)t}$ at k around k_c, where

$$t k_c^2 = \frac{(D-1)}{2} \mu \left[1 + O\left(\frac{1}{\mu t} \right) \right],$$

and we have assumed μ and m_0 to be comparable. The effect of changing β_0 is only visible in the $O(1/\mu t)$ term. In the region $|\mathbf{x}| < \sqrt{t/\mu}$ the integral is dominated by the saddle point at k_c, to give

$$-- iG(\mathbf{x}; t, t) = W(x; t)$$
$$\approx W(0; t) \times \exp\left(\frac{-\mu x^2}{8t} \right) \mathrm{sinc}\left(\frac{x}{\sqrt{t/\mu}} \right),$$

for $D = 3$, where

$$W(0; t) \approx C \frac{e^{2\mu t}}{(\mu t)^{3/2}},$$

for some C, which we do not need to know. The exponential growth of $G(\mathbf{0}; t)$ in t reflects the way the field amplitudes fall off the hill centered at $\Phi = 0$.

After symmetry breaking to $O(N-1)$ the mass of the Higgs is $m_H = \sqrt{2}\mu$ with cold correlation length $\xi(0) =$

m_H^{-1}. On identifying $e^{-\mu x^2/8t}$ with $e^{-x^2/\xi^2(t)}$ we interpret

$$\xi(t) = (8\sqrt{2})^{1/2}\sqrt{t\,\xi(0)},$$

as the size of Higgs field domains. This $t^{1/2}$ growth behavior at early times is characteristic of relativistic systems (i.e., with a double time derivative) with a nonconserved order parameter.

To calculate the number density of defects at early times we have to insert this expression for $-iG$ or W into the equations derived earlier. Expanding $W(x;t)$ as

$$W(x;t) = W(0;t)\,\exp\left(\frac{-x^2}{\xi^2(t)}\right) \times\left[1 - \frac{4}{3}\frac{x^2}{\xi^2(t)} + \mathrm{O}\left(\frac{x^2}{\xi^2(t)}\right)\right]$$

gives

$$\rho_m(t) = \frac{1}{\pi^2}\left(\frac{\sqrt{14/3}}{\xi(t)}\right)^3 \approx \frac{1.02}{\xi(t)^3},$$

for an O(3) theory with monopoles in three-dimensions and

$$\rho_{s,i}(t) = \frac{1}{2\pi}\left(\frac{\sqrt{14/3}}{\xi(t)}\right)^2 \approx \frac{0.74}{\xi(t)^2},$$

for an O(2) theory with strings in three-dimensions. The first observation is that the dependence of the density on time t is only through the correlation length $\xi(t)$, or in other words we have a *scaling* solution. This is entirely unconnected with the scaling solutions usually studied in the context of defects, which are relevant at late times. As the domains of coherent field form and expand, the interdefect distance grows accordingly. Since the only way the defect density can decrease without the background space-time expanding is by defect-antidefect annihilation, we deduce that the coalescence of domains proceeds by the annihilation of defects on the boundaries. In the case of monopoles, this is simple annihilation, whereas in the case of strings, it is the annihilation of small loops of string. However, since the density of defects only depends on $\xi(t)$ in this early stage, the fraction of string in "infinite" string remains constant. [3] Thus, at the same time as small loops disappear, other loops must rearrange themselves so that the length of "infinite" string decreases accordingly. Second, there is roughly one defect per coherence size, a long held belief for whatever mechanism.

[3]By "infinite" string we mean that string that does not intersect itself. We have not shown that vortices behave as random walks, although it seems approximately likely. If they do, then the fraction of string in "infinite" string is determined by the probability that such a walk does not self-intersect. There is more string in infinite string than in finite (which means small) loops.

It is also possible to use Halperin's results to calculate defect-defect correlation functions. For example, the monopole-monopole correlation function on scales larger than a coherence length is found to be

$$\langle\rho_{\mathrm{net}}(\mathbf{x})\,\rho_{\mathrm{net}}(\mathbf{0})\rangle_t = \rho_m(t)\delta(\mathbf{x}) + g(\mathbf{x})_t,$$

where $g(\mathbf{x})_t$ is a measure of the screening of a monopole at the origin, satisfying

$$\int d^D x\, g(\mathbf{x})_t = -\rho_m(t)\ .$$

Explicit calculation yields

$$g(\mathbf{x})_t = -\frac{3\sqrt{2}\exp[-3x^2/\xi^2(t)]\sin^3[2\sqrt{2}x/\xi(t)]}{8\pi^3 x\xi^5(t)} \times\left[1 + \mathrm{O}\left(\frac{\xi(t)}{x}\right)\right].$$

This alternation in the sign of the screening on scale $\xi(t)$ is compatible with the density result presented previously. Similar calculations have been performed for the correlations of the string densities $\rho_{net,i}(\mathbf{x})$ introduced earlier. At the moment this is more information than we need.

VII. CONCLUSIONS

Under the conditions of a symmetry-breaking phase transition, from O(N) to O($N-1$), which proceeds by a rapid quench, we have derived expressions for the evolution of the defect density during the early part of the fall from the false vacuum to the true vacuum, at least for weakly coupled theories. In particular, there is approximately one defect per correlation volume for the time during which the approximation is valid. The $t^{1/2}$ time dependence of the correlation length $\xi(t)$ that we have seen above is specific to a nonconserved order parameter in a theory with a double time derivative, but we expect the qualitative features to be similar for all defect production by quenched symmetry-breaking phase transitions.

Thus, generically, we expect the defect density to follow the correlation length during the early part of the fall from the hill. Further, the more weakly coupled the theory, the more that the correlation length will grow during the fall, since the period before which the fields feel the upturn in potential and slow down is longer. For very weakly coupled theories the correlation length can grow significantly in the time interval $t = \mathrm{O}(\ln(1/\lambda))$ available. In almost all physically realistic scenarios except inflation, however, the coupling is not small enough for this growth to be significant. For more strongly coupled theories we know less but self-consistent calculations can be performed using the methods of Boyanovsky and coworkers [9,10] and will be discussed elsewhere.

ACKNOWLEDGMENTS

The authors are grateful to R. Brandenberger, T.S. Evans, M. Hindmarsh, T.W.B. Kibble, and N. Turok for useful conversations, particularly M. Hindmarsh for his observations on the work of Halperin and T.W.B. Kibble for his understanding of the mechanism of defect formation. Both the authors would like to thank the Isaac Newton Insitute for its hospitality and A.J.G. would like to thank PPARC for its support.

[1] T.W.B. Kibble, J. Phys. A **9**, 1387 (1976).
[2] T. Vachaspati and A. Vilenkin, Phys. Rev. **D 30**, 2036 (1984).
[3] V.I. Ginzburg, Fiz. Tverd. Tela (Leningrad) **2**, 2031 (1960) [Sov. Phys. Solid State **2**, 1826 (1961)].
[4] M. Hindmarsh and R.J. Rivers, Nucl. Phys. **417**, 506 (1994).
[5] W.H. Zurek, Nature (London) **317**, 505 (1985); Acta Phys. Pol. B **24**, 1301 (1993).
[6] K.T. Mahanthappa and P.M. Bakshi, J. Math. Phys. **4**, 1 (1963); **4**, 12 (1963).
[7] J. Schwinger, J. Math. Phys. **2**, 407 (1961).
[8] L.V. Keldysh, Sov. Phys. JETP **20**, 1018 (1965).
[9] D. Boyanovsky, Da-Shin Lee, and Anupam Singh, Phys. Rev. D **48**, 800 (1993).
[10] D. Boyanovsky, H.J. de Vega, R. Holman, D-S. Lee, and A. Singh, Phys. Rev. D **51**, 4419 (1995).
[11] M. Gleiser, Phys. Rev. D **42**, 3350 (1990); M. Gleiser, E. W. Kolb, and R. Watkins, Nucl. Phys. **B364**, 411 (1991).
[12] B.I. Halperin, in *Physics of Defects*, Proceedings of Les Houches, Summer School, Les Houches, France, 1980, edited by R. Balian, M. Kleman, and J.-P. Poirer, Les Houches Summer School Proceedings Vol. XXXV (North-Holland, Amsterdam, 1981), p. 816.