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New classes of exact multistring solutions in curved spacetimes

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We find new classes of *exact* solutions of the equations describing propagation of a classical string, in a variety of given curved backgrounds. They include stationary and dynamical (open, closed, straight, finitely, and infinitely long) strings as well as *multistring* solutions, in terms of elliptic functions. The physical properties, string length, energy, and pressure are computed and analyzed. In anti–de Sitter spacetime, the solutions describe an *infinite* number of infinitely long stationary strings of equal energy but different pressures. In de Sitter spacetime, outside the horizon, they describe infinitely many *dynamical* strings infalling nonradially, scattering at the horizon and going back to spatial infinity in different directions. For special values of the constants of motion, there are families of solutions with *selected finite* numbers of different and independent strings. In black hole spacetimes (without cosmological constant), *no* multistring solutions are found. In the Schwarzschild black hole, inside the horizon, we find one straight string infalling nonradially, with *indefinitely* growing size, into the $r = 0$ singularity. In the (2+1)-black hole anti–de Sitter background, the string stops at $r = 0$ with *finite* length.

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I. INTRODUCTION AND RESULTS

In this paper we find new classes of *exact* solutions of the equations describing propagation of a classical string, in a variety of given curved backgrounds. This is a continuation of our investigations on the *exact* string dynamics in curved spacetimes, see, for example, Refs. [1–3]. In curved spacetimes, it is generally impossible to obtain the exact and complete solution to the string equations of motion and constraints. These are highly nonlinear coupled partial differential equations and are generally not integrable. However, in some curved spacetimes such as gravitational wave backgrounds [4] and cosmic string backgrounds [5], the string dynamics turns out to be exactly solvable. Moreover, the full string equations of motion and constraints have been shown to be exactly integrable in D-dimensional de Sitter spacetime [6]; they are equivalent to a generalized sinh-Gordon equation. Several new properties like the multistring solutions [7,8,1] emerged. On the other hand, in most spacetimes, quite general families of exact solutions can be found by making an appropriate ansatz, which exploits the symmetries of the underlying curved spacetime. In axially symmetric spacetimes, a convenient ansatz corresponds to circular strings. Such an ansatz effectively decouples the dependence on the spatial world-sheet coordinate σ, and the string equations of motion and constraints reduce to nonlinear coupled ordinary differential equations. They are considerably simpler to handle than the original system, and they have indeed been analyzed and solved in a number of interesting cases from gravitation [2,9], cosmology [1,3], and string theory [2,10]. In this paper we will make instead an ansatz which effectively decouples the dependence on the temporal world-sheet coordinate τ. This ansatz, which we call the "stationary string ansatz" is dual to the "circular string ansatz" in the sense that it corresponds to a formal interchange of the world-sheet coordinates (τ, σ), as well as of the azimuthal angle ϕ and the stationary time t in the target space:

$$\tau \leftrightarrow \sigma, \qquad t \leftrightarrow \phi. \tag{1.1}$$

The stationary string ansatz will describe stationary strings when t (and τ) are timelike, for instance, in anti–de Sitter spacetime (in static coordinates) and outside the horizon of a Schwarzschild black hole. On the other hand, if t (and τ) are spacelike, for instance, inside the horizon of a Schwarzschild black hole or outside the horizon of de Sitter spacetime (in static coordinates), the stationary string ansatz will describe dynamical propagating strings. Considering a static line element in the form

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1.2}$$

the stationary string ansatz reads explicitly

$$t = \tau, \quad r = r(\sigma), \quad \phi = \phi(\sigma), \quad \theta = \pi/2. \tag{1.3}$$

The string equations of motion and constraints reduce to two separated first-order ordinary differential equations:

$$\phi' = \frac{L}{r^2}, \qquad r'^2 + V(r) = 0,$$
$$V(r) = -a(r)\left(a(r) - \frac{L^2}{r^2}\right), \tag{1.4}$$

where L is an integration constant. The qualitative features of the possible string configurations can be read off directly from the shape of the potential $V(r)$. Thereafter, the detailed analysis of the quantitative features can be performed by explicitly solving the (integrable) system of Eqs. (1.4). The induced line element on the world sheet is given by

$$ds^2 = a(r)(-d\tau^2 + d\sigma^2). \tag{1.5}$$

Thus, if $a(r)$ is negative, the world-sheet coordinate τ becomes spacelike while σ becomes timelike and the stationary string ansatz (1.3) describes a dynamical string. If $a(r)$ is positive, the ansatz describes a stationary string. In this paper we solve explicitly Eqs. (1.4) and we analyze in detail the solutions and their physical interpretation in Minkowski, de Sitter, anti–de Sitter, Schwarzschild and (2+1)-black hole anti–de Sitter spacetimes. In all these cases, the solutions are expressed in terms of elliptic (or elementary) functions. We furthermore analyze the physical properties, string length, energy, and pressure of these solutions. We also give the qualitative features of the solutions in Schwarzschild–de Sitter and Schwarzschild–anti–de Sitter spacetimes.

In Minkowski spacetime (M), the potential is given by

$$V_M(r) = \frac{L^2}{r^2} - 1, \tag{1.6}$$

and the solution of Eqs. (1.4) describes one infinitely long straight string with "impact-parameter" L. The equation of state takes the well-known form

$$dE = -dP_2 = \text{const}, \quad P_1 = P_3 = 0. \tag{1.7}$$

In anti–de Sitter (AdS) spacetime, the potential is given by

$$V_{\text{AdS}}(r) = -(1 + H^2r^2)\left(1 + H^2r^2 - \frac{L^2}{r^2}\right). \tag{1.8}$$

The radial coordinate $r(\sigma)$ is periodic with finite period T_σ, which is expressed in terms of a complete elliptic integral, Eq. (4.11). For $\sigma \in [0, T_\sigma]$, the solution describes an infinitely long stationary string in the wedge $\phi \in\,]0, \Delta\phi[$, where

$$\Delta\phi = 2k\sqrt{\frac{1-2k^2}{1-k^2}}[\Pi(1-k^2,\ k) - K(k)] \ \in\ ]0,\ \pi[. \tag{1.9}$$

The elliptic modulus k parametrizes the solutions, $k \in\,]0, 1/\sqrt{2}[$. The azimuthal angle is generally not a periodic function of σ; thus, when the spacelike world-sheet coordinate σ runs through the range $]-\infty, +\infty[$, the solution describes an *infinite* number of infinitely long stationary open strings. The general solution is therefore a multistring solution. Until now multistring solutions were only found in de Sitter spacetime [1,7,8]. Our results show that multistring solutions are a general feature of spacetimes with a cosmological constant (positive or negative). The solution in anti–de Sitter spacetime describes a *finite* number of strings if the following relation holds:

$$N\Delta\phi = 2\pi M. \tag{1.10}$$

Here N and M are integers, determining the number of strings and the winding in azimuthal angle, respectively, for the multistring solution, see Fig. 2. The equation of state for a full multistring solution takes the form ($P_3 = 0$)

$$dP_1 = dP_2 = -\tfrac{1}{2}dE \quad \text{for } r \to \infty \tag{1.11}$$

corresponding to extremely unstable strings [11].

In de Sitter (dS) spacetime, the potential is given by

$$V_{\text{dS}}(r) = -(1 - H^2r^2)\left(1 - H^2r^2 - \frac{L^2}{r^2}\right). \tag{1.12}$$

In this case we have to distinguish between solutions inside the horizon (where τ is timelike) and solutions outside the horizon (where τ is spacelike). Inside the horizon, the generic solution describes one infinitely long open stationary string winding around $r = 0$. For special values of the constants of motion, corresponding to a relation, which formally takes the same form as Eq. (1.10), the solution describes a closed string of finite length $l = N\pi/H$. The integer N in this case determines the number of "leaves," see Fig. 3. The energy is positive and finite and grows with N. The pressure turns out to vanish identically, thus the equation of state corresponds to cold matter. Outside the horizon, the world-sheet coordinate τ becomes spacelike while σ becomes timelike, thus we define

$$\tilde{\tau} \equiv \sigma, \qquad \tilde{\sigma} \equiv \tau, \tag{1.13}$$

and the string solution is expressed in hyperboloid coordinates, Eqs. (5.31). The radial coordinate $r(\tilde{\tau})$ is periodic with a finite period $T_{\tilde{\tau}}$, Eq. (5.29). For $\tilde{\tau} \in [0, T_{\tilde{\tau}}]$, the solution describes a straight string incoming nonradially from spatial infinity, scattering at the horizon, and escaping toward infinity again, Fig. 4. The string length is zero at the horizon and grows indefinitely in the asymptotic regions. As in the case of anti–de Sitter spacetime, the azimuthal angle is generally not a periodic function, thus when the timelike world-sheet coordinate $\tilde{\tau}$ runs through the range $]-\infty, +\infty[$, the solution describes an *infinite* number of dynamical straight strings scattering at the horizon at different angles. The general solution is therefore a multistring solution. In particular, a multistring solution describing a *finite* number of strings is obtained if a relation of the form (1.10) is satisfied. It turns out that the solution describes *at least* three strings. The energy and pressures of a full multistring solution are also computed. In the asymptotic region they satisfy an equation of state corresponding to extremely unstable strings, i.e., as Eq. (1.11).

In the Schwarzschild (S) black hole background, the potential is given by

$$V_{\text{S}}(r) = -(1 - 2m/r)\left(1 - 2m/r - \frac{L^2}{r^2}\right). \tag{1.14}$$

No multistring solutions are found in this case. Outside the horizon the solution of Eqs. (1.4) describes one infinitely long stationary open string. This solution was derived in [12] and we shall not go into details here. In our notation the solution is given by Eqs. (6.4)–(6.6). Inside the horizon, where τ becomes spacelike while σ becomes timelike, we make the redefinitions (1.13) and the solution is expressed in terms of Kruskal coordinates, Eq. (6.16). The solution describes one straight string infalling *nonradially* toward the singularity. At the horizon, the string length is zero and it grows *indefinitely* when the string approaches the spacetime singularity. Thereafter, the solution cannot be continued.

In the (2+1)-black hole anti–de Sitter (BH-AdS) spacetime [13], the potential is given by

$$V_{\rm BH-AdS}(r) = -\left(\frac{r^2}{l^2}-1\right)\left(\frac{r^2}{l^2}-1-\frac{L^2}{r^2}\right). \quad (1.15)$$

Outside the horizon, the solutions "interpolate" between the solutions found in anti–de Sitter spacetime and outside the horizon of the Schwarzschild black hole. The solutions thus describe infinitely long stationary open strings. As in anti–de Sitter spacetime, the general solution is a *multistring* describing *infinitely* many strings. In particular, for certain values of the constants of motion, corresponding to the condition of the form (1.10), the solution describes a finite number of strings. In the simplest version of the (2+1)-black hole anti–de Sitter background ($M = 1,\ J = 0$), it turns out that the solution describes *at least* seven strings. Inside the horizon, we make the redefinitions (1.13) and the solution is expressed in terms of Kruskal-like coordinates, Eq. (7.21). The solution is similar to the solution found inside the horizon of

TABLE I. Short summary of the features of the string solutions found in this paper. In anti–de Sitter spacetime and outside the horizon of de Sitter and (2+1)-BH-AdS spacetimes, the solutions describe a *finite* number of strings provided a condition of the form $N\Delta\phi = 2\pi M$ is satisfied, where $\Delta\phi$ is the angle between the "arms" of the string and (N, M) are integers.

Line element	$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
Ansatz	$t = \tau,\ r = r(\sigma),\ \phi = \phi(\sigma),\ \theta = \pi/2$
String solution	$\phi' = \frac{L}{r^2},\ r'^2 + V(r) = 0;\ V(r) = -a(r)[a(r) - \frac{L^2}{r^2}]$
String length element	$dl = \sqrt{a(r(\sigma))}d\sigma$

Minkowski, $a(r) = 1,\ V(r) = \frac{L^2}{r^2} - 1$

The solution describes one infinitely long stationary straight string:

$\frac{dE}{dl} = -\frac{dP_2}{dl} = \text{const},$ $\quad P_1 = P_3 = 0$

Anti–de Sitter, $a(r) = 1 + H^2r^2,\ V(r) = -(1 + H^2r^2)[1 + H^2r^2 - \frac{L^2}{r^2}]$

The solution describes a finite or infinite number of infinitely long stationary strings:

$\frac{dE}{dl} = -2\frac{dP_1}{dl} = -2\frac{dP_2}{dl},$ $\quad P_3 = 0$ (asymptotically for $r \to \infty$)

de Sitter, $A(r) = 1 - H^2r^2,\ V(r) = -(1 - H^2r^2)[1 - H^2r^2 - \frac{L^2}{r^2}]$

Inside the horizon, the solution describes one finitely or infinitely long stationary string winding around $r = 0$:

$P_1 = P_2 = P_3 = 0$, as cold matter, E expressed in terms of elliptic functions.

Outside the horizone, the solution describes a finite or infinite number of dynamical straight strings "scattering at the horizon." The string length vanishes at the horizon, but stretches indefinitely at spatial infinite:

$\frac{dE}{dl} = -2\frac{dP_1}{dl} = -2\frac{dP_2}{dl},$ $\quad P_3 = 0$ (asymptotically for $r \to \infty$)

Schwarzschild, $a(r) = 1 - 2m/r,\ V(r) = -(1 - 2m/r)[1 - 2m/r - \frac{L^2}{r^2}]$

Outside the horizon, the solution describes one infinitely long stationary string.

Inside the horizon, the solution describes one dynamical straight string. The string size is zero at the horizon, and grows *indefinitely* as the string falls toward $r = 0$.

(2+1)-black hole AdS, $a(r) = \frac{r^2}{l^2},\ V(r) = -(\frac{r^2}{l^2} - 1)[\frac{r^2}{l^2} - 1 - \frac{L^2}{r^2}]$

Outside the horizon, the solution describes a finite or infinite number of infinitely long stationary string.

Inside the horizon, the solution describes one dynamical straight string. The string size is zero at the horizon, and grows *finitely* as the string falls toward $r = 0$.

the Schwarzschild black hole, but there is one important difference at $r = 0$. As in the Schwarzschild black hole background, the solution describes one straight string infalling nonradially toward $r = 0$, and beyond this point the solution cannot be continued because of the global structure of the spacetime. At the horizon the string size is zero and during the fall toward $r = 0$, the string size *grows* but stays *finite*. This should be compared with the straight string inside the horizon of the Schwarzschild black hole, where the string size grows indefinitely. The physical reason for this difference is that the point $r = 0$ is not a strong curvature singularity in the (2+1)-black hole anti–de Sitter spacetime.

Finally we consider also the Schwarzschild–de Sitter and Schwarzschild–anti–de Sitter spacetimes. These spacetimes contain all the features of the spacetimes already discussed: singularities, horizons, positive or negative cosmological constants. All the various types of string solutions found in the other spacetimes (open, closed, straight, finitely and infinitely long, multistrings) are therefore present in the different regions of the Schwarzschild–de Sitter and Schwarzschild–anti–de Sitter spacetimes. The details are given in Sec. VIII.

Throughout the paper we use sign conventions of Misner, Thorne, and Wheeler [14] and units in which, beside $G = 1$, $c = 1$, the string tension $(2\pi\alpha')^{-1} = 1$. A short summary of our results is presented in Table I.

II. GENERAL FORMALISM

In this section we derive the ordinary differential equations obtained from the generic string equations of motion and constraints using a stationary string ansatz. For simplicity we consider stationary strings embedded in static spherically symmetric spacetimes. The results, however, can be easily generalized to stationary axially symmetric spacetimes. Stationary strings in stationary background spacetimes were first discussed from a general point of view in Ref. [12]. It was shown that stationary string configurations can be described by a geodesic equation in a properly chosen "internal" space [12]. Using the same formalism, small perturbations around the stationary strings have also been considered [15,16]. In this paper we will use a different approach and we will find with it new classes of solutions. For a stationary string in a general static and spherically symmetric spacetime, we derive an effective potential in the radial spacetime coordinate. The potential provides immediate information about the stationary string configurations. To be more specific we consider the spacetime line element

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (2.1)$$

which includes as special cases Minkowski spacetime, anti–de Sitter spacetime, de Sitter spacetime, Schwarzschild black hole spacetime, and its extensions including charge and cosmological constant. The string equations of motion and constraints are

$$\ddot{x}^\mu - x''^\mu + \Gamma^\mu_{\rho\sigma}(\dot{x}^\rho \dot{x}^\sigma - x'^\rho x'^\sigma) = 0,$$
$$g_{\mu\nu}\dot{x}^\mu x'^\nu = g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + x'^\mu x'^\nu) = 0,$$

where the overdot and prime stand for derivatives with respect to τ and σ, respectively. For the metric defined by the line element (2.1), they take the form

$$\ddot{t} - t'' + \frac{a_{,r}}{a}(\dot{t}\dot{r} - t'r') = 0,$$
$$\ddot{r} - r'' - \frac{a_{,r}}{2a}(\dot{r}^2 - r'^2) + \frac{aa_{,r}}{2}(\dot{t}^2 - t'^2) - ar(\dot{\theta}^2 - \theta'^2) - ar\sin^2\theta(\dot{\phi}^2 - \phi'^2) = 0,$$
$$\ddot{\theta} - \theta'' + \frac{2}{r}(\dot{\theta}\dot{r} - \theta' r') - \sin\theta\cos\theta(\dot{\phi}^2 - \phi'^2) = 0,$$
$$\ddot{\phi} - \phi'' + \frac{2}{r}(\dot{\phi}\dot{r} - \phi' r') + 2\cot\theta(\dot{\theta}\dot{\phi} - \theta'\phi') = 0,$$
$$-a\dot{t}t' + \frac{1}{a}\dot{r}r' + r^2\dot{\theta}\theta' + r^2\sin^2\theta\,\dot{\phi}\phi' = 0,$$
$$-a(\dot{t}^2 + t'^2) + \frac{1}{a}(\dot{r}^2 + r'^2) + r^2(\dot{\theta}^2 + \theta'^2) + r^2\sin^2\theta(\dot{\phi}^2 + \phi'^2) = 0. \quad (2.2)$$

The stationary string ansatz, consistent with the symmetries of the background, is taken to be

$$t = \tau, \quad r = r(\sigma), \quad \phi = \phi(\sigma), \quad \theta = \pi/2; \quad (2.3)$$

i.e., the string is in the equatorial plane and the two functions $r(\sigma)$, $\phi(\sigma)$ are to be determined by the equations of motion and constraints, Eqs. (2.2). After inserting the ansatz, the equations of motion and constraints are consistently reduced to two first-order ordinary differential equations:

$$\phi' = \frac{L}{r^2}, \quad (2.4)$$

$$r'^2 = a(r)\left(a(r) - \frac{L^2}{r^2}\right), \quad (2.5)$$

where L is an integration constant, which without loss of generality can be taken to be non-negative. $r(\sigma)$ is obtained by inversion of the integral

$$\sigma - \sigma_0 = \int_{r_0}^{r} \frac{dx}{\sqrt{a(x)[a(x) - (L/x)^2]}}, \quad (2.6)$$

after which $\phi(\sigma)$ is obtained by integrating Eq. (2.4). In all cases under consideration in this paper, Eqs. (2.4) and (2.5) will be solved in terms of elliptic or elementary functions. It is convenient to define an effective potential $V(r)$ by

$$V(r) = -a(r)\left(a(r) - \frac{L^2}{r^2}\right), \quad (2.7)$$

such that the r equation of motion takes the form

$$r'^2 + V(r) = 0. \quad (2.8)$$

With this definition the stationary string will be located at the r axis in a $(r, V(r))$ diagram. The possible string configurations can therefore be read off from knowledge about the zeros of the potential. The exact string shape can thereafter be obtained by solving Eqs. (2.4) and (2.5). Notice that the circular string solutions of Eq. (2.8) must be considered separately. They are determined by

$$r = \text{const} \equiv r_c, \qquad \phi = \frac{L}{r_c^2}\sigma, \qquad V(r_c) = 0, \tag{2.9}$$

where $r_c \neq 0$, $L \neq 0$, but they are solutions to the original second-order differential equations (2.2) only provided

$$\left.\frac{dV(r)}{dr}\right|_{r=r_c} = 0. \tag{2.10}$$

We shall return to this point later for each of the curved backgrounds under consideration.

Insertion of the ansatz, using the results (2.4) and (2.5), in the line element (2.1), leads to

$$ds^2 = a(r)(-d\tau^2 + d\sigma^2). \tag{2.11}$$

The string length element dl is then identified as

$$dl = \sqrt{a(r(\sigma))}\, d\sigma, \tag{2.12}$$

and the physical string length is obtained by integrating over the appropriate range of σ, which may be finite or infinite. For strings in Friedmann-Robertson-Walker (FRW) universes, it is interesting to also consider the string energy and pressure that can be obtained from the spacetime energy-momentum tensor,

$$\sqrt{-g}T^{\mu\nu} = \int d\tau d\sigma (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu)\delta^{(4)}(X - X(\tau,\sigma)), \tag{2.13}$$

by integrating over a volume that completely encloses the string. The coordinates X^μ are here the comoving FRW coordinates,

$$ds^2 = -(dX^0)^2 + a^2(X^0)\,\frac{dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)}{(1 + \frac{k}{4}R^2)^2}, \tag{2.14}$$

including as special cases Minkowski, de Sitter, and anti–de Sitter spacetimes:

$$a(X^0) = 1, \quad k = 0 \quad \text{for Minkowski spacetime,}$$
$$a(X^0) = e^{HX^0}, \quad k = 0 \quad \text{for de Sitter spacetime,}$$
$$a(X^0) = \cos(HX^0), \quad k = -H^2 \quad \text{for anti–de Sitter spacetime,}$$

which can all be brought into the static spherically symmetric form of Eq. (2.1).

We close this section with the following interesting observation: the function $a(r)$ introduced in the line element (2.1) is not necessarily non-negative. In de Sitter and Schwarzschild spacetimes it changes sign at the horizon. Then the world-sheet coordinate τ becomes spacelike while σ becomes timelike, see Eq. (2.11), and the string solution must be expressed in other coordinates. In such cases the stationary string ansatz, Eq. (2.3), actually describes dynamical propagating strings. We therefore reach the interesting conclusion that the stationary string ansatz, Eq. (2.3), for different initial conditions, describes both stationary equilibrium string configurations as well as dynamical propagating strings, in different regions of the background spacetime.

In the following sections we use the general formalism of this section to describe and analyze stationary and dynamical strings in various spacetimes. For completeness we first recall the results in flat Minkowski spacetime. We then consider anti–de Sitter and de Sitter spacetimes and thereafter turn to the (2+1)- and (3+1)-dimensional black hole spacetimes with and without a cosmological constant.

III. MINKOWSKI SPACETIME

In Minkowski spacetime the situation is considerably simple. The potential, Eq. (2.7), takes the form [see Fig. 1(a)]

$$V(r) = \frac{L^2}{r^2} - 1, \tag{3.1}$$

so that stationary strings are only possible in the region $r \geq L$. Equations (2.4) and (2.5) are easily solved by

$$r(\sigma) = \sqrt{\sigma^2 + L^2}, \tag{3.2}$$

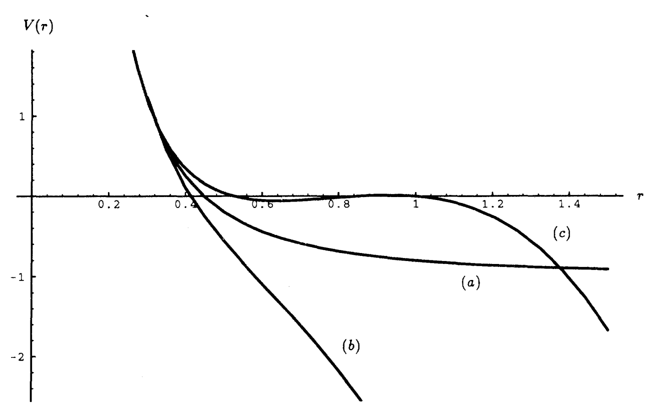

FIG. 1. The potential $V(r)$, Eq. (2.7), in three cases (a) Minkowski, (b) anti–de Sitter, (c) de Sitter spacetimes. The potential is defined such that (classical) string solutions can only exist in the regions where $V(r) \leq 0$.

$$\phi(\sigma) = \arctan(\sigma/L), \tag{3.3}$$

which for $\sigma \in]-\infty, +\infty[$ describes an infinitely long straight string parallel to the X^2 axis with "impact parameter" L. Obviously, the string length, string energy, and integrated string pressure are all infinite. For infinitely long strings it is however more interesting to consider the energy and pressure densities. From Eqs. (2.12) and (2.13) we find

$$\frac{dE}{dl} = \frac{d}{dl}\int d^3X\sqrt{-g}T^{00} = 1, \tag{3.4}$$

$$\frac{dP_2}{dl} = \frac{d}{dl}\int d^3X\sqrt{-g}T^{y}{}_{y} = -1, \tag{3.5}$$

while $P_1 = P_3 = 0$. This is the well-known result concerning stationary strings in flat Minkowski spacetime. Finally, notice that the circular string $r = L$, $\phi = \sigma/L$, which solves the first-order differential equations (2.4) and (2.5), must be excluded since it does not satisfy the original second-order differential equations of motion, Eq. (2.2), cf. the remarks related to Eqs. (2.9) and (2.10).

IV. ANTI–DE SITTER SPACETIME

In anti–de Sitter spacetime, which corresponds to $a(r) = 1 + H^2r^2$, we find the stationary string potential

$$V(r) = -(1+H^2r^2)\left(1+H^2r^2-\frac{L^2}{r^2}\right) \tag{4.1}$$

[see Fig. 1(b)]. It follows that stationary strings can only be found for $r \geq r_0$:

$$r_0 = \frac{1}{H}\left(\frac{-1+\sqrt{1+4H^2L^2}}{2}\right)^{1/2}. \tag{4.2}$$

The circular string configuration $r = r_0$, $\phi = L\sigma/r_0^2$, which solves the first-order differential equations (2.4) and (2.5), is excluded since it does not satisfy the original second-order differential equations of motion, Eq. (2.2), cf. the remarks related to Eqs. (2.9) and (2.10). This is similar to the situation in Minkowski spacetime. It turns out then, that all stationary strings in anti–de Sitter spacetime are infinitely long open strings. For $L = 0$ ($r_0 = 0$) we have the straight string on $r = 0$:

$$r = \frac{1}{H}\tan(H\sigma), \qquad \phi = \text{const}. \tag{4.3}$$

For $L \neq 0$, Eq. (2.5) is solved by a Weierstrass elliptic function

$$H^2r^2(\sigma) = \frac{1}{H^2}\wp(\sigma-\sigma_0;\, g_2,\, g_3) - \frac{2}{3}, \tag{4.4}$$

with invariants

$$g_2 = 4H^4(\tfrac{1}{3}+H^2L^2), \qquad g_3 = \tfrac{4}{3}H^6(\tfrac{2}{9}+H^2L^2), \tag{4.5}$$

discriminant

$$\Delta = 16H^{16}L^4(1+4H^2L^2) \tag{4.6}$$

and roots

$$\begin{aligned} e_1 &= \frac{H^2}{6}[1+3\sqrt{1+4H^2L^2}\,] > e_2 \\ &= -\frac{H^2}{3} > e_3 \\ &= \frac{H^2}{6}[1-3\sqrt{1+4H^2L^2}\,]. \end{aligned} \tag{4.7}$$

The integration constant σ_0 must be carefully chosen in order to obtain a real solution for real σ. In the present case it turns out that σ_0 must be real and it is convenient to take $\sigma_0 = 0$. Then, solution (4.4) can be written in terms of a Jacobi elliptic function

$$H^2r^2(\sigma) = \nu^2 - \mu^2 + \mu^2\, ns^2[\mu H\sigma,\, k], \tag{4.8}$$

where we introduced the more compact notation

$$\mu^2 \equiv \sqrt{1+4H^2L^2}, \qquad \nu^2 \equiv \tfrac{1}{2}(-1+\mu^2\,), \qquad k \equiv \frac{\nu}{\mu}. \tag{4.9}$$

It follows that the elliptic modulus $k \in]0,\, 1/\sqrt{2}[$, where $k \to 0$ corresponds to $L \to 0$, while $k \to 1/\sqrt{2}$ corresponds to $L \to \infty$. By integrating Eq. (2.4) we find

$$\phi(\sigma) = \frac{-k}{\sqrt{1-k^2}}\left\{H\sigma - \sqrt{1-2k^2}\,\Pi(1-k^2,\, \mu H\sigma,\, k)\right\}, \tag{4.10}$$

where Π is the elliptic integral of the third kind. Equations (4.8) and (4.10) provide the complete solution for stationary strings in anti–de Sitter spacetime. The radial coordinate, Eq. (4.8), is periodic with a period

$$T_\sigma = \frac{2K(k)}{\mu H}, \tag{4.11}$$

where $K(k)$ is the complete elliptic integral of the first kind. For $\sigma \in [0,\, T_\sigma]$, r goes from infinity toward $r = r_0$ and back toward infinity. In the same range of σ, the azimuthal angle ϕ goes from $\phi = 0$ to $\phi = \Delta\phi$:

$$\Delta\phi = 2k\sqrt{\frac{1-2k^2}{1-k^2}}[\Pi(1-k^2,\, k) - K(k)] \in\,]0,\, \pi[\ , \tag{4.12}$$

that is to say, $\Delta\phi$ is the angle between the two "arms" of the stationary string. Two important remarks are now in order: First, the stationary string extends from spatial infinity to spatial infinity for a *finite* range of the world-sheet coordinate σ. Second, the azimuthal angle is generically *not* a periodic function of σ with period T_σ (or an integer multiple of T_σ), not even modulo 2π. These two statements together imply that the solution described by Eqs. (4.8) and (4.10) is actually a *multi-*

string solution in the sense that one single world sheet, determined by one set of initial conditions, describes a finite or even an infinite number of different and independent strings. Multistring solutions were first found in de Sitter spacetime for *nonstationary* strings [7,8,1]. In that case, it was found that strings can contract from infinite size to a minimal size and back toward infinite size for a *finite* range of the timelike world sheet coordinate τ. The process, on the other hand, takes infinite physical time, such that when the world-sheet time τ runs from $-\infty$ to $+\infty$, the world sheet describes *infinitely* (for degenerate cases: finitely) many strings. The physical time was not periodic so the solution really described different and independent strings; not simply infinitely many (or finitely many) copies of the same string. Here, in anti–de Sitter spacetime the situation is somewhat similar, but with the roles of τ and σ as well as t and ϕ interchanged: For $\sigma \in [0,\ T_\sigma]$ the solution describes one infinitely long string in the wedge $\phi \in [0,\ \Delta\phi]$, for $\sigma \in [T_\sigma,\ 2T_\sigma]$ it describes another infinitely long string in the wedge $\phi \in [\Delta\phi,\ 2\Delta\phi]$, etc. In the general case, the solution, Eqs. (4.8) and (4.10), describes infinitely many strings. However, for certain values of the "impact-parameter" L, or alternatively of the elliptic modulus k, the azimuthal angle becomes periodic modulo 2π and then the solution describes only a finite number of strings. Clearly, this situation appears provided:

$$N\Delta\phi = 2\pi M, \tag{4.13}$$

where N and M are integers. From the exact expression of $\Delta\phi$, Eq. (4.12), it follows that $M/N \in\,]0, 1/2[$, and the solutions are then conveniently parametrized in terms of (N, M). The simplest examples are $(N, M) = (3, 1)$, $(N, M) = (4, 1)$, etc., see Fig. 2. The integer N gives the number of strings in the multistring solution while the integer M is a "winding-number" of the azimuthal angle. It should be stressed also that the general solution is a multistring solution but it can of course be truncated, in particular, to describe only one string by considering an appropriate range of the world-sheet coordinate σ.

We now consider the physical properties of the stationary strings found above. Each string has infinite length; the string-length element, Eq. (2.12), being

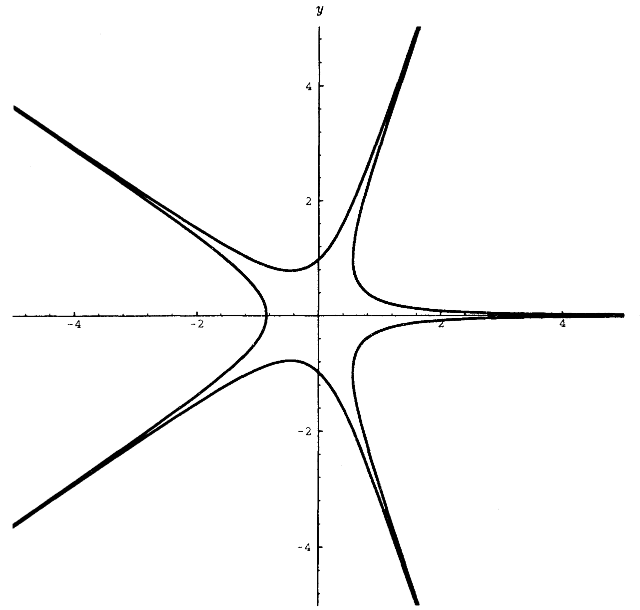

FIG. 2. The $(N, M) = (5, 1)$ multistring solution in anti–de Sitter spacetime. The (N, M) multistring solutions describe N stationary strings with M windings in the azimuthal angle ϕ, in anti–de Sitter spacetime.

$$\frac{dl}{d\sigma} = \sqrt{1 + H^2 r^2(\sigma)} = \frac{\mid ds[\mu H\sigma,\ k] \mid}{\sqrt{1 - 2k^2}}, \tag{4.14}$$

which diverges for $\sigma = 0,\ \pm T_\sigma,\ \pm 2T_\sigma, ...,$ i.e., at spatial infinity. Next we consider the energy and pressure densities. The energy as a function of the cosmic time is obtained by integrating T^{00} :

$$E(X^0) = \int d^3X \sqrt{-g}\, T^{00}, \tag{4.15}$$

where $T^{\mu\nu}$ is given by Eq. (2.13), and the cosmic time X^0 is given in terms of static coordinates by

$$HX^0(\tau, \sigma) = \pm \arccos \sqrt{[1 + H^2 r^2(\sigma)] \cos^2(H\tau) - H^2 r^2(\sigma)}. \tag{4.16}$$

For a full (N, M) multistring solution we find after some algebra, the integral expression

$$E(X^0) = 2N \int_0^{T_\sigma/2} d\sigma \frac{\mid \cos(HX^0) \mid \{[1 + H^2 r^2(\sigma)]^2 + H^2 L^2 \tan^2(HX^0)\}}{[1 + H^2 r^2(\sigma)] \sqrt{H^2 r^2(\sigma) + \cos^2(HX^0)}}. \tag{4.17}$$

Not surprisingly, the total energy is infinite. Using also Eq. (4.14) we find the energy density

$$\frac{dE(X^0)}{dl} = 2N \frac{\mid \cos(HX^0) \mid \{[1 + H^2 r^2(\sigma)]^2 + H^2 L^2 \tan^2(HX^0)\}}{[1 + H^2 r^2(\sigma)]^{3/2} \sqrt{H^2 r^2(\sigma) + \cos^2(HX^0)}}, \tag{4.18}$$

with the asymptotic value

$$\frac{dE(X^0)}{dl} = 2N \mid \cos(HX^0) \mid \quad \text{for } r \to \infty \,. \tag{4.19}$$

Notice that the right-hand side of Eq. (4.18) can be expressed in terms of the physical length l (and X^0), by integrating Eq. (4.14), and that the asymptotic region $l \to \infty$ corresponds to $r \to \infty$. Each individual string in the multistring solution gives the same contribution to the energy density but different contributions to the pressure densities in the different directions. However, if we consider the full (N, M) multistring solution we find $P_3 = 0$, $P_1 = P_2 \equiv P$:

$$\begin{aligned} P(X^0) &= \int d^3X\sqrt{-g}\, T^1{}_1 = \int d^3X\sqrt{-g}\, T^2{}_2 \\ &= N\int d\tau \int_0^{T_\sigma/2} d\sigma \frac{\cos^2(HX^0)[\dot{R}^2 - R'^2 - R^2\phi'^2]}{(1-H^2R^2/4)^2}\delta(X^0 - X^0(\tau,\sigma)), \end{aligned} \tag{4.20}$$

where $X^1 = R\cos\phi$, $X^2 = R\sin\phi$ (in the equatorial plane), and R is expressed in terms of static coordinates by

$$HR(\tau,\sigma) = \frac{2}{Hr(\sigma)}\{\sqrt{1+H^2r^2(\sigma)}\,\cos(H\tau) - \sqrt{[1+H^2r^2(\sigma)]\cos^2(H\tau) - H^2r^2(\sigma)}\,\}, \tag{4.21}$$

see Eqs. (2.1) and (2.14). After some algebra we find

$$P(X^0) = -N\int_0^{T_\sigma/2} d\sigma \frac{\mid \cos(HX^0) \mid \{[1+H^2r^2(\sigma)]^2 - H^2L^2\tan^2(HX^0)\}}{[1+H^2r^2(\sigma)]\sqrt{H^2r^2(\sigma)+\cos^2(HX^0)}}, \tag{4.22}$$

which is infinite. The pressure density is finite

$$\frac{dP(X^0)}{dl} = -N\,\frac{\mid \cos(HX^0) \mid \{[1+H^2r^2(\sigma)]^2 - H^2L^2\tan^2(HX^0)\}}{[1+H^2r^2(\sigma)]^{3/2}\sqrt{H^2r^2(\sigma)+\cos^2(HX^0)}}, \tag{4.23}$$

with the asymptotic value

$$\frac{dP(X^0)}{dl} = -N \mid \cos(HX^0) \mid \quad \text{for } r \to \infty \,. \tag{4.24}$$

Comparing with Eq. (4.19), the energy and pressure densities in the asymptotic region satisfy

$$dP(X^0) = -\tfrac{1}{2}dE(X^0) \quad \text{for } r \to \infty \,. \tag{4.25}$$

This type of "equation of state" is quite typical for strings in (2+1)-dimensional FRW universes [3,11,17], so it is not surprising that we recover it here for stationary strings in the equatorial plane of (3+1)-dimensional anti–de Sitter spacetime. Generally, for arbitrary $r(\sigma)$, there is no simple expression for the equation of state and the pressure density can take both positive and negative values depending on X^0 and $r(\sigma)$. An exception is the case when $L = 0$ where the string solution (4.3) describes a straight string. If the string is oriented along the X^2 axis we find

$$\begin{aligned} E(X^0) &= -P_2(X^0) \\ &= 2\int_0^{\pi/2} d\sigma\, \frac{|\cos(HX^0)|[1+H^2r^2(\sigma)]}{\sqrt{H^2r^2(\sigma)+\cos^2(HX^0)}}, \end{aligned} \tag{4.26}$$

while $P_1 = P_3 = 0$. The integrated energy and pressure are infinite but the densities satisfy the same equation of state as the straight string in flat Minkowski spacetime, compare with Eqs. (3.4) and (3.5).

V. DE SITTER SPACETIME

We now come to the cosmologically interesting case of de Sitter spacetime, corresponding to $a(r) = 1 - H^2r^2$ in Eq. (2.1). The stationary string potential in this case is

$$V(r) = -(1-H^2r^2)\left(1 - H^2r^2 - \frac{L^2}{r^2}\right), \tag{5.1}$$

see Fig. 1(c). The potential vanishes at the horizon $r = 1/H$ and for $r = r_{0\pm}$:

$$r_{0\pm} = \frac{1}{H}\left(\frac{1 \pm \sqrt{1-4H^2L^2}}{2}\right)^{1/2}, \tag{5.2}$$

and we must consider separately the two cases: $HL > 1/2$ and $HL < 1/2$. The first-order differential equations (2.4) and (2.5) are solved by $r = r_{0\pm}$, $\phi = L\sigma/r_{0\pm}^2$ as well as by $r = 1/H$, $\phi = H^2L\sigma$, but these three solutions must be excluded [cf. Eqs. (2.9) and (2.10)] since they do not satisfy the original second-order differential equations (2.2) except in the case $HL = 1/2$, where one of them survives:

$$r = \frac{1}{\sqrt{2}H}, \qquad \phi(\sigma) = H\sigma. \tag{5.3}$$

This is the stationary circular string configuration in de Sitter spacetime already discussed in Refs. [1,7,8,15]. Another "degenerate" case appears for $L = 0$ where we find the straight string configuration:

$$r(\sigma) = \frac{1}{H}\tanh(H\sigma), \qquad \phi = \text{const.} \tag{5.4}$$

Let us now consider the general case $L \neq 0, \;\; HL \neq 1/2$.

A. $0 < HL < 1/2$

In this case the potential has two different real zeros inside the horizon besides the zero at the horizon $r = 1/H$. It follows from the potential, Fig. 1(c), that solutions will exist both inside the horizon and outside the horizon, but they can never actually cross the horizon. The solutions outside the horizon must be expressed in terms of comoving coordinates or hyperboloid coordinates (say), since the static coordinates, Eq. (2.1), are only appropriate to cover the region of de Sitter spacetime inside the horizon; we will return to that question later. Equation (2.5) is solved by a Weierstrass elliptic function [compare with Eqs. (4.4)–(4.7)]

$$H^2r^2(\sigma) = \frac{1}{H^2}\wp(\sigma - \sigma_0;\, g_2,\, g_3) + \tfrac{2}{3}, \tag{5.5}$$

with invariants

$$g_2 = 4H^4(\tfrac{1}{3} - H^2L^2), \qquad g_3 = \tfrac{4}{3}H^6(-\tfrac{2}{9} + H^2L^2), \tag{5.6}$$

discriminant

$$\Delta = 16H^{16}L^4(1 - 4H^2L^2)\ , \tag{5.7}$$

and roots

$$\begin{aligned} e_1 &= \frac{H^2}{3} > e_2 \\ &= \frac{H^2}{6}[-1 + 3\sqrt{1 - 4H^2L^2}\ ] > e_3 \\ &= \frac{H^2}{6}[-1 - 3\sqrt{1 - 4H^2L^2}\ ]. \end{aligned} \tag{5.8}$$

The solution, Eqs. (5.5) and (5.8), was originally derived in Ref. [15], but it was only analyzed in the two "degenerate" cases corresponding to the solutions given by Eqs. (5.3) and (5.4). Here we shall analyze the general solution. We can write Eq. (5.5) in terms of a Jacobi elliptic function

$$H^2r^2(\sigma) = \nu^2 + \mu^2\ ns^2[\mu H(\sigma - \sigma_0),\ k], \tag{5.9}$$

where we introduced the more compact notation [compare with Eqs. (4.8) and (4.9)]

$$\begin{aligned} \mu^2 &\equiv \frac{1}{2}(1 + \sqrt{1 - 4H^2L^2}), \\ \nu^2 &\equiv \frac{1}{2}(1 - \sqrt{1 - 4H^2L^2}), \\ k &\equiv \frac{\sqrt{\mu^2 - \nu^2}}{\mu}. \end{aligned} \tag{5.10}$$

It follows that the elliptic modulus $k \in\,]0,\ 1[$, where $k \to 0$ corresponds to $HL \to 1/2$, while $k \to 1$ corresponds to $L \to 0$. We still have not fixed the integration constant σ_0. It turns out that two qualitatively different families of real solutions appear depending on the choice of σ_0. For $\sigma_0 = 0$ we find

$$H^2r_+^2(\sigma) = \nu^2 + \mu^2\ ns^2[\mu H\sigma,\ k], \tag{5.11}$$

while, for $\sigma_0 = \omega' = iK'(k)/(\mu H)$,

$$H^2r_-^2(\sigma) = \nu^2 + (\mu^2 - \nu^2)\ sn^2[\mu H\sigma,\ k]. \tag{5.12}$$

Notice that $H^2r_+^2(\sigma) \geq 1$, thus this solution is always outside the horizon and it must be expressed in a different coordinate system. For the r_- solution we find that $\mu^2 \geq H^2r_-^2(\sigma) \geq \nu^2$, thus this solution is always inside the horizon, in fact, it oscillates (in the spacelike world-sheet coordinate σ) between $\nu^2 > 0$ and $\mu^2 < 1$.

Let us first consider the solution r_- in a little more detail. The corresponding azimuthal angle is obtained from Eq. (2.4):

$$\phi_-(\sigma) = \sqrt{\frac{2 - k^2}{1 - k^2}}\,\Pi\left(\frac{-k^2}{1 - k^2},\ \mu H\sigma,\ k\right), \tag{5.13}$$

where Π is the elliptic integral of the third kind. The radial coordinate, Eq. (5.12), is periodic with a period T_σ as formally given by Eq. (4.11), but with μ and k given by Eq. (5.10). For $\sigma \in [0,\ T_\sigma]$, r_- increases from $r_- = \nu/H$ to $r_- = \mu/H$ and then decreases back to $r_- = \nu/H$. In the same range of σ, the azimuthal angle increases from $\phi_- = 0$ to $\phi_- = \Delta\phi_-$:

$$\Delta\phi_- = 2\sqrt{\frac{2 - k^2}{1 - k^2}}\,\Pi\left(\frac{-k^2}{1 - k^2},\ k\right)\ \in\ ]\pi,\ \sqrt{2}\pi[\ . \tag{5.14}$$

Generally the solution described by Eqs. (5.12) and (5.13) represents an *infinitely long open* string winding around $r = 0$ between $r = \nu/H$ and $r = \mu/H$. However, in the special case where the azimuthal angle is periodic modulo 2π with a period which is an integer multiple of T_σ, it describes a closed string of *finite length.* This closed string condition takes the form

$$N\Delta\phi_- = 2\pi M, \tag{5.15}$$

which is similar to Eq. (4.13) but with $\Delta\phi$ replaced by $\Delta\phi_-$. The closed strings inside the horizon of de Sitter spacetime are then parametrized in terms of the two integers (N, M), and we find, from Eq. (5.14),

$$M/N \in\,]1/2,\ 1/\sqrt{2}[. \tag{5.16}$$

The simplest examples are $(N, M) = (3, 2)$, $(N, M) = (5, 3)$, etc., see Fig. 3. The integer N gives the number of "leaves" (see Fig. 3) of the solution, while the integer M is a winding number of the azimuthal angle. At this point it is interesting also to determine which types of stationary closed string solutions are *not* allowed because of the conditions, Eqs. (5.15) and (5.16). It is, for instance, impossible to have a solution with 4 "leaves" ($N = 4$), while 3 "leaves" ($N = 3$) or 5 "leaves" ($N = 5$) are perfectly allowed. Notice also that the closed circular string,

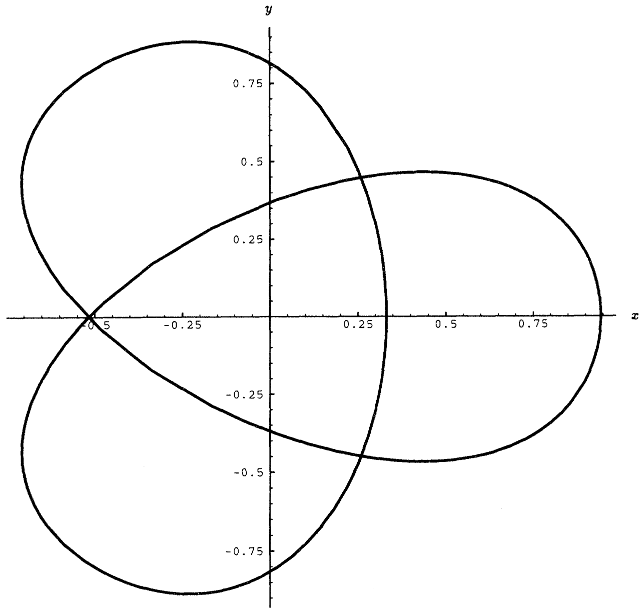

FIG. 3. The $(N, M) = (3, 2)$ stationary string solution inside the horizon of de Sitter spacetime. In addition to the circular string, this is the simplest stationary closed string configuration in de Sitter spacetime.

Eq. (5.3), in a pure mathematical sense, corresponds to $(N, M) = (\sqrt{2}, 1)$.

Using Eqs. (2.12) and (2.13) we can calculate the physical length, energy, and pressure of the (N, M) strings described above. The string length is given by

$$l_- = \int_0^{2NK(k)/(\mu H)} d\sigma\, \sqrt{1 - H^2 r_-^2(\sigma)} = N\pi/H. \quad (5.17)$$

This result holds for the circular string also (taking formally $N = \sqrt{2}$, as discussed above). The string energy is given by an expression of the form (4.15) where

$$HX^0(\tau, \sigma) = \frac{1}{2} \ln[1 - H^2 r_-^2(\sigma)] + H\tau. \quad (5.18)$$

Using (2.13) the integrals are easily evaluated and we find

$$E_-(X^0) = E_- = \frac{4N}{H} \frac{E(k)}{\sqrt{2 - k^2}}, \quad (5.19)$$

where $E(k)$ is the complete elliptic integral of the second kind. It should be stressed that N and k are not independent quantities. For each (N, M) string, the elliptic modulus must be calculated by solving Eq. (5.15). Numerically we find that the energy is an increasing function of N, i.e., the energy grows with the number of "leaves." Notice that the result, (5.19), holds also for the circular string $(N = \sqrt{2})$, in agreement with [1] [in units where $(2\pi\alpha')^{-1} = 1$]. Let us finally calculate the pressure of a generic (N, M) string. From the symmetries of the problem, it follows that $P_3 = 0$, $P_1 = P_2 \equiv P_-$, where

$$P_-(X^0) = \int d^3X \sqrt{-g}\, T^1{}_1 = \int d^3X \sqrt{-g}\, T^2{}_2$$
$$= e^{2HX^0} \int d\tau \int_0^{2NK(k)/(\mu H)} d\sigma [\dot{X}^1 \dot{X}^1 - X'^1 X'^1] \delta(X^0 - X^0(\tau, \sigma)) \quad (5.20)$$

and where

$$X^1(\tau, \sigma) = \frac{r_-(\sigma) \cos \phi_-(\sigma)}{\sqrt{1 - H^2 r_-^2(\sigma)}} e^{-H\tau}, \quad (5.21)$$

see Eqs. (2.1) and (2.14). After some algebra the integral is reduced to

$$P_-(X^0) = P_- = N \int_{\nu/H}^{\mu/H} dr \frac{(H^2 r^2 - 1)^2 + H^2 L^2}{(1 - H^2 r^2)^{3/2} \sqrt{1 - H^2 r^2 - (L/r)^2}}. \quad (5.22)$$

This integral can be evaluated in terms of complete elliptic integrals

$$P_- = \frac{N}{H\sqrt{2 - k^2}} [(1 - k^2)\Pi(k^2, k) - E(k)], \quad (5.23)$$

which vanishes identically by an identity between elliptic integrals (Ref. [18], formula 17.7.24):

$$\Pi(k^2, \arcsin(\mathrm{sn}[u, k]), k) = \frac{E[\arcsin(\mathrm{sn}[u, k]), k]}{1 - k^2} - \frac{k^2}{1 - k^2} \frac{\mathrm{sn}[u, k] \sqrt{1 - \mathrm{sn}^2[u, k]}}{\sqrt{1 - k^2 \mathrm{sn}^2[u, k]}}.$$

For $u = K(k)$ (in our notation), this is exactly the combination of elliptic integrals appearing in Eq. (5.23). The closed stationary strings inside the horizon in de Sitter spacetime thus satisfy an equation of state of the cold matter type. A similar result was found recently [3] for oscillating circular strings in de Sitter spacetime.

We now consider in more detail the solution $r_+(\sigma)$, Eq. (5.11). Although obtained from the stationary string ansatz, Eq. (2.3), this solution is actually not stationary since it is always outside the horizon where τ is spacelike while σ is timelike, as follows from Eq. (2.11) when $a(r) = 1 - H^2r^2$. Therefore, we define

$$\tilde{\sigma} \equiv \tau, \qquad \tilde{\tau} \equiv \sigma, \tag{5.24}$$

and identify the physical string length element outside the horizon as:

$$dl = \sqrt{H^2r^2 - 1}\, d\tilde{\sigma}, \tag{5.25}$$

which vanishes at the horizon and diverges at infinity. The actual string length is obtained by integrating over the range of $\tilde{\sigma}$, which is now a finite range, for instance, $\tilde{\sigma} \in [0,\ \pi]$. Since the radial coordinate now depends on the timelike world-sheet coordinate $\tilde{\tau}$, the string length is then simply

$$l_+(\tilde{\tau}) = \pi\sqrt{H^2r_+^2(\tilde{\tau}) - 1}. \tag{5.26}$$

In our further analysis we shall not need to specify the exact range of $\tilde{\sigma}$. The solution (5.11) is written as

$$H^2r_+^2(\tilde{\tau}) = \nu^2 + \mu^2\, ns^2[\mu H\tilde{\tau},\ k]. \tag{5.27}$$

The corresponding azimuthal angle is obtained from Eqs. (2.4) and (5.27):

$$\phi_+(\tilde{\tau}) = \frac{1}{\sqrt{1-k^2}} \left\{ H\tilde{\tau} - \sqrt{2-k^2}\, \Pi(\,-(1-k^2),\ \mu H\tilde{\tau}, k) \right\}, \tag{5.28}$$

thus it also depends on the timelike world-sheet coordinate $\tilde{\tau}$, only. The radial coordinate, Eq. (5.27), oscillates between the horizon and infinity with a period

$$T_{\tilde{\tau}} = \frac{2K(k)}{\mu H}. \tag{5.29}$$

For $\tilde{\tau} \in [0,\ T_{\tilde{\tau}}]$, r goes from infinity toward the horizon and back toward infinity. In the same $\tilde{\tau}$ range, the string size, Eq. (5.26), contracts from infinite size to zero size and then expands back to infinite size. The azimuthal angle increases from $\phi_+ = 0$ to $\phi_+ = \Delta\phi_+$:

$$\Delta\phi_+ = 2\sqrt{\frac{2-k^2}{1-k^2}}[K(k) - \Pi(\,-(1-k^2),\ k)]\ \in\ ]0,\ \pi(\sqrt{2}-1)[\ . \tag{5.30}$$

In order to better obtain the physical interpretation of this solution, it must be expressed in a coordinate system which covers the complete de Sitter manifold. In hyperboloid coordinates, the solution takes the form

$$\begin{aligned} q^0 &= q^0(\tilde{\tau},\tilde{\sigma}) = \sqrt{H^2r_+^2(\tilde{\tau}) - 1}\cosh(H\tilde{\sigma}),\\ q^1 &= q^1(\tilde{\tau},\tilde{\sigma}) = \sqrt{H^2r_+^2(\tilde{\tau}) - 1}\sinh(H\tilde{\sigma}),\\ q^2 &= q^2(\tilde{\tau}) = Hr_+(\tilde{\tau})\cos\phi_+(\tilde{\tau}),\\ q^3 &= q^3(\tilde{\tau}) = Hr_+(\tilde{\tau})\sin\phi_+(\tilde{\tau}), \end{aligned} \tag{5.31}$$

and we have dropped q^4, which is identically zero. As usual, we should take another copy of (q^0, q^1) to cover the full de Sitter manifold; the coordinates, Eq. (5.31), only describe the expanding de Sitter universe ($q^0 \geq 0$). Notice that the spatial coordinates q^i depend on the spacelike world-sheet coordinate $\tilde{\sigma}$ through q^1 only, and that all $\tilde{\sigma}$ dependence disappears at the horizon. The solution r_+ thus describes a straight string which at $q^0 = -\infty$ starts at spatial infinity with infinite length. As the de Sitter universe contracts, the straight string falls nonradially toward the horizon while it contracts and eventually becomes a point at the horizon, when the de Sitter universe takes its minimal size (for $q^0 = 0$). As the de Sitter universe expands, the straight string also expands and travels nonradially away toward infinity, where its length grows indefinitely. Since the string travels from spatial infinity toward the horizon and back toward spatial infinity for $q^0 \in\ ]-\infty,\ +\infty[$, but for a *finite* range of the world-sheet coordinate $\tilde{\tau}$ (say $\tilde{\tau} \in [0,\ T_{\tilde{\tau}}]$), the full solution is actually a multistring. In the general case, the solution describes infinitely many straight strings incoming from different directions, "scattering" at the horizon, and then going out again in different directions toward infinity. An exceptional case is provided by the limit $L = 0$, where the solution describes only one string falling radially in from infinity and going radially out again. In all other cases, the solution describes finitely or infinitely many strings. Following the analysis of the stationary multistrings in anti–de Sitter spacetime, as done in Sec. IV, we find the following condition for the solution to describe only finitely many strings in de Sitter spacetime:

$$N\Delta\phi_+ = 2\pi M, \tag{5.32}$$

where N and M are again integers. It follows that $M/N \in\ ]0,\ 1/\sqrt{2} - 1/2[$ and the simplest examples are then $(N, M) = (5, 1)$, $(N, M) = (6, 1)$, etc., see, Fig. 4. Again, the integer N describes the number of strings in the multistring solution while the integer M is a winding number of the azimuthal angle.

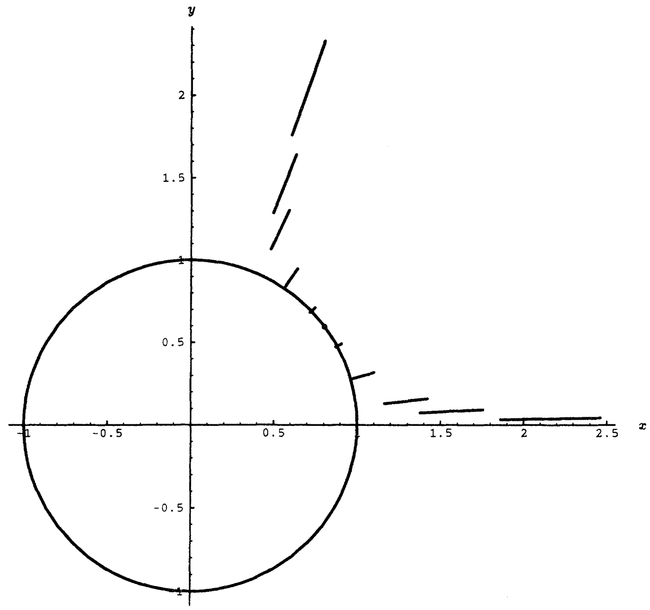

FIG. 4. Schematic representation of the time evolution of the $(N,M) = (5,1)$ dynamical multistring solution, outside the horizon of de Sitter spacetime. Only one of the five strings is shown; the others are obtained by rotating the figure by the angles $2\pi/5$, $4\pi/5$, $6\pi/5$, and $8\pi/5$. During the "scattering" at the horizon, the strings collapse to a point and reexpand.

In the limit $HL = 1/2$, the solution, Eqs. (5.27) and (5.28), reduces to elementary functions

$$H^2 r_+^2(\tilde\tau) = 1 + \frac{1}{2}\cot^2\frac{H\tilde\tau}{\sqrt{2}}, \tag{5.33}$$

$$\phi_+(\tilde\tau) = H\tilde\tau - \arctan\left(\sqrt{2}\tan\frac{H\tilde\tau}{\sqrt{2}}\right). \tag{5.34}$$

For this solution we have $\Delta\phi_+ = \pi(\sqrt{2}-1)$, and it describes infinitely many strings.

We already considered the physical length of the string solutions, Eqs. (5.27) and (5.28). Let us now consider the energy and pressure. The string energy is given by an expression of the form (4.15) where

$$HX^0(\tilde\tau,\tilde\sigma) = \tfrac{1}{2}\ln[H^2 r_+^2(\tilde\tau) - 1] + H\tilde\sigma. \tag{5.35}$$

For a full (N,M) multistring we find, using Eq. (2.13),

$$E_+(X^0) = 2N\int d\tilde\sigma\,\frac{e^{3H(X^0-\tilde\sigma)} + H^2L^2 e^{-H(X^0-\tilde\sigma)}}{\sqrt{e^{4H(X^0-\tilde\sigma)} + e^{2H(X^0-\tilde\sigma)} + H^2L^2}}. \tag{5.36}$$

By the substitution $z = \exp(H\tilde\sigma)$, the integral can be evaluated in terms of elliptic integrals but we shall not give the explicit expression here. Asymptotically, when the string travels toward spatial infinity, we find from Eqs. (5.25) and (5.36) the following expression for the energy density:

$$\frac{dE_+(X^0)}{dl_+} = 2N \quad \text{for } X^0 \to \infty\,. \tag{5.37}$$

It follows that the energy diverges in this limit, since the physical length diverges. Concerning the pressures we find that $P_3 = 0$, $P_1 = P_2 \equiv P_+$:

$$\begin{aligned} P_+(X^0) &= \int d^3X\sqrt{-g}\,T^1{}_1 = \int d^3X\sqrt{-g}\,T^2{}_2 \\ &= e^{2HX^0}\int d\tilde\sigma\int_0^{2NK(k)/(\mu H)} d\tilde\tau\left[\left(\frac{dX^1}{d\tilde\tau}\right)^2 - \left(\frac{dX^1}{d\tilde\sigma}\right)^2\right]\delta(X^0 - X^0(\tilde\tau,\tilde\sigma)), \end{aligned} \tag{5.38}$$

where

$$X^1(\tilde\tau,\tilde\sigma) = \frac{r_+(\tilde\tau)\cos\phi_+(\tilde\tau)}{\sqrt{H^2r_+^2(\tilde\tau) - 1}}e^{-H\tilde\sigma}, \tag{5.39}$$

compare with Eqs. (5.20) and (5.21). After some algebra we find the integral expression

$$P_+(X^0) = -N\int d\tilde\sigma\,\frac{e^{3H(X^0-\tilde\sigma)} - H^2L^2e^{-H(X^0-\tilde\sigma)}}{\sqrt{e^{4H(X^0-\tilde\sigma)}e^{2H(X^0-\tilde\sigma)} + H^2L^2}}, \tag{5.40}$$

which is again of elliptic type. Asymptotically we find the pressure density, using also Eq. (5.25):

$$\frac{dP_+(X^0)}{dl_+} = -N \quad \text{for } X^0 \to \infty\,. \tag{5.41}$$

Thus, in this limit the strings satisfy an equation of state such as (4.25):

$$dP_+(X^0) = -\tfrac{1}{2}dE_+(X^0) \quad \text{for } X^0 \to \infty\,, \tag{5.42}$$

i.e., such as extremely unstable strings [11].

B. $HL > 1/2$

In this case we only find string solutions outside the horizon, as follows by inspection of the potential, Eq. (5.1). The solutions obtained from the stationary string ansatz are all dynamical and they are in fact of the same type as the r_+ strings discussed above. We shall therefore not go into too much detail here. We perform from the beginning the redefinitions (5.24). Then Eq. (2.5) is solved by the Weierstrass function [compare with Eqs. (5.5)–(5.8)]:

$$H^2r^2(\tilde{\tau}) = \frac{1}{H^2}\wp(\tilde{\tau}-\tilde{\tau}_0;\ g_2,\ g_3) + \frac{2}{3}, \tag{5.43}$$

with invariants

$$g_2 = 4H^4(\tfrac{1}{3} - H^2L^2), \qquad g_3 = \tfrac{4}{3}H^6(-\tfrac{2}{9} + H^2L^2), \tag{5.44}$$

discriminant

$$\Delta = 16H^{16}L^4(1-4H^2L^2)\ , \tag{5.45}$$

and roots

$$\begin{aligned} e_1 &= \frac{H^2}{6}[-1+3i\sqrt{4H^2L^2-1}],\\ e_2 &= \frac{H^2}{3},\\ e_3 &= \frac{H^2}{6}[-1-3i\sqrt{4H^2L^2-1}]. \end{aligned} \tag{5.46}$$

The real solutions in terms of Jacobi elliptic functions take the form

$$H^2r^2(\tilde{\tau}) = 1 + HL\frac{1+cn[2H\sqrt{HL}\tilde{\tau},\ k]}{1-cn[2H\sqrt{HL}\tilde{\tau},\ k]}, \tag{5.47}$$

where the elliptic modulus is now defined by

$$k = \sqrt{\frac{1}{2} - \frac{1}{4HL}}. \tag{5.48}$$

It follows that $k \in]0,\ 1/\sqrt{2}[$, where $k \to 0$ corresponds to $HL \to 1/2$, while $k \to 1/\sqrt{2}$ corresponds to $HL \to \infty$. The azimuthal angle is obtained by integration of Eq. (2.4). The result is most easily expressed in terms of elliptic θ functions,

$$\phi(\tilde{\tau}) = \frac{1}{2i}\left\{\frac{\pi\tilde{\tau}}{\omega_2}\frac{\theta_1'}{\theta_1}\left(\frac{\pi a}{2\omega_2}\right) + \ln\left|\frac{\theta_1(\frac{\pi(\tilde{\tau}-a)}{2\omega_2})}{\theta_1(\frac{\pi(\tilde{\tau}+a)}{2\omega_2})}\right|\right\}, \tag{5.49}$$

where $\omega_2 = K(k)/(H\sqrt{HL})$ and a is an imaginary constant satisfying

$$a = \frac{iy}{H\sqrt{HL}}, \qquad cn[2y,\ k'] = \frac{1-4k^2}{3-4k^2}, \tag{5.50}$$

i.e., y can be expressed as an incomplete elliptic integral. The further analysis of the solution, Eqs. (5.47) and (5.49), follows closely the analysis of the r_+-solution for $HL < 1/2$. The radial coordinate, Eq. (5.47), is periodic with period:

$$T_{\tilde{\tau}} = \frac{2K(k)}{H\sqrt{HL}}. \tag{5.51}$$

For $\tilde{\tau} \in [0,\ T_{\tilde{\tau}}]$, r goes from infinity toward the horizon and back toward infinity. In the same $\tilde{\tau}$ range, the string size which is also here in the form of Eq. (5.26), contracts from infinite size to zero size and then expands back to infinite size. The azimuthal angle increases from $\phi = 0$ to $\phi = \Delta\phi$:

$$\Delta\phi = 2\frac{\sqrt{2-4k^2}}{1-4k^2}\left\{K(k) - \frac{2}{(3-4k^2)\sqrt{1-k^2}}\Pi\left[\left(\frac{1-4k^2}{3-4k^2}\right)^2,\ \frac{k^2}{k^2-1}\right]\right\}. \tag{5.52}$$

Generally the solution describes infinitely many strings. The condition for the solution to describe only finitely many strings is of the form (5.32), with $\Delta\phi_+$ substituted by $\Delta\phi$, and leads to $M/N \in]1/\sqrt{2} - 1/2,\ 1/2[$. The simplest examples are $(N,M) = (3,1)$, $(N,M) = (4,1)$, etc., and the physical interpretation of these string solutions is similar to the physical interpretation of the r_+ solution discussed before for $HL < 1/2$.

We close this section with a small remark on the line element (2.11) for $a(r) = 1 - H^2r^2$. It is convenient to introduce the fundamental quadratic form $\alpha(\sigma)$,

$$\tfrac{1}{2}e^{\alpha} = 1 - H^2r^2, \tag{5.53}$$

determining the physical string size (inside the horizon). From the r equation of motion in de Sitter spacetime, Eqs. (2.8) and (5.1), follows

$$\frac{d^2\alpha}{d\sigma^2} + H^2e^{\alpha} - 4H^2L^2e^{-\alpha} = 0. \tag{5.54}$$

The redefinitions $\sigma' = \sqrt{2HL}\ H\sigma$, $\alpha(\sigma) = \ln(2HL) + \tilde{\alpha}(\sigma')$ yield

$$\frac{d^2\tilde{\alpha}}{d\sigma'^2} + e^{\tilde{\alpha}} - e^{-\tilde{\alpha}} = 0, \tag{5.55}$$

that is, the sinh-Gordon equation, as was proved more generally by de Vega and Sánchez [6].

VI. SCHWARZSCHILD BLACK HOLE

Stationary strings in the background of a Schwarzschild black hole (and its charged and rotating generalizations) were already considered in Ref. [12]. In this section we will describe *all* string solutions in the

Schwarzschild black hole background obtained from the ansatz (2.3). These will include also *dynamical* strings inside the event horizon. The string potential is obtained from Eq. (2.7), for $a(r) = 1 - 2m/r$:

$$V(r) = -\left(1 - \frac{2m}{r}\right)\left(1 - \frac{2m}{r} - \frac{L^2}{r^2}\right), \tag{6.1}$$

see Fig. 5(a). The potential vanishes at the horizon $r = 2m$ and for $r = r_0$:

$$r_0 = m + \sqrt{m^2 + L^2}, \tag{6.2}$$

but it is easily seen from Eqs. (2.9) and (2.10) that none of the corresponding circular string solutions satisfy the original second-order differential equations (2.2). Thus there are no stationary circular strings in the background of a Schwarzschild black hole. The line element for the string solutions take the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)(-d\tau^2 + d\sigma^2), \tag{6.3}$$

which defines the invariant string length element. Notice that inside the horizon τ becomes spacelike while σ becomes timelike, so that the world-sheet coordinates must be redefined and the string solutions must be expressed in better well-behaved spacetime coordinates. Only the string solutions outside the horizon will be stationary. Inside the horizon, the ansatz (2.3) will describe dynamical propagating strings.

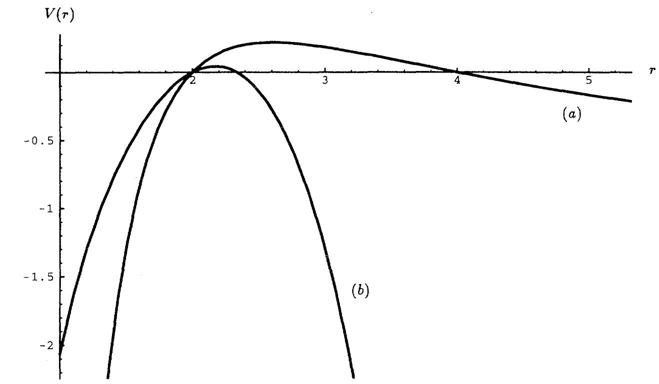

FIG. 5. The potential $V(r)$, Eq. (2.7), in two cases (a) Schwarzschild and (b) (2+1)-black hole anti–de Sitter spacetimes. Notice the different asymptotic behavior in the two cases.

Let us first consider the solutions outside the horizon. From the potential $V(r)$ follows that they can only exist for $r \geq r_0$. The solutions were already described in [12] but let us restate the results here in a somewhat different way. Equation (2.5) is solved by the function $r_- = r_-(\sigma)$, defined by inversion of the identity:

$$\pm\sigma = \frac{2m^2 + L^2}{\sqrt{m^2 + L^2}} F\left(\psi_-(r_-), \frac{m^2}{m^2 + L^2}\right) - \sqrt{m^2 + L^2}\, E\left(\psi_-(r_-), \frac{m^2}{m^2 + L^2}\right)$$
$$+ m \ln\left(\frac{2}{L^2}\sqrt{(r_-^2 - 2mr_- - L^2)(r_-^2 - 2mr_-)} + \frac{2}{L^2}(r_-^2 - 2mr_-) - 1\right)$$
$$+ (r_- - m)\sqrt{\frac{r_-^2 - 2mr_- - L^2}{r_-^2 - 2mr_-}}, \tag{6.4}$$

where

$$\psi_-(r_-) = \arcsin\left(\sqrt{\frac{r_-^2 - 2mr_- - L^2}{r_-^2 - 2mr_-}}\right) \tag{6.5}$$

and F and E are the incomplete elliptic integrals of the first and second kinds, respectively. The azimuthal angle is then obtained from Eq. (2.4):

$$\phi_-(\sigma) = L\int_0^\sigma \frac{dx}{r_-^2(x)}$$
$$= \pm\sqrt{\frac{L^2}{m^2 + L^2}} F\left(\psi_-[r_-(\sigma)], \sqrt{\frac{m^2}{m^2 + L^2}}\right). \tag{6.6}$$

Notice the special values

$$r_-(-\infty) = \infty, \quad r_-(0) = r_0, \quad r_-(+\infty) = \infty\ , \tag{6.7}$$

with the corresponding angles

$$\phi_-(\pm\infty) = \pm\sqrt{\frac{L^2}{m^2 + L^2}} K\left(\sqrt{\frac{m^2}{m^2 + L^2}}\right),$$
$$\phi_-(0) = 0. \tag{6.8}$$

The solution describes an infinitely long string extending from spatial infinity toward the minimal distance r_0 from the black hole, and out toward spatial infinity again. The angle between the two "arms" is given by

$$\Delta\phi_- = 2\sqrt{\frac{L^2}{m^2 + L^2}} K\left(\sqrt{\frac{m^2}{m^2 + L^2}}\right), \tag{6.9}$$

which changes continuously from 0 to π when L changes from 0 to ∞. There are no multistring solutions in this case; when σ runs from $-\infty$ to $+\infty$, the solution describes only one string. In the degenerate case $L = 0$, the formulas simplify considerably. Equation (6.4) for the radial coordinate becomes

$$-\sigma = r_- - 4m + 2m\ln\frac{r_- - 2m}{2m}. \tag{6.10}$$

This string configuration extends from spatial infinity to the horizon $r = 2m$, for $\sigma \in]-\infty, +\infty]$, and the angle (6.9) between the two arms vanishes, that is, the string is straight.

Let us now consider the solutions inside the horizon. We make the redefinitions (5.24) and introduce Kruskal coordinates

$$u = \sqrt{1-\tfrac{r}{2m}}\, e^{r/(4m)}\sinh\tfrac{t}{4m},$$
$$v = \sqrt{1-\tfrac{r}{2m}}\, e^{r/(4m)}\cosh\tfrac{t}{4m}, \tag{6.11}$$

so that (in the equatorial plane)

$$ds^2 = \frac{32m^3}{r}e^{-r/(2m)}(-dv^2+du^2)+r^2d\phi^2. \tag{6.12}$$

The radial coordinate $r_+ = r_+(\tilde\tau)$ is obtained from Eq. (2.5):

$$\tilde\tau = \frac{2m^2+L^2}{\sqrt{m^2+L^2}}F\left(\psi_+(r_+),\ \frac{m^2}{m^2+L^2}\right) - \sqrt{m^2+L^2}\,E\left(\psi_+(r_+),\ \frac{m^2}{m^2+L^2}\right)$$
$$-m\ln\left(\frac{-2}{L^2}\sqrt{(r_+^2-2mr_+-L^2)(r_+^2-2mr_+)} - \frac{2}{L^2}(r_+^2-2mr_+)+1\right) + (r_+-m)\sqrt{\frac{r_+^2-2mr_+}{r_+^2-2mr_+-L^2}}, \tag{6.13}$$

where

$$\psi_+(r_+) = \arcsin\left(\sqrt{\frac{m^2+L^2}{m^2}}\sqrt{\frac{r_+^2-2mr_+}{r_+^2-2mr_+-L^2}}\right). \tag{6.14}$$

The azimuthal angle takes the form

$$\phi_+(\tilde\tau) = L\int_0^{\tilde\tau}\frac{dx}{r_+^2(x)}$$
$$= \sqrt{\frac{L^2}{m^2+L^2}}F\left(\psi_+[r_+(\tilde\tau)],\ \sqrt{\frac{m^2}{m^2+L^2}}\right). \tag{6.15}$$

Both the radial coordinate and the azimuthal angle depend on the timelike world-sheet coordinate $\tilde\tau$, only. In Kruskal coordinates the solution is written as

$$u(\tilde\tau,\tilde\sigma) = \sqrt{1-\frac{r_+(\tilde\tau)}{2m}}\ e^{r_+(\tilde\tau)/(4m)}\sinh\frac{\tilde\sigma}{4m},$$
$$v(\tilde\tau,\tilde\sigma) = \sqrt{1-\frac{r_+(\tilde\tau)}{2m}}\ e^{r_+(\tilde\tau)/(4m)}\cosh\frac{\tilde\sigma}{4m},$$
$$\phi = \phi_+(\tilde\tau). \tag{6.16}$$

That is,

$$v^2-u^2 = \left(1-\frac{r_+(\tilde\tau)}{2m}\right)e^{r_+(\tilde\tau)/(2m)},$$
$$\frac{u}{v} = \tanh\frac{\tilde\sigma}{4m}, \tag{6.17}$$

and the line element becomes

$$ds^2 = \left(\frac{2m}{r_+}-1\right)(-d\tilde\tau^2+d\tilde\sigma^2). \tag{6.18}$$

When $\tilde\tau$ goes from $\tilde\tau=0$ to $\tilde\tau=\tilde\tau_0$, where

$$\tilde\tau_0 = 2\frac{2m^2+L^2}{\sqrt{m^2+L^2}}K\left(\frac{m^2}{m^2+L^2}\right)$$
$$-2\sqrt{m^2+L^2}\,E\left(\frac{m^2}{m^2+L^2}\right), \tag{6.19}$$

the radial coordinate $r_+(\tilde\tau)$ goes from the horizon $r_+(0)=2m$ to the spacetime singularity $r_+(\tilde\tau_0)=0$ and the solution cannot be continued. In the same range of $\tilde\tau$, the azimuthal angle goes from $\phi_+(0)=0$ to

$$\phi_+(\tilde\tau_0) = 2\sqrt{\frac{L^2}{m^2+L^2}}K\left(\sqrt{\frac{m^2}{m^2+L^2}}\right). \tag{6.20}$$

It is interesting to consider also the string length element, obtained from Eq. (6.18), during the fall of the string toward the singularity:

$$dl_+ = \sqrt{2m/r_+-1}\,d\tilde\sigma. \tag{6.21}$$

The physical string length is obtained by integrating this equation over the range of $\tilde\sigma$, but since r_+ depends on $\tilde\tau$ only, the string length is proportional to $\sqrt{2m/r_+-1}$. At the horizon the string length is therefore zero, i.e., the string starts as a point. During the fall toward the singularity, the string length grows proportionally to $1/\sqrt{r_+}$ and eventually grows indefinitely, see Fig. 6. This kind of behavior for strings near the singularity of a Schwarzschild black hole was originally found using a string series perturbation approach [19,2]. Notice that the string is straight for all $\tilde\tau$ but because of the $\tilde\tau$ dependence of the azimuthal angle, the string falls toward the singularity in a nonradial way. An exceptional case is obtained for $L=0$. In that case the azimuthal angle (6.14) is constant, while the radial coordinate is given by

$$\tilde\tau = r_+ + 2m\ln\frac{2m-r_+}{2m}. \tag{6.22}$$

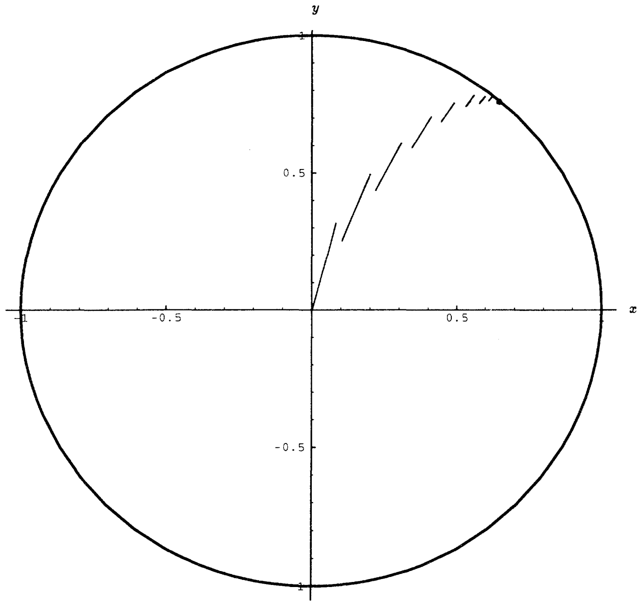

FIG. 6. The dynamical straight string inside the horizon of the Schwarzschild black hole. When the string falls (nonradially) toward the singularity, the string length grows indefinitely.

The string thus falls radially from the horizon toward the spacetime singularity for $\tilde{\tau} \in]-\infty,\ 0]$. The dynamics, for arbitrary L, is in many respects similar to what was found for the $r_+(\tilde{\tau})$-solution in de Sitter spacetime (in the expanding phase of the de Sitter spacetime), however, the solution in the Schwarzschild–black hole background describes only *one string*; no *multistring* solutions has been found in this background.

VII. (2+1)-DIMENSIONAL BLACK HOLE-AdS

It is interesting to consider also the stationary string ansatz in the (2+1)-dimensional black hole anti–de Sitter (BH-AdS) spacetime found by Banados *et al.* [13]. A general analysis of string propagation in the BH-AdS spacetime was performed recently by the present authors [2], based on the string perturbation series approach [20] as well as on exact circular string configurations. Here we will consider the new (stationary and dynamical) string solutions (outside and inside the horizon, respectively), obtained from the ansatz, Eq. (2.3). We will compare these results with those obtained in Secs. IV and VI in the equatorial plane of ordinary anti–de Sitter and Schwarzschild spacetimes.

The line element of the (2+1)-dimensional BH-AdS spacetime takes, in its simplest version, the form

$$ds^2 = -\left(\frac{r^2}{l^2} - 1\right) dt^2 + \left(\frac{r^2}{l^2} - 1\right)^{-1} dr^2 + r^2 d\phi^2. \quad (7.1)$$

This line element, where ϕ is identified with $\phi + 2\pi$, describes a black hole spacetime with mass $M = 1$ and angular momentum $J = 0$. There is a horizon at $r = l$ and the spacetime has constant curvature. Asymptotically it approaches anti–de Sitter spacetime with negative cosmological constant $\Lambda = -1/l^2$. A two-parameter (mass M and angular momentum J) family of black holes is obtained by periodically identifying a linear combination of t and ϕ [13,21], but the simpler case described by Eq. (7.1) is general enough for our purposes here. Notice that the line element (7.1) is in the general form of Eq. (2.1) with $a(r) = r^2/l^2 - 1$ (and $\theta = \pi/2$). We thus obtain the string potential, Eq. (2.7)

$$V(r) = -\left(\frac{r^2}{l^2} - 1\right)\left(\frac{r^2}{l^2} - 1 - \frac{L^2}{r^2}\right) \quad (7.2)$$

[see Fig. 5(b)]. The potential vanishes at the horizon $r = l$ and for $r = r_0$,

$$r_0 = l\left(\frac{1 + \sqrt{1 + 4L^2/l^2}}{2}\right)^{1/2}, \quad (7.3)$$

and the situation looks quite similar to the case of the Schwarzschild black hole background. In particular, it is easily seen from Eqs. (2.9) and (2.10) that there can be no stationary circular strings in the BH-AdS spacetime. The line element for the string solutions reads

$$ds^2 = \left(\frac{r^2}{l^2} - 1\right)(-d\tau^2 + d\sigma^2), \quad (7.4)$$

so that τ becomes spacelike inside the horizon while σ becomes timelike. As in the background of the Schwarzschild black hole, only the strings outside the horizon will be stationary; the string solutions inside the horizon will be dynamical. Equation (2.5) is solved by a Weierstrass elliptic function [compare with Eqs. (4.4)–(4.7)]

$$r^2(\sigma) = l^4 \wp(\sigma - \sigma_0;\ g_2,\ g_3) + \tfrac{2}{3}l^2, \quad (7.5)$$

with invariants

$$g_2 = \frac{4}{l^6}\left(\frac{l^2}{3} + L^2\right), \qquad g_3 = \frac{-4}{3l^8}\left(\frac{2l^2}{9} + L^2\right), \quad (7.6)$$

discriminant

$$\Delta = \frac{16L^4}{l^{18}}(l^2 + 4L^2), \quad (7.7)$$

and roots

$$\begin{aligned} e_1 &= \frac{1}{6l^2}[-1 + 3\sqrt{1 + 4L^2/l^2}\,] \geq e_2 \\ &= \frac{1}{3l^2} \geq e_3 \\ &= \frac{1}{6l^2}[-1 - 3\sqrt{1 + 4L^2/l^2}\,]. \end{aligned} \quad (7.8)$$

We can write Eq. (7.5) in terms of a Jacobi elliptic function

$$r^2(\sigma) = l^2(\nu^2 - \mu^2) + l^2\mu^2 ns^2[\mu(\sigma - \sigma_0)/l,\ k], \quad (7.9)$$

where we introduced the more compact notation [compare with Eqs. (4.8) and (4.9)]

$$\mu^2 \equiv \sqrt{1+4L^2/l^2}, \qquad \nu^2 \equiv \tfrac{1}{2}(1+\mu^2), \qquad k \equiv \frac{\nu}{\mu}. \quad (7.10)$$

It follows that the elliptic modulus $k \in]1/\sqrt{2},\ 1]$, where $k \to 1/\sqrt{2}$ corresponds to $L/l \to \infty$, while $k \to 1$ corresponds to $L \to 0$. We still have not fixed the integration constant σ_0. It turns out that two qualitatively different families of real solutions appear depending on the choice of σ_0. For $\sigma_0 = 0$ we find

$$r_-^2(\sigma) = l^2(\nu^2-\mu^2) + l^2\mu^2 ns^2[\mu\sigma/l,\ k], \quad (7.11)$$

while, for $\sigma_0 = \omega' = ilK'(k)/\mu$,

$$r_+^2(\sigma) = l^2(\nu^2-\mu^2) + l^2\nu^2\ sn^2[\mu\sigma/l,\ k]. \quad (7.12)$$

The solution $r_-^2(\sigma)$ is always outside the horizon and describes stationary strings. The solution $r_+^2(\sigma)$, on the other hand, is always inside the horizon and must be expressed in a different coordinate system. Let us consider first the solution $r_-(\sigma)$ outside the horizon. The corresponding azimuthal angle is obtained from Eq. (2.4):

$$\phi_-(\sigma) = \frac{-k}{\sqrt{1-k^2}}\left\{\frac{\sigma}{l} - \sqrt{2k^2-1}\right. \\ \left. \times\Pi(1-k^2,\ \mu\sigma/l,\ k)\right\}. \quad (7.13)$$

The expressions found here for the radial coordinate and the azimuthal angle are formally quite similar to the expressions found for the stationary strings in the ordinary anti–de Sitter spacetime, compare with Eqs. (4.8) and (4.10), but the elliptic modulus takes different values here and the string configurations are actually very different. The radial coordinate, Eq. (7.11), is periodic with a period

$$T_\sigma = \frac{2lK(k)}{\mu}, \quad (7.14)$$

where $K(k)$ is the complete elliptic integral of the first kind. For $\sigma \in [0,\ T_\sigma]$, r goes from infinity toward $r = r_0$ and back toward infinity. In the same range of σ, the azimuthal angle ϕ_- goes from $\phi_- = 0$ to $\phi_- = \Delta\phi_-$:

$$\Delta\phi_- = 2k\sqrt{\frac{2k^2-1}{1-k^2}}\ [\Pi(1-k^2,\ k) - K(k)], \quad (7.15)$$

compare with Eq. (4.12), i.e., $\Delta\phi_-$ is the angle between the two "arms" of the stationary string. It is interesting that $\Delta\phi_-$ goes to zero in both limits $k \to 1/\sqrt{2}$ and $k \to 1$. In this sense the stationary strings in the (2+1)-dimensional BH-AdS spacetime interpolates between the stationary strings in ordinary anti–de Sitter and Schwarzschild spacetimes. The maximal angle between the arms is obtained for an intermediate value of the elliptic modulus:

$$\text{Max}(\Delta\phi_-) = 1.0023\ldots \quad \text{for } k = 0.909\ldots. \quad (7.16)$$

As in ordinary de Sitter spacetime, the solution is actually a *multistring* solution. In the general case the solution, Eqs. (7.11) and (7.13), describes infinitely many strings. In particular, following the argument of Secs. IV–VI, the solution describes a *finite* number of strings when an equation of the form (4.13), with $\Delta\phi_-$ given by Eq. (7.15), is satisfied. It follows that $M/N \in [0,\ 0.1596...]$, i.e., the simplest examples are $(N, M) = (7, 1)$, $(N, M) = (8, 1)$, etc. The solution therefore describes at *least* 7 different strings.

We now turn to the r_+ solution, Eq. (7.12), which is always inside the horizon. As in the case of the Schwarzschild black hole (inside the horizon) we make the redefinition Eq. (5.24) and introduce the Kruskal-like coordinates:

$$u = \sqrt{\frac{l-r}{l+r}}\ \sinh\frac{t}{l}, \quad v = \sqrt{\frac{l-r}{l+r}}\ \cosh\frac{t}{l}\ , \quad (7.17)$$

so that

$$ds^2 = (l+r)^2(-dv^2+du^2) + r^2 d\phi^2. \quad (7.18)$$

The radial coordinate now takes the form

$$r_+^2(\tilde\tau) = l^2(\nu^2-\mu^2) + l^2\nu^2\ sn^2[\mu\tilde\tau/l,\ k], \quad (7.19)$$

and the corresponding azimuthal angle is obtained from Eq. (2.4):

$$\phi_+(\tilde\tau) = -k\sqrt{\frac{2k^2-1}{1-k^2}}\ \Pi\left(\frac{k^2}{1-k^2},\ \mu\tilde\tau/l,\ k\right). \quad (7.20)$$

We now follow closely the analysis of the $r_+(\tilde\tau)$ solution inside the horizon of the Schwarzschild black hole, see Sec. VI. In Kruskal coordinates the solution is written

$$u(\tilde\tau,\tilde\sigma) = \sqrt{\frac{l-r_+(\tilde\tau)}{l+r_+(\tilde\tau)}}\ \sinh\frac{\tilde\sigma}{l}, \\ v(\tilde\tau,\tilde\sigma) = \sqrt{\frac{l-r_+(\tilde\tau)}{l+r_+(\tilde\tau)}}\ \cosh\frac{\tilde\sigma}{l}, \\ \phi = \phi_+(\tilde\tau). \quad (7.21)$$

That is,

$$v^2 - u^2 = \frac{l-r_+(\tilde\tau)}{l+r_+(\tilde\tau)}, \quad \frac{u}{v} = \tanh\frac{\tilde\sigma}{l}, \quad (7.22)$$

and the line element becomes

$$ds^2 = \left(1-\frac{r_+^2}{l^2}\right)(-d\tilde\tau^2 + d\tilde\sigma^2). \quad (7.23)$$

When $\tilde\tau$ goes from $\tilde\tau = lK(k)/\mu$ to $\tilde\tau = \tilde\tau_0$, where

$$\tilde\tau_0 = \frac{l}{\mu}F\left[\arcsin\left(\frac{\sqrt{1-k^2}}{k}\right),\ k\right], \quad (7.24)$$

the radial coordinate $r_+(\tilde\tau)$ goes from the horizon $r_+[lK(k)/\mu] = l$ to $r_+(\tilde\tau_0) = 0$, and the solution cannot be continued because of the global causal structure of the spacetime. In the same range of $\tilde\tau$, the azimuthal angle goes from

$$\phi_+[lK(k)/\mu] = -k\sqrt{\frac{2k^2-1}{1-k^2}}\,\Pi\left(\frac{k^2}{1-k^2},\, k\right) \qquad (7.25)$$

to

$$\phi_+(\tilde{\tau}_0) = -k\sqrt{\frac{2k^2-1}{1-k^2}}\,\Pi\left(\frac{k^2}{1-k^2},\, \mu\tilde{\tau}_0/l,\, k\right). \qquad (7.26)$$

The string length element is given by

$$dl_+ = \sqrt{1 - r_+^2/l^2}\, d\tilde{\sigma}. \qquad (7.27)$$

At the horizon the string length is zero, i.e., the string starts as a point. When the string propagates toward $r = 0$, the string length grows but is always *finite*. This illustrates the important difference between the point $r = 0$ in the Schwarzschild black hole and in the (2+1)-dimensional BH-AdS spacetime. For the Schwarzschild black hole the point $r = 0$ is a physical curvature singularity, expressed by a power-law singularity in curvature scalars. For the (2+1)-dimensional BH-AdS spacetime, on the other hand, the curvature is constant $R_{\mu\nu} = -(2/l^2)g_{\mu\nu}$ everywhere, except probably at $r = 0$, where there is at most a delta-function singularity. This difference shows up clearly in the string solutions close to $r = 0$: In the Schwarzschild–black hole background we found that the string *stretches indefinitely* near $r = 0$, which is not the case in the (2+1) dimensional BH-AdS spacetime. Notice however that in *both* cases the string is *straight* during the nonradial fall toward $r = 0$. The infinite string stretching is a typical feature of string instability [1,6–8,11,20] and a generic characteristic behavior of strings near "strong enough" (stronger than delta-function type) spacetime singularities [22].

VIII. SCHWARZSCHILD–dS AND SCHWARZSCHILDS–AdS

Following the analysis of the preceding sections, it is straightforward to consider also more complicated curved spacetimes from general relativity and string theory. The mathematics will however in most cases be quite complicated but the qualitative results can in any case be read off directly from the potential, Eq. (2.7). Let us illustrate this by the two examples of ordinary Schwarzschild–anti–de Sitter (S-AdS) and Schwarzschild–de Sitter (S-dS) spacetimes.

The line element of S-AdS spacetime is given by

$$ds^2 = -\left(1 - \frac{2m}{r} + H^2r^2\right)dt^2 + \left(1 - \frac{2m}{r} + H^2r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2); \qquad (8.1)$$

i.e., it corresponds to $a(r) = 1 - 2m/r + H^2r^2$ in the notation of Eq. (2.1). The potential (2.7) takes the form

$$V(r) = -\left(1 - \frac{2m}{r} + H^2r^2\right)\left(1 - \frac{2m}{r} + H^2r^2 - \frac{L^2}{r^2}\right), \qquad (8.2)$$

see Fig. 7(a). The potential vanishes at the horizon $r = r_h$ and for $r = r_0$, where r_h and r_0 are the *unique* positive zeros of the equations

$$1 - \frac{2m}{r_h} + H^2r_h^2 = 0, \qquad 1 - \frac{2m}{r_0} + H^2r_0^2 - \frac{L^2}{r_0^2} = 0. \qquad (8.3)$$

From the potential, Fig. 7(a), and the analysis of Secs. IV, VI, and VII we can now easily describe the qualitative features of the stationary and dynamical string solutions obtained from the ansatz (2.3). Inside the horizon the solution will describe one single straight string falling nonradially toward the physical singularity. The string starts as a point at the horizon and stretches indefinitely for $r \to 0$. For $r \geq r_0$, outside the horizon, the solution is a *multistring* solution of the same type as found outside the horizon of the (2+1)-dimensional BH-AdS spacetime. The solution will describe infinitely or finitely many strings depending on the value of the parameter L.

In the case of S-dS spacetime, the line element is obtained from Eq. (8.1) by changing the sign of H^2. The corresponding string potential is then given by

$$V(r) = -\left(1 - \frac{2m}{r} - H^2r^2\right)\left(1 - \frac{2m}{r} - H^2r^2 - \frac{L^2}{r^2}\right), \qquad (8.4)$$

see Fig. 7(b). The S-dS spacetime has two horizons provided $\sqrt{27}Hm < 1$. We will consider only that case. Explicit expressions for the two horizons are given, for

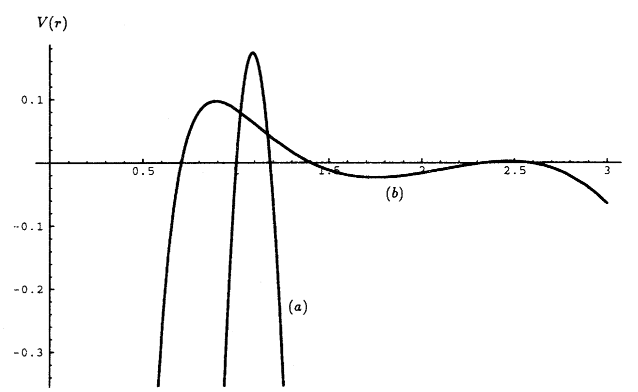

FIG. 7. The potential $V(r)$, Eq. (2.7), in the two cases (a) Schwarzschild–anti–de Sitter (S-AdS) and (b) Schwarzschild–de Sitter (S-dS) spacetimes. In S-dS spacetime, degenerate cases with less structure exist, while in S-AdS the potential takes form (a) for all nonzero values of the parameters (m, H, L).

instance, in Ref. [2]. The potential, Eq. (8.4), obviously vanishes at the two horizons, and depending on the value of the parameter L, there can be additional (one or two more) zeros between the horizons. In Fig. 7(b) we show the most general case where the potential has four different zeros, and all the different types of string solutions considered until now are present. Inside the inner (Schwarzschild-like) horizon, the string solution in S-dS is similar to the dynamical solution $r_+(\tilde{\tau})$ inside the horizon of Schwarzschild or Schwarzschild–anti–de Sitter spacetimes, Sec. VI. In the region between the two horizons a truly stationary string of the type $r_-(\sigma)$ found inside the horizon of de Sitter spacetime will be present, Sec. V. Finally, outside the outer (de Sitter–like) horizon the solution describes dynamical *multistrings* of the type $r_+(\tilde{\tau})$ found outside the horizon of ordinary de Sitter spacetime, Sec. V.

IX. CONCLUSION

In this paper we have studied the exact string solutions obtained by the stationary string ansatz, Eq. (2.3), in a variety of curved backgrounds including Schwarzschild, de Sitter, and anti–de Sitter spacetimes. Many different types of solutions have been found: closed stationary strings, infinitely long stationary strings, dynamical straight strings, and multistring solutions describing finitely or infinitely many stationary or dynamical strings. In all cases we have obtained the exact solutions in terms of either elementary or elliptic functions. Furthermore, we have analyzed the physical properties (length, energy, pressure) of the string solutions, thus this paper supplements earlier investigations on generic (based on approximative methods) and exact circular string solutions, important for the general understanding of the string dynamics in curved spacetimes.

We close with a few remarks on the stability of the solutions. Generally, the question of stability must be addressed by considering small perturbations around the exact solutions. In Ref. [15] a covariant formalism describing physical perturbations propagating along an arbitrary string configuration embedded in an arbitrary curved spacetime, was developed. The resulting equations determining the evolution of the perturbations are however very complicated in the general case, although partial (analytical) results have been obtained in special cases for de Sitter [9,23] and Schwarzschild–black hole [9,15,16] spacetimes. The exact solutions found in this paper fall essentially into two classes: dynamical and stationary. The dynamical string solutions outside the horizon of de Sitter (or S-dS) and inside the horizon of Schwarzschild (or S-dS, S-AdS) spacetimes, are already unstable at the zeroth-order approximation (i.e., without including small perturbations), in the sense that their physical length grows indefinitely. For the stationary string solutions the situation is more delicate. The existence of the stationary configurations is based on an exact balance between the string tension and the local attractive or repulsive gravity. For that reason, it can be expected that the configurations are actually unstable for certain modes of perturbation, especially in strong curvature regions. This question could deserve further investigations, but is out of the scope of this paper.

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