

Semiclassical quantization of circular strings in de Sitter and anti-de Sitter spacetimes

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We compute the *exact* equation of state of circular strings in the (2+1)-dimensional de Sitter (dS) and anti-de Sitter (AdS) spacetimes, and analyze its properties for the different (oscillating, contracting, and expanding) strings. The string equation of state has the perfect fluid form $P = (\gamma - 1)E$, with the pressure and energy expressed closely and completely in terms of elliptic functions, the instantaneous coefficient γ depending on the elliptic modulus. We semiclassically quantize the oscillating circular strings. The string mass is $m = \sqrt{C}/(\pi H \alpha')$, C being the Casimir operator, $C = -L_{\mu\nu}L^{\mu\nu}$, of the $O(3,1)$ -dS [$O(2,2)$ -AdS] group, and H is the Hubble constant. We find $\alpha' m_{\text{dS}}^2 \approx 4n - 5H^2 \alpha' n^2$, ($n \in N_0$), and a *finite* number of states $N_{\text{dS}} \approx 0.34/(H^2 \alpha')$ in de Sitter spacetime; $m_{\text{AdS}}^2 \approx H^2 n^2$ (large $n \in N_0$) and $N_{\text{AdS}} = \infty$ in anti-de Sitter spacetime. The level spacing grows with n in AdS spacetime, while it is approximately constant (although smaller than in Minkowski spacetime and slightly decreasing) in dS spacetime. The massive states in dS spacetime decay through the tunnel effect and the semiclassical decay probability is computed. The semiclassical quantization of *exact* (circular) strings and the canonical quantization of generic string perturbations around the string center of mass qualitatively agree.

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I. INTRODUCTION AND RESULTS

The systematic investigation of string dynamics in curved spacetimes started in Ref. [1] has given new insight and produced new physical phenomena with respect to string propagation in flat spacetime (and with respect to quantum fields in curved spacetime) [2]. These results are relevant both for fundamental (quantum) strings and for cosmic strings, which behave, essentially, in a classical way.

Among the cosmological backgrounds, de Sitter spacetime occupies a special place. On the one hand, it is relevant for inflation, and on the other hand, string propagation turns out to be particularly interesting there [1–6].

Recently, a novel feature for strings in de Sitter spacetime was found: exact multistring solutions. That is, one single world sheet generically describes two strings [4], several strings [5], and even *infinitely many* [6] (different and independent) strings.

Circular strings are specially suited for detailed investigation. Since the string equations of motion become separable, one has to deal with nonlinear ordinary differential equations instead of nonlinear partial differential equations. In order to obtain generic noncircular string solutions the full power of the inverse scattering method is needed in de Sitter spacetime [5].

Cosmological spacetimes are not Ricci flat and hence they are not string vacua even at first order in α' . Strings are there noncritical and quantization will presumably lead to features such as ghost states. No definite answer

is available at present to such conformal anomaly effects.

We think it is important in this context to investigate the quantum aspects in the semiclassical regime, where anomaly effects are practically irrelevant. Semiclassically, in this context, means the regime in which $H^2 \alpha' \ll 1$, where H is the Hubble constant. We proceed in this paper to semiclassically quantize time-periodic string solutions in de Sitter and anti-de Sitter spacetimes after dealing with Minkowski spacetime as an instructive exercise. Time-periodic string solutions here include all the circular string solutions in Minkowski and anti-de Sitter spacetimes, as well as the oscillating string solutions in de Sitter spacetime.

In this paper, we also complete the physical characterization of all circular string solutions found recently in de Sitter [6] and in anti-de Sitter spacetimes [9], by computing the corresponding equations of state from the exact string dynamics.

The circular string solutions in de Sitter and anti-de Sitter spacetimes depend on elliptic moduli k and \bar{k} , respectively. From the exact solutions we find their energy-momentum tensor. It turns out to have the perfect fluid form with an equation of state

$$P = (\gamma - 1)E, \quad (1.1)$$

where γ in general is time dependent and depends on the elliptic modulus as well. We analyze the equation of state for all circular string solutions in de Sitter and anti-de Sitter spacetimes. In de Sitter spacetime, for strings ex-

panding from zero radius towards infinity, the equation of state changes continuously from an ultrarelativistic matter type when $r \approx 0$, $P = +E/2$ (in 2 + 1 dimensions) to an unstable string type, $P = -E/2$, when $r \rightarrow \infty$. On the other hand, for an oscillating stable string in de Sitter spacetime, γ oscillates between $\gamma(r = 0) = 3/2$ and $\gamma(r = r_{\max}) = 1/2 + k^2/(1 + k^2)$, where $k \in [0, 1]$. Averaging over one oscillation period, the pressure *vanishes*. That is, these stable string solutions actually describe *cold matter*.

In anti-de Sitter spacetime, only oscillating (stable) circular string solutions exist. We find that γ oscillates between $\gamma(r = 0) = 3/2$ and $\gamma(r = r_{\max}) = 1/2$; i.e., the equation of state “oscillates” between $P = +E/2$ and $P = -E/2$. This is similar to the situation in flat Minkowski spacetime. When averaging over an oscillation period in anti-de Sitter spacetime, we find that γ takes values from 1 to $1 + 1/\pi^2$ for the allowed range of the elliptic modulus. That is, the average pressure over one oscillation period is always *positive* in anti-de Sitter spacetime.

In general, positive pressure characterizes the regime in which the string radius is small relative to the string maximal size, while negative pressure is characteristic for the regime in which the string radius is large. In Minkowski

spacetime, the two regimes are of equal “size,” in the sense that the average pressure is identically zero. The influence of the spacetime curvature is among other effects to modify the relative size of these two regimes.

In order to semiclassically quantize these string solutions, we compute the classical action S_{cl} as a function of the string mass m :

$$m \equiv -\frac{dS_{\text{cl}}}{dT}, \quad (1.2)$$

where we choose T as the period in the *physical* time variable (in general different from the world-sheet time). The quantization condition takes the form

$$W(m) \equiv S_{\text{cl}}(T(m)) + m T(m) = 2\pi n, \quad n \in \mathbb{N}_0. \quad (1.3)$$

In Minkowski spacetime, this formula reproduces the exact mass spectrum except for the intercept [see Eq. (3.11)].

We find for de Sitter (anti-de Sitter) spacetime that the mass is exactly proportional to the square root of the Casimir operator $C = -L_{\mu\nu}L^{\mu\nu}$ of the $O(3,1)$ -de Sitter [$O(2,2)$ -anti-de Sitter] group:

$$m = \frac{\sqrt{C}}{\pi H \alpha'}. \quad (1.4)$$

TABLE I. Circular string energy and pressure in Minkowski, de Sitter, and anti-de Sitter spacetimes.

Circular string	Energy	Pressure	Equation of state
Minkowski			
$r(\tau) = \sqrt{b}\alpha' \cos\tau $			$P(r \rightarrow 0) = +E/2$, i.e.,
Oscillating between 0 and $r_{\max} = \sqrt{b}\alpha'$	$E = \sqrt{b} = \text{const.}$	$P = -\frac{\sqrt{b}}{2}\cos 2\tau$	ultrarel. in 2+1 dim. $P(r \rightarrow r_{\max}) = -E/2$
Average values	$\langle E \rangle = E = \sqrt{b}$	$\langle P \rangle = 0$	Cold matter
de Sitter			
(i) $H^2\alpha'^2 b > 1/4$	$E(r \rightarrow 0) = \sqrt{b}$,	$P(r \rightarrow 0) = \sqrt{b}/2$,	$P(r \rightarrow 0) = +E/2$
Strings expanding from 0 towards $r = \infty$	$E(r \rightarrow \infty) = r/\alpha'$,	$P(r \rightarrow \infty) = -r/2\alpha'$,	$P(r \rightarrow \infty) = -E/2$
	E increases and P decreases as strings expand, P becoming negative		
(ii) $H^2\alpha'^2 b \leq 1/4$	$E(r \rightarrow 0) = \sqrt{b}$,	$P(r \rightarrow 0) = \sqrt{b}/2$,	$P(r \rightarrow 0) = +E/2$
Strings oscillating between 0 and $Hr_{\max} \equiv \nu = \sqrt{1 - \sqrt{1 - 4H^2\alpha'^2 b/2}}$	$E(r_{\max}) = \frac{\nu}{H\alpha'\sqrt{1-\nu^2}}$,	$P(r_{\max}) = \frac{\nu(\nu^2-1/2)}{H\alpha'\sqrt{1-\nu^2}}$,	$P(r_{\max}) = (\gamma - 1)E$
	$\nu \ll 1$ (as in Minkowski space)		$\gamma = \nu^2 + 1/2$ $P(r_{\max}) = -E/2$
	$\nu \rightarrow 1/\sqrt{2}$ (as cold matter)		$P(r_{\max}) = 0$
Average values	$\langle E \rangle = \frac{2\nu}{H\alpha'T_r} \Pi\left(\nu^2, \frac{\nu}{\sqrt{1-\nu^2}}\right)$,	$\langle P \rangle = 0$	Cold matter
Anti-de Sitter			
Strings oscillating between 0 and $Hr_{\max} \equiv \bar{\nu} = \sqrt{-1 + \sqrt{1 + 4H^2\alpha'^2 b/2}}$	$E(r \rightarrow 0) = \frac{\bar{\nu}\sqrt{1+\bar{\nu}^2}}{H\alpha'}$,	$P(r \rightarrow 0) = \frac{\bar{\nu}\sqrt{1+\bar{\nu}^2}}{2H\alpha'}$,	$P(r \rightarrow 0) = +E/2$
	$E(r_{\max}) = \frac{\bar{\nu}}{H\alpha'}$,	$P(r_{\max}) = -\frac{\bar{\nu}}{2H\alpha'}$,	$P(r_{\max}) = -E/2$
Average values	$\langle E \rangle = \frac{2\bar{k}}{H\alpha'T_r} \sqrt{(1-2\bar{k}^2)/(1-\bar{k}^2)} \Pi\left(\frac{\bar{k}^2}{1-\bar{k}^2}, \bar{k}\right)$,	$\langle P \rangle = (\gamma - 1)\langle E \rangle \neq 0$	
$(\bar{k} \equiv \bar{\nu}/\sqrt{1+2\bar{\nu}^2}) \in [0, 1/\sqrt{2}]$	$\langle E \rangle_{\bar{k} \ll 1} \approx \frac{\bar{k}}{H\alpha'}$,	$\langle P \rangle_{\bar{k} \ll 1} \approx \frac{\bar{k}^3}{32H\alpha'}$,	$\gamma_{\bar{k} \ll 1} \approx 1 + \frac{\bar{k}^2}{32}$
	$\langle E \rangle_{\bar{k} \rightarrow 1/\sqrt{2}} \approx \frac{\pi}{2H\alpha'K(1/\sqrt{2})\sqrt{1-2\bar{k}^2}}$,		$\langle P \rangle_{\bar{k} \rightarrow 1/\sqrt{2}} \approx \frac{1}{\pi^2} \langle E \rangle_{\bar{k} \rightarrow 1/\sqrt{2}}$

In Figs. 4 and 5, we give parametric plots of $H^2\alpha'W$ as a function of $H^2m^2\alpha'^2$ for $k \in [0, 1]$ for de Sitter spacetime and for $\bar{k} \in [0, 1/\sqrt{2}]$ for anti-de Sitter spacetime, respectively. We find, for de Sitter spacetime,

$$\alpha'm_{\text{dS}}^2 \approx 4n - 5H^2\alpha' n^2, \quad n \in N_0. \quad (1.5)$$

This is different from the mass spectrum in Minkowski spacetime. The level spacing is, however, still approximately constant, but the levels are less separated than in Minkowski spacetime. Notice in particular that the level spacing slightly decreases for larger and larger n . In de Sitter spacetime there is only a *finite* number of levels as can be seen from Fig. 4. The number of quantized circular string states can be estimated to be

$$N_q \approx \frac{0.34}{H^2\alpha'}. \quad (1.6)$$

It is interesting to compare this result with the number of particle states obtained using canonical quantization [1]. One finds in this way a maximum number of states:

$$N_{\text{max}} \approx \frac{0.15}{H^2\alpha'}, \quad (1.7)$$

which is of the same order as the semiclassical value (1.6).

It must be noticed that in de Sitter spacetime these states can *decay* quantum mechanically due to the possibility of quantum-mechanical tunneling through the potential barrier; see Fig. 1(b). Semiclassically, the decay probability is, however, highly suppressed for $H^2\alpha' \ll 1$ and for any value of the elliptic modulus k , except near $k = 1$ where the barrier disappears, and for which the tunneling probability is close to 1.

In anti-de Sitter spacetime arbitrary high mass states exist. The quantization of the high mass states yields

$$\alpha'm_{\text{AdS}}^2 \approx H^2\alpha' n^2. \quad (1.8)$$

Thus the (mass)² grows like n^2 and the level spacing grows proportionally to n . This is a completely different behavior as compared to Minkowski spacetime where the level spacing is constant. A similar result was recently found, using canonical quantization of generic strings in anti-de Sitter spacetime [9,10]. The physical consequences, especially the nonexistence of a critical string temperature (Hagedorn temperature), of this kind of behavior is discussed in detail in Ref. [10].

For both de Sitter and anti-de Sitter spacetimes we find thus a strong qualitative agreement between the results obtained using canonical quantization, based on

TABLE II. Semiclassical quantization of oscillating circular strings in Minkowski, de Sitter, and anti-de Sitter spacetimes.

Stationary phase	$W(M) = S_{\text{cl}}(T(M)) + MT(M)$
Stationary phase point	$\frac{\partial S_{\text{cl}}}{\partial T} = -M$, determines the period T
Quantization condition	$W(M) = 2\pi n, n \in N_0$

Minkowski

Circular string	$r(\tau) = \frac{AT}{2\pi} \cos \frac{2\pi\tau}{T}$
Classical action	$S_{\text{cl}} = -\frac{A^2 T^2}{2\pi\alpha'}$
Mass	$m = \frac{AT}{\pi\alpha'}$
Quant. condition	$\alpha'm^2 = 4n, n \in N_0$

de Sitter

Circular string	$Hr(\tau) = \frac{k}{\sqrt{1+k^2}} \text{sn} \left[\frac{\tau}{\sqrt{1+k^2}}, k \right]$
Classical action	$S_{\text{cl}} = \frac{8}{H^2\alpha'} \frac{E(k) - K(k)}{\sqrt{1+k^2}}$
Mass	$m = \frac{2}{H\alpha'} \frac{k}{\sqrt{1+k^2}}$
Quant. condition	$\alpha'm^2 \approx 4n - 5H^2\alpha' n^2, n \in n_0$

Level spacing is still approximately constant, but levels are less separated than in Minkowski spacetime. Furthermore, it turns out that $H^2\alpha'^2 m^2 \in [0, 1]$ and $H^2\alpha'W \in [0, 4\sqrt{2} - 4\ln(1+\sqrt{2})]$, so that there is only a *finite* number of states:

$$N_q \approx 1 + \text{Int} \left(\frac{2\sqrt{2} - 2\ln(1+\sqrt{2})}{\pi H^2\alpha'} \right) \approx \frac{0.34}{H^2\alpha'} \quad \text{for } H^2\alpha' \ll 1$$

The states decay quantum mechanically with probability

$$\mathcal{T} \propto \exp \left[\frac{-4}{H^2\alpha'} \frac{E(k') - \Pi(-(k'/k)^2, k')}{\sqrt{1+k^2}} \right]$$

Anti-de Sitter

Circular string	$Hr(\tau) = \frac{\bar{k}}{\sqrt{1-2\bar{k}^2}} \text{cn} \left[\frac{\tau}{\sqrt{1-2\bar{k}^2}}, \bar{k} \right]$
Classical action	$S_{\text{cl}} = \frac{8}{H^2\alpha'} \frac{(1-\bar{k}^2)K(\bar{k}) - E(\bar{k})}{\sqrt{1-2\bar{k}^2}}$
Mass	$m = \frac{2\bar{k}}{H\alpha'} \frac{\sqrt{1-\bar{k}^2}}{\sqrt{1-2\bar{k}^2}}$
Quant. condition	$\alpha'm^2 \approx H^2\alpha' n^2, n \in N_0$ (for high states, $\bar{k} \rightarrow \sqrt{1/2}$)

Level spacing grows as n . Notice that $H^2\alpha'^2 m^2 \in [0, \infty]$ and $H^2\alpha'W \in [0, \infty]$, so that there are infinitely many states.

Agreement with canonical quantization in the string perturbation series approach.

generic string solutions (string perturbation approach), and the results obtained using the semiclassical approach, based on oscillating circular string configurations.

This paper is organized as follows. In Sec. II we describe the time-periodic string solutions in Minkowski, de Sitter, and anti-de Sitter spacetimes. We derive the corresponding equations of state and give the physical interpretation in the various regimes for the different kinds of string solutions (oscillating and nonoscillating). In Sec. III we proceed to quantize the oscillating strings semi-classically, deriving the quantum mass spectrum, and we compare with the results obtained using canonical quantization. A summary of our results and conclusions is presented in Sec. IV and in Tables I and II.

II. PERIODIC STRING SOLUTIONS AND THEIR PHYSICAL INTERPRETATION

The evolution of circular strings in curved spacetimes has recently been discussed from both gravitational and cosmological points of view [3–9]. For completeness and comparison we first consider flat Minkowski spacetime. We then investigate the string dynamics in de Sitter spacetime (negative local gravity) and finally consider anti-de Sitter spacetime (positive local gravity). We investigate the effects of positive and negative local gravity in the energy-momentum tensor of such circular strings.

The string equations of motion and constraints for a circular string are most easily solved using static coordinates:

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\phi^2. \quad (2.1)$$

For simplicity we consider the string dynamics in a (2+1)-dimensional spacetime. All our solutions can, however, be embedded in a higher-dimensional spacetime, where they will describe plane circular strings. The circular string ansatz ($t = t(\tau)$, $r = r(\tau)$, $\phi = \sigma$) leads, after one integration, to the following set of first-order ordinary differential equations:

$$\dot{t} = \frac{\sqrt{b} \alpha'}{a(r)}, \quad (2.2)$$

$$\dot{r}^2 + r^2 a(r) = b \alpha'^2, \quad (2.3)$$

where b is a non-negative integration constant with the dimension of (mass)². The left-hand side of Eq. (2.3) is in the form of “kinetic” + “potential” energy. The potential is given by $V(r) = r^2 a(r)$, where

$$\begin{aligned} a(r) &= 1 && \text{for Minkowski spacetime,} \\ a(r) &= 1 - H^2 r^2 && \text{for de Sitter spacetime,} \\ a(r) &= 1 + H^2 r^2 && \text{for anti-de Sitter spacetime.} \end{aligned}$$

Properties such as the energy and pressure of the strings are more conveniently discussed in comoving (cosmological) coordinates,

$$ds^2 = -(dX^0)^2 + a^2(X^0) \frac{dR^2 + R^2 d\phi^2}{(1 + \frac{k}{4} R^2)^2}, \quad (2.4)$$

including as special cases Minkowski, de Sitter, and anti-de Sitter spacetimes:

$$\begin{aligned} a(X^0) &= 1, \quad k = 0 && \text{for Minkowski spacetime,} \\ a(X^0) &= e^{HX^0}, \quad k = 0 && \text{for de Sitter spacetime,} \\ a(X^0) &= \cos HX^0, \quad k = -H^2 && \text{for anti-de Sitter spacetime.} \end{aligned}$$

The spacetime energy-momentum tensor is given in (2+1)-dimensional spacetime by

$$\sqrt{-g} T^{\mu\nu} = \frac{1}{2\pi\alpha'} \int d\tau d\sigma (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) \delta^{(3)}(X - X(\tau, \sigma)). \quad (2.5)$$

After integration over a spatial volume that completely encloses the string [8], the energy-momentum tensor for a circular string takes the form of a fluid:

$$T_\nu^\mu = \text{diag}(-E, P, P), \quad (2.6)$$

where, in the comoving coordinates (2.4),

$$E(X) = \frac{1}{\alpha'} \dot{X}^0, \quad (2.7)$$

$$P(X) = \frac{1}{2\alpha'} \frac{a^2(X^0)}{(1 + \frac{k}{4} R^2)^2} \frac{\dot{R}^2 - R^2}{\dot{X}^0} \quad (2.8)$$

represent the string energy and pressure, respectively.

A. Minkowski spacetime

In this case Eqs. (2.2), (2.3) are solved by

$$r(\tau) = \sqrt{b} \alpha' \cos \tau, \quad t(\tau) = \sqrt{b} \alpha' \tau; \quad (2.9)$$

i.e., the string radius follows a pure harmonic motion with a period in world-sheet time $T_\tau = 2\pi$. The energy and pressure, Eqs. (2.7), (2.8), are

$$E = \sqrt{b}, \quad (2.10)$$

$$P = \frac{b\alpha'^2 - 2r^2}{2\sqrt{b}\alpha'} = -\frac{\sqrt{b}}{2} \cos 2\tau. \quad (2.11)$$

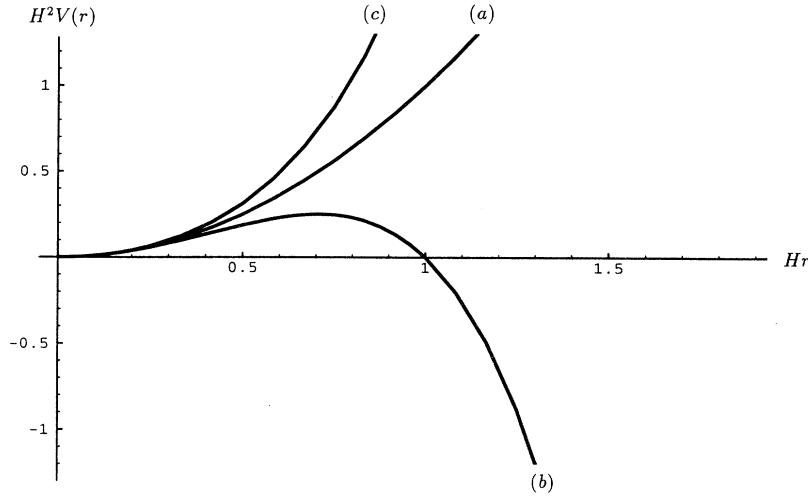


FIG. 1. The potential $V(r) = r^2 a(r)$ introduced after Eq. (2.3) for a circular string in the three spacetimes: (a) Minkowski spacetime, (b) de Sitter spacetime, (c) anti-de Sitter spacetime.

The energy is constant while the pressure depends on the string radius. For $r \rightarrow 0$ ($\tau \rightarrow \pi/2$), that is, when the string is collapsed, we find the equation of state $P = E/2$ corresponding to ultrarelativistic matter in $2 + 1$ dimensions. For $r \rightarrow \sqrt{b}\alpha'$ ($\tau \rightarrow \pi$), that is, when the string has its maximal size, the pressure is negative and we find $P = -E/2$. This is the same equation of state that was found for extremely unstable strings in inflationary universes [11]. The circular string thus oscillates between these two limiting types of equation of state. This illustrates that instantaneous negative pressure is a generic feature of strings, not only for unstable strings in inflationary universes, but even for stable oscillating strings in flat Minkowski spacetime. Even in flat Minkowski spacetime we see that there is a positive pressure regime (when the string radius is small relative to its maximal size) and a negative pressure regime (when the string radius is large). For the circular strings in Minkowski spacetime the two regimes are of equal size in the sense that the average pressure equals zero [as can be easily shown by integrating Eq. (2.11) over a full period]. The strings thus, on average, obey an equation of state of the cold matter type. The influence of the curvature of spacetime is, among other effects, to change the relative “size” of the positive pressure regime to the negative pressure regime, as we will see in the following subsections.

B. de Sitter spacetime

In de Sitter spacetime the solution of Eqs. (2.2), (2.3) involves elliptic functions. As can be seen from the potential, Fig. 1(b), the dynamics in de Sitter spacetime is completely different from the dynamics in Minkowski and anti-de Sitter spacetime. The inflation of the background here gives rise to a finite barrier, implying the existence of oscillating strings as well as contracting and expanding strings, whose exact dynamics was discussed in detail in Ref. [6]. The energy and pressure have been discussed in Ref. [6]. In this subsection we further analyze the string energy and pressure in de Sitter spacetime. The coordinate transformation relating the line elements (2.1) and (2.4), in the case of de Sitter spacetime, is given by

$$X^0 = \frac{1}{2H} \ln |1 - H^2 r^2| + t, \quad (2.12)$$

$$R = \frac{r e^{-Ht}}{\sqrt{1 - H^2 r^2}}. \quad (2.13)$$

The energy and pressure then take the form

$$H\alpha' E = \frac{H^2 r \dot{r} - H\sqrt{b}\alpha'}{H^2 r^2 - 1}, \quad (2.14)$$

$$H\alpha' P = \frac{(H^4 r^4 - 2H^2 r^2 - H^2 b\alpha'^2)H^2 r \dot{r} + (3H^4 r^4 - 2H^2 r^2 + H^2 b\alpha'^2)H\sqrt{b}\alpha'}{2(1 - H^2 r^2)(H^4 r^4 + H^2 b\alpha'^2)}. \quad (2.15)$$

Both the energy and the pressure now depend on the string radius r and the velocity \dot{r} . The latter can, however, be eliminated using Eq. (2.3).

Let us first consider a string expanding from $r = 0$ towards infinity. This corresponds to a string with $\dot{r} > 0$ and $H^2 b\alpha'^2 > 1/4$; see Fig. 1(b). For $r = 0$ we find

$E = \sqrt{b}$ and $P = \sqrt{b}/2$; thus, the equation of state $P = E/2$. This is the same result as in Minkowski spacetime, i.e., like ultrarelativistic matter. As the string expands, the energy soon starts to increase while the pressure starts to decrease and becomes negative; see Fig. 2. For $r \rightarrow \infty$ we find $E = r/\alpha'$ and $P = -r/(2\alpha')$; thus,

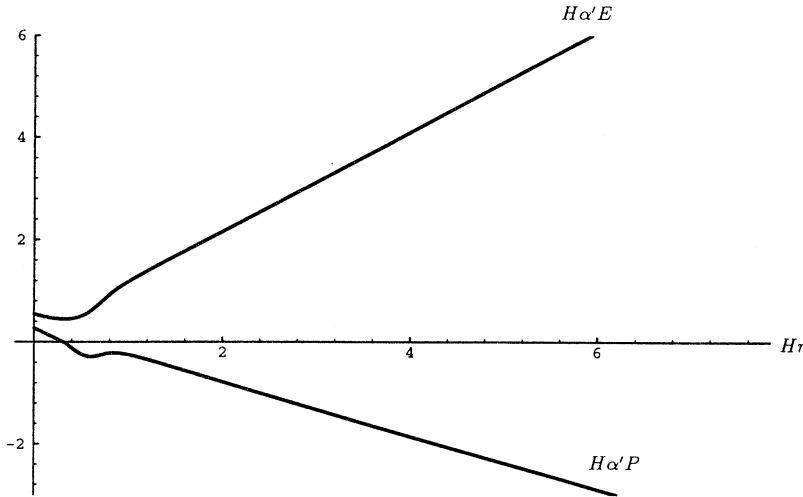


FIG. 2. The energy and pressure, Eqs. (2.14), (2.15), for a string expanding from $r = 0$ towards infinity (unstable string) in de Sitter spacetime. The curves are drawn for the case $H^2 b \alpha'^2 = 0.3$.

not surprisingly, we have recovered the equation of state of extremely unstable strings [11], $P = -E/2$.

Now consider an oscillating string, i.e., a string with $H^2 b \alpha'^2 \leq 1/4$ in the region to the left of the potential barrier, Fig. 1(b). The equation of state near $r = 0$ is the same as for the expanding string, but now the string has a maximal radius:

$$Hr_{\max} \equiv \nu = \sqrt{\frac{1 - \sqrt{1 - 4H^2 b \alpha'^2}}{2}}. \quad (2.16)$$

For $r = r_{\max}$ we find:

$$H\alpha'E = \frac{\nu}{\sqrt{1-\nu^2}} \equiv k, \quad H\alpha'P = \frac{k}{2} \left(-1 + \frac{2k^2}{1+k^2} \right), \quad (2.17)$$

corresponding to a perfect-fluid-type equation of state:

$$P = (\gamma - 1)E, \quad \gamma = \frac{k^2}{1+k^2} + \frac{1}{2}. \quad (2.18)$$

Notice that $k \in [0, 1]$ with $k = 0$ describing a string at the bottom of the potential while $k = 1$ describes a string oscillating between $r = 0$ and the top of the potential

barrier (in this case the string actually only makes one oscillation [4,6]). For $k \ll 1$ the equation of state (2.18) near the maximal radius reduces to $P = -E/2$ which is the same result we found in Minkowski spacetime. In the other limit $k \rightarrow 1$ we find, however, $P = 0$ corresponding to cold matter.

For the oscillating strings we can also calculate the average values of energy and pressure by integrating Eqs. (2.14), (2.15) over a full period, Fig. 3. Since a τ integral can be converted into a r integral, using Eq. (2.3), the average values can be obtained without using the exact τ dependence of the string radius [the solution of Eq. (2.3)]. The average energy becomes [6]

$$H\alpha'\langle E \rangle = \frac{2k}{T_\tau \sqrt{1+k^2}} \Pi \left(\frac{k^2}{1+k^2}, k \right), \quad (2.19)$$

where T_τ is the period in the world-sheet time τ and Π is the complete elliptic integral of the third kind. The average pressure can be expressed here as a combination of complete elliptic integrals of the third kind. Numerical evaluation gives zero with high accuracy. We have checked in addition that the first orders in the expansion in the elliptic modulus k identically vanish. We conclude

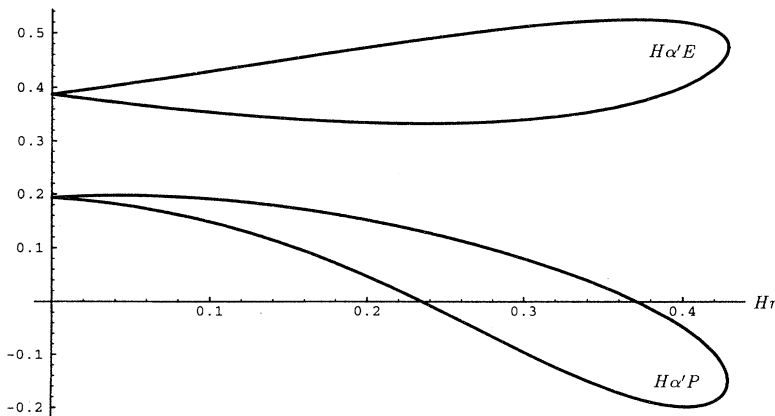


FIG. 3. The energy and pressure, Eqs. (2.14), (2.15), for an oscillating (stable) string in de Sitter spacetime. The curves describe one period of oscillation in the case $H^2 b \alpha'^2 = 0.15$.

that the average pressure is zero as in Minkowski spacetime. Thus, in average the *oscillating strings* describe *cold matter*.

C. Anti-de Sitter spacetime

As an example of a Friedmann-Robertson-Walker (FRW) universe with positive local gravity we now consider anti-de Sitter spacetime. Here the circular string potential goes to infinity even faster than in Minkowski

spacetime [see Fig. 1(c)], and so we can only have oscillating string solutions. The maximal string radius is given by

$$Hr_{\max} \equiv \bar{\nu} = \sqrt{\frac{-1 + \sqrt{1 + 4H^2 b \alpha'^2}}{2}}, \quad (2.20)$$

which can take any non-negative value. The comoving coordinates, Eq. (2.4), are in the case of anti-de Sitter spacetime given by

$$HX^0 = \pm \arccos \sqrt{(1 + H^2 r^2) \cos^2 Ht - H^2 r^2}, \quad (2.21)$$

$$HR = \frac{2}{Hr} [\sqrt{1 + H^2 r^2} \cos Ht - \sqrt{(1 + H^2 r^2) \cos^2 Ht - H^2 r^2}]. \quad (2.22)$$

The energy and pressure can be written as

$$H\alpha' E = \frac{H^2 r \dot{r} \sin Ht + H\sqrt{b\alpha'} \cos Ht}{\sqrt{1 + H^2 r^2} \sqrt{(1 + H^2 r^2) \cos^2 Ht - H^2 r^2}}, \quad (2.23)$$

$$H\alpha' P = \frac{[H\dot{r} \cos Ht + H^2 \sqrt{b\alpha'} r \sin Ht]^2 - H^2 r^2 (1 + H^2 r^2) [(1 + H^2 r^2) \cos^2 Ht - H^2 r^2]}{2[H^2 r \dot{r} \sin Ht + H\sqrt{b\alpha'} \cos Ht] \sqrt{1 + H^2 r^2} \sqrt{(1 + H^2 r^2) \cos^2 Ht - H^2 r^2}}. \quad (2.24)$$

Using Eqs. (2.2), (2.3) we can then write down the energy and pressure explicitly as functions of the string radius r , only. For the present purposes it is, however, sufficient to consider some limiting cases. It is convenient to introduce the parameter \bar{k} :

$$\bar{k} \equiv \frac{\bar{\nu}}{\sqrt{1 + 2\bar{\nu}^2}} \in [0, \sqrt{1/2}]. \quad (2.25)$$

For $\bar{k} \rightarrow 0$ the string oscillates near the bottom of the potential, while the other extreme corresponds to $\bar{k} \rightarrow \sqrt{1/2}$. For $r = 0$ we find

$$H\alpha' E = \frac{\bar{k} \sqrt{1 - \bar{k}^2}}{1 - 2\bar{k}^2} = 2H\alpha' P, \quad (2.26)$$

thus the ultrarelativistic matter equation of state.

For $r = r_{\max}$ we find

$$H\alpha' E = \frac{\bar{k}}{\sqrt{1 - 2\bar{k}^2}}, \quad H\alpha' P = -\frac{\bar{k}}{2\sqrt{1 - 2\bar{k}^2}}, \quad (2.27)$$

corresponding to the equation of state $P = -E/2$. This is exactly as in Minkowski spacetime: The string oscillates between the two limiting types of equation of state, $P = E/2$ and $P = -E/2$. A new phenomenon appears, however, when we calculate the average values over a full period. The average energy becomes

$$H\alpha' \langle E \rangle = \frac{2\bar{k}}{T_\tau} \sqrt{\frac{1 - 2\bar{k}^2}{1 - \bar{k}^2}} \Pi\left(\frac{\bar{k}^2}{1 - \bar{k}^2}, \bar{k}\right), \quad (2.28)$$

where T_τ is the period in the world-sheet time τ and Π is the complete elliptic integral of the third kind. The

average pressure for oscillating strings in anti-de Sitter spacetime is, contrary to Minkowski and de Sitter spacetime, nonzero. No simple analytic expression for it has been found for arbitrary \bar{k} . The equation of state is of perfect fluid type $\langle P \rangle = (\gamma - 1)\langle E \rangle$ where γ depends on \bar{k} . Approximate results can be obtained in the two extreme limits. For $\bar{k} \ll 1$ we find

$$H\alpha' \langle P \rangle = \frac{\bar{k}^3}{32} + O(\bar{k}^5), \quad (2.29)$$

while Eq. (2.28) gives

$$H\alpha' \langle E \rangle = \bar{k} + O(\bar{k}^3), \quad (2.30)$$

and, therefore,

$$\gamma = 1 + \frac{\bar{k}^2}{32} + O(\bar{k}^4). \quad (2.31)$$

In the limit $\bar{k} \rightarrow \sqrt{1/2}$ we find

$$H\alpha' \langle P \rangle = \frac{1}{2\pi K(\sqrt{1/2}) \sqrt{1 - 2\bar{k}^2}} + O(\sqrt{1 - 2\bar{k}^2}), \quad (2.32)$$

and, from Eq. (2.28),

$$H\alpha' \langle E \rangle = \frac{\pi}{2K(\sqrt{1/2}) \sqrt{1 - 2\bar{k}^2}} + O(\sqrt{1 - 2\bar{k}^2}), \quad (2.33)$$

where K is the complete elliptic integral of the first kind. This corresponds to

$$\gamma = 1 + 1/\pi^2 + O(1 - 2\bar{k}^2). \quad (2.34)$$

Numerical evaluation of $\langle E \rangle$ and $\langle P \rangle$ shows that γ monotonically grows from $\gamma = 1$ until $\gamma = 1 + 1/\pi^2$ when \bar{k} grows from zero to $1/\sqrt{2}$, so that the average pressure is always positive.

This concludes our analysis of the various types of equation of state for circular strings in Minkowski, de Sitter, and anti-de Sitter spacetimes. The results are summarized in Table I.

III. SEMICLASSICAL QUANTIZATION

In this section we perform a semiclassical quantization of the circular string configurations discussed in the previous section. We use an approach developed in field theory by Dashen and co-workers [12,13], based on the stationary phase approximation of the partition function. The method can be only used for time-periodic solutions of the classical equations of motion. In our string problem, these solutions, however, include all the circular string solutions in Minkowski and in anti-de Sitter spacetimes, as well as the oscillating circular strings ($H^2 b \alpha'^2 \leq 1/4$; cf. Sec. IIB) in de Sitter spacetime.

The result of the stationary phase integration is expressed in terms of the function

$$W(m) \equiv S_{\text{cl}}(T(m)) + m T(m), \quad (3.1)$$

where S_{cl} is the action of the classical solution, m is the mass, and the period $T(m)$ is implicitly given by

$$\frac{dS_{\text{cl}}}{dT} = -m. \quad (3.2)$$

In string theory we must choose T to be the period in a physical time variable. For example, when a light cone gauge exists, T is the period in $X^0 = \alpha' p \tau$. The bound state quantization condition then becomes [12,13]

$$W(m) = 2\pi n, \quad n \in N_0, \quad (3.3)$$

for n “large.” The method has been successfully used in many cases from quantum mechanics to quantum field theory. For integrable field theories the semiclassical quantization happens in fact, to be exact. It must be noticed that string theory in de Sitter spacetime is exactly integrable [14].

To demonstrate the method and to fix the normalization we first consider circular strings in flat Minkowski spacetime. We then perform the same analysis for de Sitter and anti-de Sitter spacetimes, and then after, we compare with approximate results obtained using canonical quantization [1,10].

A. Minkowski spacetime

The string action in the conformal gauge in Minkowski spacetime is given by

$$S = \frac{1}{2\pi\alpha'} \int_0^T d\sigma \int_0^T d\tau g_{\mu\nu} (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu), \quad (3.4)$$

where the world-sheet coordinate σ runs from 0 to T . That is,

$$X^\mu(\sigma + T, \tau) = X^\mu(\sigma, \tau).$$

In this parametrization the circular string solution (2.9) takes the form

$$X^0 = A \tau, \quad (3.5)$$

$$X^1 = \frac{AT}{2\pi} \cos\left(\frac{2\pi\sigma}{T}\right) \cos\left(\frac{2\pi\tau}{T}\right), \quad (3.6)$$

$$X^2 = \frac{AT}{2\pi} \sin\left(\frac{2\pi\sigma}{T}\right) \cos\left(\frac{2\pi\tau}{T}\right), \quad (3.7)$$

where A is an arbitrary constant [$A = \sqrt{b}\alpha'$ in the notation of Eq. (2.9)] and T became the period in the τ variable too. For this solution in Minkowski spacetime, we find, from Eq. (3.4),

$$S_{\text{cl}} = -\frac{A^2}{2\pi\alpha'} T^2. \quad (3.8)$$

Equation (3.2) then takes the form

$$M = \frac{A^2}{\pi\alpha'} T, \quad (3.9)$$

and then the quantization condition (3.3) yields

$$M^2 = \frac{4A^2}{\alpha'} n, \quad n \in N_0. \quad (3.10)$$

We must identify the mass with the variable conjugated to the physical time X^0 . Since M is conjugated to τ and $X^0 = A \tau$, $m \equiv M/A$ is the string mass. Therefore the semiclassical string spectrum results:

$$\alpha' m^2 = 4n, \quad n \in N_0. \quad (3.11)$$

If we subtract the intercept -4 in Eq. (3.11), this is the well-known (exact) mass formula for closed bosonic strings in flat Minkowski spacetime.

B. de Sitter spacetime

Using the notation introduced in Eqs. (2.16), (2.17), the oscillating strings in de Sitter spacetime are given by [6]

$$Hr(\tau) = \frac{k}{\sqrt{1+k^2}} \text{sn}\left[\frac{\tau}{\sqrt{1+k^2}}, k\right]. \quad (3.12)$$

Equation (2.2) is then integrated to

$$Ht(\tau) = \frac{k}{\sqrt{1+k^2}} \Pi\left(\frac{k^2}{1+k^2}, \frac{\tau}{\sqrt{1+k^2}}, k\right). \quad (3.13)$$

The period in comoving time, which from Eq. (2.12) equals the period in static coordinate time, is then given by

$$HT = \frac{4k}{\sqrt{1+k^2}} \Pi\left(\frac{k^2}{1+k^2}, k\right). \quad (3.14)$$

Notice that the expressions for the periods in the physical time X^0 and in the world-sheet time τ are different. The period in the physical time can be further rewritten in terms of incomplete elliptic integrals of the first and second kinds:

$$HT = \frac{4kK(k)}{\sqrt{1+k^2}} + 4K(k)E(\phi, k) - 4E(k)F(\phi, k), \quad (3.15)$$

where

$$\phi = \arcsin \frac{1}{\sqrt{1+k^2}}. \quad (3.16)$$

The classical action over one period becomes

$$S_{cl} = \frac{8}{H^2\alpha'} \frac{E(k) - K(k)}{\sqrt{1+k^2}}. \quad (3.17)$$

A straightforward calculation gives

$$H \frac{dT}{dk} = -\frac{4}{\sqrt{1+k^2}} \left[K(k) - \frac{2E(k)}{1-k^2} \right], \quad (3.18)$$

as well as

$$\frac{dS_{cl}}{dk} = \frac{8}{H^2\alpha'} \frac{k}{(1+k^2)^{3/2}} \left[K(k) - \frac{2E(k)}{1-k^2} \right]. \quad (3.19)$$

Then, identifying the string mass m as the conjugate to the comoving time X^0 , Eq. (3.2) leads to

$$m = \frac{2}{H\alpha'} \frac{k}{1+k^2}. \quad (3.20)$$

The string solutions in de Sitter spacetime enjoy conserved quantities associated with the $O(3,1)$ rotations on the hyperboloid. Using hyperboloid coordinates, the only nonzero component for the circular string solutions under consideration here is given by $L_{10} = -L_{01} = \sqrt{C}$ [6]:

$$L_{10} = \sqrt{C} = 2\pi \frac{k}{1+k^2}, \quad (3.21)$$

where $C = -L_{\mu\nu}L^{\mu\nu}$ is the Casimir operator of the group. Hence, the mass is exactly *linear* in \sqrt{C}

$$m = \frac{\sqrt{C}}{\pi H\alpha'}. \quad (3.22)$$

The physical meaning of such a type of “linear” Regge trajectory deserves further investigation.

The quantization condition (3.3) finally gives

$$W = \frac{8}{H^2\alpha'\sqrt{1+k^2}} \left[E(k) - \frac{K(k)}{1+k^2} + \frac{k[K(k)E(\phi, k) - E(k)F(\phi, k)]}{\sqrt{1+k^2}} \right] = 2\pi n. \quad (3.23)$$

This equation determines a quantization of the parameter k , which by Eq. (3.20) gives a quantization of the mass. A full parametric plot of $H^2\alpha'W$ as a function of $H^2m^2\alpha'^2$ for $k \in [0, 1]$ is shown in Fig. 4. In the whole k range a good approximation is provided by the second-order polynomial:

$$W \approx 1.6 m^2\alpha' + 0.53 H^2\alpha'(m^2\alpha')^2. \quad (3.24)$$

Expanding in $H^2\alpha'$, the quantization of the mass becomes

$$\alpha'm^2 \approx 4n - 5H^2\alpha' n^2 + \dots, \quad n \in N_0. \quad (3.25)$$

This is different from the result obtained in Minkowski spacetime. The level spacing is, however, still approximately constant, but the levels are less separated than in Minkowski spacetime. Notice also that the level spacing slightly decreases for larger and larger n , and that the Minkowski result is recovered for $H = 0$.¹

In de Sitter spacetime there is only a *finite* number of levels as can be seen from Fig. 4. This is due to the *finite* height of the potential barrier. The number of quantized circular string states is easily estimated using Eqs. (3.23), (3.24) and Fig. 4:

$$N_q \approx 1 + \text{Int} \left(\frac{2\sqrt{2} - 2\ln(1+\sqrt{2})}{\pi H^2\alpha'} \right). \quad (3.26)$$

For $H^2\alpha' \ll 1$, which is clearly satisfied in most interesting cases, we get

$$N_q \approx \frac{0.34}{H^2\alpha'}. \quad (3.27)$$

It should be stressed, however, that these states are not truly stable stationary states because of the possibility of quantum-mechanical tunneling through the barrier. The probability of decay is given by

$$\mathcal{T} \propto e^{-S_E}, \quad (3.28)$$

where S_E is the Euclidean action of the classical solution in the classically forbidden region. Defining $t = it_E$, $\tau = i\tau_E$, and $S_E = iS$, we find from Eqs. (2.2) and (2.3),

$$\frac{dt_E}{d\tau_E} = \frac{\sqrt{b}\alpha'}{1-H^2r^2}, \quad (3.29)$$

$$\left(\frac{dr}{d\tau_E} \right)^2 = r^2(1-H^2r^2) - b\alpha'^2. \quad (3.30)$$

Then the Euclidean action takes the form

¹We thank J. Garriga for raising this question to us.

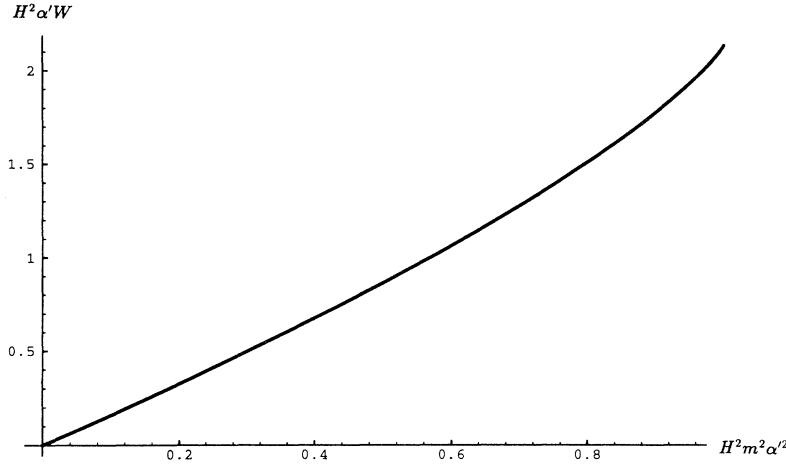


FIG. 4. Parametric plot of $H^2 \alpha' W$ as a function of $H^2 m^2 \alpha'^2$, Eqs. (3.20), (3.23), for $k \in [0, 1]$ in de Sitter space-time. Notice that $H^2 m^2 \alpha'^2 \in [0, 1]$ and $H^2 \alpha' W \in [0, 4\sqrt{2} - 4\ln(1 + \sqrt{2})]$. For $W = 2\pi n$ ($n \geq 0$) there can only be a *finite* number of states.

$$S_E = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \int_{\text{period}} d\tau_E \left\{ (1 - H^2 r^2) \left(\frac{dt_E}{d\tau_E} \right)^2 - \frac{1}{1 - H^2 r^2} \left(\frac{dr}{d\tau_E} \right)^2 - r^2 \left(\frac{d\phi}{d\sigma} \right)^2 \right\}. \quad (3.31)$$

The integral can be expressed in terms of complete elliptic integrals of second and third kinds:

$$S_E = \frac{4}{H^2 \alpha'} \frac{E(k') - \Pi(- (k'/k)^2, k')}{\sqrt{1 + k^2}}, \quad (3.32)$$

where $k' = \sqrt{1 - k^2}$. For $H^2 \alpha' \ll 1$ the quantum-mechanical tunneling is highly suppressed for any value of k except near $k = 1$, as follows by analyzing Eq. (3.32) in a little more detail. For $k \rightarrow 1$, where the barrier disappears, the decay probability becomes unity. For $k \rightarrow 0$, near the bottom of the potential, the decay probability is

$$\mathcal{T}_{k \rightarrow 0} \propto \exp \left[-\frac{4}{H^2 \alpha'} \left\{ 1 - \frac{\pi}{2} k - \frac{k^2}{2} \ln \frac{4}{k} + O(k^2) \right\} \right]. \quad (3.33)$$

For k identically zero, the decay process can be interpreted as a creation of strings with probability $\mathcal{T}_{k=0} \propto \exp[-\frac{4}{H^2 \alpha'}]$. This $k=0$ term coincides with the result found by Basu, Guth, and Vilenkin [15] in the context of a cosmic string nucleation scenario.

Let us now return to the number of states, Eq. (3.27). It is interesting to compare the results here with the results obtained using canonical quantization of generic strings. By using a string perturbation series approach for $H^2 \alpha' \ll 1$, it was shown by de Vega and Sánchez [1] that the mass formula in de Sitter spacetime takes the form

$$\alpha' m^2 = 24 \sum_{n>0} \frac{2n^2 - H^2 m^2 \alpha'^2}{\sqrt{n^2 - H^2 m^2 \alpha'^2}} + \sum_{n>0} \frac{2n^2 - H^2 m^2 \alpha'^2}{\sqrt{n^2 - H^2 m^2 \alpha'^2}} \sum_R [(\alpha_n^R)^\dagger \alpha_n^R + (\tilde{\alpha}_n^R)^\dagger \tilde{\alpha}_n^R], \quad (3.34)$$

where

$$[\alpha_n^R, (\alpha_n^R)^\dagger] = [\tilde{\alpha}_n^R, (\tilde{\alpha}_n^R)^\dagger] = 1. \quad (3.35)$$

It follows that real mass solutions can only be defined up to some maximal mass of the order $\alpha' m^2 \approx 1/(H^2 \alpha')$. To be a little more specific consider physical states in the form

$$(\tilde{\alpha}_1^{R_1})^\dagger (\alpha_1^{S_1})^\dagger \dots (\tilde{\alpha}_1^{R_N})^\dagger (\alpha_1^{S_N})^\dagger | 0 \rangle, \quad (3.36)$$

with mass implicitly given by

$$\alpha' m^2 = 24 \sum_{n>0} \frac{2n^2 - H^2 m^2 \alpha'^2}{\sqrt{n^2 - H^2 m^2 \alpha'^2}} + 2N \frac{2 - H^2 m^2 \alpha'^2}{\sqrt{1 - H^2 m^2 \alpha'^2}}. \quad (3.37)$$

For $H^2 \alpha' \ll 1$ we find real mass solutions to this equation only for

$$N \leq N_{\max} \approx \frac{0.15}{H^2 \alpha'}. \quad (3.38)$$

Thus, along a trajectory in the Regge plot, we find only N_{\max} states. This is the relevant quantity to be compared with the number of *exact* circular string states N_q in the potential, and the two numbers are in fact of the same order; compare with Eq. (3.27).

C. Anti-de Sitter spacetime

The calculations here are very similar to the calculations of Sec. IIIB, but the results will turn out to

be completely different. In the notation of Eqs. (2.20), (2.25) the oscillating strings in anti-de Sitter spacetime are given by [9]

$$Hr(\tau) = \frac{\bar{k}}{\sqrt{1-2\bar{k}^2}} \text{cn} \left[\frac{\tau}{\sqrt{1-2\bar{k}^2}}, \bar{k} \right]. \quad (3.39)$$

In this case Eq. (2.2) leads to:

$$Ht(\tau) = \bar{k} \sqrt{\frac{1-2\bar{k}^2}{1-\bar{k}^2}} \Pi \left(\frac{\bar{k}^2}{1-\bar{k}^2}, \frac{\tau}{\sqrt{1-2\bar{k}^2}}, \bar{k} \right). \quad (3.40)$$

Also in anti-de Sitter spacetime, from Eq. (2.21), the period in comoving time equals the period in static coordinate time:

$$HT = 4\bar{k} \sqrt{\frac{1-2\bar{k}^2}{1-\bar{k}^2}} \Pi \left(\frac{\bar{k}^2}{1-\bar{k}^2}, \bar{k} \right). \quad (3.41)$$

It is rewritten in terms of incomplete elliptic integrals of the first and second kinds:

$$HT = 2\pi + 4\bar{k}K(\bar{k}) \sqrt{\frac{1-2\bar{k}^2}{1-\bar{k}^2}} - 4K(\bar{k})E(\bar{\phi}, \bar{k}') + 4[K(\bar{k}) - E(\bar{k})]F(\bar{\phi}, \bar{k}'), \quad (3.42)$$

where

$$W = \frac{8}{H^2\alpha'} \left\{ \frac{\pi \bar{k} \sqrt{1-\bar{k}^2}}{2(1-2\bar{k}^2)} + \frac{K(\bar{k}) - E(\bar{k})}{\sqrt{1-2\bar{k}^2}} - \frac{\bar{k} \sqrt{1-\bar{k}^2}}{1-2\bar{k}^2} [K(\bar{k})E(\bar{\phi}, \bar{k}') - (K(\bar{k}) - E(\bar{k}))F(\bar{\phi}, \bar{k}')] \right\} = 2\pi n. \quad (3.49)$$

This equation determines a quantization of the parameter \bar{k} , which by Eq. (3.47) gives a quantization of the mass. A parametric plot of $H^2\alpha'W$ as a function of $H^2m^2\alpha'^2$ for $\bar{k} \in [0, 1/\sqrt{2}]$ is shown in Fig. 5. The curve continues forever to the right (contrary to the case of de Sitter spacetime, Fig. 4), so that arbitrarily high mass states exist. In anti-de Sitter spacetime, this is also clear from the potential, Fig. 1(c). Asymptotically ($\bar{k} \rightarrow \sqrt{1/2}$) we find, from Eqs. (3.47), (3.49),

$$m = \frac{1}{H\alpha'(1-2\bar{k}^2)} + O(1), \quad (3.50)$$

$$W = \frac{2\pi}{H^2\alpha'(1-2\bar{k}^2)} + O\left(\frac{1}{\sqrt{1-2\bar{k}^2}}\right); \quad (3.51)$$

i.e.,

$$W \approx \frac{2\pi}{H} m. \quad (3.52)$$

The quantization of the high mass states then takes the form

$$\bar{\phi} = \arcsin \frac{\sqrt{1-2\bar{k}^2}}{1-\bar{k}^2}, \quad \bar{k}' = \sqrt{1-\bar{k}^2}. \quad (3.43)$$

The classical action over one period becomes

$$S_{\text{cl}} = \frac{8}{H^2\alpha'} \frac{(1-\bar{k}^2)K(\bar{k}) - E(\bar{k})}{\sqrt{1-2\bar{k}^2}}. \quad (3.44)$$

A straightforward calculation gives

$$H \frac{dT}{d\bar{k}} = \frac{4}{\sqrt{(1-\bar{k}^2)(1-2\bar{k}^2)}} [2E(\bar{k}) - K(\bar{k})], \quad (3.45)$$

as well as

$$\frac{dS_{\text{cl}}}{d\bar{k}} = -\frac{8\bar{k}}{H^2\alpha'} \frac{2E(\bar{k}) - K(\bar{k})}{(1-2\bar{k}^2)^{3/2}}. \quad (3.46)$$

The mass is obtained from Eq. (3.2):

$$m = \frac{2\bar{k}}{H\alpha'} \frac{\sqrt{1-\bar{k}^2}}{1-2\bar{k}^2}. \quad (3.47)$$

As in de Sitter spacetime, we find here an exact *linear* relation between the mass and the square root of the Casimir operator $\sqrt{C} = L_{10}$ (in hyperboloid coordinates) of the group:

$$m = \frac{\sqrt{C}}{\pi H\alpha'}. \quad (3.48)$$

The quantization condition (3.3) finally gives

$$\alpha' m^2 \approx H^2 \alpha' n^2. \quad (3.53)$$

Thus the (mass)² grows like n^2 and the level spacing grows proportionally to n . This is a completely different behavior as compared to Minkowski spacetime where the level spacing is constant. A similar result was found recently, using canonical quantization of generic strings in anti-de Sitter spacetime [10]. The mass formula in anti-de Sitter spacetime takes the form (3.34) but with $H^2 < 0$ (reminiscent of the formal relation between de Sitter and anti-de Sitter line elements in static coordinates). Then, the square roots in the denominators are well defined for any value of $\alpha'm^2$ and arbitrary high mass states can be constructed. By considering states of the form (3.36) for very large N [$N \gg 1/(H^2\alpha')$] it was shown that [10]

$$\alpha' m^2 \approx 4H^2\alpha' N^2, \quad (3.54)$$

in qualitative agreement with the result obtained here for circular strings, Eq. (3.53). It should be noticed that the circular string oscillations in anti-de Sitter spacetime (and in de Sitter spacetime) do not follow a pure harmonic motion as in flat Minkowski spacetime, since

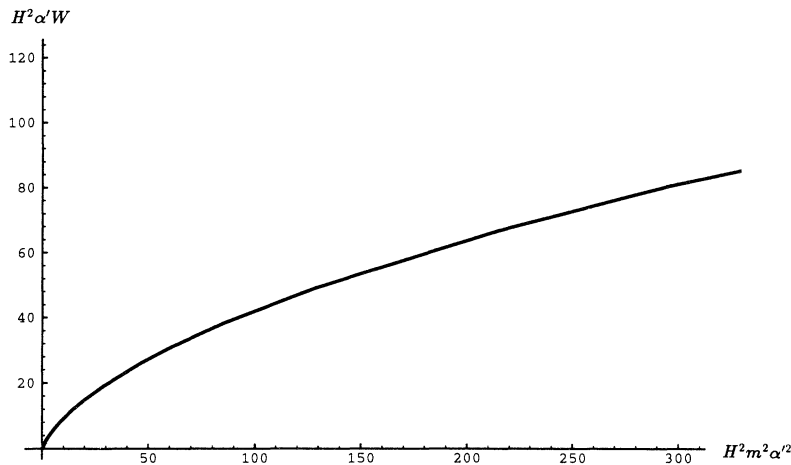


FIG. 5. Parametric plot of $H^2 \alpha' W$ as a function of $H^2 m^2 \alpha'^2$, Eqs. (3.44), (3.46), in anti-de Sitter spacetime. Notice that $H^2 m^2 \alpha'^2 \in [0, \infty[$ and $H^2 \alpha' W \in [0, \infty[$. For $W = 2\pi n$ ($n \geq 0$) there are *infinitely many* states.

expressed in terms of Jacobi elliptic functions they are in fact very precise superpositions of all frequencies. The states Eq. (3.36), involving only one frequency, should therefore not have exactly the same mass as a circular string, and so we can only expect a qualitative agreement for the results obtained using the two different approaches, and that was indeed what we found.

IV. CONCLUSION

We have computed *exactly* the equation of state of the circular string solutions recently found in de Sitter [6] and anti-de Sitter [9] spacetimes. The string equation of state has the perfect fluid form $P = (\gamma - 1)E$, with P and E expressed closely and completely in terms of elliptic functions and the instantaneous parameter γ

depending on the elliptic modulus. We have quantized the time-periodic (oscillating) string solutions within the semiclassical (stationary phase approximation) approach.

The main results of this paper are summarized in Tables I and II. The semiclassical quantization of the *exact* (circular) string solutions and the canonical quantization in the string perturbation series approach of the generic strings, give the same qualitative results.

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