Generalized uncertainty relations and long-time limits for quantum Brownian motion models

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We study the time evolution of the reduced density operator for a class of quantum Brownian motion models consisting of a particle moving in a potential $V(x)$ and coupled to an environment of harmonic oscillators in a thermal state. Our principal tool is the Wigner function of the reduced density operator and for linear systems we derive an explicit expression for the Wigner function propagator. We use it to derive two generalized uncertainty relations. The first consists of a sharp lower bound on the uncertainty function $U = (\Delta p)^2(\Delta q)^2$ after evolution for time t in the presence of an environment. The second, a stronger and simpler result, consists of a lower bound at time t on the quantity $\mathcal{A}^2 = U - C_{pq}^2$, where $C_{pq} = \frac{1}{2}\langle \Delta\hat{p}\Delta\hat{q} + \Delta\hat{q}\Delta\hat{p}\rangle$. ($\mathcal{A}$ is essentially the area enclosed by the $1-\sigma$ contour of the Wigner function.) In both cases the minimizing initial state is a correlated coherent state (a nonminimal Gaussian pure state), and in the first case the lower bound is only an envelope. These generalized uncertainty relations supply a measure of the comparative size of quantum and thermal fluctuations. We prove two simple inequalities, relating uncertainty to von Neumann entropy, $-\mathrm{Tr}(\rho \ln \rho)$, and the von Neumann entropy to linear entropy, $1 - \mathrm{Tr}\rho^2$. We also prove some results on the long-time limit of the Wigner function for arbitrary initial states. For the harmonic oscillator the Wigner function for all initial states becomes a Gaussian at large times (often, but not always, a thermal state). We derive the explicit forms of the long-time limit for the free particle (which does not in general go to a Gaussian), and also for more general potentials in the approximation of high temperature. We discuss connections with previous work by Hu and Zhang and by Paz and Zurek.

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I. INTRODUCTION

Quantum Brownian motion (QBM) models have been the subject of a number of studies over many years [1–15]. Many reasons for the interest in these models may be found: they permit the possibility of studying in some detail the approach to equilibrium in nonequilibrium system; they arise in studies of macroscopic quantum effects; and they are related to the question of dissipation in tunneling. Most recently, they have been studied in the contexts of quantum measurement theory, decoherence, and the quantum to classical transition. It is these contexts towards which the present work is directed. We are, in particular, interested in the emergence of classical behavior in open quantum systems.

The quantum Brownian motion models belong to an important class of non equilibrium systems in which there is a natural separation into a distinguished subsystem $\mathcal{S}$ and the rest (the environment). The distinguished subsystem $\mathcal{S}$ is often referred to as an open quantum system. One is interested in the behavior of $\mathcal{S}$, but not in the detailed behavior of the environment. $\mathcal{S}$ is most completely described by the reduced density operator ρ obtained by tracing out over the environment states. One's goal is then to obtain an evolution equation for ρ, which will in general by nonunitary, from which one may calculate the probabilities of any observables referring to $\mathcal{S}$ only. In the one-dimensional QBM models studied in this paper, $\mathcal{S}$ consists of a particle of mass M moving in a potential $V(x)$. $\mathcal{S}$ is linearly coupled to an environment, consisting of a large number of harmonic oscillators in a thermal state at temperature T, and characterized by a dissipation coefficient γ.

Given such a model, there are many interesting questions one can then ask about it. Under what conditions is there suppression of interference between localized wave packets? Under what conditions does the Brownian particle evolve approximately classically? How big are the fluctuations about classical predictability? Are these fluctuations larger than the inescapable quantum fluctuations? Is there a generalization of the uncertainty principle to include environmentally induced fluctuations, representing the smallest amount of noise the system must suffer? What sort of states are most stable in the face of these fluctuations? Does entropy or uncertainty increase as time evolves? Toward what sort of states does the system evolve in the long-time limit? Does the system tend towards thermal equilibrium in the long-time limit?

Many of the above questions have been addressed before [16,14,17–19,15,20,21] and some of them will be the topic of the present paper. We are particularly interested in generalizations of the uncertainty principle. The methods we will develop, however, will be applicable to most of the other questions listed above, and in fact, a spin off of our work is a simple method for finding the long-time

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limits of arbitrary initial states, for linear systems.

There are a number of reasons why it is of interest to obtain generalizations of the uncertainty principle. First, on general grounds it is desirable to know how the usual uncertainty principle is modified in the face of environmentally induced fluctuations. This paper therefore contributes to the extensive body of work which is concerned with the fundamental question of modifications and generalizations of the uncertainty principle (see, for example, [22] and references therein). A modified uncertainty relation which incorporates the effects of environmentally induced fluctuations was previously derived in [16]. The uncertainty relation derived there was an information-theoretic relation in terms of the so-called Wehrl entropy. The aim of the present work is to derive such a relation in terms of the usual uncertainty function

$$U = (\Delta p)^2(\Delta q)^2 \tag{1.1}$$

and also related functions.

Second, in a number of recent papers, Hu and Zhang have computed the time evolution of the uncertainty function (1.1) in the presence of a thermal environment, and thus estimated the comparative size of quantum and environmentally induced effects [14]. Their computation concerned only Gaussian initial states, and it therefore becomes an interesting issue to see what can be said about arbitrary initial states.

Third, in the context of the decoherence program it is of interest to know what sort of states suffer the least amount of noise under evolution in the presence of an environment [18,21]. Such states are regarded as the most "predictable" or "stable" states. These states were found, in [18,21], by seeking the class of states that generated the least amount of von Neumann entropy entropy (or linear entropy) after evolution for a fixed period of time. The present work offers an alternative characterization of these states—they are the states that minimize the generalized uncertainty relations considered here. A further question of interest is to discover the connection between these two different measures of uncertainty, and this we also consider.

Finally, as shown by Hu and Matacz [23], the influence functional formalism (used here) has applications to statistical processes in cosmological and black hole spacetimes. In particular a quantum Brownian motion model can describe a particle detector coupled to a scalar field around a black hole, and can describe the dynamics of the field modes of a gravitational wave or inflation field in cosmological models. Our results may therefore be of relevance to these considerations.

Our starting point is a study of the dynamical evolution of the reduced density operator, and this is most conveniently carried out in the Wigner representation. In particular, the tool we found to be of greatest use in the Wigner function propagator. The reason why the Wigner representation is so clarifying is that, at least for linear systems, the unitary part of the evolution corresponds to transporting the initial Wigner function along the classical phase-space trajectories. It is therefore possible to cleanly separate the unitary effects from the nonunitary effects induced by the environment.

In the presence of an environment, the Wigner function propagator may be calculated exactly for linear systems, and it is just a Gaussian. It permits the explicit calculation of all moments of p and q at any time for arbitrary initial states in terms of the moments at the initial time. We describe the features of quantum Brownian motion models in Sec. II, and we calculate the Wigner function propagator for linear systems in Sec. III.

In Sec. IV, we use the results of Sec. III to derive generalized uncertainty relations for quantum Brownian motion models. Two relations are derived. The first is a lower bound over all initial states on the uncertainty function (1.1), at an arbitrary time t. This lower bound represents the least amount of noise, both quantum and environmentally induced, the system must suffer after evolution for time t in the presence of an environment. The lower bound is not the time evolution of a particular initial state, but is actually an envelope—the initial state achieving the lower bound at time t is different for each time. The second relation is a lower bound on the related quantity,

$$\mathcal{A}^2 = (\Delta p)^2(\Delta q)^2 - \tfrac{1}{4}(\Delta\hat{p}\Delta\hat{q} + \Delta\hat{q}\Delta\hat{p})^2 \ , \tag{1.2}$$

where $\Delta\hat{q} = \hat{q} - \langle\hat{q}\rangle$ and likewise for $\Delta\hat{p}$. $\mathcal{A}$ is essentially the area enclosed by the $1-\sigma$ contour of the Wigner function, and satisfies a strengthened version of the usual uncertainty principle, $\mathcal{A} \geq \hbar/2$ [24]. The time evolution of $\mathcal{A}$ turns out to be much simpler than that of U, and the lower bound on it at any time *is* the time evolution of a particular initial state, hence this result is stronger than the first one. In both cases the minimizing initial states is a *correlated coherent state*, a nonminimal Gaussian pure state, of the form $\psi(x) = e^{-(a+ib)x^2}$, where a, b are real. (This is clearly some kind of *squeezed* coherent state, but we choose to follow the nomenclature of [24].) The detailed forms of the lower bounds on U and $\mathcal{A}$ are discussed in Sec. V, in the case of an Ohmic environment and the connection with the work of Hu and Zhang [14] is discussed.

In Sec. VI, we discuss the connection of these measures of uncertainty with von Neumann entropy, $S[\rho] = -\mathrm{Tr}(\rho \ln \rho)$. We prove an inequality relating uncertainty to von Neumann entropy, which in the regime of large uncertainty has the form, $U \geq \mathcal{A}^2 \geq \hbar^2 e^{2S}$. We also exhibit the connection with the linear entropy,$S_L = 1 - \mathrm{Tr}\rho^2$. These results make the connection with the work of Zurek *et al.* [18,21].

In Sec. VII, we show that our expression for the time evolution of the Wigner function, for linear systems, readily permits the computation of the long-time limit of an arbitrary initial state. For the harmonic oscillator in a certain class of environments, all initial states go to a Gaussian Wigner function in the long-time limit, not always a thermal state. We discuss how some of our results may be generalized to systems with more general potentials, and we consider the case of the long-time limit of arbitrary initial states for the damped free particle. We summarize and discuss our results in Sec. VIII.

II. QUANTUM BROWNIAN MOTION MODELS

We are concerned in this paper with the class of quantum Brownian models consisting of a particle of large mass M moving in a potential $V(x)$ and linearly coupled to a bath of harmonic oscillators. (This section is entirely review of standard material [25,8,12,26].) The total system is therefore described by the action

$$S_{\rm tot}[x(t),q_n(t)] = \int dt[\tfrac{1}{2}M\dot{x}^2 - V(x)] + \sum_n \int dt[\tfrac{1}{2}m_n\dot{q}_n^2 - \tfrac{1}{2}m_n\omega_n^2 q_n^2 - C_n q_n x] \ . \tag{2.1}$$

Quantum mechanically, the total system will be described most completely by the density matrix, $\rho_t(x,q_n,x',q_n')$. However, we are interested solely in the behavior of the large particle, and hence the relevant quantity is the reduced density matrix, obtained by tracing over the environmental coordinates:

$$\rho_t(x,x') = \prod_n \int dq_n \rho_t(x,q_n,x'q_n) \ . \tag{2.2}$$

It is convenient to introduce the (nonunitary) reduced density matrix propagator J defined by the relation

$$\rho_t(x,y) = \int dx_0 dy_0 J(x,y,t|x_0,y_0,0)\rho_0(x_0,y_0) \ . \tag{2.3}$$

Under the assumption that the initial density operator for the total system factorizes, $\rho = \rho_{\rm system}\otimes\rho_{\rm bath}$, the reduced density operator propagator is given by the path-integral expression

$$J(x_f,y_f,t|x_0,y_0,0) = \int \mathcal{D}x\mathcal{D}y \exp\left(\frac{i}{\hbar}S[x] - \frac{i}{\hbar}S[y] + \frac{i}{\hbar}W[x,y]\right) \ , \tag{2.4}$$

where

$$S[x] = \int dt[\tfrac{1}{2}M\dot{x}^2 - V(x)] \tag{2.5}$$

and $W[x(t),y(t)]$ is the Feynman-Vernon influence functional phase:

$$W[x(t),y(t)] = -\int_0^t ds \int_0^s ds'[x(s)-y(s)]\eta(s-s')[x(s')+y(s')] + i\int_0^t ds\int_0^s ds'[x(s)-y(s)]\nu(s-s')[x(s')-y(s')] \ . \tag{2.6}$$

The kernels $\eta(s)$ and $\nu(s)$ are defined by

$$\nu(s) = \int_0^\infty \frac{d\omega}{\pi} I(\omega)\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\omega s \ , \tag{2.7}$$

$$\eta(s) = \frac{d}{ds}\gamma(s) \ . \tag{2.8}$$

Here

$$\gamma(s) = \int_0^\infty \frac{d\omega}{\pi}\frac{I(\omega)}{\omega}\cos\omega s \tag{2.9}$$

and $I(\omega)$ is the spectral density

$$I(\omega) = \sum_n \delta(\omega-\omega_n)\frac{\pi C_n^2}{2m_n\omega_n} \ . \tag{2.10}$$

The kernel $\nu(s)$ contributes a phase to the path integral (2.4), effectively modifying the action of the distinguished system. It leads to dissipation and frequency renormalization in the effective equations of motion. The kernel $\eta(s)$ damps contributions from differing values of x and y. It is responsible for noise (and also for the process of decoherence discussed elsewhere). These two kernels are completely determined once a form for the spectral density (2.10) has been specified. A convenient class of choices is

$$I(\omega) = M\gamma\omega\left(\frac{\omega}{\tilde{\omega}}\right)^{s-1}\exp\left(-\frac{\omega^2}{\Lambda^2}\right) \ . \tag{2.11}$$

Here, Λ is a cutoff, which will generally be taken to be very large, sometimes infinite, and $\tilde{\omega}$ is a frequency scale, which may be taken to be Λ. We will concentrate almost entirely on the case of the Ohmic environment, $s=1$, with occasional reference to the supra-Ohmic and sub-Ohmic cases, $s>1$ and $s<1$, respectively.

In the Ohmic case, (2.8) is

$$\gamma(s) = M\gamma\frac{\Lambda}{2\pi^{1/2}}\exp(-\tfrac{1}{4}\Lambda^2 s^2) \ , \tag{2.12}$$

and thus, when Λ is very large,

$$\gamma(s) \approx M\gamma\delta(s) \ . \tag{2.13}$$

We will work in this limit, unless otherwise stated. It should be noted, however, that this limit sometimes leads to a violation of positivity of the density operator at very short timescales (of order Λ^{-1}) [27]. We will say more about this later.

In the limit in which (2.13) holds, the real part of the influence functional phase (2.6) may be written

$$\mathrm{Re}W = \int_0^t ds M\gamma(x-y)(\dot{x}-\dot{y}) - \int_0^t ds M\gamma\delta(0)(x^2-y^2) + M\gamma[x(0)^2 - y(0)^2] \ . \tag{2.14}$$

Here, the $\delta(0)$ is understood in terms of (2.12) at $s = 0$ for large Λ. It is easily seen that the terms involving it may be absorbed by defining a renormalized potential in the path integral (2.4):

$$V_R(x) = V(x) - M\gamma\delta(0)x^2 \ . \tag{2.15}$$

For a harmonic oscillator of frequency ω_0, considered below, we therefore define a renormalized frequency, ω_R, with $\omega_R^2 = \omega_0^2 - 2\gamma\delta(0)$. The terms involving the end points $x(0)$, $y(0)$ are clearly negligible and will be dropped.

The noise kernel (2.7) is nonlocal for large Λ, except in the so-called Fokker-Planck limit, $kT \gg \hbar\Lambda$, in which case one has

$$\nu(s) = \frac{2M\gamma kT}{\hbar}\delta(s) \ . \tag{2.16}$$

An evolution equation for ρ may be derived. Its expected most general form is

$$\begin{aligned} i\hbar\frac{\partial\rho}{\partial t} = &-\frac{\hbar^2}{2M}\left(\frac{\partial^2\rho}{\partial x^2} - \frac{\partial^2\rho}{\partial y^2}\right) + [V_R(x) - V_R(y)]\rho \\ &-i\hbar\Gamma(t)(x-y)\left(\frac{\partial\rho}{\partial x} - \frac{\partial\rho}{\partial y}\right) \\ &-i\Gamma(t)h(t)(x-y)^2\rho \\ &+\hbar\Gamma(t)f(t)(x-y)\left(\frac{\partial\rho}{\partial x} + \frac{\partial\rho}{\partial y}\right) \ . \end{aligned} \tag{2.17}$$

The coefficients $\Gamma(t)$, $f(t)$, $h(t)$, are in general rather complicated nonlocal functions of time, and appear to be known only in two particular cases. First, explicit expressions for them may be found in [26] for the case of linear systems. Second, in the Fokker-Planck limit, for any potential $V(x)$, one has [8],

$$\Gamma(t) = \gamma, \quad h(t) = \frac{2MkT}{\hbar}, \quad f(t) = 0 \ . \tag{2.18}$$

The propagator J may be evaluated exactly for the case of the simple harmonic oscillator, $V(x) = \frac{1}{2}M\omega_0^2x^2$. We will also be interested in the free particle, $\omega_0 = 0$. Introducing $X = x + y, \xi = x - y$, it may be shown that [8,26]

$$J(X_f,\xi_f,t|X_0,\xi_0,0) = \frac{N}{\pi\hbar}\exp\left(\frac{i}{\hbar}\tilde{S} - \frac{\phi}{\hbar}\right) \ , \tag{2.19}$$

where

$$\tilde{S} = \tilde{K}(t)X_f\xi_f + \hat{K}(t)X_0\xi_0 - L(t)X_0\xi_f - N(t)X_f\xi_0 \tag{2.20}$$

and

$$\phi = A(t)\xi_f^2 + B(t)\xi_f\xi_0 + C(t)\xi_0^2 \ . \tag{2.21}$$

Explicit expressions for the coefficients A, B, C and $\tilde{K}$, $\hat{K}$, L, N are given in the Appendix.

III. THE WIGNER FUNCTION PROPAGATOR

Instead of the density operator, it is often convenient to work with the Wigner function, defined by

$$W(p,q) = \frac{1}{2\pi\hbar}\int d\xi \exp\left(-\frac{i}{\hbar}p\xi\right)\rho(q + \tfrac{1}{2}\xi, q - \tfrac{1}{2}\xi) \ . \tag{3.1}$$

Its inverse is

$$\rho(x,y) = \int dp \exp\left(\frac{i}{\hbar}p(x-y)\right) W\left(p, \frac{x+y}{2}\right) \ . \tag{3.2}$$

(See [28,29] for properties of the Wigner function.)

The evolution of the reduced density operator is described by the (generally nonunitary) propagator, J, Eq. (2.4). Correspondingly, one may introduce the Wigner function propagator, $K(p,q,t|p_0,q_0,0)$, defined by

$$\begin{aligned} K(p,q,t|p_0,q_0,0) = &\frac{1}{2\pi\hbar}\int d\xi d\xi_0 e^{-(i/\hbar)p\xi}e^{(i/\hbar)p_0\xi_0} \\ &\times J(q + \tfrac{1}{2}\xi, q - \tfrac{1}{2}\xi, t|q_0 \\ &+\tfrac{1}{2}\xi_0, q_0 - \tfrac{1}{2}\xi_0, 0) \end{aligned} \tag{3.3}$$

and one has

$$W(p,q,t) = \int dp_0 dq_0 K(p,q,t|p_0,q_0,0)W(p,q,0). \tag{3.4}$$

By performing the Wigner transform of the evolution equation for ρ, Eq. (2.17), one may derive an analogous equation for the Wigner function. It is

$$\begin{aligned} \frac{\partial W}{\partial t} = &-\frac{p}{M}\frac{\partial W}{\partial q} + V_R'(q)\frac{\partial W}{\partial p} + 2\Gamma(t)\frac{\partial}{\partial p}(pW) \\ &+\hbar\Gamma(t)h(t)\frac{\partial^2 W}{\partial p^2} + \hbar\Gamma(t)f(t)\frac{\partial^2 W}{\partial q\partial p} \\ &+\sum_{k=1}^{\infty}\left(\frac{i\hbar}{2}\right)^{2k}\frac{1}{(2k+1)!}V^{(2k+1)}(q)\frac{\partial^{2k+1}W}{\partial p^{2k+1}} \ . \end{aligned} \tag{3.5}$$

It follows that the Wigner function propagator also obeys this equation, together with the initial conditions

$$K(p,q,0|p_0,q_0,0) = \delta(p-p_0)\delta(q-q_0) \ . \tag{3.6}$$

Consider the right-hand side of the Wigner function evolution equation (3.5). The first two terms are a Liouville operator, and if the equation had only these terms, W would evolve along the classical flow in phase space. The third term is the dissipative term. It modifies the flow along which W evolves, and also causes a contrac-

tion of each area element. The fourth and fifth terms are diffusive terms, and produce an expansion of each area element. They are responsible for noise, and also for the destruction of interference, by erasing the structure of the Wigner function on small scales. The last term, the power series, together with the first two terms make up the unitary part of the evolution. Hence, up to corrections of order $\hbar^2$, unitary evolution corresponds to approximately classical evolution of the Wigner function. It is partly for this reason that the evolution of a quantum system is most conveniently undertaken in the Wigner representation. The higher corrections can often be argued to be negligible, e.g., if the Wigner function does not develop too much detailed structure on small scales. There are, however, important examples where they cannot be neglected, e.g., in chaotic systems [30].

To illustrate the simplicity of evolution in the Wigner representation, we write down the solution for the propagator, in the case in which the diffusive terms, dissipative terms, and higher-order terms in $\hbar$ are ignored. It is

$$K(p,q,t|p_0,q_0 0) = \delta(p-p^{\rm cl})\delta(q-q^{\rm cl}) \ , \tag{3.7}$$

where $p^{\rm cl} = p^{\rm cl}(p_0,q_0,t)$ and $q^{\rm cl} = q^{\rm cl}(p_0,q_0,t)$ are the solutions to the classical equations of motion,

$$\dot{p} = -V'(q) \ , \tag{3.8}$$

$$\dot{q} = \frac{p}{M} \ , \tag{3.9}$$

satisfying the initial conditions $p^{\rm cl} = p_0$, $q^{\rm cl} = q_0$ at $t = 0$.

For the case of linear systems coupled to an environment, described in the previous section, the density operator propagator is given explicitly by (2.19)–(2.21). The Wigner function propagator may be computed, yielding

$$\begin{aligned} K(p,q,t|p_0,q_0 0) = {} & \frac{N}{\pi\hbar(4AC-B^2)^{1/2}} \\ & \times \exp[-\alpha(p-p^{\rm cl})^2 - \beta(q-q^{\rm cl})^2 \\ & -\epsilon(p-p^{\rm cl})(q-q^{\rm cl})] \end{aligned} \tag{3.10}$$

where

$$\alpha = \frac{C}{\hbar(4AC-B^2)} \ , \tag{3.11}$$

$$\beta = \frac{4(AN^2+BN\tilde{K}+C\tilde{K}^2)}{\hbar(4AC-B^2)} \ , \tag{3.12}$$

$$\epsilon = -\frac{2(NB+2C\tilde{K})}{\hbar(4AC-B^2)} \ , \tag{3.13}$$

and $q^{\rm cl}, p^{\rm cl} = M\dot{q}^{\rm cl}$, are the solution to the classical field equations with dissipation and with a renormalized frequency,

$$\ddot{q} + 2\gamma\dot{q} + \omega_R^2 q = 0 \tag{3.14}$$

matching p_0, q_0 at $t = 0$. Explicitly,

$$q^{\rm cl} = \frac{p_0}{2N} + \frac{\hat{K}}{N}q_0 \ , \tag{3.15}$$

$$p^{\rm cl} = \frac{\tilde{K}}{N}p_0 - 2(LN-\tilde{K}\hat{K})\frac{q_0}{N} \ . \tag{3.16}$$

Now, for what follows, it is important that the transformation from p_0, q_0 to $p^{\rm cl}, q^{\rm cl}$, defined by Eqs. (3.15) and (3.16) is not a canonical transformation, since the Jacobian of the transformation is

$$\frac{\partial(p^{\rm cl},q^{\rm cl})}{\partial(p_0,q_0)} = \frac{L}{N} \ . \tag{3.17}$$

This is because the classical evolution is dissipative, which tends to cause the phase-space cell to shrink. This shrinking effect can be compensated for by a scaling transformation of $p^{\rm cl}, q^{\rm cl}$. In particular, the variables,

$$p' = \lambda^{-1}p^{\rm cl}, \quad q' = \lambda^{-1}q^{\rm cl} \tag{3.18}$$

where $\lambda = (L/N)^{1/2}$, are related to p_0, q_0 by a canonical transformation. Using these new variables, the propagation of the Wigner function may be written

$$\begin{aligned} W(p,q,t) = {} & \int dp_0 dq_0 \frac{1}{\hbar(4AC-B^2)^{1/2}} W(p_0,q_0,0) \\ & \times \exp[-\alpha(p-\lambda p')^2 - \beta(q-\lambda q')^2 \\ & -\epsilon(p-\lambda p')(q-\lambda q')] \ . \end{aligned} \tag{3.19}$$

Now one may perform the canonical transformation of integration variables from p_0, q_0 to $p'q'$,

$$\begin{aligned} W(p,q,t) = {} & \int dp' dq' \frac{1}{\hbar(4AC-B^2)^{1/2}} W'(p',q',0) \\ & \times \exp[-\alpha(p-\lambda p')^2 - \beta(q-\lambda q')^2 \\ & -\epsilon(p-\lambda p')(q-\lambda q')] \ , \end{aligned} \tag{3.20}$$

where

$$W'(p',q',0) = W(p_0,q_0,0) \ . \tag{3.21}$$

Now the point is that the quantity W' defined by this transformation is still a Wigner function, i.e., it is the Wigner transform of density operator, ρ', say. In fact, it is readily shown that ρ' is related to the original ρ by a unitary transformation. This would not be true of the transformation from p_0, q_0 to $p^{\rm cl}, q^{\rm cl}$.

Equation (3.20) is the main result of this section: a simple expression for the evolution of the Wigner function for linear systems from an arbitrary initial state. The (nondissipative part of the) classical evolution has been absorbed into a canonical transformation of the initial state, and the effects of dissipation and diffusion are contained in the coefficients $\alpha, \beta, \epsilon, \lambda$.

It is sometimes convenient to write the Wigner function propagator in the alternative form

$$\begin{aligned} K(p,q,t|p_0,q_0,0) = {} & \frac{N}{\pi\hbar(4AC-B^2)^{1/2}} \\ & \times \exp[-\mu(p_0-p_0^{\rm cl})^2 - \nu(q_0-q_0^{\rm cl})^2 \\ & -\sigma(p_0-p_0^{\rm cl})(q_0-q_0^{\rm cl})] \ , \end{aligned} \tag{3.22}$$

where

$$\mu = \frac{A}{\hbar(4AC - B^2)} , \tag{3.23}$$

$$\nu = \frac{4(A\hat{K}^2 + BL\hat{K} + CL^2)}{\hbar(4AC - B^2)} , \tag{3.24}$$

$$\sigma = \frac{2(LB + 2A\hat{K})}{\hbar(4AC - B^2)} , \tag{3.25}$$

$p_0^{\rm cl}, q_0^{\rm cl}$ are again solutions to the classical field equations with dissipation, but now matching the final conditions p, q at time t:

$$q_0^{\rm cl} = -\frac{p}{2L} + \frac{\hat{K}}{L} q , \tag{3.26}$$

$$p_0^{\rm cl} = \frac{\hat{K}}{L} p - 2(LN - \tilde{K}\hat{K})\frac{q}{L} . \tag{3.27}$$

We will use these expressions in the following sections.

IV. GENERALIZED UNCERTAINTY RELATIONS

We now show how the results of the previous section may be used to derive generalized uncertainty relations for quantum Brownian motion models, for linear systems.

A. A lower bound on the uncertainty function

From the Wigner function propagator, one may obtain expressions for the distributions at p and q at time t:

$$\begin{aligned}\rho(q,q,t) &= \int dp W(p,q,t) \\ &= \int dp_0 dq_0 \frac{1}{(2\pi s_q^2)^{1/2}} \exp\left(-\frac{(q-\lambda q')^2}{2s_q^2}\right) \\ &\quad \times W'(p',q',0) ,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\tilde{\rho}(p,p,t) &= \int dq W(p,q,t) \\ &= \int dp_0 dq_0 \frac{1}{(2\pi s_q^2)^{1/2}} \exp\left(-\frac{(p-\lambda p')^2}{2s_p^2}\right) \\ &\quad \times W'(p',q',0) ,\end{aligned} \tag{4.2}$$

where

$$s_q^2 = \frac{2\alpha}{4\alpha\beta - \epsilon^2}, \quad s_p^2 = \frac{2\beta}{4\alpha\beta - \epsilon^2} . \tag{4.3}$$

From these results, it is straightforward to compute the variances of p and q at any time t:

$$(\Delta q_t)^2 = \lambda^2 (\Delta q')^2 + s_q^2 , \tag{4.4}$$

$$(\Delta p_t)^2 = \lambda^2 (\Delta p')^2 + s_p^2 , \tag{4.5}$$

where $(\Delta q')^2, (\Delta p')^2$ denote the variances of q and p in the canonically transformed initial Wigner function (3.21), or equivalently, in the unitarily transformed initial density operator ρ_0'. (The variances are generally not invariant under such transformations of the state.)

We have the usual uncertainty principle

$$(\Delta q')(\Delta p') \geq \frac{\hbar}{2} \tag{4.6}$$

with equality if and only if ρ' is a coherent state, i.e., if and only if

$$W'(p',q',0) = \frac{1}{\pi\hbar} \exp\left(-\frac{2(\Delta q')^2}{\hbar^2} p'^2 - \frac{q'^2}{2(\Delta q')^2}\right) \tag{4.7}$$

(where $\Delta q'$ is, so far, an arbitrary parameter). It follows that the uncertainty function at time t satisfies

$$\begin{aligned}U_t &\equiv (\Delta q_t)^2 (\Delta p_t)^2 \\ &\geq \frac{\hbar^2\lambda^2}{4} + \lambda(\Delta q')^2 s_p^2 + \frac{\hbar^2 \lambda s_q^2}{4(\Delta q')^2} + s_q^2 s_p^2\end{aligned} \tag{4.8}$$

with equality if and only if the initial state is given by (4.7). Since $\Delta q'$ is arbitrary, we may minimize over it. The minimum is at

$$(\Delta q')^2 = \frac{\hbar s_q}{2 s_p} \tag{4.9}$$

and inserting the minimum value in (4.7), we obtain

$$(\Delta q_t)(\Delta p_t) \geq \frac{\lambda\hbar}{2} + s_p s_q . \tag{4.10}$$

In terms of the coefficients of the nonunitary propagator, (3.11)–(3.13), this reads

$$\begin{aligned}(\Delta q_t)(\Delta p_t) \geq \frac{\hbar L}{2N} + \frac{\hbar}{2N^2}(ACN^2 + BCN\tilde{K} \\ + C^2\tilde{K}^2)^{1/2} .\end{aligned} \tag{4.11}$$

This the first main result of this section: the uncertainty at time t satisfies the generalized uncertainty relation (4.11), with equality if and only if the initial state is given by (4.7) with $\Delta q'$ given by (4.9).

Consider now the conditions for equality. From (4.7) and (3.21), the minimizing initial Wigner function is

$$\begin{aligned}W(p_0,q_0,0) = \frac{1}{\pi\hbar} \exp\Bigg[-\frac{2\sigma^2}{\hbar^2 LN}[\tilde{K}p_0 - 2(LN - \tilde{K}\hat{K})q_0]^2 \\ - \frac{1}{2\sigma^2 LN}\left(\frac{1}{2}p_0 + \hat{K}q_0\right)^2\Bigg] .\end{aligned} \tag{4.12}$$

Inverting the Wigner transform, one finds that the minimizing state is a pure state:

$$\Psi(x) = \left(\frac{2a}{\pi}\right)^{1/4} \exp[-(a+ib)x^2] . \tag{4.13}$$

This is a so-called correlated coherent state [24], discussed below. The coefficients a and b may be computed from (4.12) but will not be needed here so are not given.

It is important to note that the initial state (4.13) minimizing the uncertainty at time t is *different for each moment of time.* That is, the lower bound (4.11), is *not* the

time evolution of a particular initial state. It is actually an envelope. The initial state (4.13), which depends on t, will minimize the uncertainty at time t but generally not at other times. This is because, although the effect of diffusion is generally to increase the uncertainty as time goes on, there are competing effects that may reduce it. In particular, wave-packet reassembly (the time reverse of wave-packet spreading) and dissipation may cause the uncertainty to decrease. This point is discussed further in [16], where an analogous lower bound on the Wehrl entropy (discussed below) was proved.

B. A sharper lower bound

As stated, the lower bound on the uncertainty (4.11) is sharp, in the sense that at each moment of time there exists an initial state at which the lower bound is achieved. But it is only an envelope because the initial state achieving it is different for each moment of time. It turns out that an improvement of this result is possible.

First, we note that the conventional uncertainty relation, $\Delta p \Delta q \geq \hbar/2$, may be generalized to the stronger result [24]

$$\mathcal{A}^2 \equiv (\Delta p)^2 (\Delta q)^2 - C_{pq}^2 \geq \frac{\hbar^2}{4} , \tag{4.14}$$

where C_{pq} is the correlation between p and q:

$$\begin{aligned} C_{pq} &= \tfrac{1}{2}\langle \Delta\hat{p}\Delta\hat{q} + \Delta\hat{q}\Delta\hat{p}\rangle \\ &= \int dpdq\, pqW(p,q) - \langle\hat{p}\rangle\langle\hat{q}\rangle , \end{aligned} \tag{4.15}$$

where $\Delta\hat{q} = \hat{q} - \langle\hat{q}\rangle$, $\Delta\hat{p} = \hat{p} - \langle\hat{p}\rangle$. The quantity $\mathcal{A}$ defined by (4.14) is essentially the area enclosed by the $1-\sigma$ contour of the Wigner function. Equality in (4.14) is achieved when the state is a correlated coherent state,

$$\begin{aligned} \Psi(x) = \frac{1}{(2\pi\eta^2)^{1/4}} \exp\Bigg(-\frac{x^2}{4\eta^2}\left[1 - \frac{ir}{(1-r^2)^{1/2}}\right] \\ + \frac{\alpha x}{\eta} - \frac{1}{2}(\alpha^2 + |\alpha|^2)\Bigg) , \end{aligned} \tag{4.16}$$

where η, r are real parameters [$r = C_{pq}/(\Delta p \Delta q)$ and $\eta = \Delta q$], and α is a complex parameter.

We have, however, already seen a correlated coherent state—it is the state minimizing the uncertainty relation (4.11). It is therefore plausible that a stronger version of (4.11) might be possible in terms of the Wigner function area (4.14). Indeed, computing the correlation from (3.19), one finds the remarkably simple result

$$\mathcal{A}_t^2 = \frac{L^2}{N^2}\mathcal{A}_0^2 + \frac{\hbar^2}{4N^2}(4AC - B^2) . \tag{4.17}$$

This give the Wigner function area $\mathcal{A}$ at an arbitrary time t, $\mathcal{A}_t$, in terms of its value at time $t = 0$, $\mathcal{A}_0$. It is remarkable because $\mathcal{A}$ at time t depends on *only* its value at $t = 0$, and not on any of the other moments of the initial state. Now at $t = 0$, $\mathcal{A}_0$ satisfies the uncertainty principle (4.14). We thus obtain, for the Wigner function area at time t, the inequality

$$\mathcal{A}_t^2 \geq \frac{\hbar^2 L^2}{4N^2} + \frac{\hbar^2}{4N^2}(4AC - B^2) . \tag{4.18}$$

Equality is achieved when the initial state belongs to the four-parameter family of correlated coherent states. Most importantly, the inequality is sharp at every moment of time for one and the same initial state. It is not an envelope. Equation (4.18) is the second main result of this section.

The simplicity of the time evolution in terms of the Wigner area is easy to understand. For linear systems, the area is preserved under unitary time evolution. For the nonunitary case considered here, therefore (4.17) expresses entirely the effects of the environment, with the effects of unitary evolution completely factored out. In this respect it is similar to the von Neumann entropy discussed in the next section. Indeed, for Gaussian states, the von Neumann entropy is a function only of the Wigner area. Some rather different, but not unrelated generalizations of the uncertainty principle may be found in [22,31].

V. EXPLICIT FORMS OF THE LOWER BOUNDS

We now give the explicit forms of the lower bounds (4.11) and (4.18), in a variety of situations. There are a very large number of different situations and regimes that our results may cover, depending on the choice of t, T, γ, and ω (harmonic oscillator in the under- and overdamped case, inverted harmonic oscillator, free particle). We will not present here a completely exhaustive search of the parameter space, but concentrate on the cases that contain some interesting physical results. In particular, most of the following results are the underdamped case. The cases not covered here are readily derived, should the reader be interested.

Consider first the case of short times, $t \ll \gamma^{-1}$. Then for high temperatures, (4.11) and (4.18) become, respectively,

$$(\Delta q_t)(\Delta p_t) \geq \frac{\hbar}{2}\left(1 - 2\gamma t + \frac{8\sqrt{3}}{3}\frac{\gamma k T}{\hbar}t^2\right) , \tag{5.1}$$

$$\mathcal{A}_t^2 \geq \frac{\hbar^2}{4}\left(1 - 4\gamma t + \frac{8}{3}\frac{\gamma^2 k^2 T^2}{\hbar^2}t^4\right) . \tag{5.2}$$

These relations are in fact valid for any potential. They represent the initial growth of fluctuations, starting from the purely quantum fluctuations at $t = 0$.

The γt term in each expression indicates an initial decrease of fluctuations, in apparent violation of the uncertainty principle. This violation occurs, in Eq. (5.1), for example, on a time scale less than $\hbar/kT$. This is because these expressions have been derived by taking the infinite cutoff limit in Eq. (2.13), which as previously stated can lead to violations of positivity of the density operator on time scales less than Λ^{-1} [27]. The expressions are therefore not valid for $t < \Lambda^{-1}$. The high-temperature limit used in deriving (5.1) means $kT \gg \hbar\Lambda$. Combining

these inequalities, we see that (5.1) is valid only for times $t > \Lambda^{-1} \gg \hbar/kT$, for which violations of the uncertainty principle will not arise.

To illustrate explicitly that it is the infinite cutoff limit that is responsible for violations of the uncertainty principle, we compute the lower bound (4.11) leaving the cutoff finite in the expressions for the various coefficients in the propagator (2.19)–(2.21). The important point is to keep Λ finite in Eq. (2.12). Expressions (A1)–(A4) and (A14) are no longer valid, and the correct expressions for these coefficients, in the short-time limit, are given by Eqs. (A28)–(A32). One thus obtains, in the short-time limit, the lower bound

$$(\Delta q_t)(\Delta p_t) \geq \frac{\hbar}{2} + \frac{\gamma kT\Lambda}{2\pi} t^3 + \cdots \tag{5.3}$$

for which there is clearly no violation of the uncertainty principle.

Ignoring the positivity-violating terms in (5.1) and (5.2), the remaining terms given an indication of the time scale on which the thermal fluctuations become important in comparison to the quantum ones. It is

$$t \sim \left(\frac{\hbar}{\gamma kT}\right)^{1/2} \tag{5.4}$$

As noted in [32], this is the time scale on which quantum fluctuations become comparable to Nyquist noise.

A related result was derived by Hu and Zhang [14]. They showed that for the initial state consisting of a Gaussian of width σ, the thermal fluctuations become comparable with the quantum ones on a time scale of order

$$t_d \sim \frac{\hbar^2}{M\gamma kT\sigma^2} \tag{5.5}$$

(see also [16]). This was observed to coincide with the time scale characteristic of decoherence for that initial state [33].

The two results (5.4) and (5.5) are, in fact, consistent, and this may be seen as follows. Note that (5.5) contains the initial width σ but (5.4) does not. As discussed earlier, the initial state minimizing the uncertainty U at time t is actually an envelope. There is a different initial state doing the job for each t, so the state depends on t. The minimizing initial state is the Gaussian (4.7) with width σ given by (4.9). For small t one has

$$\sigma^2 = \frac{\hbar s_q}{2s_p} \sim \frac{\hbar t}{m} . \tag{5.6}$$

Hence the width of the initial state which minimizes the uncertainty at the time (5.4) when thermal and quantum fluctuations are comparable is

$$\sigma^2 \sim \frac{\hbar}{m}\left(\frac{\hbar}{\gamma kT}\right)^{1/2} . \tag{5.7}$$

Inserting this value of σ into (5.5), one finds that t_d coincides with (5.4). Our results are thus in agreement with Hu and Zhang [14].

In the weak-coupling regime, for all temperatures and times (except very short times), Eq. (4.18) becomes, using (A18)–(A20),

$$A_t^2 \geq \frac{\hbar^2}{4}\left[e^{-4\gamma t} + \coth^2\left(\frac{\hbar\omega}{2kT}\right)(1 - e^{-2\gamma t})^2\right] . \tag{5.8}$$

An expression may also be derived for arbitrary coupling and high temperatures, using (A11)–(A13), but this is rather complicated and will not be given.

In the long-time limit, for any coupling and temperature,

$$U_t = \mathcal{A}_t^2 \geq \frac{\hbar^2}{\omega_R^2}\left(F_1^2 - \frac{\gamma^2}{\omega^2}F_2^2\right)^{1/2} . \tag{5.9}$$

In the high-temperature limit this becomes

$$U_t = \mathcal{A}_t^2 \geq \frac{k^2T^2}{\omega_R^2} \tag{5.10}$$

and, in the weak-coupling limit,

$$U_t = \mathcal{A}_t^2 \geq \frac{\hbar^2}{4}\coth^2\left(\frac{\hbar\omega}{2kT}\right) . \tag{5.11}$$

VI. INEQUALITIES RELATING UNCERTAINTY TO ENTROPY AND LINEAR ENTROPY

The von Neumann entropy of the reduced density operator,

$$S[\rho] = -\mathrm{Tr}(\rho \ln \rho) \tag{6.1}$$

is often discussed in the present context, in association with uncertainty, decoherence, and correlations of the distinguished systems with its environment [18,19,21,34,17]. Zurek, Paz, and Habib, for example, looked for classes of initial states, which under evolution according to a master equation of the form (2.17), generate the least amount of entropy at time t. They regarded such states as the most stable under evolution in the presence of an environment. They argued that the initial states doing the job are coherent states, at least approximately.

One of the reasons for looking at the von Neumann entropy is that it is constant for unitary evolution, thus for open systems such as those considered here, it is principally a measure of environmentally induced effects. The Wigner area considered in Sec. IV also has this property. The results of Sec. IV thus agree with the claims of Zurek *et al.* (except that it is really the correlated coherent states, rather than the ordinary coherent states which are the most stable, although this distinction is not very important). It would be useful to find a connection between these two measures of stability or fluctuations.

A. Uncertainty vs von Neumann entropy

The connection between uncertainty and the von Neumann entropy may be found indirectly, by considering

first of all the phase-space distribution

$$\mu(p,q) = \langle z|\rho|z\rangle \ , \tag{6.2}$$

where

$$\begin{aligned}\langle x|z\rangle &= \langle x|p,q\rangle \\ &= \left(\frac{1}{2\pi\sigma_q^2}\right)^{1/4} \exp\left(-\frac{(x-q)^2}{4\sigma_q^2} + \frac{i}{\hbar}px\right)\end{aligned} \tag{6.3}$$

are the standard coherent states. The function $\mu(p,q)$ is normalized according to

$$\int \frac{dpdq}{2\pi\hbar}\mu(q,p) = 1 \ . \tag{6.4}$$

It is readily shown that $\mu(p,q)$ is also equal to

$$\begin{aligned}\mu(p,q) = 2\int dp'dq' \exp\Bigg(&-\frac{(p-p')^2}{2\sigma_p^2} \\ &-\frac{(q-q')^2}{2\sigma_q^2}\Bigg) W_\rho(p',q') \ ,\end{aligned} \tag{6.5}$$

where $W_\rho(p,q)$ is the Wigner function of ρ and $\sigma_p\sigma_q = \frac{1}{2}\hbar$. Equation (6.5) is sometimes known as the Husimi distribution [35], and is positive even though the Wigner function is not in general (see also [36]).

There exists an information-theoretic measure of the uncertainty or spread in phase space contained in the distribution (6.5), namely the so-called Wehrl entropy [37]

$$I(P,Q) = -\int \frac{dpdq}{2\pi\hbar}\mu(p,q)\ln\mu(p,q) \ . \tag{6.6}$$

This is the Shannon information of (6.2). $I(P,Q)$ is large for spread out distributions, and small for very concentrated ones. Because of the uncertainty principle, one would expect a limit on the degree to which $\mu(p,q)$ may be concentrated about a small region of phase space, and hence a lower bound on (6.6). Indeed, a nontrivial theorem due to Lieb shows that

$$I(P,Q) \geq 1 \tag{6.7}$$

with equality if and only if ρ is the density matrix of a coherent state, $|z'\rangle\langle z'|$ [38]. In [16], the Wehrl entropy was used as a measure of both quantum and thermal fluctuations, and a lower bound analogous to (4.11) was derived.

The reason for studying this quantity is that it provides the link between the von Neumann entropy and the uncertainty measures U and $\mathcal{A}$. On the one hand, an elementary property of Shannon information is

$$I(P,Q) \leq \ln\left(\frac{e}{\hbar}(\det K)^{1/2}\right) \ , \tag{6.8}$$

where K is the 2×2 covariance matrix of the distribution $\mu(p,q)$ [39]. Equality holds if and only if $\mu(p,q)$ is a Gaussian. From (6.5), one has

$$\det K = \left((\Delta q)^2 + \sigma_q^2\right)\left((\Delta p)^2 + \sigma_p^2\right) - C_{pq}^2 \ , \tag{6.9}$$

where Δq, Δp are the quantum-mechanical variances of the state ρ. On the other hand, an elementary property of the Wehrl entropy, following from the concavity of Shannon information, is [37]

$$I(P,Q) \geq S[\rho] \ . \tag{6.10}$$

Hence combining the upper and lower limits (6.8) and (6.10), one obtains

$$\left((\Delta q)^2 + \sigma_q^2\right)\left((\Delta p)^2 + \sigma_p^2\right) - C_{pq}^2 \geq \hbar^2 e^{2(S-1)} \ . \tag{6.11}$$

Finally, since the parameter σ_q is arbitrary (and $\sigma_p\sigma_q = \hbar/2$), we may minimize the left-hand side over it, with the result

$$(\Delta p\Delta q + \tfrac{1}{2}\hbar)^2 - C_{pq}^2 \geq \hbar^2 e^{2(S-1)} \ . \tag{6.12}$$

This is the exact form of the connection between the uncertainty and entropy for a general mixed state ρ.

In the regime where quantum fluctuations are more significant than thermal ones, it is appropriate to use the lower bound (6.7) rather than (6.10) (since $S[\rho]$ goes to zero if the state is pure), and this is formally achieved by setting $S = 1$ in (6.12). One then deduces the usual uncertainty principle form (6.12) [although not the generalized version including the correlation, (4.14)].

In the regime where thermal (or environmentally induced) fluctuations are dominant, one would expect $\Delta p\Delta q \gg \hbar/2$ and $S \gg 1$, and (6.12) then gives

$$\frac{\Delta p\Delta q}{\hbar} \geq \frac{\mathcal{A}}{\hbar} \geq e^S \ . \tag{6.13}$$

This is the simplest form of the connection between the uncertainty and the von Neumann entropy: the entropy is bounded from above by the logarithm of the number of phase-space cells the state occupies.

To see how sharp these equalities can be, consider the case of a Gaussian Wigner function. It may be shown that the von Neumann entropy of a Gaussian is

$$S[\rho] = -\ln\left(\frac{2}{1+\mu}\right) - \frac{\mu-1}{2}\ln\left(\frac{\mu-1}{\mu+1}\right) \ , \tag{6.14}$$

where $\mu = 2\mathcal{A}/\hbar$ and $\mathcal{A}$ is the area of the Gaussian Wigner function (see Ref. [17] for example). For large $\mathcal{A}$,

$$S[\rho] \approx \ln\left(\frac{\mathcal{A}}{\hbar}\right) \ , \tag{6.15}$$

and hence we have equality in (6.13).

B. von Neumann entropy vs linear entropy

In practice, the discussion of Zurek *et al.*, and indeed many discussions of von Neumann entropy, often concern the so-called linear entropy,

$$S_L = 1 - \mathrm{Tr}\rho^2 \ . \tag{6.16}$$

How is this related to von Neumann entropy? Let the density operator be

$$\rho = \sum_n p_n |n\rangle\langle n| \ . \tag{6.17}$$

We use Jensen's inequality (see Ref. [39], for example), which states that if f is a convex function, and X a random variable,

$$\langle f(X)\rangle \geq f(\langle X\rangle) \tag{6.18}$$

where $\langle\cdots\rangle$ denotes the mean over X. Take f to be the exponential function, and X to be $\ln p_n$. Then,

$$\sum_n p_n^2 \geq \exp\left(\sum_n p_n \ln p_n\right) \tag{6.19}$$

and thus

$$\mathrm{Tr}\rho^2 \geq e^{-S[\rho]} \ . \tag{6.20}$$

In terms of the linear entropy

$$S_L = 1 - \mathrm{Tr}\rho^2 \geq 1 - e^{-S} \ . \tag{6.21}$$

Equality in (6.21) is reached for pure states, when $S = S_L = 0$, and for very mixed states, when S is very large and $S_L \approx 1$.

VII. LONG-TIME LIMITS

One of the particularly interesting questions for nonequilibrium systems of the type considered here is whether they settle down to a unique state after a long period of time. It turns out to be particularly straightforward to answer this question for linear systems, using the Wigner function propagator described in Sec. III. Combining (3.4) and (3.10), we have the expression

$$\begin{aligned} W(p,q,t) = {} & \frac{N}{\pi\hbar(4AC-B^2)^{1/2}} \int dp_0 dq_0 W(p_0,q_0,0) \\ & \times \exp[-\alpha(p-p^{\mathrm{cl}})^2 - \beta(q-q^{\mathrm{cl}})^2 \\ & -\epsilon(p-p^{\mathrm{cl}})(q-q^{\mathrm{cl}})] \ , \end{aligned} \tag{7.1}$$

where, recall, $p^{\mathrm{cl}}, q^{\mathrm{cl}}$ are the value of the classical solutions (with dissipation) at time t matching the initial data p_0, q_0 at $t = 0$. This expression allows us to compute the moments at any time t in terms of the initial moments. By computing the long-time limits of these moments, the form of the long-time limit of the Wigner function may be obtained, since it is completely determined by its moments.

A. The harmonic oscillator

For the harmonic oscillator, simplifications occur. Consider first the case of the harmonic oscillator in an ohmic environment. One has $p^{\mathrm{cl}} = M\dot{q}^{\mathrm{cl}}$ and q^{cl} satisfies Eq. (3.14). Either from Eq. (3.14), or from the explicit solution (3.15) and (3.16), it is easily seen that $p^{\mathrm{cl}} \to 0$ and $q^{\mathrm{cl}} \to 0$ in the long-time limit. All dependence on p_0 and q_0 drops out of the exponential in (7.1), and one obtains the following expression for the asymptotic value of the Wigner function:

$$W_\infty(p,q) = \frac{N}{\pi\hbar(4AC-B^2)^{1/2}} \exp(-\alpha p^2 - \beta q^2 - \epsilon pq) \ . \tag{7.2}$$

The coefficients α, β, ϵ are given by the long-time limits of (3.11)–(3.13). Using results (A21)–(A23) in the Appendix, one finds

$$\alpha = \frac{1}{2M\hbar[F_1 - (\gamma/\omega)F_2]} \ , \tag{7.3}$$

$$\beta = \frac{M\omega_R^2}{2\hbar[F_1 + (\gamma/\omega)F_2]} \ ,$$

$$\epsilon = 0 \ , \tag{7.4}$$

with F_1 and F_2 given by (A24) and (A25). These relations represent the *exact* form of the long-time limit in the ohmic case.

It is useful to compare the result (7.2) with the Wigner function of the harmonic oscillator in a thermal state:

$$W_T(p,q) = \frac{1}{\pi\hbar}\tanh\left(\frac{\hbar\omega_R}{2kT}\right)\exp\left\{-\tanh\left(\frac{\hbar\omega_R}{2kT}\right)\left[\frac{M\omega_R}{2kT}q^2 + \frac{1}{M\hbar\omega_R}p^2\right]\right\} \ . \tag{7.5}$$

The long-time limit (7.2) coincides with the thermal Wigner function (7.5) in the high-temperature limit, and in the weak-coupling limit, as may be seen from Eqs. (A26) and (A27) in the Appendix. Note, however, that the asymptotic state is not always a thermal state.

It is also of interest to compute the uncertainty in q in the long-time limit. It is

$$\begin{aligned} (\Delta q)_\infty^2 &= \frac{2\hbar}{M\omega_R^2}\left(F_1 + \frac{\gamma}{\omega}F_2\right) \\ &= \frac{\hbar}{\pi}\int_0^\infty d\nu \coth\left(\frac{\hbar\nu}{2kT}\right) \\ &\quad \times \frac{1}{M}\frac{2\gamma\nu}{(\omega_R^2-\nu^2)^2 + 4\gamma^2\nu^2} \ . \end{aligned} \tag{7.6}$$

This is in agreement with the fluctuation-dissipation relation [40]. Equation (7.6) was given in [8], for the special case of a Gaussian initial state, but not surprisingly coincides with this more general result because every initial state goes to a Gaussian.

For the harmonic oscillator with a nonohmic environment, it may be shown, using the more general treatment of [26], that $q^{\rm cl}$ in the Wigner function propagator in (7.1) is the solution to the integrodifferential equation

$$\ddot{q}(t) + \omega_0^2 q(t) - \int_0^t ds\, \eta(t-s) q(s) = 0 \ , \tag{7.7}$$

where $\eta(t-s)$ is dissipation kernel (2.10) for an arbitrary spectral density $I(\omega)$. An explicit representation of the solutions may be obtained using Laplace transform, and the solution is given in terms of the contour integrals

$$q^{\rm cl}(t) = \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} dz \frac{e^{zt}}{z^2 + M^{-1} z\hat{\gamma}(z) + \omega^2} \left(\frac{p_0}{M} - z q_0 \right) \ , \tag{7.8}$$

$$p^{\rm cl}(t) = \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} dz \frac{e^{zt}}{z^2 + M^{-1} z\hat{\gamma}(z) + \omega^2} \times \left(z p_0 - z^2 M q_0 \right) \ , \tag{7.9}$$

where $\hat{\gamma}(z)$ is the Laplace transform of $\gamma(t)$ defined by Eqs. (2.8) and (2.9). Instead of the class of spectral densities (2.11), let

$$I(\omega) = M g_r \omega^r \theta(\Lambda - \omega) \ . \tag{7.10}$$

It has been argued that this class of spectral densities actually gives a very general picture of the long-time limit [12], and it may then be shown that, to leading order in ω/Λ,

$$\hat{\gamma}(\omega) = \begin{cases} \frac{2M g_r}{\sin(\frac{1}{2}\pi r)} \omega^{r-1} & \text{if } 0 < r < 2 \ , \\ \frac{M g_r}{\pi} \omega \ln\left(1 + \frac{\Lambda^2}{\omega^2}\right) & \text{if } r = 2 \ , \\ \frac{2M g_r \Lambda^{r-2}}{\pi(r-2)} \omega & \text{if } r > 2 \ . \end{cases} \tag{7.11}$$

Using these results, and the change of variables $zt = y$, it is then straightforward to show that $q^{\rm cl}(t)$ and $p^{\rm cl}(t)$ both go to zero at $t \to \infty$. The long-time limit of the Wigner function is therefore again the Gaussian (7.2) (although the coefficients α, β, ϵ, are not the same as before, and we do not give them here).

These results are consistent with the proof in [15] that the Wigner function of a single member of a chain of coupled harmonic oscillators tends towards a Gaussian Wigner function in the long-time limit, under certain reasonable conditions on the environment. We believe that the proof given here, however, is somewhat simpler and more direct.

B. Long-time limit for the damped free particle

In the free-particle case, using the general methods of [41], it is readily shown that the phase-space operator on the right-hand side of Eq. (3.5) does not have any zero eigenvalues. It follows that there are no stationary solutions. One cannot therefore necessarily expect all initial states to tend towards a unique final state in the long-time limit. Still, it is of interest to ask whether one can say anything at all about the general form of the solution for long times.

Evolution of the damped free particle has been considered extensively before (see, e.g., [12,42]). Here, however, we are interested in one particular point that has not previously been addressed, namely the long-time limit of arbitrary initial states. Because we are interested in understanding this general point, rather than a detailed calculation, we will for simplicity work in the high-temperature limit (although our conclusions can be generalized to other cases).

Consider Eq. (7.1) for the damped free particle. For large t, one has, in the high-temperature limit,

$$\alpha = \frac{1}{2MkT}, \quad \beta = \frac{M\gamma}{2kTt}, \quad \epsilon = -\frac{1}{2kTt} \ , \tag{7.12}$$

$$A = B = \frac{MkT}{2\hbar}, \quad C = \frac{2MkT\gamma t}{\hbar} \ , \tag{7.13}$$

$$p^{\rm cl} = 0, \quad q^{\rm cl} = q_0 + \frac{p_0}{2M\gamma} \ . \tag{7.14}$$

Introducing $\tilde{q} = q_0 + p_0/(2M\gamma)$, Eq. (7.1) may be written

$$W(p,q,t) = \frac{N}{\pi\hbar(4AC - B^2)^{1/2}} \exp\left[-\left(\alpha - \frac{\epsilon^2}{4\beta} \right) p^2 \right] \times \int d\tilde{q}\, g_0(\tilde{q}) \exp\left[-\beta \left(q + \frac{\epsilon}{2\beta} p - \tilde{q} \right)^2 \right] \ , \tag{7.15}$$

where

$$q_0(\tilde{q}) = \int dp_0 W\left(p_0, \tilde{q} - \frac{p_0}{2M\gamma}, 0 \right) \ . \tag{7.16}$$

The integrand of (7.16) is still a Wigner function, since the shift in the q argument can be compensated for by a unitary transformation of the density operator.

If we integrate out q, and noting that $\epsilon^2/4\beta \to 0$ for large t, we obtain

$$\int dq W(p,q,t) = \frac{1}{(2\pi MkT)^{1/2}} \exp\left(-\frac{p^2}{2MkT} \right) \ . \tag{7.17}$$

Hence the distribution of momenta approaches a Boltzmann distribution for all initial states. The remaining question is, what we can say about the q distribution in the long-time limit?

Let $y = q + (\epsilon/2\beta)p = q - (p/2M\gamma)$. Then the integral in the expression for the Wigner function (7.15) is

$$g(y,t) \equiv \int d\tilde{q}\, g_0(\tilde{q}) \exp\left(-\beta \left(q + \frac{\epsilon}{2\beta} p - \tilde{q} \right)^2 \right) = \int d\tilde{q}\, g_0(\tilde{q}) \exp\left(-\frac{(y - \tilde{q})^2}{4Dt} \right) \ , \tag{7.18}$$

where $D = kT/(2M\gamma)$. $g(y,t)$ obeys the diffusion equation

$$\frac{\partial g}{\partial t} = D\frac{\partial^2 g}{\partial y^2} . \tag{7.19}$$

From (7.18) or (7.19), one may compute all the moments of y at time t in terms of their initial moments. One has, for example,

$$\langle y^4\rangle_t = 12D^2t^2 + 12Dt\langle y^2\rangle_0 + \langle y^4\rangle_0 . \tag{7.20}$$

From the explicit expressions for the moments such as this, it is straightforward to show that the leading-order behavior of all moments of the form $\langle y^n\rangle$ as $t \to \infty$ are correctly reproduced by the Gaussian distribution

$$g(y) = \frac{1}{(4\pi Dt)^{1/2}} \exp\left(-\frac{(y-\langle y\rangle)^2}{4Dt}\right) . \tag{7.21}$$

This suggests that, for large times, the Wigner function approaches the asymptotic form

$$W(p,q,t) = \exp\left(-\frac{p^2}{2MkT}\right) \exp\left(-\frac{M\gamma}{2kTt}\left(q - \frac{p}{2M\gamma} - \langle q\rangle_0 - \frac{\langle p\rangle_0}{2M\gamma}\right)^2\right) . \tag{7.22}$$

Hence the distribution of p is a Boltzmann distribution, as noted above, and the q distribution, obtained by integrating out p, is peaked about the value $q = \langle q\rangle_0 + \langle p\rangle_0/(2M\gamma)$, which is the asymptotic value of q under classical evolution, starting from the initial values $\langle q\rangle_0, \langle p\rangle_0$.

The result (7.22) can, however, be rather misleading, and has limited value as an approximation to the Wigner function at large times. To understand this, write the Wigner function for large times as

$$W(p,q,t) = W_S(p,q,t) + W_1(p,q,t) + \cdots , \tag{7.23}$$

where W_S is the leading order approximation to the Wigner function for large t and W_1 is the next to leading order term. When a stationary solution exists, W_S is the stationary solution, and hence is independent of time. Furthermore, the next term W_1 is proportional to $e^{-\lambda_1 t}$ where $\lambda_1 > 0$ is the first nonzero eigenvalue of the operator appearing on the right-hand side of (3.5) [41]. It follows that all moments of (7.23) are given by their moments in W_S plus an exponentially decaying term. Even if the moment of W_S vanishes, the correction given by W_1 goes to zero for large t. All moments of (7.23) approach the stationary moments like $e^{-\lambda_1 t}$.

Now consider the free-particle case. Here there is no stationary solution, so W_S is time dependent. The next correction W_1 does not decay exponentially fast. There are certain moments [e.g., $\langle(y-\langle y\rangle)^3\rangle$] that are zero for W_S and take their leading contribution from W_1. Such moments *grow* with time, as may be seen from (7.20), so unlike the stationary solution case, they are not well approximated by their moments in W_S. Also, even for the moments which are well approximated by the moments in W_S for large t, the rate of approach to the regime in which that approximation is valid depends on the initial conditions. It does not proceed at a universal rate.

Therefore, although Eq. (7.23) is the formal solution to the Wigner function equation for large t, it is not very useful. In practice it will be more useful to work directly with the equations for the moments.

C. General potentials

Finally, we sketch the case of more general potentials. We have calculated the Wigner function propagator only for linear systems. To compute the long-time limit of the Wigner function for more general potentials, we resort to the evolution equation (3.5). It is sometimes argued that the power series involving third and higher derivative terms may be neglected. For example, in the unitary case, Omnès has considered the evolution of phase-space projectors according to (3.5) [43]. He has shown that when the corresponding classical dynamics is regular, their evolution is described to a good approximation by the first two terms on the right-hand side of (3.5) if the phase-space cell projected onto is sufficiently large and regularly shaped. Also, Paz and Zurek have argued that the diffusive terms may smooth out the Wigner function, suppressing contributions from the higher-order terms [30]. The most general conditions under which these extra terms may be neglected is not known, but when they can be neglected, and in the high-temperature limit, the Wigner function evolution equation then is

$$\frac{\partial W}{\partial t} = -\frac{p}{M}\frac{\partial W}{\partial q} + V_R'(q)\frac{\partial W}{\partial p} + 2\gamma\frac{\partial}{\partial p}(pW) + 2M\gamma kT\frac{\partial^2 W}{\partial p^2} . \tag{7.24}$$

This is the Kramers equation, whose properties are well known [41]. For example, it possesses the stationary solution

$$W(p,q) = \tilde{N} \exp\left(-\frac{p^2}{2MkT} - \frac{V_R(q)}{kT}\right) , \tag{7.25}$$

where $\tilde{N}$ is a normalization factor. This will be an admissable solution, i.e., a Wigner function, only if the potential is such that $\exp[-V_R(q)/kT]$ is normalizable. This requires $V_R(q) \to \infty$ as $q \to \pm\infty$ faster than $\ln|q|$. In that case, the stationary distribution is then the Wigner transform of a thermal state, $\rho = Z^{-1}e^{-H/kT}$, where $Z = \mathrm{Tr}(e^{-H/kT})$, for large temperatures. The general results of [41] then show that all solutions to (7.24) tend to the stationary solution (7.25) as $t \to \infty$. Hence, to the extent that Eq. (7.24) is valid, all initial states tend towards the thermal state in the long-time limit. The question of the validity of (7.24) is not likely to be straightforward, and will be considered in more detail elsewhere.

These claims are substantiated by the results of [44],

in which it was argued (using the quantum state diffusion approach to open systems), that a wide class of initial density operators evolving according to (2.17) in the high-temperature limit approach the form

$$\rho(t) = \int dp\, dq\, f(p,q,t)|\psi_{pq}\rangle\langle\psi_{pq}| \tag{7.26}$$

on a very short time scale related to the decoherence time. Here, $|\psi_{pq}\rangle$ is a generalized Gaussian coherent state and $f(p,q,t)$ is a positive, normalized solution to the Fokker-Planck equation (7.24). This result is valid even for general potentials, as long as the length scale on which the potential varies is much greater than the width of the coherent states. This result may also be of use in extending the results of Sec. IV, so far valid only for linear systems, to more general potentials. These possibilities will be explored elsewhere.

VIII. SUMMARY AND DISCUSSION

We have studied the evolution of open quantum systems described by the evolution equation (2.17). We were concerned with two particular questions: generalized uncertainty relations for this class of nonequilibrium systems and long-time limits. Our results may be summarized as follows.

(1) For any linear system whose evolution is described by the propagator (2.19)–(2.21), the uncertainty $\mathcal{U}$ and the Wigner function area $\mathcal{A}$ have the sharp lower bounds (4.11) and (4.18), respectively. These represent the least possible amount of noise the system must suffer after evolution for time t in the presence of an environment. These expressions are valid for all types of environment (i.e., for all choices of spectral density). Also worthy of note is the particularly simple expression (4.17) of the evolution of the Wigner function area $\mathcal{A}$.

(2) For the particular case of the Ohmic environment, the explicit form of the lower bounds is given in Sec. V. These explicit expressions give the comparative sizes of quantum and thermal fluctuations, generalizing the work of Hu and Zhang [14].

(3) For the linear systems considered here, these generalized uncertainty relations achieve equality for Gaussian pure initial states of the form (4.16). Such states are therefore the ones that suffer the least amount of noise under evolution in the presence of an environment, substantiating the results of Zurek *et al.* [21,18,19].

(4) The uncertainty is connected to the von Neumann entropy via the relation (6.12), and the entropy is connected to the linear entropy by the relation (6.21).

(5) For a harmonic oscillator in a wide class of environments all initial states tend towards a Gaussian Wigner function in the long-time limit. It is the same as a thermal state for an ohmic environment in the high-temperature limit or the weak-coupling limit. This proof of the long-time limit is much simpler than a previous proof by Tegmark and Shapiro [15] (although not quite as general).

(6) For the free particle, the Wigner function tends to a nonstationary Gaussian state, although this is not a very useful expression because it does not give a correct approximation to all the moments for large times.

The reason we were able to prove results (1) and (4) with such ease was our use of the Wigner function propagator, in terms of which the quantum evolution takes a particularly transparent form. We comment that the detailed methods used here could well be of use in related calculations. For example, it might be possible to discuss decoherence of arbitrary initial states using the Wigner function propagator derived in Sec. III. These and other related questions will be pursued in future publications.

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APPENDIX

In this appendix we give the explicit forms of the coefficients A, B, C and $\tilde{K}, \hat{K}, L, N$ appearing in the explicit expression for the propagator, (2.17). The following results are taken from Caldeira and Leggett [8], and from Hu and Zhang [14], with minor elaborations and extensions.

We first give the forms of the coefficients for the harmonic oscillator in the underdamped case, $\omega_R > \gamma$. Let $\omega^2 = \omega_R^2 - \gamma^2$. We work in the underdamped case, $\gamma < \omega_R$.

Then we have

$$\tilde{K}(t) = -\tfrac{1}{2}M\gamma + \tfrac{1}{2}M\omega\, \cot\omega t\ , \tag{A1}$$

$$\hat{K}(t) = +\tfrac{1}{2}M\gamma + \tfrac{1}{2}M\omega\, \cot\omega t\ , \tag{A2}$$

$$L(t) = \frac{M\omega e^{-\gamma t}}{2\sin\omega t}\ , \tag{A3}$$

$$N(t) = \frac{M\omega e^{\gamma t}}{2\sin\omega t}\ . \tag{A4}$$

Also,

$$A(t) = \frac{M\gamma}{\pi}\int_0^\infty d\nu\, \exp\left(-\frac{\nu^2}{\Lambda^2}\right)\nu\, \coth\left(\frac{\hbar\nu}{2kT}\right) A_\nu(t)\ , \tag{A5}$$

where

$$A_\nu(t) = \frac{e^{-2\gamma t}}{\sin^2\omega t}\int_0^t d\tau \int_0^t ds\, \sin\omega\tau\, \cos\nu(\tau - s) \times \sin\omega s e^{\gamma(\tau+s)}\ . \tag{A6}$$

Similarly,

$$B(t)=\frac{M\gamma}{\pi}\int_0^\infty d\nu\,\exp\left(-\frac{\nu^2}{\Lambda^2}\right)\nu\,\coth\left(\frac{\hbar\nu}{2kT}\right)B_\nu(t)\ , \tag{A7}$$

where

$$B_\nu(t)=\frac{2e^{-2\gamma t}}{\sin^2\omega t}\int_0^t d\tau\int_0^t ds\,\sin\omega\tau\,\cos\nu(\tau-s)\times\sin\omega(t-s)e^{\gamma(\tau+s)}\ , \tag{A8}$$

and

$$C(t)=\frac{M\gamma}{\pi}\int_0^\infty d\nu\,\exp\left(-\frac{\nu^2}{\Lambda^2}\right)\nu\,\coth\left(\frac{\hbar\nu}{2kT}\right)C_\nu(t)\ , \tag{A9}$$

where

$$C_\nu(t)=\frac{1}{\sin^2\omega t}\int_0^t d\tau\int_0^t ds\,\sin\omega(t-\tau)\cos\nu(\tau-s)\sin\omega(t-s)e^{\gamma(\tau+s)}\ . \tag{A10}$$

We have included, for completeness, the explicit dependence on the cutoff in the expressions for A, B, and C, and this is sometimes required, although we will generally work with the case $\Lambda\to\infty$. The integrals for A_ν, B_ν, and C_ν have been evaluated by Hu and Zhang [14], and we will make heavy use of their results. The remaining integrations over ν to yield A, B, and C cannot be carried out in general, but asymptotic expansions are possible in various regimes of interest, and these we now give.

1. High-temperature limit

In the much-studied high-temperature (Fokker-Planck) limit, one has $\coth(\hbar\nu/kT)\approx kT/\hbar\nu$, and the integrals (A5)–(A10) may be evaluated exactly for any t, with the results

$$A=\frac{MkT}{2\hbar\sin^2\omega t}\left[1-e^{-2\gamma t}-\frac{1}{\gamma^2+\omega^2}(\gamma^2\cos2\omega t+\omega\gamma\sin2\omega t)-\frac{\gamma}{\gamma^2+\omega^2}e^{-2\gamma t}\right]\ , \tag{A11}$$

$$B=\frac{MkTe^{\gamma t}}{\hbar\sin^2\omega t}\left[-(1-e^{-2\gamma t})\cos\omega t+\frac{1}{\gamma^2+\omega^2}(\gamma^2(1+e^{-2\gamma t})\cos\omega t+\omega\gamma(1-e^{-2\gamma t})\sin\omega t)\right]\ , \tag{A12}$$

$$C=\frac{MkTe^{2\gamma t}}{2\hbar\sin^2\omega t}\left[\frac{\omega^2}{\gamma^2+\omega^2}-e^{-2\gamma t}+\frac{e^{-2\gamma t}}{\gamma^2+\omega^2}(\gamma^2\cos\omega t-\omega\gamma\sin2\omega t)\right]\ . \tag{A13}$$

At short times, one has

$$A=B=C=\frac{2M\gamma kTt}{3\hbar} \tag{A14}$$

and, for long times,

$$A=\frac{MkT}{2\hbar\sin^2\omega t}\left[1-\frac{1}{\gamma^2+\omega^2}(\gamma^2\cos2\omega t+\omega\gamma\sin2\omega t)\right]\ , \tag{A15}$$

$$B=\frac{MkTe^{\gamma t}}{\hbar\sin^2\omega t}\left[-\frac{\omega^2}{\gamma^2+\omega^2}\cos\omega t+\frac{\omega\gamma}{\gamma^2+\omega^2}\sin\omega t\right]\ , \tag{A16}$$

$$C=\frac{MkTe^{2\gamma t}}{2\hbar\sin^2\omega t}\frac{\omega^2}{(\gamma^2+\omega^2)}\ . \tag{A17}$$

2. Weak-coupling limit

In the weak-coupling regime, $\gamma\ll\omega$, but for arbitrary temperature, one has, from Hu and Zhang [14],

$$A=\frac{M\omega}{4\sin^2\omega t}\coth\left(\frac{\hbar\omega}{2kT}\right)\left[1-\frac{\gamma}{\omega}\sin2\omega t-e^{-2\gamma t}\right]\ , \tag{A18}$$

$$B=\frac{M\omega e^{\gamma t}}{2\sin^2\omega t}\coth\left(\frac{\hbar\omega}{2kT}\right)\left[\frac{\gamma}{\omega}\sin\omega t-\cos\omega t+\left(\frac{\gamma}{\omega}\sin\omega t+\cos\omega t\right)e^{-2\gamma t}\right]\ , \tag{A19}$$

$$C=\frac{M\omega e^{2\gamma t}}{4\sin^2\omega t}\coth\left(\frac{\hbar\omega}{2kT}\right)\left[1-\left(1+\frac{\gamma}{\omega}\sin\omega t\right)e^{-2\gamma t}\right]\ , \tag{A20}$$

where terms of order γ^2/ω^2 have been neglected. The long-time limits of these expressions are easily seen.

3. Long-time limit for arbitrary temperature and coupling

It is also possible to determine the exact form of the long-time limits of A, B, and C, without assuming high temperature or weak coupling. This is necessary in order

to give a completely general statement about the long-time limits of arbitrary initial states, as in Sec. VII. From Hu and Zhang, they are

$$A = \frac{M}{2\omega_R^2 \sin^2\omega t}\Big\{(\omega^2\cos^2\omega t - \gamma\omega\,\sin 2\omega t)\left(F_1 + \frac{\gamma}{\omega}F_2\right) + \omega^2\sin^2\omega t\left[\left(1+\frac{2\gamma^2}{\omega^2}\right)F_1 - \frac{\gamma}{\omega}F_2\right]\Big\}\,, \tag{A21}$$

$$B = \frac{Me^{\gamma t}}{\omega_R^2\sin^2\omega t}(\gamma\omega\,\sin\omega t - \omega^2\cos\omega t)\left(F_1 + \frac{\gamma}{\omega}F_2\right)\,, \tag{A22}$$

$$C = \frac{M\omega^2 e^{2\gamma t}}{2\omega_R^2\sin^2\omega t}\left(F_1 + \frac{\gamma}{\omega}F_2\right)\,, \tag{A23}$$

where

$$F_1 = \frac{1}{2\pi}\int_0^\infty d\nu\,\nu\,\coth\left(\frac{\hbar\nu}{2kT}\right)\left(\frac{\gamma}{\gamma^2+(\omega+\nu)^2} + \frac{\gamma}{\gamma^2+(\omega-\nu)^2}\right)\,, \tag{A24}$$

$$F_2 = \frac{1}{2\pi}\int_0^\infty d\nu\,\nu\,\coth\left(\frac{\hbar\nu}{2kT}\right)\left(\frac{\omega+\nu}{\gamma^2+(\omega+\nu)^2} + \frac{\omega-\nu}{\gamma^2+(\omega-\nu)^2}\right)\,. \tag{A25}$$

The integrals F_1 and F_2 do not appear to be exactly soluble, but can be evaluated in the high-temperature and weak-coupling regimes. In the high-temperature regime,

$$F_1 \approx \frac{kT}{\hbar}\,,\quad F_2 \approx 0 \tag{A26}$$

and the results (A11)–(A13) are recovered. In the weak-coupling regime,

$$F_1 \approx \tfrac{1}{2}\omega\,\coth\left(\frac{\hbar\omega}{2kT}\right)\,,\quad F_2 \approx 0 \tag{A27}$$

and the results (A18)–(A20) are recovered.

4. Very short-time limit with finite cutoff

For finite cutoff, and in the very short-time limit, $t \ll \Lambda^{-1}$, and at high temperature, $kT \gg \hbar\omega$, one has

$$A = 2B = C = \frac{M\gamma kT\Lambda}{4\pi\hbar}t^2 \tag{A28}$$

and

$$\tilde{K} = \frac{M}{2t} + \frac{M\gamma\Lambda^3}{45\pi}t^3\,, \tag{A29}$$

$$\hat{K} = \frac{M}{2t} + \frac{M\gamma\Lambda^3}{4\pi}t^3\,, \tag{A30}$$

$$L = \frac{M}{2t} - \frac{11M\gamma\Lambda^3}{180\pi}t^3\,, \tag{A31}$$

$$N = \frac{M}{2t} - \frac{M\gamma\Lambda^3}{180\pi}t^3\,. \tag{A32}$$

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