Black holes in three-dimensional topological gravity

S. Carlip*
Department of Physics, University of California, Davis, California 95616

J. Gegenberg†
Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick, Canada E3B 5A3

R. B. Mann‡
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge University, Cambridge, CB2 9EWf, United Kingdom
and Department of Physics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

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We investigate the black hole solution to (2+1)-dimensional gravity coupled to topological matter, with a vanishing cosmological constant. We calculate the total energy, angular momentum, and entropy of the black hole in this model and compare with results obtained in Einstein gravity. We find that the theory with topological matter reverses the identification of energy and angular momentum with the parameters in the metric, compared with general relativity, and that the entropy is determined by the circumference of the inner rather than the outer horizon. We speculate that this results from the contribution of the topological matter fields to the conserved currents. We also briefly discuss two new possible (2+1)-dimensional black holes.

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I. THE (2+1)-DIMENSIONAL BLACK HOLE

It is a peculiar feature of general relativity in 2+1 dimensions that any solution of the Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \ , \tag{1.1}$$

with a vanishing stress-energy tensor, has a constant curvature. Despite this limitation, Bañados *et al.* [1] have made the interesting observation that when $\Lambda = -1/l^2 < 0$, the field equations have a black hole solution, characterized by the metric

$$\begin{gathered} ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2 \ , \\ -\infty < t < \infty, \quad 0 < r < \infty, \quad 0 \leq \phi \leq 2\pi \ , \end{gathered} \tag{1.2}$$

with lapse and angular shift functions

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\phi(r) = -\frac{J}{2r^2} \ . \tag{1.3}$$

As a space of constant curvature, this geometry can be obtained directly from anti–de Sitter (AdS) space by means of appropriate identifications, as discussed in [2]. When $M > 0$ and $|J| \leq Ml$, the solution has an outer event horizon at $r = r_+$, where

$$r_+^2 = \frac{Ml^2}{2} \left\{ 1 + \left[1 - \left(\frac{J}{Ml} \right)^2 \right]^{1/2} \right\} \ , \tag{1.4}$$

and an inner horizon at $r_- = Jl/2r_+$, i.e.,

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+ r_-}{l} \ . \tag{1.5}$$

[Note that in 2+1 dimensions G has dimensions of an inverse mass, so $M = G\times$(conventional mass) is dimensionless.]

The parameters J and M have been shown to be the quasilocal angular momentum and mass of the black hole [3]; alternatively they can be expressed in terms of Casimir invariants in a gauge-theoretical formulation of (2+1)-dimensional gravity [4]. The parameter M can also be expressed in terms of the initial energy density of a disk of collapsing dust in AdS space [5].

For later reference, it will be useful to display the first-order formulation of Einstein gravity and the first-order form of the black hole solution. We suppose that M^3 is a smooth orientable three-manifold whose cotangent bundle has SO(2,1) as its structure group. The fibers of T^*M^3 are three-dimensional vector spaces which come equipped with a "natural" metric η_{ab} and volume element ϵ_{abc}. A smooth frame field, or triad, on M^3 is a set of three independent one-form fields E^a, and a spin connection ω_a on M^3 is an SO(2,1) connection. In terms of these fields, the Einstein action takes the simple form [6]

*Electronic address: carlip@dirac.ucdavis.edu
†Electronic address: lenin@math.unb.ca
‡Electronic address: rbm20@damtp-cambridge.ac.uk

$$I = \int \mathbf{L} = -\frac{1}{2}\int_{M^3}\bigg(E^a \wedge R_a[\omega] - \frac{\Lambda}{3}\epsilon_{abc}E^a \wedge E^b \wedge E^c\bigg) , \quad (1.6)$$

where the curvature $R_a[\omega]$ is

$$R_a[\omega] = d\omega_a + \tfrac{1}{4}\epsilon_{abc}\omega^b \wedge \omega^c . \quad (1.7)$$

We use the convention $\epsilon^{012} = +1$, and our units are such that $8\pi G = 1$. Variation of I with respect to ω_a yields the condition

$$D_\omega E^a = dE^a + \tfrac{1}{2}\epsilon^{abc}\omega_b \wedge E_c = 0 , \quad (1.8)$$

which is the usual relationship between E^a and the connection (that is, the condition that ω^a be torsion-free). Variation with respect to the triad yields

$$R_a = \Lambda\epsilon_{abc}E^b \wedge E^c , \quad (1.9)$$

which is equivalent to Eq. (1.1) upon insertion of Eq. (1.8).

The black hole (1.2) can be now described by the spacetime triad

$$\begin{aligned} E^0 &= \sqrt{\nu^2(r)-1}\left(\frac{r_+}{l}dt - r_-d\phi\right) , \\ E^1 &= \frac{l}{\nu}d[\sqrt{\nu^2(r)-1}] , \\ E^2 &= \nu(r)\left(r_+d\phi - \frac{r_-}{l}dt\right) , \end{aligned} \quad (1.10)$$

and the compatible spin connection

$$\begin{aligned} \omega^0 &= -2\sqrt{\nu^2(r)-1}\left(\frac{r_+}{l}d\phi - \frac{r_-}{l^2}dt\right) , \\ \omega^1 &= 0 , \\ \omega^2 &= -2\nu(r)\left(\frac{r_+}{l^2}dt - \frac{r_-}{l}d\phi\right) , \end{aligned} \quad (1.11)$$

where $\nu^2(r) := r^2 - r_-^2/r_+^2 - r_-^2$.

II. *BCEA* GRAVITY

There is another three-dimensional theory of gravity that admits the asymptotically AdS black hole as a solution. This theory, developed by two of the present authors [7], minimally couples topological matter to Einstein gravity in 2+1 dimensions, in effect replacing the cosmological constant by a simple set of matter fields. We shall call this model "*BCEA* theory."

We start with a triad E^a and a spin connection A^a as in the last section, but without imposing the torsion-free condition (1.8). Now let B^a and C^a be two additional one-form "matter" fields. The action for *BCEA* theory is

$$I = \int \mathbf{L} = -\frac{1}{2}\int_{M^3}(E^a \wedge R_a[A] + B^a \wedge D_{(A)}C_a) , \quad (2.1)$$

where $D_{(A)}$ is the covariant derivative with respect to the connection A^a and the curvature $R_a[A]$ is given by (1.7).

The stationary points of I are determined by the field equations

$$\begin{aligned} R_a[A] &= 0 , \\ D_{(A)}B^a &= 0 , \\ D_{(A)}C_a &= 0 , \\ D_{(A)}E^a + \tfrac{1}{2}\epsilon^{abc}B_b \wedge C_c &= 0 . \end{aligned} \quad (2.2)$$

Because of the term in $B_b \wedge C_c$ in the last equation of motion, the triad E^a is not, in general, compatible with the spin connection A_a. Nevertheless, the equations of motion above determine a Lorentzian geometry on TM^3: if we define a one-form field Q_a by the requirement

$$\epsilon^{abc}(Q_b \wedge E_c - B_b \wedge C_c) = 0 , \quad (2.3)$$

then the equation of motion for the E^a can be written as

$$dE^a + \tfrac{1}{2}\epsilon^{abc}\omega_b \wedge E_c = 0, \quad \omega_a := A_a + Q_a . \quad (2.4)$$

Equation (2.4) may be recognized as the condition that the frame field E^a be compatible with the (nonflat) spin connection ω_a. We may thus interpret *BCEA* theory as a model of (2+1)-dimensional gravity with a triad E^a and a connection ω_a coupled to matter fields B^a and C^a. Alternatively, the model may be viewed as a "teleparallel" theory of gravity, with a triad E^a and a flat, but not torsion-free, connection A^a, again coupled to matter fields B^a and C^a. In either case, the geometry is determined by the metric $g_{\mu\nu} = \eta_{ab}E^a{}_\mu E^b{}_\nu$.

The action functional I is invariant under a 12-parameter group whose infinitesimal generators are [7]

$$\begin{aligned} \delta B^a &= D_{(A)}\rho^a + \tfrac{1}{2}\epsilon^{abc}B_b\tau_c , \\ \delta C^a &= D_{(A)}\lambda^a + \tfrac{1}{2}\epsilon^{abc}C_b\tau_c , \\ \delta E^a &= D_{(A)}\xi^a + \tfrac{1}{2}\epsilon^{abc}(E_b\tau_c + B_b\lambda_c + C_b\rho_c) , \\ \delta A^a &= D_{(A)}\tau^a . \end{aligned} \quad (2.5)$$

This group may be recognized as I(ISO(2,1)), where the notation IG denotes the semidirect product of the Lie group G with its own Lie algebra $\mathcal{L}_G$. Like the action for ordinary Einstein gravity in three dimensions [6], the *BCEA* action can be obtained from a Chern-Simons functional, now for the gauge group I(ISO(2,1)).

III. THE BLACK HOLE SOLUTION IN *BCEA* THEORY

We shall now demonstrate that the asymptotically anti–de Sitter black hole of Sec. I is also a solution of *BCEA* theory. The computation is simplest in the gauge $A^a = 0$, for which the field equations for B and C reduce to the condition that the one-forms B^a and C^a be closed. It is then easy to show that

$$\begin{aligned} B^0 &= -r_+d\phi + \frac{r_-}{l}dt , \\ B^1 &= -ld(\nu + \sqrt{\nu^2-1}) , \\ B^2 &= \frac{r_+}{l}dt - r_-d\phi , \end{aligned} \quad (3.1)$$

and

$$C^0 = -\frac{1}{l}B^0 \ ,$$
$$C^1 = d(\nu - \sqrt{\nu^2 - 1}) \ ,$$
$$C^2 = \frac{1}{l}B^2 \ , \tag{3.2}$$

together with the black hole triad of Eq. (1.10) and the connection $A^a = 0$, satisfy the $BCEA$ equations of motion.

Recall that for pure gravity, $l = |\Lambda|^{-1/2}$ is a parameter appearing in the action. In $BCEA$ theory, on the other hand, l appears as a constant of integration. The need for such a dimensionful constant is clear—without it, the two terms in (for example) B^0 would not have the same dimension—but its value is arbitrary. It is an interesting open question whether other integration constants appear in the most general circularly symmetric stationary solution of $BCEA$ theory. We know that two parameters (r_+ and r_-) appear in pure gravity; since the dimensionality of the moduli space of $BCEA$ theory is twice that of pure gravity [7], we might expect one more parameter in addition to l, but we do not yet have a proof of this assertion.

As discussed above, the triad E^a determines the spacetime geometry, so the solution given by Eqs. (1.10), (3.1), and (3.2) still has an interpretation as a black hole. To obtain more information, we can investigate the conserved quantities, or Noether currents, associated with the symmetries of the solution. For the conventional black hole, for example, the charge and mass are the conserved charges associated with spatial translations and rotations at infinity, and the entropy is the charge associated with an appropriate Killing vector at the horizon. It is instructive to understand the analogous statements for the $BCEA$ solution.

IV. NOETHER CURRENTS

We shall follow the methods of Wald, which we briefly summarize for the case of a three-dimensional spacetime (see [8] for details).

A. Noether charges in the manner of Wald

Let $\mathbf{L}[\phi]$ be a Lagrangian three-form (in three-dimensional spacetime), where ϕ represents an arbitrary set of fields. Under a variation $\delta\phi$,

$$\delta\mathbf{L} = \Xi\delta\phi + d\Theta[\phi, \delta\phi] \ , \tag{4.1}$$

where the field equations are $\Xi = 0$ and Θ is a boundary term, constructed locally from ϕ and $\delta\phi$, that determines the symplectic structure of the theory. For the action to be invariant under a symmetry transformation

$$\phi \to \phi + \delta_g\phi, \quad g \in G \ , \tag{4.2}$$

it is clearly necessary that

$$\delta_g\mathbf{L} = d\alpha[\phi, \delta_g\phi] \ , \tag{4.3}$$

for some two-form α. Combining Eq. (4.2) and Eq. (4.3), we see that $d\mathbf{j}[g] = 0$ when the equations of motion are satisfied, where

$$\mathbf{j}[g] = \Theta[\phi, \delta_g\phi] - \alpha[\phi, \delta_g\phi] \ . \tag{4.4}$$

The two-form $\mathbf{j}[g]$ is the Hodge dual of the usual Noether current associated with the symmetry generated by g; its integral over a Cauchy surface $\mathcal{C}$ gives a conserved charge $q[g]$. Note that when $\mathbf{j}$ is exact, that is, $\mathbf{j} = d\mathbf{Q}$, then the integral that gives $q[g]$ reduces to an integral at $\partial\mathcal{C}$. This is the case when the symmetry group is the group of diffeomorphisms, and explains why the energy in general relativitity can be written as an integral at spatial infinity.

We now specialize to the case of diffeomorphism invariance. Let $\delta_\zeta\phi = \mathcal{L}_\zeta\phi$ be a diffeomorphism generated by a vector field ζ. It is then easy to show that

$$\mathbf{j}[\zeta] = \Theta[\phi, \mathcal{L}_\zeta\phi] - \zeta \cdot \mathbf{L} \ . \tag{4.5}$$

($\mathcal{L}$ is the Lie derivative, and the centered dot denotes contraction of a vector with the first index of a form.) The Noether charges now have simple physical interpretations: if t^μ generates an asymptotic time translation and φ^μ generates an asymptotic rotation, then Wald has shown [8] that the canonical energy and angular momentum are

$$\mathcal{E} = \int_\infty (\mathbf{Q}[t] - t \cdot G) \ ,$$
$$\mathcal{J} = -\int_\infty \mathbf{Q}[\varphi] \ . \tag{4.6}$$

Here the integrals are taken along a circle of constant time and infinite radius, and G is defined by the condition

$$\delta_0 \int_\infty t \cdot G = \int_\infty t \cdot \Theta[\phi, \delta_0\phi] \ , \tag{4.7}$$

for variations δ_0 lying within the space of solutions of the equations of motion.

The derivation summarized here is valid for an arbitrary diffeomorphism-invariant Lagrangian. In particular, Noether charges can be obtained for both first-order (triad) and second-order (metric) gravity. As we shall see below, however, Wald's further derivation of black hole entropy as a Noether charge requires added assumptions, which do not hold for the first-order systems we are considering.

B. Noether charges for Einstein gravity

For the first-order Einstein action of Eq. (1.6), it is easy to see that

$$\Theta = \tfrac{1}{2}E^a \wedge \delta\omega_a \ , \tag{4.8}$$

and a simple computation gives a current (4.5) of

$$\mathbf{j}[\zeta] = d\mathbf{Q}[\zeta], \quad \mathbf{Q}[\zeta] = -\tfrac{1}{2}E^a\zeta \cdot \omega_a \ . \tag{4.9}$$

The canonical energy (4.6) for the black hole solution of Eqs. (1.10) and (1.11) is then easy to calculate: we obtain

$$\mathcal{E} = \pi M \ , \tag{4.10}$$

while the canonical angular momentum $\mathcal{J}$ is

$$\mathcal{J} = \pi J \ . \tag{4.11}$$

C. Noether charges for the $BCEA$ system

For the $BCEA$ system, there are two sets of symmetries that may give rise to interesting Noether charges, the diffeomorphisms and the I(ISO(2,1)) transformations (2.5). These symmetries are not independent. On shell,

$$\begin{aligned}
\mathcal{L}_\zeta A^a &= D(\zeta \cdot A^a) \ , \\
\mathcal{L}_\zeta B^a &= D(\zeta \cdot B^a) + \tfrac{1}{2}\epsilon^{abc}B_b(\zeta \cdot A_c) \ , \\
\mathcal{L}_\zeta C^a &= D(\zeta \cdot C^a) + \tfrac{1}{2}\epsilon^{abc}C_b(\zeta \cdot A_c) \ , \\
\mathcal{L}_\zeta E^a &= D(\zeta \cdot E^a) + \tfrac{1}{2}\epsilon^{abc}[E_b(\zeta \cdot A_c) + C_b(\zeta \cdot B_c) \\
&\quad + B_b(\zeta \cdot C_c)] \ ,
\end{aligned} \tag{4.12}$$

and the transformation (4.12) are precisely the gauge transformations (2.5) with parameters

$$\tau^a = \zeta \cdot A^a, \quad \rho^a = \zeta \cdot B^a, \quad \lambda^a = \zeta \cdot C^a, \quad \hat{\zeta}^a = \zeta \cdot E^a \ . \tag{4.13}$$

The diffeomorphisms are thus equivalent to I(ISO(2,1)) gauge transformations on shell, as one expects in a topological theory.

From the Lagrangian (2.1), it is evident that

$$\Theta = \tfrac{1}{2}(E^a \wedge \delta A_a + B^a \wedge \delta C_a) \ . \tag{4.14}$$

For the transformations (2.5), it then follows that

$$\mathbf{j}[g] = d\mathbf{Q}[g], \quad \mathbf{Q}[g] = -\tfrac{1}{2}(E^a\tau_a + B^a\lambda_a) \ , \tag{4.15}$$

on shell. The conserved charge is thus

$$q[g] = \int_C \Theta[g] = -\frac{1}{2}\int_\infty (E^a\tau_a + B^a\lambda_a) \ . \tag{4.16}$$

In particular, consider the $BCEA$ black hole. For the integral (4.16) to exist, we must restrict ourselves to parameters with the asymptotic behavior

$$\tau_a \sim \frac{1}{r}\hat{\tau}_a, \quad \lambda_a \sim \hat{\lambda}_a \ , \tag{4.17}$$

where $\hat{\tau}_a$ and $\hat{\lambda}_a$ are constants (or possibly functions of ϕ). Then

$$\begin{aligned}
q[\hat{\tau}_a, \hat{\lambda}_a] &= \frac{-\pi(-\hat{\tau}_0 r_- + \hat{\tau}_2 r_+)}{(r_+^2 - r_-^2)^{1/2}} \\
&\quad + \pi(\hat{\lambda}_0 r_+ + \hat{\lambda}_2 r_-) \ .
\end{aligned} \tag{4.18}$$

The charges are thus determined by the constants $r_\pm$, as one might expect.

As a special case, we may consider transformations parametrized as in (4.13), that is, I(ISO(2,1)) transformations that correspond to diffeomorphisms. We find conserved charges

$$\begin{aligned}
q[\zeta^\phi, \zeta^t] &= -\frac{\pi(r_+^2 + r_-^2)}{l}\zeta^\phi + \frac{2\pi r_+ r_-}{l^2}\zeta^t \\
&= -\pi M l \zeta^\phi + \frac{\pi J}{l}\zeta^t \ .
\end{aligned} \tag{4.19}$$

To determine the canonical energy, we must add in the term $t \cdot G$ of Eq. (4.6), but it is not hard to check that this gives no further contribution. The charges associated with asymptotic time translations and rotations are the mass and angular momentum, as expected, but in the wrong order—the mass appears as the charge for ζ^ϕ, while the angular momentum is the charge for ζ^t. We note here that the same values of the total energy and angular momentum emerge from a Regge-Teitelboim analysis [9].

A possible interpretation of this result is to attribute the mass and charge of Eq. (4.19) to a combination of the black hole and the "matter" fields B and C. That is, we can write

$$\begin{aligned}
M_{\text{total}} &= M_{bh} + M_{B-C} \ , \\
J_{\text{total}} &= J_{bh} + J_{B-C} \ ,
\end{aligned} \tag{4.20}$$

with

$$M_{B-C} = -\frac{J_{B-C}}{l} = \frac{J}{l} - M \ . \tag{4.21}$$

It is perhaps not surprising to find that the B and C fields carry angular momentum—they are, after all, sources of torsion in the field equations (2.2). We do not, however, have a good explanation for the extremal condition $M_{B-C} = -J_{B-C}/l$.

D. Thermodynamics

In the first item of [8], Wald shows that black hole entropy can also be derived as a Noether charge. Unfortunately, that derivation breaks down in the first-order formulation. In particular, Eq. (24) in [8] has no nontrivial solutions for $\tilde{\mathbf{Q}}$. Indeed, for the ϕ component of $\mathbf{Q}$ in pure first-order gravity, we must solve the variational equation:

$$\delta\tilde{\mathbf{Q}} = -\frac{2r_+}{r_+^2 - r_-^2}(r_+\delta r_+ - r_-\delta r_-) \ . \tag{4.22}$$

This is not integrable.

However, we can still use the first law of thermodynamics to determine an entropy for the $BCEA$ system. To do so, we choose a constant Ω_H such that the Killing vector

$$\chi^\mu = t^\mu + \Omega_H \varphi^\mu \tag{4.23}$$

vanishes at the horizon. The surface gravity κ is then

$$\kappa^2 = -\tfrac{1}{2}\nabla^\mu\chi^\nu\nabla_\mu\chi_\nu \ , \tag{4.24}$$

and the first law of black hole dynamics takes the form

$$\frac{\kappa}{2\pi}\delta S = \delta\mathcal{E} - \Omega_H\delta\mathcal{J} \ . \tag{4.25}$$

For the (2+1)-dimensional black hole (1.2), in particular, it is easily checked that

$$\Omega_H = -N^\phi(r_+), \quad \kappa = \frac{r_+^2 - r_-^2}{r_+ l^2} \ . \tag{4.26}$$

The relationship to standard thermodynamics may then be established by means of the Euclidean path integral: it is shown in Refs. [1,10] that the black hole temperature obtained from the path integral is $\beta = 2\pi/\kappa$, so (4.25) is just the first law of thermodynamics.

For the black hole in Einstein gravity, (4.10) and (4.11) gives

$$\begin{aligned}\delta\mathcal{E} - \Omega_H\delta\mathcal{J} &= \frac{2\pi}{l^2}\left\{ r_+\delta r_+ + r_-\delta r_- \right. \\ &\qquad \left. -\frac{r_-}{r_+}(r_+\delta r_- + r_-\delta r_+) \right\} \\ &= 2\pi\kappa\delta r_+ \ . \end{aligned} \tag{4.27}$$

The entropy (4.25) is thus

$$S = 4\pi^2 r_+ \ , \tag{4.28}$$

agreeing with the conventional expression in our choice of units. For the $BCEA$ black hole, on the other hand, (4.19) gives

$$\begin{aligned}\delta\mathcal{E} - \Omega_H\delta\mathcal{J} &= \frac{2\pi}{l^2}\left\{ r_+\delta r_- + r_-\delta r_+ \right. \\ &\qquad \left. -\frac{r_-}{r_+}(r_+\delta r_+ + r_-\delta r_-) \right\} \\ &= 2\pi\kappa\delta r_- \ . \end{aligned} \tag{4.29}$$

Hence

$$S = 4\pi^2 r_- \ . \tag{4.30}$$

Once again, the parameters of the conventional black hole have been interchanged: this time, the roles of r_+ and r_- are reversed.

V. CONCLUSION

We have seen that the (2+1)-dimensional black hole does not require a negative cosmological constant. Rather, the cosmological constant can be replaced by a suitable distribution of "matter." In itself, this is perhaps not a very surprising observation, although the division of mass and angular momentum between the metric and the B and C fields is rather unexpected. But it is certainly interesting that even such a simple system of topological matter can give rise to a black hole.

One may also examine the $BCEA$ model in second-order form, using (2.3) and (2.4) to express the connection A^a in terms of the fields E^a, B^a, and C^a in the action. In this case, it turns out that the B and C fields contribute to the Noether current at spatial infinity, but not at the event horizon. The result is that, as in the first-order form, the locally constructed expression for the entropy does not satisfy the first law of thermodynamics. This is probably generic in models where matter fields are coupled to the spin connection, i.e., to first derivatives of the metric.

The $BCEA$ system is, in fact, quite powerful, and our results suggest an interesting direction to search for additional black hole solutions. As noted in Sec. I, the standard (2+1)-dimensional black hole can be obtained from anti–de Sitter space by a set of identifications. It is natural to ask whether this procedure can lead to black hole solutions if we begin with a different homogeneous geometry.

Homogeneous Riemannian metrics have been classified by Thurston [11], and their Wick rotated metrics provide a useful starting point. Our preliminary results are that "black-hole-like" solutions can be obtained from at least three of these geometries, H^3 (which gives the usual black hole), $H^2 \times E^1$, and SOL. In particular, if we let

$$f(r) := \frac{r^2}{l^2} - M \ , \tag{5.1}$$

the metrics

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + d\phi^2 \tag{5.2}$$

(obtained from $H^2 \times E^1$ by identifications) and

$$ds^2 = -\frac{f}{Ml^2}dt^2 + \frac{1}{l^2 f}dr^2 + \frac{M}{f}e^{-2t/l}d\phi^2 \tag{5.3}$$

(obtained from SOL) satisfy the $BCEA$ equations of motion for suitable choices of B and C. We do not yet understand the detailed behavior of these solutions, but we believe them to be of some interest.

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