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Wormhole cosmic strings

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We construct regular multiwormhole solutions to a gravitating σ model in three space-time dimensions, and extend these solutions to cylindrical traversable wormholes in four and five dimensions. We then discuss the possibility of identifying wormhole mouths in pairs to give rise to Wheeler wormholes. Such an identification is consistent with the original field equations only in the absence of the σ-model source, but with possible naked cosmic string sources. The resulting Wheeler wormhole space-times are flat outside the sources and may be asymptotically Minkowskian.

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I. INTRODUCTION

The intriguing possibility that we might live in a multiply connected Universe and be able to travel to distant galaxies through traversable wormholes has been popularized by the analysis of Morris and Thorne [1]. Traversable wormholes may occur as solutions to the Einstein field equations with suitable sources violating the weak energy condition. While most Lorentzian wormhole solutions discussed in the literature are spherically symmetric [2], this is an unnecessary limitation, as stressed by Visser [3]. For instance, if the weak energy condition is relaxed there might occur cylindrical wormholes, which from afar would appear as cosmic strings. In [4], generalizing previous work in 2+1 dimensions [5] (see also [6]), we have shown that an infinite cylinder of exotic matter with equal negative energy density and longitudinal stresses ($\mu = \tau_z < 0$) and vanishing azimuthal stress (τ_φ=0) generates a symmetrical wormhole space-time, with two axes at spacelike infinity. The metric both "outside" and "inside" the matter cylinder is the well-known conical cosmic string metric [7], with a deficit angle (the same on both sides) which can be chosen at will, independently of the values of the cylinder parameters.

In the present work we wish to investigate cylindrical wormholes in a more fundamental, purely field-theoretical model, that of an O(3) nonlinear σ-model field coupled repulsively to gravity. We have previously shown [8] that in three space-time dimensions this model admits static multiwormhole solutions with two points at spacelike infinity. These solutions can be promoted in a straightforward fashion to four-dimensional "multiwormhole cosmic string" space-times with two axes at spacelike infinity. However, the would-be regular multiwormhole solutions constructed in [8] actually have conical singularities, which give rise to naked cosmic strings in the four-dimensional case. In this paper, we show how these extra singularities may be removed to yield genuine regular multiwormhole solutions. The corresponding four-dimensional space-time metrics are asymptotic to the conical cosmic string metric, with a deficit angle which can be positive, zero, or negative according to the value of the σ-model fundamental length. From our σ-model wormhole space-times, we derive multiwormhole solutions to sourceless five-dimensional general relativity, and discuss briefly the structure of the four-dimensional metric and electromagnetic fields which result from Kaluza-Klein dimensional reduction. We also discuss the possibility of identifying wormhole mouths in pairs to give rise to Wheeler wormholes in a space-time with only one axis at spacelike infinity. We find that such an identification is consistent with the original field equations only in the absence of the σ-model source, but with possible naked cosmic string sources. Finally, we discuss the extension of our results to the case where the σ-model field is minimally coupled to a Chern-Simons gauge field [9].

II. THREE-DIMENSIONAL σ-MODEL WORMHOLES

The O(3) nonlinear σ model in three space-time dimensions is defined by the action

$$S = \int d^3x\sqrt{|g|}\,\tfrac{1}{2}[g^{\mu\nu}\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi} + \lambda(\boldsymbol{\phi}^2 - \nu^2)] \;, \quad (2.1)$$

where the Lagrange multiplier λ constrains the isovector field $\boldsymbol{\phi}$ to vary on the two-sphere $\boldsymbol{\phi}^2 = \nu^2$. As first shown in [10], this model admits static multisoliton solutions in a flat background space-time. We showed [11] that these solutions are actually independent of the background metric, which may be curved, and we derived [11,8] the soliton solutions to the coupled Einstein-σ system

$$S = \int d^3x\sqrt{|g|}\frac{1}{2}\left[-\frac{1}{\kappa}g^{\mu\nu}R_{\mu\nu} \pm g^{\mu\nu}\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi} \pm\lambda(\boldsymbol{\phi}^2 - \nu^2)\right] \quad (2.2)$$

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($\kappa = 8\pi G$), where the field $\boldsymbol{\phi}$ may be coupled attractively (upper sign) or repulsively (lower sign) to gravity [both signs are possible in three-dimensional gravity [12]; the lower sign arises naturally in the case where the action (2.2) is obtained by dimensional reduction from a five-dimensional Kaluza-Klein theory [13]]. These gravitating σ-model solitons were independently constructed in [14].

We briefly recall the construction of Ref. [8]. The stereographic map

$$\phi_1 + i\phi_2 = \frac{2\nu\psi}{1+|\psi|^2}, \quad \phi_3 = \nu\frac{1-|\psi|^2}{1+|\psi|^2} , \qquad (2.3)$$

projects the sphere $\boldsymbol{\phi}^2 = \nu^2$ on the complex ψ plane. The field equations derived from the action (2.2) may then be written

$$R_{\mu\nu} = \pm 2\kappa\nu^2 F(\partial_\mu\psi^*\partial_\nu\psi + \partial_\nu\psi^*\partial_\mu\psi) ,$$
$$\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi) = 2F^{1/2}\psi^* g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi , \qquad (2.4)$$

where $F(|\psi|) \equiv (1+|\psi|^2)^{-2}$. We search for static solutions such that ψ is time independent, and the metric may be written

$$ds^2 = h^2 dt^2 - e^{2u} d\mathbf{x}^2 \qquad (2.5)$$

in isotropic spatial coordinates. The Einstein equations (2.4) then reduce to the system

$$\frac{\partial^2 h}{\partial\zeta\partial\zeta^*} = 0 ,$$
$$\frac{\partial^2 u}{\partial\zeta\partial\zeta^*} = \mp\kappa\nu^2 F\left(\left|\frac{\partial\psi}{\partial\zeta}\right|^2 + \left|\frac{\partial\psi}{\partial\zeta^*}\right|^2\right) , \qquad (2.6)$$
$$\frac{\partial}{\partial\zeta}\left(e^{-2u}\frac{\partial h}{\partial\zeta}\right) \pm 4\kappa\nu^2 h e^{-2u} F\frac{\partial\psi^*}{\partial\zeta}\frac{\partial\psi}{\partial\zeta} = 0 ,$$

where $\zeta \equiv x + iy$. In the case of multisoliton solutions, the metric should be asymptotic to that generated by a system of point particles. This implies that the harmonic function h is constant; we choose h=1. The last equation (2.6) then shows that ψ must be an analytic or antianalytic function, which also solves the last equation (2.4) (both sides vanish). We assume for definiteness ψ to be analytic, $\psi = \psi(\zeta)$, and, without loss of generality, we choose the south pole of the sphere $\boldsymbol{\phi}^2 = \nu^2$ [the center of the stereographic projection (2.3)] to be the image of the point at infinity of the ζ plane, i.e., $\psi(\infty) = \infty$. Finally, the integration of the second equation (2.6) leads to the metric function

$$e^{2u} = \frac{(1+|\psi|^2)^{\mp 2\kappa\nu^2}}{|f(\psi)|^2} , \qquad (2.7)$$

where f is an arbitrary analytic function of ψ.

If this function is constant then the metric

$$ds^2 = dt^2 - (1+|\psi|^2)^{\mp 2\kappa\nu^2} d\zeta\, d\zeta^* \qquad (2.8)$$

is everywhere regular provided the point $\zeta = \infty$ is indeed at spatial infinity. Assuming ψ to be of the order of ζ^n (n integer) for $\zeta \to \infty$, we find that for the upper sign in (2.8) the spatial metric is asymptotically conical (or cylindrical) if $n\kappa\nu^2 \leq \frac{1}{2}$, and compact if $n\kappa\nu^2 = 1$ (in the case where ψ is linear in ζ, this is the well-known σ-model monopole compactification mechanism [15]), while for the lower sign the metric is asymptotically pseudoconical (the deficit angle is negative).

If on the other hand the function $f(\psi)$ is not constant, then it has at least one zero ψ_0, which leads to a metric singularity unless the point $\psi = \psi_0$ is at spatial infinity. There are then generically two points at spatial infinity, $\psi = \infty$ and $\psi = \psi_0$. Following Ref. [8], we assume that these two points are respectively the south and north poles of the sphere $\boldsymbol{\phi}^2 = \nu^2$ (implying $\psi_0 = 0$), and that the north-south reflection

$$\psi \to (\psi^*)^{-1} \qquad (2.9)$$

about the equatorial plane ϕ_3=0 is an isometry of our space-time. Putting

$$\psi = e^Z , \qquad (2.10)$$

with $Z = X + iY$, this condition leads to the space-time metric in Z coordinates

$$ds^2 = dt^2 - (\cosh X)^{\mp 2\kappa\nu^2}\frac{dZ\, dZ^*}{|g(Z)|^2} , \qquad (2.11)$$

which is invariant under (2.9) if $|g(-Z)| = |g(Z)|$.

Now, because of the isometry (2.9), we may take $\psi(\zeta)$ to be a conformal map of the ζ plane on the south (or north) hemisphere, i.e., the exterior (or the interior) of the circle $|\psi|$=1 (X=0). Such a map is given by the transformation [16]

$$\cosh Z \equiv \frac{1}{2}\left(\psi + \frac{1}{\psi}\right) = \zeta_1 , \qquad (2.12)$$

leading to the one-soliton metric

$$ds^2 = dt^2 - (\cosh X)^{\mp 2\kappa\nu^2}|g(Z)|^{-2}\frac{d\zeta_1 d\zeta_1^*}{|\zeta_1^2 - 1|} . \qquad (2.13)$$

This is singular at the zeros or poles of $g(Z)$, so that a necessary condition for regularity is $g(Z)$=const. However, even with this choice, the metric (2.13) still has two conical singularities (branch points) with angular deficit π located at the two points P ($\zeta_1 = -1$) and P' ($\zeta_1 = 1$). In the conventional interpretation of conical singularities in three-dimensional gravity [17,12], each of these singularities would be associated with a point mass $m = \pi/\kappa$. Our point of view here is that these singularities are spurious, and may be removed by transforming to a suitable coordinate system, thereby revealing the wormhole structure of our space-time. The transformation (2.12) maps the region $|\psi| > 1$ of the ψ plane on the ζ_1 plane cut along the segment PP'. The inverse transformation defines the bivalued function $\psi(\zeta_1)$ which becomes single valued on the Riemann surface made of two copies of the ζ_1 plane connected along the cut X=0. The metric transformed from (2.13) [with $g(Z) = l^{-1}$, l constant] by $\psi(\zeta_1)$ is

$$ds_1^2 = dt^2 - l^2(1+r^2)^{\mp\kappa\nu^2-1}[dr^2 + (1+r^2)d\theta^2] \ , \tag{2.14}$$

where we have used the polar representation $\psi = Re^{i\theta}$, and $r = \frac{1}{2}(R - 1/R)$ varies from $-\infty$ to $+\infty$. In the case of the upper sign, the points $r = \pm\infty$ are actually at a finite distance, and the spatial sections are compact; they are regular only for $\kappa\nu^2{=}1$, in which case we again recover the σ-model spherical compactification. In the case of the lower sign, the spatial sections have the two points at infinity $r = \pm\infty$, the regular metric (2.14) being asymptotically conical (for $\kappa\nu^2 < 1$) or pseudoconical (for $\kappa\nu^2 > 1$), and locally cylindrical on the equator $r{=}0$.

In the limit $\kappa\nu^2 \to 0$ we recover the cylindrical spacetime $ds^2 = dt^2 - dr^2 - l^2 d\theta^2$. Conversely, the metric (2.13) with $g = l^{-1}$, $\kappa\nu^2{=}0$ is obtained from the cylindrical metric by pinching the cylinder of radius l along a parallel. By cutting the cylinder along the resulting segment, we obtain two copies of the bicone generated by two point particles P and P' of mass π/κ. The generic bicone is singular because it can be flattened only by making two cuts extended from each particle to infinity; our construction shows that the bicone can be maximally extended to a regular surface, a cylinder, when the two deficit angles are equal to π (Fig. 1).

The multisoliton solution is obtained from (2.12) by the conformal map

$$\zeta_1 = \prod_{i=1}^{n}(\zeta - a_i) \ , \tag{2.15}$$

depending on n complex constants a_i. The associated metric (2.13) is singular at the $(n-1)$ zeros of the polynomial $\partial\zeta_1/\partial\zeta$, unless the even function $g(Z)$ is chosen precisely so as to compensate these zeros, $g(Z) = l^{-1}\partial\zeta_1/\partial\zeta$ [in [8], $g(Z)$ was implicitly assumed to be constant, so that the multisoliton metric was actually singular]. The resulting multisoliton, multiwormhole solution is given by Eqs. (2.12) and (2.15), and

$$ds^2 = dt^2 - l^2(\cosh X)^{2\kappa\nu^2}\frac{d\zeta\, d\zeta^*}{|\zeta_1^2 - 1|} \ , \tag{2.16}$$

where we have taken the lower sign in (2.13) (the upper sign leads to singular solutions for all $n > 1$). The metric

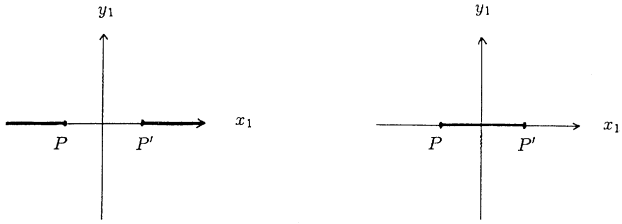

FIG. 1. Two possible definitions of the domain of analyticity of the metric (2.13). For $\kappa\nu^2{=}0$ the first possibility leads to a bicone, while the second possibility leads, after analytical continuation, to a cylinder.

(2.16) contains $2n$ conical singularities located at the zeros $b_i^{\pm}$ of $(\zeta_1^2 - 1)$, each with angular deficit π. As in the case $n{=}1$, these metrical singularities are characteristic of the wormhole topology, and may be removed pairwise by transforming to a suitable coordinate system. The spatial sections of the n-wormhole space-time are Riemann surfaces made of two copies of the ζ_1 plane joined along n cuts, the n components (assumed to be disjoint) of the equator $X{=}0$, each of which connects two singularities $\zeta = b_i^+$ $(\zeta_1 = +1)$ and $\zeta = b_i^-$ $(\zeta_1 = -1)$. To remove any given pair of singularities (b_i^-, b_i^+), we make the coordinate transformation from ζ to $\tilde{\psi}$, defined by

$$\zeta - \bar{b}_i = \tfrac{1}{2}(b_i^+ - b_i^-)\tilde{\zeta}_1 = \tfrac{1}{4}(b_i^+ - b_i^-)\left(\tilde{\psi} + \frac{1}{\tilde{\psi}}\right) \tag{2.17}$$

[where $\bar{b}_i = \frac{1}{2}(b_i^+ + b_i^-)$]; this maps the ith cut (which we now choose to be the straight segment connecting b_i^+ to b_i^-) into the circle $|\tilde{\psi}|{=}1$, in the vicinity of which the transformed metric is regular and can be extended from $|\tilde{\psi}| > 1$ to $|\tilde{\psi}| < 1$. The asymptotic behavior of the metric function in (2.16) is

$$e^{2u} \sim \rho^{2n(\kappa\nu^2-1)} \quad (\zeta \to \infty) \tag{2.18}$$

$(\rho = |\zeta|)$. It follows that the spatial sections are, in each Riemann sheet, asymptotically pseudoconical for $\kappa\nu^2 > 1$, asymptotically Euclidean for $\kappa\nu^2 = 1$, and asymptotically conical (cylindrical) for $1{-}1/n \le \kappa\nu^2 < 1$. The values $\kappa\nu^2 = 1 - 2/n$ yield regular compact spatial sections of genus $n-1$. For $\kappa\nu^2 = 0$, $n = 2$, the maximally extended spatial sections of (2.16) are flat tori, as may be checked by transforming the metric (2.16) (where we can choose $\zeta_1 = \zeta^2 - a^2$, with $a > 1$) to the Minkowski form

$$ds^2 = dt^2 - dw\, dw^* \tag{2.19}$$

with

$$dw = \frac{l\, d\zeta}{\sqrt{(\zeta^2 - a^2)^2 - 1}} \ , \tag{2.20}$$

and noting that the inverse function $\zeta(w)$ [16] is a biperiodical Jacobi function,

$$\zeta = \sqrt{a^2+1}\ \mathrm{sn}\left(\frac{\sqrt{a^2+1}}{l}w, k\right) \tag{2.21}$$

with $k^2 = (a^2 - 1)/(a^2 + 1)$, which implies that the real and imaginary parts of w should both be periodically identified; conversely, the metric (2.16) with $n = 2$, $\kappa\nu^2 = 0$ may be obtained by pinching the flat torus $S^1 \times S^1$ along two opposite circles, yielding two copies of the tetracone.

The total energy associated with a static metric of the form (2.5) with $h{=}1$ may be determined from the asymptotic behavior of the metric function u by [11,12]

$$M = -\frac{2\pi}{\kappa}\lim_{\rho\to\infty}\rho\frac{\partial u}{\partial\rho} \ . \tag{2.22}$$

In the case of a space-time with n wormholes, this total energy is related to the Euler invariant

$$I = \frac{1}{16\pi}\int_\Sigma d^2x\sqrt{|g|}g^{ij}R_{ij} \qquad (2.23)$$

(where the integral extends over both sheets of the Riemann surface Σ) by[1] [8]

$$2M = \frac{4\pi}{\kappa}(n - 2I) \ . \qquad (2.24)$$

For our σ-model regular wormhole metrics (2.16), the Euler invariant I is, from the first equation (2.4), proportional to the soliton number [11] (degree of the map $\bar{\Sigma} \to S^2$, where $\bar{\Sigma}$ is the closure of Σ), equal to the wormhole number:

$$I = \frac{\kappa\nu^2}{2}\frac{1}{\pi}\int_\Sigma d^2x\, F\left|\frac{\partial\psi}{\partial\zeta}\right|^2 = n\frac{\kappa\nu^2}{2} \ . \qquad (2.25)$$

The total energy given by Eq. (2.24) is therefore the energy of a system of n noninteracting particles:

$$M = n(1-\kappa\nu^2)\frac{2\pi}{\kappa} \ , \qquad (2.26)$$

in accordance with the asymptotic behavior (2.18).

III. WORMHOLE COSMIC STRINGS IN FOUR AND FIVE DIMENSIONS

Our three-dimensional multiwormhole space-times with metric (2.16) can be factored by the z axis, leading to four-dimensional multiwormhole cosmic string space-times with metric:

$$ds^2 = dt^2 - l^2(\cosh X)^{2\kappa\nu^2}\frac{d\zeta\, d\zeta^*}{|\zeta_1^2 - 1|} - dz^2 \ . \qquad (3.1)$$

For an observer at spacelike infinity, this appears to be a static system of n parallel cosmic strings with mass per unit length and longitudinal tension both equal to $(1-\kappa\nu^2)2\pi/\kappa$. However, at short range each individual "cosmic string" turns out to be a cylindrical wormhole leading to the other sheet of three-dimensional space.

The regular metric (3.1) may also be generalized to a hybrid system of n wormhole cosmic strings together with p naked cosmic strings (line singularities) in each sheet of three-dimensional space by choosing appropriately the function $g(Z)$ in Eq. (2.13). These systems survive in the limit of a vanishing σ-model source ($\kappa\nu^2 \to 0$), the naked cosmic strings acting as sources for the multiwormhole configuration; the total energy per unit length (tension) is then

$$M = \frac{2n\pi}{\kappa} + \sum_{i=1}^{p} m_i \qquad (3.2)$$

(where m_i is the mass per unit length of the ith naked cosmic string), so that a necessary condition for three-dimensional space to be open ($M \le 2\pi/\kappa$) is $\sum m_i \le 0$.

For the special value $\kappa\nu^2$=2, Kaluza-Klein cosmic string space-times with negative deficit angle may also be derived from our σ-model wormhole space-times. Making for five-dimensional general relativity with three commuting Killing vectors the ansatz [13,8].

$$ds^2 = g_{ij}(x^k)dx^i dx^j + [2\phi_a(x^k)\phi_b(x^k) - \delta_{ab}]dx^a dx^b \qquad (3.3)$$

(i, j, k=1,2; a, b=3,4,5) where $\boldsymbol{\phi}$ varies on the unit sphere ($\boldsymbol{\phi}^2 = 1$), we reduce the five-dimensional Einstein-Hilbert action to

$$S = \int d^3x \int d^2x\sqrt{|g|}\frac{1}{2}g^{ij}\left[-\frac{1}{\kappa}R_{ij} - \frac{2}{\kappa}\partial_i\boldsymbol{\phi}\cdot\partial_j\boldsymbol{\phi}\right] \ . \qquad (3.4)$$

After a suitable rescaling of $\boldsymbol{\phi}$, this is equivalent to the static restriction of the action (2.2) where the lower sign is taken, and $\kappa\nu^2$=2. Accordingly, to each multiwormhole solution (2.15) and (2.16), there corresponds a solution of sourceless five-dimensional general relativity. The resulting metric (3.3) has the signature $- - - + -$ at spacelike infinity if the stereographic map is chosen to be, for instance,

$$\phi_5 + i\phi_3 = \frac{2\psi}{1+|\psi|^2}, \quad \phi_4 = \frac{1-|\psi|^2}{1+|\psi|^2} \ . \qquad (3.5)$$

When ψ varies from $\psi = \infty$ to ψ=0 (the points at infinity of the two Riemann sheets of Σ), ϕ_4 varies from -1 to $+1$, so that light cones tumble over from future oriented for, e.g., $\psi = \infty$, to space oriented (in the plane x^3, x^5) on the cuts $|\psi|$=1, and to past oriented for ψ=0. This shows that our Kaluza-Klein wormhole space-times are metrical kinks [18], which kink number n.

The Kaluza-Klein projection of the five-dimensional metric (3.3) leads to the four-dimensional metric components, the electromagnetic potentials, and the scalar field

$$\bar{g}_{ab} = \frac{2\phi_a\phi_b}{1-2\phi_5^2} - \delta_{ab}, \quad A_a = \frac{2\phi_a\phi_5}{1-2\phi_5^2}, \quad \sigma = 1 - 2\phi_5^2 \ . \qquad (3.6)$$

The n=1 fields are not axisymmetric. The electric potential A_4 and the metric tensor component $\bar{g}_{34}$ are asymptotically "cylindrical dipole" fields (gradients of the two-dimensional monopole harmonic field $\ln\rho$), while the other fields are asymptotically "cylindrical quadrupole." While the five-dimensional metric (3.3) is everywhere regular, the Kaluza-Klein projection procedure breaks down for $\phi_5^2 = \frac{1}{2}$, i.e., in the case n=1 on the two oval cylinders [which may be thought of as the two ef-

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[1] Equation (2.24), obtained by using the Gauss-Bonnet theorem (see also [14]), is valid for the case where Σ has two asymptotically flat regions with the same angular deficit. In the case of only one asymptotically flat region, the left-hand side of (2.24) should be replaced by M.

fective "sources" for the four-dimensional fields (3.6)] $\cos\theta = \pm\sqrt{2}(1+\rho^2)/3\rho$ (where $\zeta = \rho e^{i\theta}$), inside which the five-dimensional geometry with compactified fifth dimension admits the closed timelike curves x^μ=const ($\mu \neq 5$).

IV. WHEELER WORMHOLES

In the second section we have explained how the spatial part of the metric (2.16) may be maximally extended to a manifold with two asymptotically flat regions connected by n traversable wormholes. Wheeler wormholes [19], by contrast, connect two distant regions of a spatial manifold with only one asymptotically flat region (Fig. 2). It is often taken for granted that two-sided wormhole systems may be transformed to Wheeler wormhole systems by suitably identifying together the two asymptotically flat regions, although it is far from obvious that this can be done consistently. Let us discuss how such an identification, which gives rise to a new maximal extension of the metric (2.16), may be carried out in our model for the case n=2.

For n=2, we may always choose a coordinate system such that the map (2.15) simplifies to

$$\zeta_1 = \zeta^2 - a^2 \tag{4.1}$$

($a > 1$). With this parametrization the metric (2.16) is manifestly invariant under the symmetry $\zeta \to -\zeta$, which exchanges the two cuts. Combining this isometry with the complex inversion $\psi \to 1/\psi$, which exchanges the two Riemann sheets, we obtain an isometry which maps a neighborhood of the left-hand cut in the second Riemann sheet into a neighborhood of the right-hand cut in the first Riemann sheet, and vice versa. This means that we can do away with the second Riemann sheet altogether, so that the spatial sections are obtained from the first Riemann sheet alone by identifying the two cuts. We thus arrive at the reinterpretation of our wormhole pair as a single Wheeler wormhole with only one point at spatial infinity.

This interpretation may be checked out at the geodesic level in the special case $\kappa\nu^2$=0. We have shown that in this case the two-sheeted extension of the metric (2.16) leads to the toroidal space-time (2.19), with geodesics $w - w_0 = \beta t$. A large-circle geodesic Imw=const crosses the two cuts (two opposite small circles of the torus), going, for instance, from the right-hand cut to the left-hand cut in the upper half of the first Riemann sheet, then back to the right-hand cut in the lower half of the second Riemann sheet (Fig. 3). It follows from the identity $\text{sn}(u+2K) = -\text{sn}u$, where $4K$ is the real period of the Jacobi function in Eq. (2.21), that the transformation $\zeta \to -\zeta$, $\psi \to 1/\psi$ maps each point P' of the second half of the geodesic into the symmetrical point P on the first half of the geodesic, i.e., (Fig. 4), maps the torus pinched along two symmetrical circles into a smaller torus pinched along a single circle (the tetracone viewed as its own maximal extension). The topological picture is the same in the noncompact case ($\kappa\nu^2 \geq \frac{1}{2}$), the two-sheeted spatial sections being compactified by adding two symmetrical points at infinity, which are identified together in the Wheeler wormhole interpretation.

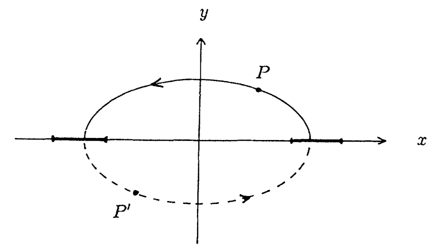

FIG. 3. An Imw=const large circle of the torus crosses the two cuts. The full portion of the geodesic is in the first Riemann sheet, the dashed portion in the second Riemann sheet. The identification of the symmetrical points P' and P results in a Wheeler wormhole.

However, there is a price to pay for this reinterpretation. The complex inversion $\psi \to 1/\psi$ corresponds, from Eq. (2.3), to the transformation of the spherical scalar field

$$(\phi_1, \phi_2, \phi_3) \to (\phi_1, -\phi_2, -\phi_3)\ , \tag{4.2}$$

so that our isometrical identification between the two Riemann sheets would lead to an identification between two inequivalent matter field configurations. In other words, the identification we have just described is possible geometrically, but not, in the case of a σ-model source, physically. Neither is this identification possible for the purely geometrical five-dimensional model of Sec. III ($\kappa\nu^2 = 2$), as it does not lead to an isometry for the five-dimensional geometry (3.3). Indeed, on account of Eqs. (3.5) and (3.6) the transformation $\psi \to 1/\psi$ reverses the electromagnetic potentials, i.e., is equivalent to charge conjugation, so that our hypothetical Kaluza-Klein Wheeler wormhole would not conserve electric charge.

We conclude that our construction of Wheeler wormholes is feasible only in the absence of the σ-model source

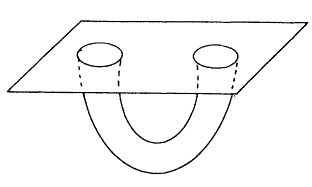

FIG. 2. A Wheeler wormhole.

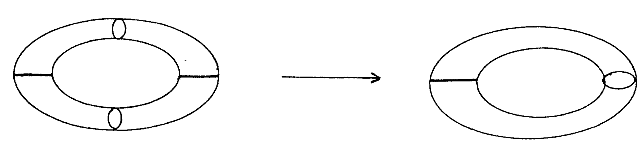

FIG. 4. The isometrical identification of the two halves of the torus results in a smaller torus with only one pinch.

considered in this paper, $\kappa\nu^2=0$. For instance, the four-dimensional metric

$$ds^2 = dt^2 - |g(\zeta)|^2 \frac{d\zeta\, d\zeta^*}{|(\zeta^2 - a^2)^2 - 1|} - dz^2 \qquad (4.3)$$

[where $g(\zeta)$ has p zeros, and $|g(-\zeta)| = |g(\zeta)|$] can thus be maximally extended to a Wheeler wormhole cosmic string generated by p naked cosmic strings of negative masses per unit length (or tensions) m_i. The net mass per unit length of this Wheeler wormhole (defined from the asymptotic deficit angle) is

$$M = \frac{4\pi}{\kappa} + \sum_{i=1}^{p} m_i \ . \qquad (4.4)$$

The simplest case p=1, $g(\zeta) = \text{const} \times \zeta^{-\kappa m/2\pi}$ corresponds to a single naked straight string generating a Wheeler wormhole with a net mass which may be positive provided $-4\pi/\kappa < m \leq -2\pi/\kappa$, and zero for $m = -4\pi/\kappa$ [the flat metric (4.3) is then asymptotically Minkowskian].

V. DISCUSSION

We have constructed regular multiwormhole solutions to an antigravitating σ model in three space-time dimensions, and extended these solutions to cylindrical wormholes in four and five dimensions. We have also discussed how a pair of two-sided wormholes may be reinterpreted as a Wheeler wormhole. However, this reinterpretation is consistent with the field equations only in the absence of the σ-model source (three or four dimensions).

While our emphasis in this paper was on regular solutions, an interesting by-product of our analysis is the construction of cylindrical wormholes in four dimensions with naked cosmic string sources. Visser [3] previously suggested the existence of flat-space wormholes framed by cosmic string configurations. Our construction goes beyond Visser's in two respects. First, our flat-space wormholes are not framed by the naked cosmic strings, which can be far from the wormhole mouths if $a \gg 1$. Second, we are able to construct not only two-sided wormholes, but also a Wheeler wormhole (with only one asymptotically Minkowskian region) generated by a single naked straight string.

Finally, let us mention that our construction of multiwormhole solutions may be straightforwardly extended to the case of a σ model gauged with a Chern-Simons gauge field [9]. This model is defined by the action, which replaces (2.1):

$$S = \int d^3x \sqrt{|g|} \frac{1}{2} \left[g^{\mu\nu} D_\mu \boldsymbol{\phi} \cdot D_\nu \boldsymbol{\phi} + \lambda(\boldsymbol{\phi}^2 - \nu^2) - \mu \frac{1}{\sqrt{|g|}} \varepsilon^{\mu\nu\rho} (\partial_\mu \mathbf{A}_\nu \cdot \mathbf{A}_\rho + \tfrac{1}{3} \varepsilon^{abc} A^a_\mu A^b_\nu A^c_\rho) \right] , \qquad (5.1)$$

where $\varepsilon^{\mu\nu\rho}$ is the antisymmetric symbol, and $D_\mu \phi^a = \partial_\mu \phi^a + \varepsilon^{abc} A^b_\mu \phi^c$ the gauge covariant derivative. As shown in [9] for the case of a flat background space-time, the static finite-energy solutions of the model (5.1) are given, up to a gauge transformation, by

$$\mathbf{A}_i = 0, \quad \mathbf{A}_0 = \eta \frac{1}{\mu} \boldsymbol{\phi} \ , \qquad (5.2)$$

where $\boldsymbol{\phi}(\mathbf{x})$ is a static finite-energy solution of the model (2.1), and $\eta = \pm 1$. This result extends trivially to the case of the gravitating gauged σ model,

$$S = \int d^3x \frac{1}{2} \left\{ \sqrt{|g|} \left[-\frac{1}{\kappa} g^{\mu\nu} R_{\mu\nu} \pm g^{\mu\nu} D_\mu \boldsymbol{\phi} \cdot D_\nu \boldsymbol{\phi} \pm \lambda(\boldsymbol{\phi}^2 - \nu^2) \right] \mp \mu \varepsilon^{\mu\nu\rho} \times (\partial_\mu \mathbf{A}_\nu \cdot \mathbf{A}_\rho + \tfrac{1}{3} \varepsilon^{abc} A^a_\mu A^b_\nu A^c_\rho) \right\} , \qquad (5.3)$$

because the Chern-Simons term in (5.3) is not coupled to gravity, while from Eq. (5.2) $D_\mu \boldsymbol{\phi} = \partial_\mu \boldsymbol{\phi}$, so that the gauged field equations for the gravitational and scalar fields reduce to the ungauged equations (2.4). It follows that [in the case of the lower sign in (5.3)] our multisoliton multiwormhole solutions (2.15) and (2.16) yield multimagnetic vortex configurations [9] for the effective Abelian electromagnetic field

$$\mathcal{F}_{\mu\nu} = \boldsymbol{\phi} \cdot \mathbf{F}_{\mu\nu} - \varepsilon^{abc} D_\mu \phi^a D_\nu \phi^b \phi^c \qquad (5.4)$$

(a four-dimensional equivalent is the Einstein-Yang-Mills-Higgs wormhole monopole constructed in [20]).

[1] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988); M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).

[2] H. G. Ellis, J. Math. Phys. **14**, 104 (1973); K. A. Bronnikov, Acta Phys. Pol. B **4**, 251 (1973); T. Kodama, Phys. Rev. D **18**, 3529 (1978); G. Clément, Gen. Relativ. Gravit. **13**, 763 (1981); A. Chodos and S. Detweiler, *ibid.* **14**, 879 (1982); M. Visser, Nucl. Phys. **B328**, 203 (1989).

[3] M. Visser, Phys. Rev. D **39**, 3182 (1989); J. G. Cramer, R. L. Forward, M. S. Morris, M. Visser, G. Benford, and G. A. Landis, *ibid.* **51**, 3117 (1995).

[4] G. Clément and I. Zouzou, Phys. Rev. D **50**, 7271 (1994).

[5] G. Clément, Ann. Phys. (N.Y.) **201**, 241 (1990).

[6] G. P. Perry and R. B. Mann, Gen. Relativ. Gravit. **24**, 305 (1992).

[7] J. Gott, Astrophys. J. **288**, 422 (1985); W. A. Hiscock, Phys. Rev. D **31**, 3288 (1985); B. Linet, Gen. Relativ. Gravit. **17**, 1109 (1985).

[8] G. Clément, Gen. Relativ. Gravit. **16**, 1095 (1984).

[9] G. Nardelli, Phys. Rev. Lett. **73**, 2524 (1994).

[10] A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975).
[11] G. Clément, Nucl. Phys. **B114**, 437 (1976).
[12] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).
[13] G. Clément, Gen. Relativ. Gravit. **16**, 131 (1984).
[14] A. Comtet and G. Gibbons, Nucl. Phys. **B299**, 719 (1988).
[15] C. Omero and R. Percacci, Nucl. Phys. **B165**, 351 (1980); M. Gell-Mann and B. Zwiebach, Phys. Lett. **141B**, 333 (1984).
[16] M. Lavrentiev and B. Chabat, *Méthodes de la théorie des fonctions d'une variable complexe* (Mir, Moscow, 1972).
[17] A. Staruszkiewicz, Acta Phys. Pol. **24**, 735 (1963).
[18] D. Finkelstein and C. W. Misner, Ann. Phys. (N.Y.) **6**, 230 (1959); J. G. Williams and R. K. P. Zia, J. Phys. A **6**, 1 (1973); J. G. Williams, Lett. Nuovo Cimento **43**, 282 (1985).
[19] J. A. Wheeler, *Geometrodynamics* (Academic, New York, 1962).
[20] G. Clément, Gen. Relativ. Gravit. **13**, 763 (1981).