Geometry of deformations of relativistic membranes

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A kinematical description of infinitesimal deformations of the world sheet spanned in spacetime by a relativistic membrane is presented. This provides a framework for obtaining both the classical equations of motion and the equations describing infinitesimal deformations about solutions of these equations when the action describing the dynamics of this membrane is constructed using *any* local geometrical world sheet scalars. As examples, we consider a Nambu membrane, and an action quadratic in the extrinsic curvature of the world sheet.

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I. INTRODUCTION

A useful approximate description of the dynamics of many physical systems is to model them as relativistic membranes of an appropriate dimension. The construction of the corresponding phenomenological action determining the dynamics of the membrane involves the selection of an appropriate linear combination of the geometrical scalars of its world sheet. At lowest order, this action is proportional to the intrinsic volume of the world sheet, and it has come to be known as the Nambu action. If the approximation stops here, the classical trajectory of the membrane will be an extremal surface of the background spacetime. A large body of information has accumulated on the dynamics of geometrically symmetrical extremal solutions (see, e.g., Ref. [1], for a review in the context of cosmology). To place these solutions in proper context, however, their stability needs to be examined both with respect to classical and quantum mechanical perturbations propagating on the world sheet [2]. What becomes clear when this is attempted for even the simplest models is that a manifestly covariant formalism to describe the evolution of these perturbations which is also independent of the particular symmetry of the membrane is desirable. This problem was approached for Nambu membranes by one of the authors in Ref. [3] and, independently, using similar techniques[1] in Refs. [4,5]. The perturbation is described by a system of coupled linear wave equations, one for the projection of the infinitesimal deformation in the world sheet onto each normal direction, which can be considered as scalar fields living on the world sheet. In this way, a perturbative framework for examining the stability of any system described by the Nambu action is provided.

The analysis presented in Refs. [3–5] was tailored to describe extremal surfaces. For some time, however, it has been realized that extrinsic curvature additions or corrections to the Nambu action can have a dramatic influence on the dynamics on short length scales [7,8]. In particular, when such corrections are introduced the development of cusps or kinks on the membrane appears to be inhibited on these scales. These corrections may arise in a more realistic truncation of the underlying field theory, or may be induced by quantum mechanical fluctuations.

The equations of motion which correspond to a generic action which is quadratic in the extrinsic curvature, are typically hyperbolic equations which are fourth order in derivatives of the embedding functions describing the world sheet. To derive these equations and their subsequent linearizations, one could attempt to imitate the analysis applied earlier to extremal surfaces. However, by following this case by case approach, one can easily lose sight of the fact that there is a solid kinematical structure underpinning these equations which is entirely independent of the underlying dynamics. In this paper we develop such a kinematical framework for describing deformations of an arbitrary world sheet. Relevant earlier investigations in this direction are Refs. [5,8–10]. In Ref. [10] Hartley and Tucker exploit very elegant exterior differential techniques to derive the equations of motion for relativistic membranes. This language could, potentially, provide a very powerful geometrical approach to the description of deformations.

Our effort divides naturally into an examination of the deformation of the intrinsic and of the extrinsic geometry of the world sheet. Once this is done, both the equations of motion, and the equations describing the dynamics of deformations about classical solutions can be constructed in lego block fashion, by assembling the various kinematical ingredients.

The paper is organized as follows. To establish our no-

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[1]The relevant mathematical formalism was developed by mathematicians much earlier in the context of minimal surfaces [6].

tation we begin in Sec. II by summarizing the well-known classical kinematical description provided by the Gauss-Weingarten equations of an embedded timelike world sheet of dimension D in a spacetime of dimension N, in terms of its intrinsic and extrinsic geometries [11–13]. There are two structures which describe the extrinsic geometry. One of these is given by the extrinsic curvatures and is well understood. The other structure, which we call the extrinsic twist potential, only features when the codimension of the world sheet is two or higher. The extrinsic twist plays a subtle role related to the covariance of the description of the geometry under rotations of the normals to the world sheet.

In this same kinematical spirit, in Secs. III and IV we describe the deformation of the world sheet. There are analogues of the Gauss-Weingarten equations which are useful for identifying the structures associated with such deformations. The description of the deformation divides naturally into two parts. The deformation of the intrinsic geometry is very simple to describe. Indeed, for example, the deformations of the world-sheet metric provide a geometrically satisfying definition of the extrinsic curvatures. The description of the deformation of the extrinsic geometry is less simple. One reason for this is because the naive deformation of the structures associated with the extrinsic geometry do not transform covariantly under normal frame rotations. By examining the deformation of the normal vectors (the analogue of the Weingarten equations) we can identify a "connection" which guarantees the manifest covariance of the deformation of such structures under normal frame rotations. It turns out, however, that this connection does not appear in any physical quantity, and thus it does not need to be calculated.

In Sec. V we apply this kinematical framework to some phenomenological actions of physical interest. We consider first the familiar Nambu action, to show how our analysis reproduces the results of Refs. [3,4]. Next, we consider an action quadratic in the extrinsic curvature. We derive the equations of motion, and the equations that describe the dynamics of deformations about classical solutions in the case the background spacetime is Minkowski space. We conclude in Sec. VI with a brief discussion.

We confine our attention to closed membranes without physical boundaries.

II. MATHEMATICS OF THE EMBEDDING OF THE WORLD SHEET

In this section we provide an overview of the well-known mathematical description of the world sheet of a membrane viewed as an embedded surface in a fixed background spacetime [11–13].

Let us consider an oriented *timelike* world sheet m of dimension D described by the embedding

$$x^\mu = X^\mu(\xi^a) \,, \tag{2.1}$$

$\mu = 0,\ldots,N-1$, and $a = 0,\ldots,D-1$, in an N-dimensional spacetime M, endowed with the metric $g_{\mu\nu}$. The D vectors

$$e_a := X^\mu_{,a}\partial_\mu \tag{2.2}$$

form a basis of tangent vectors to m at each point of m. The metric induced on the world sheet is then given by

$$\gamma_{ab} = X^\mu_{,a}X^\nu_{,b}g_{\mu\nu} = g(e_a,e_b) \,. \tag{2.3}$$

The signature of γ_{ab} is taken to be $\{-,+,\ldots,+\}$.

Let n^i denote the ith unit normal to the world sheet, $i = 1,\ldots,N-D$, defined by

$$g(n^i,n^j) = \delta^{ij}, \quad g(e_a,n^i) = 0 \,. \tag{2.4}$$

It is important to emphasize that these equations define the normal vector fields n^i only up to a $\mathrm{O}(N-D)$ rotation, and up to a sign.

Normal vielbein indices are raised and lowered with δ^{ij} and δ_{ij}, respectively, whereas tangential indices are raised and lowered with γ^{ab} and γ_{ab}, respectively.

The collection of vectors $\{e_a, n^i\}$ can be used as a basis for the spacetime vectors appropriate for the geometry under consideration.

We define the world-sheet projections of the spacetime covariant derivatives with $D_a := e^\mu_a D_\mu$, where D_μ is the (torsionless) covariant derivative compatible with $g_{\mu\nu}$. Let us now consider the world-sheet gradients of the basis vectors $\{e_a, n^i\}$, $D_a e_b$ and $D_a n^i$. These spacetime vectors can always be decomposed with respect to the basis;[2]

$$D_a e_b = \gamma_{ab}{}^c e_c - K_{ab}{}^i n_i \,, \tag{2.5a}$$
$$D_a n^i = K_{ab}{}^i e^b + \omega_a{}^{ij} n_j \,. \tag{2.5b}$$

These kinematical expressions, generalizing the classical Gauss-Weingarten equations, describe completely the extrinsic geometry of the world sheet.

The $\gamma_{ab}{}^c$ are the connection coefficients compatible with the world-sheet metric γ_{ab}:

$$\gamma_{ab}{}^c = g(D_a e_b, e^c) = \gamma_{ba}{}^c \,. \tag{2.6}$$

The quantity $K_{ab}{}^i$ is the ith extrinsic curvature of the world sheet:

$$K_{ab}{}^i = -g(D_a e_b, n^i) = K_{ba}{}^i \,. \tag{2.7}$$

The symmetry in the tangential indices of these quantities is a consequence of the integrability of the tangential basis $\{e_a\}$.

The normal fundamental form, or extrinsic twist potential, of the world sheet is defined by

$$\omega_a{}^{ij} = g(D_a n^i, n^j) = -\omega_a{}^{ji} \,. \tag{2.8}$$

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[2]To avoid confusion we adopt the notation ω (adopted by Maeda and Turok in [6]) instead of T as was used in [3] for the twist potential, and Ω (adopted by Carter in [5]) for the corresponding curvature.

In the familiar case of a hypersurface embedding, $D = N - 1$, the extrinsic twist vanishes identically. The geometrical meaning of $w_a{}^{ij}$ can be understood by recalling that there is the freedom to perform local rotations of the normal frame $\{n^i\}$. With respect to the rotation, $n^i \to O^i{}_j n^j$, $\gamma_{ab}{}^c$ transforms as a scalar, and $K_{ab}{}^i$ transforms as a vector. The extrinsic twist potential, $\omega_a{}^{ij}$, transforms as a connection:

$$\omega_a \to O\omega_a O^{-1} + O_{,a}O^{-1} \ . \tag{2.9}$$

As discussed in detail in Ref. [3], for example, it can be considered as the gauge field associated with the normal frame rotation group. It is desirable to implement this covariance in a manifest way. Let ∇_a be the (torsionless) covariant derivative compatible with γ_{ab} induced on m. We introduce a new world-sheet covariant derivative $\tilde{\nabla}_a$ defined on fields transforming as tensors under normal frame rotations as

$$\tilde{\nabla}_a \Phi^i{}_j := \nabla_a \Phi^i{}_j - \omega_a{}^{ik}\Phi_{kj} - \omega_{ajk}\Phi^{ik} \ . \tag{2.10}$$

We also introduce the curvature associated with $\omega_a{}^{ij}$ with

$$\Omega_{ab}{}^{ij} := \nabla_b \omega_a{}^{ij} - \nabla_a \omega_b{}^{ij} + \omega_a{}^{ik}\omega_{bk}{}^j - \omega_b{}^{ik}\omega_{ak}{}^j \ . \tag{2.11}$$

Note that when $D = 1$, $\Omega_{ab}{}^{ij} = 0$ so that $\omega_a{}^{ij}$ is pure gauge, at least locally. We also know that when $D = N - 1$, $\omega_a{}^{ij} = 0$. When $D = N - 2$, the gauge group is O(2) with a single generator. We conclude that the non-Abelian (or nonlinear) character of $\omega_a{}^{ij}$ displayed in an arbitrary dimension does not manifest itself in spacetime dimensions lower than five.

Given some specification of intrinsic and extrinsic geometries Eqs. (2.5) will not generally be consistent with any embedding because $X^\mu(\xi)$ is over specified by these equations. Consistency will require that the intrinsic and extrinsic geometries satisfy the Gauss-Codazzi, Codazzi-Mainardi, and Ricci integrability conditions:

$$g(R(e_b, e_a)e_c, e_d) = \mathcal{R}_{abcd} - K_{ac}{}^i K_{bdi} + K_{ad}{}^i K_{bci} \ , \tag{2.12a}$$

$$g(R(e_b, e_a)e_c, n^i) = \tilde{\nabla}_a K_{bc}{}^i - \tilde{\nabla}_b K_{ac}{}^i \ , \tag{2.12b}$$

$$g(R(e_b, e_a)n^i, n^j) = \Omega_{ab}{}^{ij} - K_{ac}{}^i K_b{}^{cj} + K_{bc}{}^i K_a{}^{cj} \ . \tag{2.12c}$$

We use the notation $g(R(Y_1, Y_2)Y_3, Y_4) = R_{\alpha\beta\mu\nu} Y_2^\alpha Y_1^\beta Y_3^\mu Y_4^\nu$. $R^\alpha{}_{\beta\mu\nu}$ is the Riemann tensor of the spacetime covariant derivative D_μ, whereas $\mathcal{R}^a{}_{bcd}$ is the Riemann tensor of the world-sheet covariant derivative ∇_a. Note that Eq. (2.12c) possesses no nontrivial contractions. In particular, it is vacuous when $D = 1$, and when $D = N - 1$.

These equations can be obtained directly from the Gauss-Weingarten equations, by taking a second spacetime covariant derivative projected onto the world sheet, subtracting the same equation with the indices on the derivatives switched, and exploiting the spacetime Ricci identity.

In de Sitter spacetime,

$$R_{\mu\nu\alpha\beta} = H^2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \ , \tag{2.13}$$

so that

$$g(R(e_b, e_a)e_c, e_d) = H^2(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}) \ . \tag{2.14}$$

The right-hand sides of Eqs. (2.12b) and (2.12c) both vanish.[3]

The appearance of the curvature of the extrinsic twist potential, $\Omega_{ab}{}^{ij}$, in the Ricci integrability condition, Eq. (2.12c), provides us with additional information about the extrinsic twist itself. First, note that, for a given spacetime, Eq. (2.12c) implies that the curvature $\Omega_{ab}{}^{ij}$ is completely determined once the intrinsic geometry and the extrinsic curvatures are specified. In fact, as was emphasized by Carter, $\Omega_{ab}{}^{ij}$ is the conformally invariant trace-free part of the squared extrinsic curvature [13,14]. This equation also provides the necessary and sufficient conditions that the extrinsic twist can be gauged away:

$$g(R(e_a, e_b)n^i, n^j) + K_{ac}{}^i K_b{}^{cj} - K_{bc}{}^i K_a{}^{cj} = 0 \ . \tag{2.15}$$

In particular, in de Sitter spacetime, if all but one K^i_{ab} vanish, then the antisymmetric product of extrinsic curvature tensors vanishes, and the integrability condition is satisfied automatically.

III. DEFORMATIONS OF THE INTRINSIC GEOMETRY

In the previous section we described the characterization of a single embedded surface in spacetime, in terms of its intrinsic and extrinsic geometry.

Let us now consider the neighboring surface described by a deformation of m:

$$x^\mu = X^\mu(\xi^a) + \delta X^\mu(\xi^a) \ .$$

We can decompose the infinitesimal deformation vector field δX^μ with respect to the spacetime basis $\{e_a, n^i\}$, as

$$\delta X = \Phi^a e_a + \Phi^i n_i \ . \tag{3.1}$$

The tangential projection can always be identified with the action of a world-sheet diffeomorphism, $\delta X^\mu = \Phi^a X^\mu_a$, and so will subsequently be ignored. The physically observable measure of the deformation is therefore

[3] From Eq. (2.12a), it is clear that the necessary and sufficient condition that the world sheet will also be a de Sitter space is that

$$K_{ac}{}^i K_{bdi} - K_{ad}{}^i K_{bci} \propto (\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}).$$

provided by the projection of δX^μ orthogonal to m, characterized by the $N-D$ scalar field Φ^i.

Our task will be to express the deformation of the geometrical structures introduced in Sec. II as linear combinations of the scalar fields Φ^i, and their covariant derivatives, $\tilde\nabla_a\Phi^i, \tilde\nabla_a\tilde\nabla_b\Phi^i, \ldots$. We make use of the covariant derivative defined in (2.10), because Φ^i transforms as a vector under normal frame rotations.

In this section we consider the deformation of the *intrinsic* geometry of the world sheet under a deformation in the embedding. The displacement δX^μ in the embedding induces a displacement in the tangent basis $\{e_a\}$. In light of the discussion above, let $\delta = \Phi^i n_i$, and consider the gradients of $\{e_a\}$ along the vector field δ, defined with $D_\delta := \delta^\mu D_\mu$. We can always expand $D_\delta e_a$ with respect to the spacetime basis $\{e_a, n_i\}$, in a way analogous to the Gauss equation (2.4a) as

$$D_\delta e_a = \beta_{ab} e^b + J_{aj} n^j \; . \tag{3.2}$$

Comparison with the Gauss equation shows that the quantity β_{ab}, defined by

$$\beta_{ab} = g(D_\delta e_a, e_b) = \beta_{ba} \; , \tag{3.3}$$

appears in the same position as $\gamma_{ab}{}^c$. The quantities J_{ai} are defined by

$$J_{ai} = g(D_\delta e_a, n_i) \; , \tag{3.4}$$

and appear in the same position as $K_{ab}{}^i$ in the Gauss equation. We note that β_{ab} transforms as a scalar under normal frame rotations, whereas J_{ai} transforms as a vector.

In order to express β_{ab} and J_{ai} in terms of Φ^i and its covariant derivatives it is crucial to recognize that, for all infinitesimal deformations of the world sheet [15], one has

$$D_\delta e_a = D_a \delta \; . \tag{3.5}$$

In words, this equation follows from the equality of the gradient along the deformation vector field δ of the tangential basis $\{e^a\}$, with the changes of $\{e^a\}$ induced by the displacement of the world sheet.

Using Eq. (3.5), it is easy to show that

$$\begin{aligned}\beta_{ab} = g(D_\delta e_a, e_b) = g(D_a\delta, e_b) &= g(D_a n^i, e_b)\Phi_i \\ &= K_{ab}{}^i \Phi_i \; ,\end{aligned} \tag{3.6}$$

$$\begin{aligned}J_{ai} = g(D_\delta e_a, n_i) &= g(D_a\delta, n_i) \\ &= g(D_a n^j, n^i)\Phi_j + \nabla_a \Phi_i \\ &= \tilde\nabla_a \Phi_i \; .\end{aligned} \tag{3.7}$$

Therefore, the gradients along the deformation of the tangential vectors depend on the values of the scalar fields Φ^i, and on their first derivatives along the world sheet.

The deformation in the induced metric on m is just twice β_{ab}:

$$\begin{aligned}D_\delta \gamma_{ab} = D_\delta g(e_a, e_b) &= 2g(e_a, D_\delta e_b) \\ &= 2\beta_{ab} = 2K_{ab}{}^i\Phi_i \; .\end{aligned} \tag{3.8}$$

In fact, this equation encodes the geometrical role of K^i_{ab}. It is half the change induced in the world-sheet metric per unit proper deformation of the world sheet along the ith normal direction.

This is all we need to know about the deformation of the intrinsic geometry, if we are only interested in the deformation of extremal surfaces. However, one might also be interested in more general theories that contain scalars constructed with the world-sheet curvature tensor, $\mathcal{R}^a{}_{bcd}$.

To derive an expression for the deformation of $\mathcal{R}^a{}_{bcd}$, we exploit the Palatini identity, and Eq. (3.8), to write the tensor valued infinitesimal deformation of the world-sheet Christoffel symbol:

$$\begin{aligned}D_\delta \gamma_{ab}{}^c &= \tfrac{1}{2}\gamma^{cd}[\nabla_b(D_\delta\gamma_{ad}) + \nabla_a(D_\delta\gamma_{bd}) - \nabla_d(D_\delta\gamma_{ab})] \\ &= \gamma^{cd}[\nabla_b(K_{ad}{}^i\Phi_i) + \nabla_a(K_{bd}{}^i\Phi_i) \\ &\quad -\nabla_d(K_{ab}{}^i\Phi_i)] \; .\end{aligned} \tag{3.9}$$

The infinitesimal deformation in the world-sheet Riemann tensor then can be simply expressed in terms of world-sheet covariant derivatives of the $D_\delta\gamma_{ab}{}^c$:

$$D_\delta \mathcal{R}^a{}_{bcd} = \nabla_c(D_\delta\gamma_{bd}{}^a) - \nabla_d(D_\delta\gamma_{bc}{}^a) \; . \tag{3.10}$$

We see that it depends on second and first world-sheet derivatives of the scalar fields Φ^i.

The corresponding infinitesimal variations in the Ricci tensor and the scalar curvature are, respectively,

$$D_\delta \mathcal{R}_{ab} = \nabla_c(D_\delta\gamma_{ab}{}^c) - \nabla_b(D_\delta\gamma_{ac}{}^c) \; ,$$

$$D_\delta \mathcal{R} = \nabla^b(D_\delta\gamma_{ab}{}^a) - \nabla_c(\gamma^{ab}D_\delta\gamma_{ab}{}^c) - 2\mathcal{R}_{ab}K^{ab}{}_i\Phi^i \; .$$

Thus, modulo a divergence,

$$D_\delta \mathcal{R} = -2\mathcal{R}_{ab}K^{ab}{}_i\Phi^i \; . \tag{3.11}$$

We also note that

$$D_\delta(\sqrt{-\gamma}\,\mathcal{R}) = -2\mathcal{G}_{ab}K^{ab}{}_i\Phi^i \; , \tag{3.12}$$

where $\mathcal{G}_{ab}$ is the world-sheet Einstein tensor.

This concludes the analysis of the deformation of the intrinsic geometry of the world sheet.

IV. DEFORMATIONS OF THE EXTRINSIC GEOMETRY

The extrinsic geometry is characterized by the extrinsic curvatures $K_{ab}{}^i$ and the extrinsic twist $\omega_a{}^{ij}$. As a preliminary step, let us examine the gradient along the deformation vector field of the normal basis, $D_\delta n^i$, in the same way as we did for the tangent basis. We expand

$$D_\delta n_i = -J_{ai}e^a + \gamma_{ij}n^j \; . \tag{4.1}$$

This equation is the analogue for infinitesimal deformations of the Weingarten equation (2.4b). We note that J_{ai} appears in Eqs. (3.2) and (4.1) in an analogous way to that of $K_{ab}{}^i$ in Eqs. (2.4a) and (2.4b).

The normal projection of $D_\delta n_i$,

$$\gamma_{ij} = g(D_\delta n_i, n_j) = -\gamma_{ji} , \qquad (4.2)$$

is a new structure we have not encountered already. It vanishes on a hypersurface embedding, in the same way that $\omega_a{}^{ij}$ vanishes in the corresponding Weingarten equation. In contrast to J_{ai} and β_{ab}, however, there is no simple relationship between γ_{ij} and deformations of the world sheet analogous to Eqs. (3.6) and (3.7).

The analogy between Eq. (4.1) and the Weingarten equation suggests a role for γ_{ij} analogous to $\omega_a{}^{ij}$. In particular, γ_{ij}, like $\omega_a{}^{ij}$, transforms as a connection under normal frame rotations:

$$\gamma \rightarrow O\gamma O^{-1} + (D_\delta O)O^{-1} . \qquad (4.3)$$

However, by an appropriate choice of $D_\delta O$, it is always possible to gauge γ_{ij} away on the world sheet. Reflecting this fact, as we will demonstrate below, γ_{ij} will never appear explicitly in any physical quantity, although it will show up in intermediate calculations. Nonetheless, we will insist on explicit covariance under normal frame rotations. For this purpose we introduce a covariant deformation derivative as follows:

$$\tilde{D}_\delta \Psi_i = D_\delta \Psi_i - \gamma_i{}^j \Psi_j . \qquad (4.4)$$

Equation (4.1) can then be written in the form

$$\tilde{D}_\delta n_i = -J_{ai} e^a = -(\tilde{\nabla}_a \Phi_i) e^a . \qquad (4.5)$$

A. Deformations of the extrinsic curvature

Let us now evaluate the deformation of the extrinsic curvatures, $\tilde{D}_\delta K_{ab}{}^i$. Using its definition we have that

$$\tilde{D}_\delta K_{ab}{}^i = -g(\tilde{D}_\delta n^i, D_a e_b) - g(n^i, D_\delta D_a e_b) .$$

Using Eq. (4.5) and the Gauss equation (2.5a), the first term on the right-hand side is given by

$$-g(\tilde{D}_\delta n^i, D_a e_b) = \gamma_{ab}{}^c J_c{}^i .$$

The second term on the right-hand side can be developed using the Ricci identity, as

$$\begin{aligned} -g(n^i, D_\delta D_a e_b) &= -g(n^i, R(\delta, e_a)e_b) - g(n^i, D_a D_\delta e_b) \\ &= -g(n^i, R(n_j, e_a)e_b)\Phi^j - D_a g(n^i, D_\delta e_b) + g(D_a n^i, D_\delta e_b) \\ &= -g(n^i, R(n_j, e_a)e_b)\Phi^j - D_a J_b{}^i + \beta_{bc} K_a{}^{ci} + \omega_a{}^{ij} J_{bj} \\ &= -g(n^i, R(n_j, e_a)e_b)\Phi^j - \tilde{\nabla}_a \tilde{\nabla}_b \Phi^i + K_{bcj} K_a{}^{ci} \Phi^j , \end{aligned}$$

where in the last line we have used Eqs. (3.6) and (3.7).

Therefore we find

$$\begin{aligned} \tilde{D}_\delta K_{ab}{}^i = -\tilde{\nabla}_a \tilde{\nabla}_b \Phi^i + [&g(R(e_a, n_j)e_b, n^i) \\ &+ K_{ac}{}^i K^c{}_{bj}]\Phi^j . \end{aligned} \qquad (4.6)$$

Note that the change of the extrinsic curvatures under an infinitesimal deformation of the world sheet involves second derivatives of the scalar fields Φ^i.

The left-hand side of Eq. (4.6) is manifestly symmetric in the indices a and b. The apparent integrability condition on the right-hand side,

$$2\tilde{\nabla}_{[a}\tilde{\nabla}_{b]} = [g(R(e_a, n_j)e_b, n^i) + K_{ac}{}^i K^c{}_{bj} - (a \leftrightarrow b)]\Phi^j , \qquad (4.7)$$

is automatically satisfied as a consequence of the Ricci integrability condition (2.12c). To show this, one needs to use the cyclic Bianchi identities for the spacetime Riemann tensor, $R_{\alpha[\beta\mu\nu]} = 0$, and the identity $2\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}\Phi^i = \Omega_{ab}{}^i{}_j \Phi^j$.

B. Deformations of the extrinsic twist potential

We turn now to the analysis of the deformation of the extrinsic twist $\omega_a{}^{ij}$. Unfortunately, the obvious measure of the deformation $\tilde{D}_\delta \omega_a{}^{ij}$ does not transform covariantly under normal frame rotations. However, by examining $\tilde{D}_\delta \omega_a{}^{ij}$ itself, we can identify the appropriate addition that provides a covariant measure of the deformation.

By definition we have that

$$\tilde{D}_\delta \omega_a{}^{ij} = D_\delta \omega_a{}^{ij} - \gamma^i{}_k \omega_a{}^{kj} - \gamma^j{}_k \omega_a{}^{ik} , \qquad (4.8)$$

where

$$\begin{aligned} D_\delta \omega_a{}^{ij} &= D_\delta g(D_a n^i, n^g) \\ &= g(D_a n^i, D_\delta n^j) + g(D_\delta D_a n^i, n^j) . \end{aligned} \qquad (4.9)$$

The first term on the right-hand side is

$$\begin{aligned} g(D_a n^i, D_\delta n^j) &= K_{ab}{}^i g(e^b, D_\delta n^j) + \omega_a{}^{ik} g(n_k, D_\delta n^j) \\ &= -K_{ab}{}^i J^{bj} + \omega_a{}^{ik}\gamma^j{}_k \\ &= -K_{ab}{}^i \tilde{\nabla}^b \Phi^j + \omega_a{}^{ik}\gamma^j{}_k . \end{aligned} \qquad (4.10)$$

In the second term of (4.9), using the Ricci identity, we have

$$
\begin{aligned}
g(D_\delta D_a n^i, n^j) &= g(R(\delta, e_a) n^i, n^j) + g(D_a D_\delta n^i, n^j) \\
&= g(R(\delta, e_a) n^i, n^j) + D_a g(D_\delta n^i, n^j) - g(D_\delta n^i, D_a n^j) \\
&= g(R(\delta, e_a) n^i, n^j) + \nabla_a \gamma^{ij} + K_{ab}{}^j \tilde{\nabla}^b \Phi^i - \omega_a{}^{jk} \gamma^i{}_k \ ,
\end{aligned} \tag{4.11}
$$

where we have used (4.10) in the last line.

We find then that

$$
\begin{aligned}
\tilde{D}_\delta \omega_a{}^{ij} = &-K_{ab}{}^i \tilde{\nabla}^b \Phi^j + K_{ab}{}^j \tilde{\nabla}^b \Phi^i + \nabla_a \gamma^{ij} \\
&+ g(R(n_k, e_a) n^j, n^i) \Phi^k
\end{aligned} \tag{4.12a}
$$

or

$$
\begin{aligned}
\tilde{D}_\delta \omega_a{}^{ij} - \nabla_a \gamma^{ij} = &-K_{ab}{}^i \tilde{\nabla}^b \Phi^j + K_{ab}{}^j \tilde{\nabla}^b \Phi^i \\
&+ g(R(n_k, e_a) n^j, n^i) \Phi^k \ .
\end{aligned} \tag{4.12b}
$$

This result indicates that the left-hand side is the covariant measure of the deformation of the extrinsic twist potential. In fact, the right-hand side of (4.12b) is manifestly covariant, and thus so also is the left-hand side. Both sides of Eq. (4.12b) are manifestly antisymmetric in the indices i and j. Unlike for the deformation of the extrinsic curvature, Eq. (4.6), here no integrability condition need ever be invoked. We also note the identity

$$
\tilde{D}_\delta \omega_a{}^{ij} - \nabla_a \gamma^{ij} = D_\delta \omega_a{}^{ij} - \tilde{\nabla}_a \gamma^{ij} \ . \tag{4.13}
$$

The deformation of the curvature of the extrinsic twist is then given by

$$
D_\delta \Omega_{ab}{}^{ij} = \tilde{\nabla}_a (D_\delta \omega_b{}^{ij}) - \tilde{\nabla}_b (D_\delta \omega_a{}^{ij}) \ . \tag{4.14}
$$

We note that Eqs. (4.6) and (4.14) are consistent with the integrability condition, Eq. (2.12c).

This concludes the description of the deformation of the extrinsic geometry of the membrane.

C. Deformations of world-sheet derivatives of the extrinsic curvature

In theories involving terms quadratic in the extrinsic curvatures, one needs to evaluate also terms like $\tilde{D}_\delta(\tilde{\nabla}_a K^i_{cd})$, to obtain the linearized equations of motion. One would like to reexpress terms of this form as

$$
\tilde{\nabla}_a (\tilde{D}_\delta K^i_{cd}) + \text{lower order terms} \ ,
$$

and exploit the fact that we already know what $D_\delta K^i_{ab}$ is. This involves the evaluation of the commutator $[\tilde{D}_\delta \tilde{\nabla}_a]$. We will do this for the commutator operating on an arbitrary world sheet and/or normal frame vector A_{bi}:

$$
\begin{aligned}
\tilde{D}_\delta \tilde{\nabla}_a A_{bi} &= D_\delta [D_a A_{bi} - \gamma^c_{ab} A_{ci} - \omega_{ai}{}^j A_{bj}] - \gamma_i{}^j \tilde{\nabla}_a A_{bj} \\
&= D_a D_\delta A_{bi} - D_\delta [\gamma^c_{ab} A_{ci} + \omega_{ai}{}^j A_{bj}] - \gamma_i{}^j \tilde{\nabla}_a A_{bj} \\
&= D_a D_\delta A_{bi} - \gamma^c_{ab} D_\delta A_{ci} - \omega_{ai}{}^j D_\delta A_{bj} + \gamma^c_{ab} D_\delta A_{ci} + \omega_{ai}{}^j D_\delta A_{bj} - D_\delta [\gamma^c_{ab} A_{ci}] - D_\delta [\omega_{ai}{}^j A_{bj}] - \gamma_i{}^j \tilde{\nabla}_a A_{bj} \\
&= \tilde{\nabla}_a D_\delta A_{bi} - (D_\delta \gamma^c_{ab}) A_{ci} - (D_\delta \omega_{ai}{}^j) A_{bj} - \gamma_i{}^j \tilde{\nabla}_a A_{bj} \ .
\end{aligned}
$$

Therefore, we find

$$
[\tilde{D}_\delta, \tilde{\nabla}_a] A_{bi} = -\{(D_\delta \gamma^c_{ab}) \delta^j_i - [(D_\delta \omega_{ai}{}^j) - (\tilde{\nabla}_a \gamma_i{}^j)] \delta^c_b\} A_{cj} \ . \tag{4.15}
$$

Note that on the right-hand side appears the same covariant combination appearing in Eq. (4.13).

A useful application of this equation is given by considering the deformation of the d'Alembertian $\tilde{\Delta} = \tilde{\nabla}^a \tilde{\nabla}_a$. Applying the d'Alembertian to an arbitrary Ψ^i, one finds

$$
\begin{aligned}
\tilde{D}_\delta (\tilde{\Delta} \Psi^i) &= (D_\delta \gamma^{ab}) \tilde{\nabla}_a \tilde{\nabla}_b \Psi^i \gamma^{ab} [\tilde{D}_\delta, \tilde{\nabla}_a] \tilde{\nabla}_b \Psi^i + \gamma^{ab} \tilde{\nabla}_a \{[\tilde{D}_\delta, \tilde{\nabla}_b] \Psi^i\} + \tilde{\Delta}(\tilde{D}_\delta \Psi^i) \\
&= \tilde{\Delta}(\tilde{D}_\delta \Psi^i) - 2\tilde{\nabla}_a [K^{ab}{}_j \Phi^j (\tilde{\nabla}_b \Psi^i)] + [\nabla^a (K_j \Phi^j)](\tilde{\nabla}_a \Psi^i) \\
&\quad + 2K^{ab[i} (\tilde{\nabla}_a \Phi^{k]}) \tilde{\nabla}_b \Psi_k + 2\tilde{\nabla}_a [K^{ab[i} (\tilde{\nabla}_b \Phi^{k]}) \Psi_k] \\
&\quad - g(R(n_j, e^b) n^k, n^i) \Phi^j (\tilde{\nabla}_b \Psi_k) - \tilde{\nabla}_b [g(R(n_j, e^b) n^k, n^i) \Phi^j \Psi_k] \ .
\end{aligned} \tag{4.16}
$$

This expression will be useful in the following section.

V. DYNAMICS: SOME EXAMPLES

In this section we apply the kinematical framework we have developed to the derivation of the equations of motion, and of the linearized equations of motion, for two phenomenological theories of relativistic membranes of physical interest. We begin with the familiar Nambu action. This will allow us to recover the results of Refs. [3,4]. A second example we consider is a correction term quadratic in the extrinsic curvatures.

The Nambu action for a relativistic membrane is given by

$$S_0 = -\sigma \int d^D\xi \sqrt{-\gamma} \; , \tag{5.1}$$

where σ is the membrane tension.

To derive the equations of motion we can describe the deformations of the world sheet with the vector field $\delta = \Phi^i n_i$, because only motions transverse to the world sheet are physical. We have that

$$\delta S_0 = -\sigma \int d^D\xi \sqrt{-\gamma}\, \gamma^{ab} K_{ab}{}^i \Phi_i = 0 \; .$$

Therefore the equations of motion describing an extremal surface are given by

$$K^i = 0 \; , \tag{5.2}$$

and we recover the well-known result that extremal surfaces have vanishing trace of the extrinsic curvatures.

To obtain the linearized equations of motion, consider

$$\begin{aligned}\tilde{D}_\delta K^i &= \gamma^{ab} \tilde{D}_\delta K_{ab}{}^i + K_{ab}{}^i D_\delta \gamma^{ab} \\ &= -\tilde{\Delta}\Phi^i + [g(R(e_a, n_j)e^a, n^i) + K_{ab}{}^i K^{ab}{}_j]\Phi^j \; ,\end{aligned}$$

so that we find the linearized equations of motion in the form

$$\tilde{\Delta}\Phi^i + [K_{ab}{}^i K^{ab}{}_j - g(R(e_a, n_j)e^a, n^i)]\Phi^j = 0 \; , \tag{5.3}$$

in agreement with Eq. (4.1) of the second paper in Ref. [3]. This set of coupled linear equations can be seen as the equations of motion for a multiplet of scalar fields, with a "variable mass" that depends on a particular projection of the curvature of spacetime, and on the extrinsic geometry.

When the projection of the spacetime Riemann tensor vanishes, Eq. (5.3) can be written in the form, which will be used below,

$$\mathcal{O}^i{}_j \Phi^j \equiv \left[\tilde{\Delta}^i_j + K_{ab}{}^i K^{ab}{}_j\right] \Phi^j = 0 \; . \tag{5.4}$$

We now consider a less simple example involving an action quadratic in the extrinsic curvature:

$$S_2 = \alpha \int d^D\xi \sqrt{-\gamma}\, K_i K^i \; , \tag{5.5}$$

where α is a coefficient characterizing the rigidity of the membrane. This action is of some interest in that, modulo the totally contracted Gauss-Codazzi equation, when $D = 2$, and the background geometry is flat, this action represents the most general action of this order in the world-sheet geometry. For an alternative derivation of the equations of motion corresponding to higher order actions of this order, see Ref. [9].

The variation of this action with respect to normal deformations of the world sheet gives

$$\begin{aligned}\delta S_2 = \alpha \int d^D\xi \sqrt{-\gamma} \{ & K_i K^i K^j \Phi_j + 2K_i[-\tilde{\Delta}\Phi^i \\ & + g(R(e_a, n_j)e^a, n^i)\Phi^j - K_{ab}{}^i K_{abj}\Phi^j]\}.\end{aligned}$$

Thus, the Euler-Lagrange equations for S_2 are given by

$$\begin{aligned}&\tilde{\Delta}K^i + [-g(R(e_a, n^j)e^a, n^i) \\ &\qquad + (\gamma^{ac}\gamma^{bd} - \tfrac{1}{2}\gamma^{ab}\gamma^{cd})K_{ab}{}^j K_{cd}{}^i]K_j = 0 \; .\end{aligned} \tag{5.6}$$

Note that extremal surfaces are obvious nontrivial solutions of these equations.

The linearized equations of motion are considerably more complicated than in the case of an extremal surface. For the sake of simplicity we restrict ourselves to the case in which the background spacetime is Minkowski, in order to disregard the spacetime curvature projections. The generalization to an arbitrary background is straightforward. For this case, a lengthy computation, exploiting (4.16), gives the linearized equations of motion in the form

$$\begin{aligned}&-\tilde{\Delta}\tilde{\Delta}\Phi^i - 2K^{ab}{}_j K^j(\tilde{\nabla}_a\tilde{\nabla}_b\Phi^i) + \tfrac{1}{2}K^j K_j \tilde{\Delta}\Phi^i \\ &\quad + (K^i K_j - 2K_{ab}{}^i K^{ab}{}_j)\tilde{\Delta}\Phi^j - 2K^{ab}{}_j(\tilde{\nabla}_a K^j)(\tilde{\nabla}_b\Phi^i) - K_j(\tilde{\nabla}^b K^j)(\tilde{\nabla}_b\Phi^i) \\ &\quad - 2\tilde{\nabla}^c[K_{ab}{}^i K^{ab}{}_j](\tilde{\nabla}_c\Phi^j) + 2K^{abi}(\tilde{\nabla}_a K_j)(\tilde{\nabla}_b\Phi^j) - 2K^{ab}{}_j(\tilde{\nabla}_a K^i)(\tilde{\nabla}_b\Phi^j) \\ &\quad + 2K_j(\tilde{\nabla}^b K^i)(\tilde{\nabla}_b\Phi^j) - \tilde{\Delta}[K_{ab}{}^i K^{ab}{}_j]\Phi^j - (\tilde{\nabla}^a K^i)(\tilde{\nabla}_a K_j)\Phi^j \\ &\quad - 2K^{ab}{}_j(\tilde{\nabla}_a\tilde{\nabla}_b K^i)\Phi^j + 2K_{ab}{}^i K^{bc}{}_k K^a{}_{cj} K^j \Phi^k + \tfrac{1}{2}K_{ab}{}^i K^{ab}{}_k K^j K_j \Phi^k + K^i K_j K^j_{ab} K^{ab}_k \Phi^k \\ &\quad - K_{ab}{}^i K^{abj} K_{cdj} K^{cd}{}_k \Phi^k = 0 \; .\end{aligned} \tag{5.7}$$

The scalar field Φ^i satisfy then a set of coupled fourth-order linear differential equations. It is interesting to note the presence of the square of the world sheet d'Alembertian as the only term that depends only on the intrinsic geometry of the world sheet.

The linearized equations (5.7) are rather complicated. An interesting special case is given by considering linearized about an extremal surface, i.e., setting $K^i = 0$. The equations simplify considerably, and reduce to

$$-(\mathcal{O}^2)^i{}_j \Phi^j = 0 \; , \tag{5.8}$$

where the operator $\mathcal{O}$, defined in Eq. (5.4), is the operator describing small perturbations about an extremal surface induced by the Nambu action. It is remarkable that its square appears here. Thus, linear perturbations about an extremal surface which satisfy Eq. (5.4), continue to be solutions when one takes into account the modifications induced by the action (5.5).

We conclude this section with the following remarks about the deformation connection γ^{ij}. We note that the action must be a scalar under normal frame rotations. Such an action (ignoring possible contractions of world-sheet or spacetime indices) involves an integrand consisting of a totally contracted product of normal frame tensors. The most simple such product is of the form P^iQ_i, where P_i and Q_i are normal frame vectors. We note that

$$\begin{aligned} D_\delta(P^iQ_i) &= P^iD_\delta Q_i + Q^iD_\delta P_i \\ &= P^iD_\delta Q_i + Q^iD_\delta P_i - (\gamma_{ij}+\gamma_{ji})P^iQ^j \\ &= P^i\tilde{D}_\delta Q_i + Q^i\tilde{D}_\delta P_i = \tilde{D}_\delta(P^iQ_i)\ , \end{aligned}$$

where the second line follows from the first line because of the antisymmetry of γ_{ij}. The introduction of the normal frame covariant variation does not complicate the derivation of the Euler-Lagrange equations. Let us now denote these equations by

$$\mathcal{E}_i = 0\ .$$

The perturbed equations of motion are then just

$$\tilde{D}_\delta\mathcal{E}_i = 0\ .$$

Modulo the background equations of motion, these equations reduce to $D_\delta\mathcal{E}_i = 0$. In other words, the connection γ_{ij} never needs to be calculated explicitly. In perturbation theory, the normal frame covariant derivative comes for free. In light of the above remarks, one can safely always set $\gamma_{ij} = 0$.

VI. DISCUSSION

In this paper we have presented a thorough analysis of the kinematics of infinitesimal deformations of the world sheet spanned by a membrane of arbitrary dimension in any spacetime. The physical measure of the deformation is given by the normal components of the displacement vector. These normal components are scalar fields living on the world sheet. The deformation of the intrinsic geometry is straightforward. The deformation of the extrinsic geometry, however, is complicated by the requirement of covariance under normal frame rotations. We introduce a manifestly covariant deformation operator. When we do this the covariant deformations of both the extrinsic curvature and the extrinsic twist curvature are given by second-order hyperbolic partial differential operators acting on the scalar fields.

This kinematical framework is applied in Sec. V to derive the equations of motion and their linearizations both for a system described by the Nambu action and for a system involving an action quadratic in the extrinsic curvature. Specializing to Minkowski spacetime for simplicity we find that the perturbations about an extremal surface are described by a second-order hyperbolic operator for the Nambu dynamics, and by its square for the dynamics described by an action quadratic in the extrinsic curvature.

A more systematic treatment of all low order actions will be addressed in a forthcoming paper [16]. We also leave for a future publication a nonperturbative description of the deformations of a relativistic membrane. This involves a nontrivial generalization of the Raychaudhuri equations for a curve [17].

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