Scalar field potential in inflationary models: Reconstruction and further constraints

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We present quantitative constraints on the scalar field potential for a general class of inflationary models. (1) We first study aspects of the reconstruction of the inflationary potential from primordial fluctuation spectra. Specifically, we consider the case of a pure power law spectrum for the total perturbations (density modes as well as tensor gravitational modes); for this case the reconstruction of the potential can be done semianalytically. We find the solutions and present a series of figures. The figures show how the shape of the potential depends on the shape of the perturbation spectrum and on the relative contribution of tensor modes and scalar density perturbations. When tensor modes provide a significant fraction of the total, the potentials $V(\phi)$ are concave upward; when tensor modes provide a negligible contribution, the potentials are concave downward. (2) We show that the ratio $\mathcal{R}$ of the amplitude of tensor perturbations (gravity wave perturbations) to scalar density perturbations is bounded from above: $\mathcal{R} \leq 6.1$. We also show that the average ratio $\langle\mathcal{R}\rangle$ is proportional to the change $\Delta\phi$ in the field: $\langle\mathcal{R}\rangle \approx 1.6\Delta\phi/M_{\rm Pl}$. Thus, if tensor perturbations are important for the formation of structure, then the width $\Delta\phi$ must be comparable to the Planck mass. (3) We constrain the change ΔV of the potential and the change $\Delta\phi$ of the inflation field during the portion of inflation when cosmological structure is produced. These constraints are then used to derive a bound on the scale Λ of the height of the potential during the portion of inflation when cosmological perturbations are produced; we find $\Lambda \leq 10^{-2} M_{\rm Pl}$. (4) In an earlier paper, we defined a fine-tuning parameter $\lambda_{\rm FT} \equiv \Delta V/(\Delta\phi)^4$ and found an upper bound for $\lambda_{\rm FT}$. In this paper, we find a lower bound on $\lambda_{\rm FT}$. The fine-tuning parameter is thus constrained to lie in the range $4\times10^{-10}\ (\Lambda/10^{17}\ {\rm GeV})^8 \leq \lambda_{\rm FT} \leq 10^{-7}$. (5) We consider the effects of requiring a non-scale-invariant spectrum of perturbations (i.e., with a spectral index $n \neq 1$) on the fine-tuning parameter $\lambda_{\rm FT}$. (6) We also present a very rough argument which indicates that inflation at very low energy scales will encounter some difficulty: the fractional change in the height of the potential during the N=8 e-foldings of structure formation is very small when the energy scale Λ is small. It is then difficult for the potential to drop to (roughly) zero in the remaining e-foldings for a normally shaped potential.

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I. INTRODUCTION

The inflationary universe model [1] provides an elegant means of solving several cosmological problems: the horizon problem, the flatness problem, and the monopole problem. In addition, quantum fluctuations produced during the inflationary epoch may provide the initial conditions required for the formation of structure in the Universe. During the inflationary epoch, the energy density of the Universe is dominated by a (nearly constant) vacuum energy term $\rho \simeq \rho_{\rm vac}$, and the scale factor R of the Universe expands superluminally (i.e., $\ddot{R} > 0$). If the time interval of accelerated expansion satisfies $\Delta t \geq 60R/\dot{R}$, a small causally connected region of the Universe grows sufficiently to explain the observed homogeneity and isotropy of the Universe, to dilute any overdensity of magnetic monopoles, and to flatten the spatial hypersurfaces (i.e., $\Omega \to 1$). In most models, the vacuum energy term is provided by the potential of a scalar field. In this paper, we present constraints on this scalar field potential for a general class of inflationary models. This present work extends the results of a previous paper [2] where we quantified the degree of fine-tuning required for successful inflationary scenarios.

In this paper we focus on rolling models of inflation; the original models of this type were proposed by [3–6]. In this class of models, the effective potential (or free energy) of the inflation field ϕ has a very flat plateau and the field evolves sufficiently slowly for inflation to take place (i.e., the field evolves by "slowly rolling" off the plateau). Many inflationary models which are currently under study are of this type, e.g., new inflation [4,5], chaotic inflation [6], and natural inflation [7]. The evolution of the field ϕ is determined by the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + \frac{dV}{d\phi} = 0 \ , \tag{1.1}$$

where H is the Hubble parameter and V is the potential. The $\Gamma\dot{\phi}$ term describes the decay rate of the ϕ field at the end of inflation (see, e.g., Ref. [8]). In this equation of motion, spatial gradient terms have been neglected (gradients are exponentially suppressed during the inflationary epoch).

In most studies of inflation, the field ϕ is assumed to be slowly rolling during most of the inflationary epoch. The slowly rolling approximation means that the motion of the inflation field is overdamped, $\ddot{\phi}=0$, so that Eq. (1.1) becomes a first order equation; the $\Gamma\dot{\phi}$ term is also generally negligible during this part of inflation. Thus, the motion is controlled entirely by the force term $(dV/d\phi)$ and the viscous damping term $(3H\dot{\phi})$ due to the expansion of the Universe. Near the end of the inflationary epoch, the field approaches the minimum of the potential (i.e., the true vacuum) and then oscillates about it, while the $\Gamma\dot{\phi}$ term gives rise to particle and entropy production. In this manner, a "graceful exit" to inflation is achieved.

For completeness, we note that workable models of inflation which use a first order phase transition have been proposed, notably extended inflation [9] and double field inflation [10]. However, these models require an additional slowly rolling field in order to complete the phase transition. These slowly rolling fields are then subject to constraints as described below.

All known versions of inflation with slowly rolling fields produce density fluctuations, which tend to be overly large unless the potential for the slowly rolling field is very flat. In particular, these models produce (scalar) density fluctuations [11] with amplitudes given by

$$\left.\frac{\delta\rho}{\rho}\right|_{\rm scalar} \simeq \frac{1}{10}\frac{H^2}{\dot{\phi}} , \tag{1.2}$$

where $(\delta\rho/\rho)|_{\rm scalar}$ is the amplitude of a density perturbation when its wavelength crosses back inside the horizon (more precisely, the Hubble length) after inflation, and the right-hand side is evaluated at the time when the fluctuation crossed outside the Hubble length during inflation. The numerical coefficient in Eq. (1.2) is chosen to be in agreement with that of our previous work in Ref. [2]. This discussion applies to any inflationary model which has a slowly rolling field ϕ; the quantum fluctuations in the motion of the field ϕ cause the hypersurface of the phase transition to be nonuniform and result in density perturbations with magnitude given by the above expression.

In addition to the scalar perturbations described above, inflationary models can also produce tensor perturbations (gravity wave perturbations, see Ref. [12]). The size of these perturbations is determined by the overall gravitational wave power, which is usually written in the form

$$P_{\rm GW}^{1/2} = \frac{\sqrt{32\pi}}{2\pi}\frac{H}{M_{\rm Pl}} , \tag{1.3}$$

where the right-hand side is evaluated at the time when the fluctuation crossed outside the Hubble length during inflation (and where $M_{\rm Pl}$ is the Planck mass).

These two types of perturbations, scalar density perturbations and gravity wave perturbations, add in quadrature and produce a total spectrum of primordial perturbations which we denote as $\delta\rho/\rho_{\rm total}$. The total perturbation amplitude is highly constrained by measurements of the anisotropy of the microwave background. On scales of cosmological interest, these measurements [13] indicate that

$$\left.\frac{\delta\rho}{\rho}\right|_{\rm total} \le \delta \approx 2\times 10^{-5} . \tag{1.4}$$

In this expression, the left-hand side represents the total amplitude of perturbations produced by inflation. The right-hand side of Eq. (1.4) represents the experimental measurements (both detections and limits) of the cosmic microwave background. In general, these measurements are a function of the observed size scale (or angular scale); details of scale dependence will be considered later (e.g., see Sec. IV B).

For the general class of inflationary models with slowly rolling fields, the coupled constraints that the Universe must inflate sufficiently and that the density perturbations must be sufficiently small require the potential $V(\phi)$ to be very flat [2,14]. In a previous paper [2], we derived upper bounds on a "fine-tuning parameter" $\lambda_{\rm FT}$ defined by

$$\lambda_{\rm FT} \equiv \frac{\Delta V}{(\Delta\phi)^4} , \tag{1.5}$$

where ΔV is the decrease in the potential $V(\phi)$ during a given portion of the inflationary epoch and $\Delta\phi$ is the change in the value of the field ϕ over the same period. In this paper, we define ΔV and $\Delta\phi$ over the portion of inflation where cosmic structure is produced; as discussed below, this portion of inflation corresponds to the $N \approx 8$ e-foldings which begin roughly 60 e-foldings before the end of inflation. The parameter $\lambda_{\rm FT}$ is the ratio of the height of the potential to its (width)^4 for the part of the potential involved in the specified time period; $\lambda_{\rm FT}$ thus measures the required degree of flatness of the potential. In Ref. [2], we found that $\lambda_{\rm FT}$ is constrained to be very small for all inflationary models which satisfy the density perturbation constraint and which exhibit overdamped motion; in particular, we obtained the bound

$$\lambda_{\rm FT} \le \frac{2025}{8}\delta^2 \approx 10^{-7} . \tag{1.6a}$$

We also showed that if the potential is a quartic polynomial with the quartic term in the Lagrangian written as $\frac{1}{4}\lambda_q\phi^4$, then a bound on $\lambda_{\rm FT}$ implies a corresponding bound on λ_q; specifically,

$$|\lambda_q| \le 36\lambda_{\rm FT} . \tag{1.6b}$$

Thus, the bound of Eq. (1.6a) implies that the quartic coupling constant must be extremely small.

In this paper we continue a quantitative study of the constraints on the scalar-field potential for models of inflation that have a slowly rolling field. In the first part of this paper, we consider the reconstruction of the inflationary potential for given primordial density fluctuation spectra. This reconstruction process has also been considered by many recent papers [15–17]. In this paper, we show that for the particular case of total perturbation spectra which are pure power law, the rconstruction of the inflationary potential can be done semianalytically and we find the corresponding solutions [see Eq. (2.14)]. For the more general case, we show how constraints on the density fluctuation spectra imply corresponding constraints on the potential.

Our results show how the shape of the potential depends on the perturbation spectrum and on the relative contribution of tensor modes and scalar perturbations (see Figs. 1–5). For the case in which tensor perturbations produce a substantial contribution to the total (e.g., in Fig. 1), the potentials $V(\phi)$ are concave upward for all of the spectral indices $n = 0.5 - 1$ considered here. For the opposite case in which tensor modes are negligible (e.g., in Fig. 4), the potentials are concave downward and somewhat like the cosine potential used in models of natural inflation [7]. Figure 5 shows a cosine potential which has been fit to the reconstructed potential for a particular case with little contribution from tensor modes (see Sec. II). Thus, for perturbation spectra with little contribution from tensor modes (and moderate departures from scale invariance), the reconstructed potential looks very much like a cosine potential.

In the next part of this paper, we show that the ratio $\mathcal{R}$ of the amplitude of tensor perturbations (gravity wave perturbations) to scalar density perturbations is bounded from above; we find that $\mathcal{R} \leq 6.1$. Thus, tensor perturbations cannot be larger than scalar perturbations by an arbitrarily large factor. We also show that the average $\langle\mathcal{R}\rangle$ of this ratio is proportional to the change $\Delta\phi$ in the field; in particular, we find that $\langle\mathcal{R}\rangle \approx 1.6\Delta\phi/M_{\rm Pl}$. Thus, if tensor perturbations are important for the formation of cosmological structure, then the width $\Delta\phi$ must be comparable to the Planck mass.

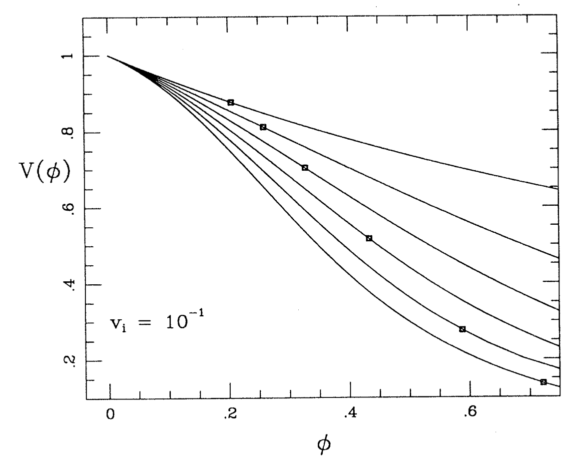

FIG. 1. Reconstructed inflationary potential for the case where tensor perturbations provide 31% of the total at x=0 (i.e., $v_i \equiv [\delta^2_{\rm GW}/\delta^2_{\rm tot}]_{x=0} = 10^{-1}$). [Note that the parameter x characterizes the number of e-foldings subsequent to the epoch x=0, which occurs ~60 e-foldings before the end of inflation; at x=0 cosmological structure on the scale of our horizon was produced.] The various curves are for indices n=0.5, 0.6, 0.7, 0.8, 0.9, and 1.0 (from bottom to top) The open symbols denote the epoch at which galaxy-sized perturbations leave the horizon during inflation. The potential V is normalized to unity at the point x=0; the field ϕ is presented in units of $M_{\rm Pl}$.

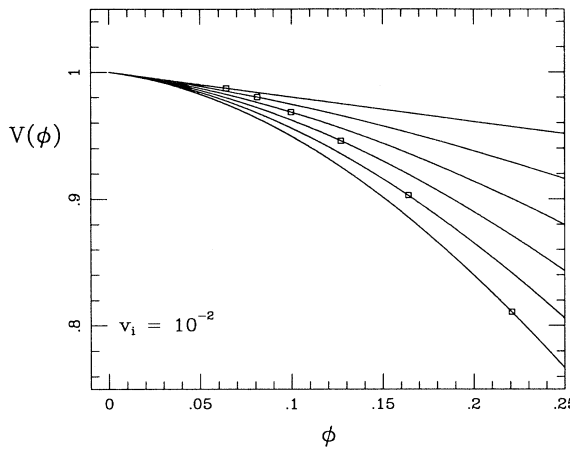

FIG. 2. Reconstructed inflationary potential for the case where tensor perturbations provide 10% of the total at x=0 (i.e., $v_i = 10^{-2}$). Other parameters and notation are as in Fig. 1.

Next, we present further constraints on the inflationary potential. In particular, we constrain both ΔV and $\Delta\phi$ individually. We show that both upper and lower bounds exist for $\Delta\phi$ and for ΔV [see Eqs. (3.21), (3.22), (3.36), and (3.40)]. In addition, these constraints are used to derive a bound on the scale Λ, i.e., the scale of the height of the potential during the portion of inflation when cosmological perturbations are produced; we obtain the bound $\Lambda \leq 10^{-2}M_{\rm Pl}$. Thus, the final ~60 e-foldings of inflation must take place after the grand unified theory (GUT) epoch. This bound on Λ is comparable to those found previously [18–20].

Finally, we consider additional bounds on the fine-tuning parameter $\lambda_{\rm FT}$. We find a lower bound on $\lambda_{\rm FT}$ [see Eq. (4.1)]. We also consider the effects of requiring a non-scale-invariant spectrum of perturbations (i.e., perturbations with spectral index $n \neq 1$) on the fine-tuning

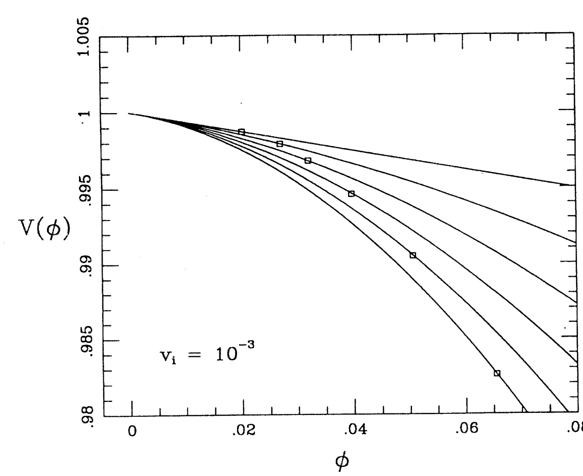

FIG. 3. Reconstructed inflationary potential for the case where tensor perturbations provide 3.1% of the total at x=0 (i.e., $v_i = 10^{-3}$). Other parameters and notation are as in Fig. 1.

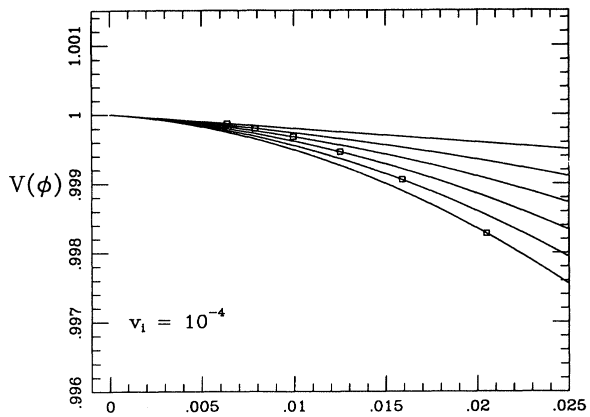

FIG. 4. Reconstructed inflationary potential for the case where tensor perturbations provide 1% of the total at x=0 (i.e., v_i = 10^{-4}). Other parameters and notation are as in Fig. 1.

parameter $\lambda_{\rm FT}$. We show that for $n < 1$, the bound on the fine-tuning parameter $\lambda_{\rm FT}$ becomes more restrictive than the n=1 case (which is effectively the case considered in Ref. [2]) by a factor of 2–5.

The constraints presented in this paper apply to inflationary models involving one or more scalar fields that are minimally coupled to gravity, and which satisfy three conditions. First, we require that the evolution during the relevant time period satisfies the density perturbation constraint, which can be written in the form

$$H^2/\dot{\phi} \leq 10\delta \ . \tag{1.7}$$

Second, we assume that during the early stages of inflation, the evolution of the field ϕ is overdamped so that the $\ddot{\phi}$ term of Eq. (1.1) is negligible (along with the $\Gamma\dot{\phi}$ term). This assumption leads to the simplified equation of motion

$$3H\frac{d\phi}{dt} = -\frac{dV}{d\phi} \ . \tag{1.8}$$

The consistency of neglecting the $\ddot{\phi}$ term implies a constraint on the potential of the form

$$\left|\frac{d}{dt}\left(\frac{1}{3H}\frac{dV}{d\phi}\right)\right| \leq \left|\frac{dV}{d\phi}\right| , \tag{1.9}$$

which we refer to as the overdamping constraint. This constraint is often called the slowly rolling condition, but we follow Ref. [2] and avoid this phrase because it suggests a constraint on $\dot{\phi}$ [see Eq. (1.10) below] rather than $\ddot{\phi}$. Notice that this constraint is a *necessary* but not a *sufficient* condition for the $\ddot{\phi}$ term to be neglected. Third, we also require that the ϕ field rolls slowly enough that its kinetic energy contribution to the energy density of the Universe is small compared to that of the vacuum. Thus, the following constraint must be satisfied during the inflationary period:

$$\frac{1}{2}\dot{\phi}^2 \leq V_{\rm tot} \ , \tag{1.10}$$

where $V_{\rm tot}$ is the *total* vacuum energy density of the Universe. Notice that additional fields (i.e., in addition to the inflation field ϕ) can be present during the inflationary epoch. Thus, the total vacuum energy density $V_{\rm tot}$ can, in general, include contributions from other scalar field potentials in addition to $V(\phi)$. The constraint (1.10) was not explicitly used in our previous work [2].

Throughout we will assume that the energy density of the Universe during inflation is dominated by the total vacuum energy. Thus the Hubble parameter is given by

$$H^2 = \frac{8\pi}{3}\frac{V_{\rm tot}}{M_{\rm Pl}^2} \ . \tag{1.11}$$

Possible alternatives to vacuum-dominated inflation, such as "gravity-driven" or "accelerated inflation" [21], have been proposed; however, we do not consider these possibilities here.

We have introduced several different potentials and energy scales and it is important to maintain the distinctions between them. The quantity $V(\phi)$ is the potential of the inflationary field ϕ and varies with time as ϕ evolves. The quantity $V_{\rm tot}$ is the total vacuum energy density of the Universe and also varies with time. The quantity ΔV is the *change* in the potential $V(\phi)$ over the portion of inflation when cosmological perturbations are produced; thus, ΔV is a given constant for a given inflationary scenario. Finally, we have defined Λ to be the energy scale of inflation when cosmological perturbations are produced; to be specific, we define

$$\Lambda^4 \equiv V_{\rm tot}|_{60} \ , \tag{1.12}$$

where the right-hand side denotes that $V_{\rm tot}$ is evaluated when the present-day horizon scale left the horizon

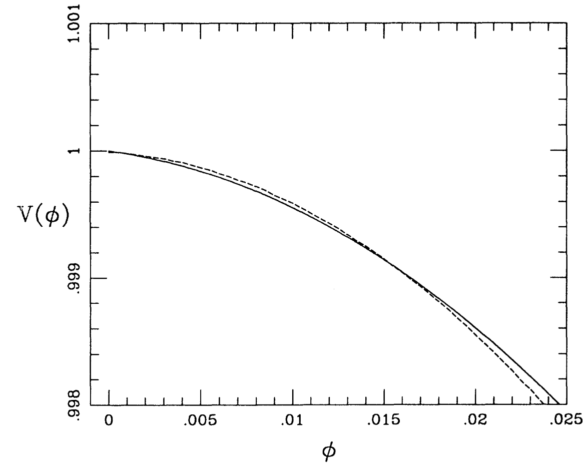

FIG. 5. Comparison of reconstructed inflationary potential and a cosine potential. The reconstructed potential was obtained using n=0.6 and v_i = 10^{-4} (tensor perturbations initially produce 1% of the total). The fit was obtained by constraining the cosine curve to agree with the reconstructed potential at the end points x=0 and x=1. The potential V is normalized to unity at the point x=0; the field ϕ is presented in units of $M_{\rm Pl}$.

during inflation (this event generally occurs about 60 e-foldings before the end of inflation).

This paper is organized as follows. In Sec. II we reconstruct the inflationary potential for the case in which the total primordial spectrum of density perturbations (the sum of both scalar and tensor contributions) is a power law. We also show how constraints on the primordial spectrum lead to corresponding constraints on the potential. In Sec. III we find constraints on the scalar field potential for a class of "standard" inflationary models, i.e., models involving any number of scalar fields that are minimally coupled to gravity, and that obey the density perturbation and overdamping constraints. In particular, we constrain $\Delta\phi$ and ΔV individually; we also derive a relationship between the width $\Delta\phi$ and the average ratio of the amplitude of tensor perturbations to scalar perturbations. We derive further constraints on the fine-tuning parameter $\lambda_{\rm FT}$ in Sec. IV; we show that $\lambda_{\rm FT}$ is also bounded from below and we show the effects of non-scale-invariant spectra of density perturbations. Finally, we conclude in Sec. V with a summary and discussion of our results.

II. RECONSTRUCTION OF INFLATIONARY POTENTIALS

In this section, we consider the problem of reconstructing the scalar field potential. As noted by many authors in the recent literature [16,17], knowledge of both the scalar perturbations and the tensor perturbations allows one to reconstruct a portion of the scalar field potential that gives rise to inflation. In this paper, we find a semianalytic solution for the potential for the case of (total) perturbation spectra which are pure power laws [see Eq. (2.14)].

To preview some of the most interesting results of this section, we refer the reader to Figs. 1–5. There we show how the shape of the potential depends on the perturbation spectrum and on the relative contribution of tensor modes. For example, when tensor modes provide a significant fraction of the total, the potentials $V(\phi)$ are concave upward for all spectral indices considered in this paper $n = 0.5-1$ (e.g., Fig. 1). For the opposite case where tensor modes provide a negligible contribution to the total, the potentials are concave downward (e.g., Fig. 4). For this latter case, the potential shape is well approximated by a cosine (see Fig. 5) as in the model of natural inflation [7].

For the rest of this section we show how these results are obtained. In addition, we comment on their usefulness for the case when the exact power law index is not known, but instead there is a range consistent with the existing status of observations. Although our knowledge of the true primordial spectrum of perturbations is not exact, constraints may be placed on the spectrum; we show how constraints on the primordial power spectrum produce corresponding constraints on the scalar field potential.

The relevant time variable for an inflationary epoch is the number of e-foldings since the beginning of the epoch. Throughout this paper, we adopt a new time variable x defined by

$$dx \equiv \frac{H\,dt}{N} \,, \tag{2.1}$$

where N is the number of e-foldings of the scalar factor during the time interval of interest, i.e.,

$$N = \int H\,dt \,, \tag{2.2}$$

where the limits of integration correspond to some portion of the inflationary epoch. As discussed below, we generally take N=8 and hence N is not the total number ($\sim$60) of e-foldings required for successful inflation. In this paper, we are mostly interested in the portion of inflation during which cosmological structure is produced. Such structure spans physical size scales (at the present epoch) in the range 3000 Mpc (the horizon size) down to about 1 Mpc (the size scale corresponding to a galactic mass). This range spans a factor of 3000 in physical size and thus corresponds to $N = \ln[3000] \approx 8$ e-foldings of the inflationary epoch [8]. The variable x defined above ranges from 0 to 1 during the relevant time period. The point x=0 corresponds to the time during inflation when perturbations on the physical size scale of the horizon at the present epoch (i.e., 3000 Mpc) were produced. Keep in mind that many additional e-foldings of the scalar factor could have taken place before x=0. If one considers $N > 8$, the constraints get tighter.

The scalar perturbations [see Eq. (1.2)] will generally be some function of the variable x introduced above: i.e.,

$$\left.\frac{\delta\rho}{\rho}\right|_{\rm scalar} = \frac{1}{10}\frac{H^2}{|\dot\phi|} \equiv \delta_S(x) \,. \tag{2.3a}$$

Similarly, the tensor perturbations (gravity wave perturbations) can be written in the form

$$\left.\frac{\delta\rho}{\rho}\right|_{\rm GW} = \frac{\sqrt{2\pi}}{2}\frac{H}{M_{\rm Pl}} \equiv \delta_{\rm GW}(x) \,, \tag{2.3b}$$

where the right-hand side is some function of x. We note that the expressions used here are correct only to leading order in the "slow-roll" approximation. Although higher order corrections to these expressions have been calculated [22], the leading order terms are adequate for our purposes. We also note that the numerical coefficients in Eqs. (2.3a) and (2.3b) are chosen so that the ratio of the tensor contribution to the scalar contribution is in agreement with the ratio given in Refs. [16] and [23].

For the particular case in which inflation arises from a single scalar field ϕ with a potential $V(\phi)$, we have

$$H^2 = (8\pi/3)V(\phi)/M_{\rm Pl}^2 \,. \tag{2.4}$$

Then we can write the above expressions in terms of the potential. The tensor modes are related to the potential through Eq. (2.3b), which can be written as the expression

$$V(x) = \frac{3}{4\pi^2} M_{\rm Pl}^4 \delta_{\rm GW}^2(x) \ . \tag{2.5}$$

Similarly, the scalar modes are related to the potential through Eq. (2.3a), which can be written using Eqs. (1.8) and (2.4) in the form

$$-\frac{1}{V^2}\frac{dV}{dx} = \frac{16\pi^2 N}{75} M_{\rm Pl}^{-4} \delta_S^{-2}(x) \ . \tag{2.6}$$

The combination of these two equations thus implies the simple differential equation

$$-\frac{1}{\delta_{\rm GW}^3}\frac{d\delta_{\rm GW}}{dx} = \frac{2N}{25}\delta_S^{-2} \equiv C\delta_S^{-2} \ , \tag{2.7}$$

where we have defined the constant $C = 2N/25$ [24].

As indicated previously, the two types of perturbations add in quadrature, so that the total spectrum of primordial perturbations, which we denote as $\delta_{\rm tot}$, can be written as the sum

$$\delta_{\rm tot}^2(x) = \delta_S^2(x) + \delta_{\rm GW}^2(x) \ . \tag{2.8}$$

If we assume that the total spectrum $\delta_{\rm tot}(x)$ is a known function, we can then combine Eqs. (2.7) and (2.8) to obtain a single differential equation for $\delta_{\rm GW}$:

$$[\delta_{\rm GW}^2 - \delta_{\rm tot}^2]\frac{d\delta_{\rm GW}}{dx} = C\delta_{\rm GW}^3 \ . \tag{2.9}$$

Thus, if the primordial spectrum $\delta_{\rm tot}$ were known exactly, we could simply solve the above differential equation for $\delta_{\rm GW}(x)$ and then solve for the scalar field potential $V(x)$ [25]. Notice that we must also specify the initial condition $\delta_{\rm GW}(0)$, i.e., the amplitude of the tensor modes at x=0. Since $\phi(x)$ is directly calculable from the equation of motion once we know $V(x)$, the usual form of the potential $V(\phi)$ as a function of the scalar field can also be obtained. This hypothetical "solution" for the potential is correct to leading order in the slow-roll approximation (see also Refs. [16,17]).

One problem with the above discussion is that we do not know the true primordial spectrum $\delta_{\rm tot}$. However, the total spectrum of perturbations is often assumed to be a power law in wave number k; i.e., the amplitudes of the perturbations vary with physical length scale L according to the law

$$\delta_{\rm tot} \equiv [\delta_S^2 + \delta_{\rm GW}^2]^{1/2} \sim L^{(1-n)/2} \ . \tag{2.10}$$

The parameter n is the power-law index of the primordial power spectrum:

$$P(k) \sim |\delta_k|^2 \sim k^n \ , \tag{2.11}$$

where k is the wave number of the perturbation [26–28]. Notice that the left-hand side of Eq. (2.10) is to be evaluated when the perturbation of length scale L enters the horizon. Notice also that $n = 1$ corresponds to a scale-invariant spectrum and that $n < 1$ corresponds to spectra with more power on large length scales. We stress that a considerable amount of processing is required to convert the primordial spectrum into observable quantities and such work is now being vigorously pursued [29]; this transformation between the primordial spectrum and actual observed quantities is generally very complicated and model dependent.

For now we take the exponent n as given and proceed to a reconstruction of the potential. Subsequently we will consider the situation when the primordial spectrum is not entirely known and not necessarily a pure power law. We define a new function

$$v(x) \equiv \delta_{\rm GW}^2/\delta_{\rm tot}^2 \ , \tag{2.12}$$

where $v(x) \le 1$ by definition. Notice that for the case in which n=1 (corresponding to a scalar-invariant perturbation spectrum), $\delta_{\rm tot}$ is independent of length scale and thus the function v is proportional to the potential V [see Eq. (2.5)]. In terms of this new function v, the differential equation (2.9) becomes

$$\frac{v-1}{2v}\frac{dv}{dx} = Cv + \alpha(v-1) \ . \tag{2.13}$$

Here we have defined $\alpha = N(1-n)/2$. For the scale invariant case of n=1 we have α=0, while for $n < 1$ we have $\alpha > 0$. For the case of α=const, Eq. (2.13) can be integrated to obtain the solution

$$\frac{-C}{C+\alpha}\ln\left[\frac{(C+\alpha)v-\alpha}{(C+\alpha)v_i-\alpha}\right] + \ln(v/v_i) = 2\alpha x \ , \tag{2.14}$$

where v_i denotes the function $v(x)$ evaluated at x=0. Keep in mind that v_i represents the ratio (squared) of the amplitude of tensor modes to the total amplitude of density fluctuations.

We can use the above results to reconstruct the inflationary potential as follows. Once the initial condition (i.e., v_i) is specified, Eq. (2.14) provides an implicit, but analytic, solution for $v(x)$. We can then use Eq. (2.12) to find $\delta_{\rm GW}^2$ for a given total spectrum $\delta_{\rm tot}$. In this work, we have taken the total spectrum to be pure power law, but we have not specified the overall normalization; we thus solve for only the shape of the potential. However, this normalization can be obtained from measurements of the Cosmic Background Explorer (COBE) [13]. Given the quantity $\delta_{\rm GW}^2$, we use Eq. (2.5) to find the potential as a function of x. Notice that we have found $V(x)$ and not $V(\phi)$. In order to make this conversion, we must also solve the equation of motion for $\phi(x)$; this equation is written in integral form in Eq. (3.14).

We have performed the reconstruction process outlined above for varying values of the initial ratio v_i and for varying choices of the index n. The results are shown in Figs. 1–4. For each choice of v_i (which determines the relative amplitude of the tensor modes), the figures show the shape of the resulting potentials $V(\phi)$ for n=0.5, 0.6, 0.7, 0.8, 0.9, and 1.0. The open symbols represent x=1, i.e., the epoch at which galaxy sized perturbations left the horizon during inflation. Keep in mind that this reconstruction process only contains information about the potential during the $N = 8$ e-foldings when structure-forming perturbations are produced. This procedure says nothing about the potential at subsequent epochs.

The results shown in Figs. 1–4 show interesting general trends. For the case in which tensor perturbations produce a substantial contribution to the total (e.g., in Fig. 1), the potential $V(\phi)$ are concave upward. For the opposite case in which tensor modes are negligible (e.g., in Fig. 4), the potentials are concave downward and somewhat reminiscent of a cosine potential. To follow up on this latter issue, we fit a cosine potential to the reconstructed potential for the specific case n=0.6 and $v_i = 10^{-4}$. The result is shown in Fig. 5. Thus, for perturbation spectra with moderate departures from scale invariance and little contribution from tensor modes, the reconstructed potential looks very much like a cosine. This type of potential is used in the model of natural inflation [7] and was first suggested for reasons of technical naturalness. In particular, the required small parameter $\lambda_{\rm FT}$ [see Eqs. (1.5) and (1.6)] occurs naturally in this model.

As mentioned above, the transformation between actual observed quantities such as the microwave anisotropy and the primordial spectrum is complicated and model dependent. Thus, a definitive prediction for $\delta_{\rm tot}(x)$ may be difficult to obtain in the near future. However, the observations can be used to determine constraints on the spectrum $\delta_{\rm tot}(x)$. For example, an analysis of the observations may imply that the true spectrum lies within some range of power laws. We can then obtain the range of possibilities for the potential from the figures by restricting ourselves to the curves corresponding to the given range of power laws.

For example, we might reasonably require that the amplitude of the perturbations does not change too much with varying length scale (wave number). In the present formulation, this statement takes the form

$$A \leq n(x) \leq B \ , \tag{2.15}$$

where n is the index of the perturbation spectrum as in Eqs. (2.10) and (2.11). For the special case in which the primordial spectrum is a pure power law, the index n is a constant independent of x and is related to the parameter α through the identity $\alpha = N(1-n)/2$. In general, the index n will not be constant, but we expect that the function $n(x)$ will be a slowly varying function.

Constraints of the form (2.15) imply corresponding constraints on the potential (for a given set of initial conditions). If the index n is constrained as in Eq. (2.15), then the amplitude $\delta_{\rm GW}$ is constrained to lie between the solutions found with $n = A$ and $n = B$. Since the potential is proportional to $\delta_{\rm GW}^2$, the potential will be similarly constrained (see Figs. 1–4). In other words, the potential is allowed to be in the range of curves corresponding to the appropriate range of indices n in the figures.

In this section, we have considered the reconstruction of the inflationary potential. Building on previous work by several groups [16,17], we found a semianalytic solution for the potential for pure power-law spectra, and plotted our results in Figs. 1–5.

III. CONSTRAINTS ON THE HEIGHT AND WIDTH OF INFLATIONARY POTENTIALS

In this section we present a series of constraints on the scalar field potential. These bounds apply to all inflationary models which belong to the general class of models which obey the density perturbation constraint (1.7), the overdamping constraint (1.9), and the condition of vacuum energy domination (1.10).

A. Relationship between the width of the potential and the relative amplitude of tensor perturbations

In this subsection, we derive a relationship between the width of the scalar field potential and the relative amplitude of tensor perturbations (i.e., gravity wave perturbations). The ratio $\mathcal{R}$ of the amplitude of tensor perturbations [Eq. (2.3b)] to the scalar perturbations [Eq. (2.3a)] can be written in the form

$$\mathcal{R} = 5\sqrt{2\pi}\frac{|\dot{\phi}|}{M_{\rm Pl}H} \ . \tag{3.1}$$

We note that the numerical coefficient in Eq. (3.1) is chosen so that the ratio of tensor to scalar fluctuations agrees with the results of Refs. [16] and [23].

The width of the potential is defined to be the change $\Delta\phi$ in the scalar field during the portion of inflation when cosmic structure can be produced. This width can be written in the form

$$\Delta\phi = \int |\dot{\phi}|dt = N\int_0^1 \frac{|\dot{\phi}|}{H}dx \ . \tag{3.2}$$

Keep in mind that $\Delta\phi$ is the change in the inflation field during the N=8 e-foldings during which cosmic structure is produced and *not* the total change in ϕ over the entire inflationary epoch.

Comparing Eqs. (3.1) for the ratio $\mathcal{R}$ with Eq. (3.2) for $\Delta\phi$, we discover the simple relationship

$$\langle\mathcal{R}\rangle = \frac{5\sqrt{2\pi}}{N}\frac{\Delta\phi}{M_{\rm Pl}} \ , \tag{3.3}$$

where we have defined $\langle\mathcal{R}\rangle$ to be the average value of $\mathcal{R}$ over the relevant time period, i.e.,

$$\langle\mathcal{R}\rangle \equiv \int_0^1 \mathcal{R}\, dx \ . \tag{3.4}$$

Thus, for the $N \approx 8$ e-foldings where density fluctuations of cosmological interest can be produced, we obtain

$$\langle\mathcal{R}\rangle \approx 1.6\Delta\phi/M_{\rm Pl} \ . \tag{3.5}$$

One important implication of this result can be stated as follows. If tensor perturbations play a major role in the formation of structure, then the width $\Delta\phi$ during the appropriate part of inflation must be comparable to the Planck mass $M_{\rm Pl}$. Notice that since this argument applies only to the average value of the ratio $\mathcal{R}$, it is logically possible for tensor modes to be significant at some particular length scale, even though the average $\langle\mathcal{R}\rangle$ is small. However, as we show next, the instantaneous value of $\mathcal{R}$ is also constrained.

We now show that the ratio $\mathcal{R}$ is bounded from above. If we use the constraint (1.10) that the kinetic energy does not dominate the vacuum energy, $\frac{1}{2}\dot{\phi}^2 \le V_{\rm tot}$, and the value of the Hubble parameter $H^2 = (8\pi/3)V_{\rm tot}/M_{\rm Pl}^2$, Eq. (3.1) implies that

$$\mathcal{R} \le 5\sqrt{3/2} \approx 6.1 \ . \qquad (3.6)$$

Although scalar perturbations can be larger than tensor modes by an arbitrarily large factor, the converse is not true: tensor modes can be *at most* a factor of $\sim$6 times larger than the scalar contribution.

B. Basic definitions and formulation

Our goal is to find general constraints on the inflationary potential (see Secs. III C and III E). For these calculations, we introduce the formulation described below. To begin, we introduce the notation

$$F(x) \equiv -\frac{dV}{d\phi} \ , \qquad (3.7)$$

where F represents a force. In terms of this notation, the overdamping constraint is written as

$$\left| \left(\frac{F}{H}\right)^{-1} \frac{d}{dx}\left(\frac{F}{H}\right) \right| \le 3N \ , \qquad (3.8)$$

and the density perturbation constraint is

$$3H^3/F \le 10\delta \ . \qquad (3.9)$$

Using the equation of motion (1.8) and the relation (1.11) which defines the Hubble parameter, we can write the condition that the Universe is dominated by potential energy rather than kinetic energy [Eq. (1.10)] in the form

$$F < \left(\frac{27}{4\pi}\right)^{1/2} H^2 M_{\rm Pl}, \qquad (3.10)$$

where $M_{\rm Pl}$ is the Planck mass.

For the sake of definiteness, we assume that the density perturbation constraint of Eq. (3.9) is saturated at the epoch x=0, i.e., when the present-day horizon scale left the horizon during inflation (notice that this assumption and the following definitions are *not* used in the reconstruction of the potential as described in Sec. II). Physically, this assumption means that scalar density perturbations are responsible for the observed fluctuations in the cosmic microwave background as measured by the COBE satellite. We thus have the relation

$$\frac{3H_i^3}{F_i} = 10\delta \ , \qquad (3.11)$$

where the subscript i denotes the epoch at which x=0 and where we consider δ to be a known number ($\sim 2\times10^{-5}$).

We note that in general tensor perturbations may produce some fraction of the total perturbations; in this case, one should replace the quantity δ in Eq. (3.11) by the corresponding smaller value δ_S which denotes only the scalar contribution: i.e.,

$$\frac{3H_i^3}{F_i} = 10\delta_S \ . \qquad (3.12)$$

In this case, the general form of the results derived below remains the same with δ replaced by δ_S. For completeness, we also note that the maximum of the density perturbation constraint need not occur at $x = 0$; this complication is considered in Ref. [2] and will not significantly affect the results of this paper.

Since we can use Eq. (3.11) to eliminate F_i in favor of δ and H_i in the upcoming equations, we are thus left with a single unknown parameter, namely, H_i. In presenting our results below, we choose to eliminate the parameter H_i in favor of the energy scale Λ at x=0; i.e., we define

$$H_i^2 \equiv \frac{8\pi}{3}\frac{\Lambda^4}{M_{\rm Pl}^2} = \frac{8\pi}{3}\frac{V_{\rm tot}(x=0)}{M_{\rm Pl}^2} \ . \qquad (3.13)$$

The quantity Λ^4 is equal to the value of the total vacuum energy density of the Universe at x=0 (which occurs $\sim$60 e-foldings before the end of inflation).

C. Constraints on the width of the potential

In this section we find both lower [Eq. (3.18)] and upper [Eq. (3.20)] bounds on $\Delta\phi$, the width of the portion of the potential responsible for perturbations on cosmologically interesting scales. In other words, $\Delta\phi$ is the width of the portion of the potential $\sim$60–50 e-foldings before the end of inflation.

Using the equation of motion (1.8) during the overdamped epoch and the definition $F = -dV/d\phi$, we can write the width $\Delta\phi$ in the form

$$\Delta\phi = \frac{N}{3}\int_0^1 (F/H^2)dx \ . \qquad (3.14)$$

In order to find bounds on $\Delta\phi$, we must find bounds on the integral appearing in the above equation. We can expand the integrand as

$$F/H^2 = (F/H)^{1/2}(F/H^3)^{1/2} \ge (F/H)^{1/2}(3/10\delta)^{1/2} \ , \qquad (3.15)$$

where we have used the density perturbation constraint [see Eq. (3.9)] to obtain the final inequality. Using this result in the integral, we thus obtain the bound

$$\Delta\phi \ge \frac{N}{(30\delta)^{1/2}}\int_0^1 (F/H)^{1/2}dx \ . \qquad (3.16)$$

A lower limit on the remaining integral can be found by using the overdamping constraint (3.8), which limits how fast the function F/H can change during the relevant portion of inflation. Notice that we have fixed the initial values F_i/H_i, and we want to find the smallest possible value for the integral. In the overdamping constraint

(3.8) we must therefore choose the sign of the derivative $d(F/H)/dx$ to be negative and as large as possible. We thus obtain

$$\Delta\phi \geq \frac{1}{(30\delta)^{1/2}}(F/H)_i^{1/2}\frac{2}{3}\{1-e^{-3N/2}\} . \qquad (3.17)$$

This expression can be simplified by noting that the quantity in curly brackets is essentially unity (the exponential term is $\sim 10^{-5}$ for N=8). Using Eqs. (3.11) and (3.13) to evaluate the quantities F_i and H_i, we obtain the desired lower bound: i.e.,

$$\frac{\Delta\phi}{M_{\rm Pl}} \geq \frac{2}{15}\left(\frac{2\pi}{3}\right)^{1/2}\delta^{-1}\frac{\Lambda^2}{M_{\rm Pl}^2} . \qquad (3.18)$$

We now derive an upper limit on the allowed width $\Delta\phi$. In this case, we use the constraint of Eq. (1.10) which implies that the kinetic energy of the rolling field does not dominate the vacuum energy density. This constraint can be written in the form

$$\frac{\dot{\phi}^2}{H^2} \leq \frac{3}{4\pi}M_{\rm Pl}^2 . \qquad (3.19)$$

Using this result in the definition for the width $\Delta\phi$ [see Eq. (3.2)], we see immediately that

$$\frac{\Delta\phi}{M_{\rm Pl}} \leq N\sqrt{\frac{3}{4\pi}} . \qquad (3.20)$$

For the standard choice N=8, this limit implies $\Delta\phi/M_{\rm Pl} \leq 3.9$. We note that this upper limit follows directly from the definition of $\Delta\phi$ and the condition (1.10) which must be met in order for inflation to take place. In particular, this bound is independent of the density perturbation constraint.

Putting all of the results of this subsection together, we find that the change $\Delta\phi$ in the field is constrained to lie in the range

$$\frac{2}{15}\left(\frac{2\pi}{3}\right)^{1/2}\delta^{-1}\frac{\Lambda^2}{M_{\rm Pl}^2} \leq \frac{\Delta\phi}{M_{\rm Pl}} \leq N\sqrt{\frac{3}{4\pi}} . \qquad (3.21)$$

Another way to write this constraint is in terms of the Hubble parameter H_i at the epoch x=0: i.e.,

$$\frac{H_i}{15\delta} \leq \Delta\phi \leq N\sqrt{\frac{3}{4\pi}}M_{\rm Pl} . \qquad (3.22)$$

The above bounds suggest that the change $\Delta\phi$ in the scalar field is rather constrained. For all cases, $\Delta\phi$ cannot be much larger than the Planck scale $M_{\rm Pl}$. The lower bound shows that the change in the field $\Delta\phi$ must be at least a factor of $\sim 10^3$ larger than the Hubble parameter. Notice that for an inflationary energy scale Λ comparable to the GUT scale, the change $\Delta\phi$ in the inflation field must be larger than $\sim M_{\rm Pl}$. In the following section, we calculate the width $\Delta\phi$ for three standard inflationary potentials and show that the condition $\Delta\phi \sim M_{\rm Pl}$ is in fact typical [30].

D. Width of the potential for examples

In this section, we calculate the width $\Delta\phi$ for several standard inflationary models, including monomial potentials (such as in the original version of chaotic inflation [6]), exponential potentials [36], and cosine potentials (such as in natural inflation [7]). Here, we write the number N of e-foldings as

$$N = \int H\,dt = \frac{8\pi}{M_{\rm Pl}^2}\int\frac{V\,d\phi}{|dV/d\phi|} , \qquad (3.23)$$

where we have used the slowly rolling version of the equation of motion to obtain the second equality. In the integral in Eq. (3.23), the range of integration corresponds to the range $\Delta\phi$ of interest. For the cases of monomial potentials and exponential potentials, we only consider the portion of inflation during which density fluctuations of cosmologically interesting sizes are produced (i.e., we take N=8 as usual). For the case of natural inflation (cosine potentials), we consider the entire overdamped phase of inflation (i.e., we take $N \approx 60$ or so).

We first consider the monomial potential of the form

$$V(\phi) = \lambda_j\phi^j , \qquad (3.24)$$

where j is an integer. For this class of models, the number of e-foldings is given [using Eq. (3.23)] by

$$N = \frac{4\pi}{j}M_{\rm Pl}^{-2}[\phi_1^2 - \phi_2^2] , \qquad (3.25)$$

where ϕ_1 and ϕ_2 are the initial and final values of the field. Without loss of generality, we take $\phi_1 > \phi_2$ [31]. For the width $\Delta\phi$, we find

$$\frac{\Delta\phi}{M_{\rm Pl}} = \frac{\phi_1-\phi_2}{M_{\rm Pl}} = [Nj/4\pi + (\phi_2/M_{\rm Pl})^2]^{1/2} - \phi_2/M_{\rm Pl} , \qquad (3.26)$$

where the first equation is just the definition of the width and the second equation is obtained by eliminating ϕ_1 using Eq. (3.25). We examine this expression in two limits, $\phi_2 \ll M_{\rm Pl}$ and $\phi_2 \geq M_{\rm Pl}$. In the first case, $\Delta\phi/M_{\rm Pl} \sim (Nj/4\pi)^{1/2} \sim 1$ for N=8. The second possibility is that the final value of the field is in the regime $\phi_2 \geq M_{\rm Pl}$. Thus, either $\Delta\phi$ is comparable to the Planck scale or ϕ_2 is larger than the Planck scale. In either case, an energy scale comparable to or larger than the Planck scale must be present in the problem.

As the next example, we consider an exponential potential [36] of the form

$$V(\phi) = V_0\exp[-\phi/\sigma] , \qquad (3.27)$$

where σ is the energy scale that characterizes the falloff of the potential. We note that this form is often used as an approximation to the true potential and is valid for only part of the inflationary epoch. However, as long as the form (3.27) holds for a few e-foldings of the scale factor, the following argument is valid. Using the definition (3.23), we obtain

$$N = 8\pi\frac{\Delta\phi}{M_{\rm Pl}}\frac{\sigma}{M_{\rm Pl}} . \qquad (3.28)$$

Solving for $\Delta\phi$, we find

$$\frac{\Delta\phi}{M_{\rm Pl}} = \frac{N}{8\pi}\frac{M_{\rm Pl}}{\sigma} \approx 0.32\frac{M_{\rm Pl}}{\sigma} \ , \tag{3.29}$$

where we have used $N = 8$ to obtain the final approximate equality. Equation (3.29) shows that either the width $\Delta\phi$ must be comparable to the Planck scale $M_{\rm Pl}$, or, the falloff scale σ must be much larger than $M_{\rm Pl}$. Once again, an energy scale comparable to or larger than the Planck scale must be present.

Finally, we consider a cosine potential, i.e.,

$$V(\phi) = \Lambda^4[1 + \cos(\phi/f)] \ , \tag{3.30}$$

such as that found in natural inflation [7]. For this case, we find

$$N_{\rm tot} = \frac{16\pi f^2}{M_{\rm Pl}^2}\ln\left\{\frac{\sin(\phi_2/2f)}{\sin(\phi_1/2f)}\right\} \ , \tag{3.31}$$

where we have denoted the number of e-foldings as $N_{\rm tot} \approx 60$ to emphasize that we do not use $N = 8$ for this case. This potential has a definite width, namely, f. Thus, in this case, we have

$$\Delta\phi_{\rm tot} \sim f \sim M_{\rm Pl}\left(\frac{N_{\rm tot}}{16\pi}\right)^{1/2}\{\ln[2f/\phi_1]\}^{-1/2} \ , \tag{3.32}$$

where $\Delta\phi_{\rm tot}$ is the width of the potential over 60 e-foldings (rather than merely the 8 of structure formation). To obtain this equation we have solved Eq. (3.31) for f and then used the fact that $\phi_2 \sim f$ (more precisely, we assume that $\ln[\sin(\phi_2/2f)]$ is of order unity). Thus, unless the remaining logarithmic factor in Eq. (3.32) becomes very far from unity, this potential has a width which is of order the Planck scale $M_{\rm Pl}$. A detailed treatment of the conditions for sufficient inflation with this potential (see Ref. [7]) confirms that f and $\Delta\phi$ must be near the Planck scale $M_{\rm Pl}$ for this model.

We thus conclude that for these particular examples, the inflationary potentials contain energy scales which are comparable to (or larger than) the Planck scale $M_{\rm Pl}$. While all of the models considered here have $\Delta\phi \sim M_{\rm Pl}$, we note that the constraint of Eq. (3.21) is much less restrictive for small values of Λ (the energy scale of inflation); for example, if $\Lambda = 10^{12}$ GeV, the bound becomes very weak, $\Delta\phi \geq 10^{-10}M_{\rm Pl}$. This apparent discrepancy is easy to understand. For many simple "well-behaved" potentials, the integral in Eq. (3.23) $\sim (\Delta\phi)^2$, and Eq. (3.23) reduces to

$$N \sim 8\pi\frac{(\Delta\phi)^2}{M_{\rm Pl}^2} \ . \tag{3.33}$$

We thus naively expect that any sufficiently well-behaved potential will have $\Delta\phi \sim M_{\rm Pl}$. However, in the general bound of Eq. (3.21), we allow the potential to take *any* form, provided only that the density perturbation constraint and the overdamping constraint are satisfied. This considerable extra freedom leads to the appreciably weaker bound.

E. Constraints on the change in height of the potential

In this section, we find both lower [Eq. (3.36)] and upper [Eq. (3.40)] bounds on ΔV, the height of the portion of the potential responsible for perturbations on cosmologically interesting scales. In other words, ΔV is the height of the portion of the potential ~60–50 e-foldings before the end of inflation. The bounds on ΔV imply an upper limit on the energy scale of inflation (during this phase).

The change in potential ΔV is given by

$$\Delta V = \frac{N}{3}\int_0^1 (F/H)^2 dx \ . \tag{3.34}$$

We have chosen our sign convention so that ΔV is a positive quantity and so that x=0 at the beginning of the constrained time period. Once again, one should keep in mind that ΔV is the change in the potential only for the N=8 e-foldings during which cosmic structure is produced and is *not* the total change over the entire inflationary epoch.

In order to constrain ΔV, we must constrain the function F/H which appears in the integrand in Eq. (3.34). We first find the lower limit on the change ΔV of the potential. Using the overdamping constraint (3.8), we find

$$\int_0^1 (F/H)^2 dx \geq \frac{F_i^2}{H_i^2}\frac{1}{6N}\{1 - e^{-6N}\} \approx \frac{F_i^2}{H_i^2}\frac{1}{6N} \ . \tag{3.35}$$

We note that this bound is the greatest lower bound for this constraint problem. We want to eliminate the quantities F_i and H_i appearing in Eq. (3.35) in favor of the density perturbation amplitude limit δ [see Eq. (3.11)] and the energy scale Λ [see Eq. (3.13)]. Using these quantities (δ and Λ), we can write the limit obtained from Eqs. (3.34) and (3.35) in the form

$$\Delta V \geq \frac{8\pi^2}{225}\delta^{-2}M_{\rm Pl}^{-4}\Lambda^8 \ . \tag{3.36}$$

Thus, the change ΔV in the potential during the portion of inflation when cosmological structure is produced is bounded to be greater than the right-hand side of Eq. (3.36).

However, the change ΔV in the potential is also bounded to be *less* than the total vacuum energy density $V_{\rm tot} = \Lambda^4$ at the beginning of the structure producing epoch; otherwise the vacuum contribution to the energy density would become negative [32]. In other words, we must also require

$$\Lambda^4 \geq \Delta V \ . \tag{3.37}$$

Combining these two limits (3.36) and (3.37) and then solving for the scale Λ, we obtain the desired limit on the energy scale Λ:

$$\frac{\Lambda}{M_{\rm Pl}} \leq \delta^{1/2}\left(\frac{15\sqrt{2}}{4\pi}\right)^{1/2} \approx 6\times 10^{-3} \ . \tag{3.38}$$

This constraint implies that the structure producing portion of inflation must take place at an energy scale less than (or roughly comparable to) the GUT scale. Although found by slightly different methods, this constraint is equivalent to that derived earlier by Lyth [18]. Notice also that this constraint is comparable to that obtained in Ref. [20] by requiring that the amplitude of tensor perturbations is not in conflict with the COBE measurement.

We note that an upper bound for ΔV also exists. This bound can be obtained by finding an upper bound for the integral in Eq. (3.34). Using the overdamping constraint with the opposite sign, we find

$$\int_0^1 (F/H)^2 dx \leq \frac{F_i^2}{H_i^2}\frac{1}{6N}\{e^{6N}-1\} \approx \frac{F_i^2}{H_i^2}\frac{e^{6N}}{6N} , \quad (3.39)$$

where the final approximate equality introduces negligible error. Once again, we eliminate the quantities F_i and H_i in favor of the density perturbation amplitude limit δ and the energy scale Λ [see Eqs. (3.11) and (3.13)]. Using the result (3.39) in the definition (3.34) thus provides an upper bound on the change ΔV of the potential: i.e.,

$$\Delta V \leq \frac{8\pi^2 e^{6N}}{225}\delta^{-2}M_{\rm Pl}^{-4}\Lambda^8 . \quad (3.40)$$

At first glance, the bound of Eq. (3.40) may not seem very stringent because of the large exponential factor e^{6N}($\sim 10^{21}$ for the usual value of N=8). However, at sufficiently small energy scales Λ, this bound becomes very severe. To illustrate this behavior, we define a new dimensionless parameter η which is the ratio of the change ΔV in the potential to the original height of the potential (Λ^4) at the beginning of the structure producing epoch: i.e.,

$$\eta \equiv \frac{\Delta V}{\Lambda^4} . \quad (3.41)$$

Our bound (3.40) on the change in the potential immediately implies a bound on the parameter η:

$$\eta \leq \frac{8\pi^2 e^{6N}}{225}\delta^{-2}\left(\frac{\Lambda}{M_{\rm Pl}}\right)^4 \sim 6\times 10^{29}\left(\frac{\Lambda}{M_{\rm Pl}}\right)^4 . \quad (3.42)$$

For example, if we consider inflationary models at low energies such as Λ=1 TeV, we obtain the bound $\eta \leq 10^{-34}$.

It is easy to see why the above result makes inflation at very low energies somewhat problematic. During the N=8 e-foldings of inflation when cosmological structure is produced, for $\Lambda \ll M_{\rm Pl}$ we must have $\eta = \Delta V/\Lambda^4 \ll 1$. However, the vacuum energy density must be essentially zero at the end of the entire inflationary epoch; thus, during the following $\sim$52 e-foldings of inflation, we must have $\Delta V/\Lambda^4 \sim 1$ [33]. It seems unlikely that particle physics models will produce a scalar field potential with such extreme curvature.

Before leaving this section, we note that the above argument defines a suggestive lower bound for the energy scale of inflation. Arguing *very* roughly, we expect that models of inflation with the parameter η very much smaller than unity are difficult to obtain. As shown by Eq. (3.42), the parameter η decreases with the energy scale Λ of inflation. We thus obtain a suggestive lower bound for Λ by requiring that η be larger than some "not too unnaturally small number," say 1/10. The requirement that $\eta > 1/10$ implies that the energy scale of inflation must obey the constraint

$$\Lambda \geq M_{\rm Pl}\,\delta^{1/2}e^{-3N/2}\left[\frac{225}{80\pi^2}\right]^{1/4}$$
$$\approx 2\times 10^{-8}M_{\rm Pl} \approx 2\times 10^{11}\text{ GeV} , \quad (3.43)$$

where the numerical value was obtained using N=8. We stress that this bound is *not* a firm lower limit on the energy scale Λ, but it is suggestive. In particular, for energy scales Λ much less than about 10^{11} GeV, the parameter η becomes very small compared to unity.

IV. CONSTRAINTS ON THE FINE-TUNING PARAMETER

In this section, we constrain the fine-tuning parameter $\lambda_{\rm FT}$ as defined by Eq. (1.5). In our previous paper [2], we found a firm upper limit on the parameter $\lambda_{\rm FT}$. In this paper, we first complete the argument by finding a lower limit on $\lambda_{\rm FT}$. Next, we show how density perturbation spectra which are not scale invariant can place slightly tighter bounds on $\lambda_{\rm FT}$. In this section, we take the total perturbation spectrum $\delta_{\rm tot}$ to be equal to the scalar density perturbation spectrum as given by Eq. (2.3a).

A. Lower bound on the fine-tuning parameter

In order to bound $\lambda_{\rm FT}$, we need bounds on both the height ΔV and the width $\Delta\phi$. We have already shown that ΔV is bounded from below by Eq. (3.36) and that $\Delta\phi$ is bounded from above by Eq. (3.20). Combining these two results thus gives us a lower bound on the ratio $\lambda_{\rm FT} = \Delta V/(\Delta\phi)^4$: i.e.,

$$\lambda_{\rm FT} \geq \frac{128}{2025}\frac{\pi^4}{N^4}\delta^{-2}\left(\frac{\Lambda}{M_{\rm Pl}}\right)^8 . \quad (4.1)$$

The bound on $\Delta\phi$ was obtained independently of the density perturbation constraint; the bound on ΔV was obtained by requiring $\delta\rho/\rho \leq \delta$.

Combining this new result with the general bound of Ref. [2], we find that $\lambda_{\rm FT}$ is confined to the range

$$\frac{128}{2025}\frac{\pi^4}{N^4}\delta^{-2}\left(\frac{\Lambda}{M_{\rm Pl}}\right)^8 \leq \lambda_{\rm FT} \leq \frac{2025}{8}\delta^2 . \quad (4.2)$$

For example, if we use representative values of $\Lambda/M_{\rm Pl} \sim 10^{-3}$, N=8, and $\delta \sim 2\times 10^{-5}$, the allowed range for the parameter $\lambda_{\rm FT}$ becomes

$$4\times 10^{-18} \leq \lambda_{\rm FT} \leq 10^{-7} . \quad (4.3)$$

We note that the bounds presented in Eq. (4.2) are applicable for all models of inflation which have slowly rolling fields and which satisfy the constraints of Eqs. (1.7), (1.9), and (1.10).

B. Effects of departures from scale invariance

For the limits presented thus far (Sec. III), we have used the density perturbation constraint in the form of Eq. (1.7), which assumes that the amplitude of the density perturbations produced by inflation must be less than a constant value (i.e., the constraint is the same for all perturbation wavelengths). However, one way to explain current cosmological data is with density perturbations with a non-scale-invariant spectrum [7,34]. In this case the departures from scale invariance imply that our universe has density perturbations which exhibit *more power on large scales.* In terms of our constraint (1.7), this result implies that we should replace the constant parameter δ with some function $\delta(x)$ which is a *decreasing* function of time (and hence a decreasing function of x) during the structure producing portion of inflation. Keep in mind that in this present discussion δ represents the upper bound on the density fluctuations and not the amplitude of the fluctuations themselves.

For this discussion, we take the spectrum of density fluctuations to be a simple power law [see Eqs. (2.10) and (2.11)]. We can incorporate this scale dependence into our density perturbation constraint by writing it in the form [35]

$$\frac{1}{10}\frac{H^2}{\dot{\phi}} \le \delta(x) = \delta_0 \exp[-\alpha x] \ , \tag{4.4}$$

where δ_0 represents the size of the allowed perturbations at the largest size scale (the present-day horizon scale) and where we have defined the parameter

$$\alpha = N(1-n)/2 \ . \tag{4.5}$$

As before,the scale invariant spectrum n=1 corresponds to α=0 while a spectrum with more power on large scales $n < 1$ corresponds to $\alpha > 0$. Notice that we have written Eq. (4.4) as an inequality; we assume that the perturbations produced during inflation (left-hand side of the equation) are smaller than (or equal to) the actual primordial perturbations (right-hand side of the equation). In previous work [2], we assumed that $\alpha = 0$ (i.e., n=1 with equal power on all length scales when the perturbation entered the horizon). For positive values of α (i.e., for $n < 1$), our new constraint is more restrictive than that used previously and hence leads to tighter bounds on $\lambda_{\rm FT}$.

We now want to show how this more restrictive constraint on the density perturbations affects our upper bound on the fine-tuning parameter $\lambda_{\rm FT}$. Although we do not present our calculations here, we find that the upper bound on the fine-tuning parameter becomes

$$\lambda_{\rm FT} \le \frac{2025}{8}\delta^2[1-2\alpha/3N]^4 \ . \tag{4.6}$$

For $n < 1$, this bound is tighter than that obtained previously in Ref. [2]. However, the factor $\mathcal{F}$ by which the bound is tighter is rather small:

$$\mathcal{F} \approx [1-2\alpha/3N]^{-4} = \left(\frac{3}{2+n}\right)^4 \ , \tag{4.7}$$

where we have used the definition of α in the second expression. Thus, for the largest expected departures from scale invariance, $n \sim 1/2$, we find $\mathcal{F} \approx 2$. Even for the rather extreme departure from scale invariance of n=0, we obtain only a modest increase in the bound with $\mathcal{F} = 81/16 \approx 5$. We therefore conclude that departures from scale invariance with $n < 1$ lead to moderately tighter constraints on the fine-tuning parameter $\lambda_{\rm FT}$.

V. SUMMARY AND DISCUSSION

In this paper, we have found constraints on the scalar field potential for a general class of inflationary models which have slowly rolling fields. These constraints apply to all models of inflation which exhibit overdamped motion of the scalar field and which obey the density perturbation constraint. This work thus extends that of Ref. [2].

(1) We have studied the reconstruction of the inflationary potential by considering both scalar and tensor modes. The simultaneous consideration of both types of perturbations leads to a differential equation which could be solved to find the potential if the total primordial spectrum of perturbations were known (see also Refs. [16,17]). For the particular case of total perturbation spectra (scalar density and tensor components) which are a pure power law, we were able to solve for the potential analytically. Figures 1–4 show the reconstructed potentials for the expected range of parameter space. When tensor modes provide a significant fraction of the total perturbations, the potentials are concave upward for all spectral indices n=0.5–1 (see Fig. 1). For the opposite case where tensor modes provide a negligible contribution to the total, the potentials are concave downward (see Fig. 4). For the case of density perturbation spectra with moderate departures from scale invariance (e.g., n=0.6) and little contribution from tensor modes, the reconstructed potential is very similar to a cosine (see Fig. 5) such as in the model of natural inflation [7]. We also showed how constraints on this spectrum imply corresponding constraints on the reconstructed potential $V(\phi)$. Thus once observations of the microwave background are at the level of giving an allowed range for the spectrum of primordial perturbations, these observations can be used to limit the potential to within a range of possibilities.

(2) We have derived a relationship between the amplitude of tensor perturbations and the width of the scalar field potential [see Eq. (3.5)]. In particular, the average ratio $\langle \mathcal{R} \rangle$ of tensor to scalar perturbations is comparable to the dimensionless width $\Delta\phi/M_{\rm Pl}$ of the potential. Thus, if tensor perturbations are important, then the width of the potential must be comparable to the Planck mass. As we discuss in item (3) below, the width $\Delta\phi$ is bounded from above; as a result, the average ratio

$\langle \mathcal{R} \rangle$ is also bounded from above. This result implies that while scalar perturbations can dominate over tensor perturbations by an arbitrarily large factor, the converse is not true: tensor perturbations can be *at most* a factor of $\sim$6 larger than scalar perturbations [see Eq. (3.6)].

(3) We have found both upper and lower limits on the change $\Delta\phi$ of the scalar field during the phase of inflation which produces cosmic structure [see Eq. (3.21)]. These limits can be summarized by the relation

$$0.6 \left(\frac{\Lambda}{10^{17}\ \text{GeV}} \right)^2 \leq \frac{\Delta\phi}{M_{\text{Pl}}} \leq 3.9 \left(\frac{N}{8} \right) .$$

The lower limit depends on the energy scale at which inflation takes place. For energy scales larger than the GUT scale, the width $\Delta\phi$ must be larger than the Planck scale M_{Pl}. The upper limit implies that the change in the scalar field during the N=8 e-foldings of structure-forming perturbations cannot be larger than $\sim 4\ M_{\text{Pl}}$.

(4) We have found both upper and lower bounds on the change ΔV of the potential during the part of inflation when cosmological structure is produced. These bounds can be used to find an upper limit on the energy scale Λ of this portion of inflation [see Eq. (3.38)]:

$$\frac{\Lambda}{M_{\text{Pl}}} \leq 6 \times 10^{-3} \left(\frac{\delta}{2 \times 10^{-5}} \right)^{1/2} ,$$

where δ is the maximum allowed amplitude of density perturbations. This limit shows that the epoch of structure-forming perturbations must take place at an energy scale less than about the GUT scale. This bound is almost identical to those found earlier from the consideration of scalar perturbations [18,19]. The bound is comparable to that obtained from the consideration of tensor perturbations [20].

(5) We have also presented a very rough argument which indicates that inflation at very low energy scales will encounter some difficulty: the fractional change in the height of the potential during the N=8 e-foldings of structure formation is very small when the energy scale Λ is small: i.e.,

$$\eta = \frac{\Delta V}{V_i} \leq 6 \times 10^{29} \left(\frac{\Lambda}{M_{\text{Pl}}} \right)^4 \sim \frac{1}{10} \left(\frac{\Lambda}{2 \times 10^{11}\ \text{GeV}} \right)^4 .$$

This bound shows that if Λ is small compared to $\sim 10^{11}$ GeV, then $\eta \ll 1$. It is then difficult for the potential to drop to (roughly) zero in the remaining e-foldings for a normally shaped potential.

(6)We have found a *lower* bound on the fine-tuning parameter λ_{FT}. Our previous bound [2] showed that the parameter λ_{FT} must be quite small ($\lesssim 10^{-7}$); this new bound shows that λ_{FT} cannot be made arbitrarily small. These bounds thus confine the fine-tuning parameter to the range

$$4 \times 10^{-10} \left(\frac{\Lambda}{10^{17}\ \text{GeV}} \right)^8 \leq \lambda_{\text{FT}} \leq 10^{-7} .$$

Again, Λ is the energy scale of the portion of inflation where cosmological structure is produced; its value is restricted in [4] above.

(7) We have explored the effects of non-scale-invariance of density perturbations on the fine-tuning parameter λ_{FT} of Ref. [2]. If the density perturbations are non-scale-invariant with $n < 1$, then we obtain a stronger bound on λ_{FT}. However, for the departures from scale invariance proposed as an explanation of recent observations of cosmological data on large scales (e.g., $n \approx 0.6$; see, e.g., Ref. [7]), the bound is improved by a rather modest factor ($\mathcal{F} \sim 2$).

The results of this paper, together with previous related results, show that the properties of the scalar field potential in inflationary universe models are very highly constrained.

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