Relativistic two fluid plasmas in the vicinity of a Schwarzschild black hole—Local approximation. I

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The $3 + 1$ split of general relativity is used in a preliminary investigation of waves propagating in a plasma influenced by the gravitational field of a Schwarzschild black hole. The relativistic two fluid equations have been reformulated to take account of gravitational effects due to the event horizon. The set of simultaneous linear equations for the perturbations are displayed but the numerical solution in Schwarzschild coordinates is reserved for a subsequent paper. Here, a local approximation is used to investigate the one-dimensional radial propagation of Alfvén and high frequency electromagnetic waves. The dispersion relation is obtained for these waves and solved numerically for the wave number k.

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I. INTRODUCTION

Although there still exist no convincing observational data which prove *conclusively* that black holes exist in the Universe, there is certainly sufficient evidence to make the study of such objects and the effects on their environment a matter of great importance to astrophysics. Black holes, however, cannot be observed directly and so must be observed indirectly through the effects they exert on their environment. With their enormous gravitational fields, they will greatly affect the surrounding plasma medium, so that plasma physics in the vicinity of a black hole becomes a subject of obvious interest in astrophysics. In the immediate vicinity of a black hole general relativity applies, and it is therefore of interest to formulate plasma physics problems in the context of general relativity. Once the general equations have been formulated, it is necessary to look for solutions which will be the counterparts of solutions from ordinary relativistic plasma physics.

The aim of the present work is to investigate the behavior of plasma waves near a Schwarzschild black hole. It would seem, in the first instance, to demand a covariant formulation based on the fluid equations of general relativity and Maxwell's equations in curved spacetime, but this approach has so far proved unproductive because of the curvature of four-dimensional spacetime in the region surrounding a black hole. The development of the $3 + 1$ formulation of general relativity by Thorne *et al.* [1] provides a method in which the electrodynamic equations and the plasma physics at least look somewhat similar to the usual formulations in flat spacetime while taking accurate account of general relativistic effects such as curvature.

The study of plasma waves in the presence of strong gravitational fields, using this approach, is still in its early stages. To the authors' knowledge, there have been two relevant attempts to exploit the $3 + 1$ formalism. Zhang [2] has considered the case of ideal magnetohydrodynamics (MHD) waves near a Kerr black hole, accounting for the effects of the hole's angular momentum but ignoring the effects due to the black hole horizon. Holcomb and Tajima [3], Holcomb [4], and Dettmann *et al.* [5] have considered some properties of wave propagation in a Friedmann universe.

The study of plasmas in the black hole environment is important because a successful study of the waves and emissions from plasma falling into a black hole will be of great value in aiding the observational identification of black hole candidates. Therefore it seems essential for the understanding of radiation processes and concomitant emission spectra in the vicinity of black holes to develop a program of black hole plasma physics culminating in the interaction of particles described by a kinetic equation including radiation. It will be by means of the observation of such radiation that the existence of black holes will ultimately be verified unambiguously. Such a program must be developed in stages. It is initiated in the present paper by the study of linear waves using the fluid theory of plasmas.

The $3 + 1$ spacetime split in the formulation of general relativity is particularly appropriate for applications to black holes as described by Thorne *et al.* [6]. In this monograph, work connected with black holes has been facilitated by the replacement of the hole's event horizon by a membrane endowed with electric charge, electrical conductivity, and finite temperature and entropy. Mathematically the membrane paradigm is equivalent to the standard, full general relativistic theory of black holes so far as physics outside the event horizon is concerned but the formulation of all physics in this region turns out to be very much simpler than it would be using the standard covariant approach of general relativity.

In the present paper a general relativistic version of the two fluid formulation of plasma physics is considered using the $3+1$ formalism. A linearized treatment of plasma waves is developed, in analogy with the special relativistic formulation by Sakai and Kawata [7] (SK), and used to investigate the nature of the waves close to the horizon of a Schwarzschild black hole. Transverse electromagnetic waves are investigated using the linearized two fluid equations. Longitudinal waves together with the two-stream instability are treated in the following paper, referred to hereafter as paper II. Such an investigation of wave propagation in a relativistic two fluid plasma in the environment close to the event horizon is important for an understanding of plasma processes near a black hole, in particular the details of what happens close to the horizon as material accretes onto the black hole. In this paper an initial attempt is made to identify the effect of the gravitational field of a Schwarzschild black hole on the properties of plasma waves in a region close to the event horizon. The main effects are due to the presence of the lapse function, that is to say, in view of Eq. (6), to the gravitational acceleration as measured by a fiducial observer (FIDO).

The present work on black hole plasma physics is confined to the linear theory but the ultimate challenge for fresh discovery requires the study of nonlinear and shock waves as well as an understanding of particle acceleration by plasma waves and the consequent emission of radiation. In this sense the present paper will form the essential basis of these future, more complicated, investigations.

In the present paper Sec. II summarizes the $3+1$ formulation of general relativity and prepares the ground for the development in Sec. III of the nonlinear two fluid equations expressing continuity and conservation of energy and momentum in which the two fluids are coupled together through Maxwell's equations for the electromagnetic fields. These equations are shown to reduce to the corresponding special relativistic expressions for zero gravitational field. They not only form the basis for the present work on the linear wave theory, but are needed also for the nonlinear treatment of the waves.

Section IV restricts consideration to the dominant radial coordinate r (or z in the more amenable Rindler coordinate system) and the equations for wave propagation are linearized in Sec. V, following brief discussions, first of the way in which the unperturbed fields and fluid parameters depend on distance from the black hole horizon, z, and second the evaluation of the derivatives with respect to z of the unperturbed quantities.

In this paper and in paper II the aim is to obtain numerical solutions for the wave dispersion relations in the local approximation. This is done without taking account of the full dependence of the equations on the radial coordinate. Thin layers are considered, each with its own appropriate mean value of the lapse function. This local, or mean-field, approximation is discussed in Sec. VI. Section VII presents the dispersion relation for the transverse waves. The numerical procedure for determining the roots of the dispersion relation is explained in Sec. VIII and the numerical solutions for the wave number k are presented in Sec. IX.

II. FORMALISM

As mentioned above, the work in this paper is based on the $3+1$ formulation of general relativity as developed by Thorne, Price, and Macdonald (TPM) [1]. The $3+1$ approach was originally developed by Arnowitt, Deser, and Misner [8] in order to study the quantization of the gravitational field. Since then, it has mostly been used in numerical relativity [9]. TPM extended the $3+1$ formalism to include electromagnetism and applied it to the Kerr metric in order to study electromagnetic effects near a rotating black hole. As a consequence, their work has also opened up many possibilities for studying electromagnetic effects on plasmas in the black hole environment.

The basic concept behind the $3+1$ split of spacetime is to select a preferred set of spacelike hypersurfaces which form the level surfaces of a congruence of timelike curves. The choice of a particular set of these hypersurfaces constitutes a time slicing of spacetime. In this case the hypersurfaces are chosen to be those of constant universal time t. In TPM notation, the Schwarzschild spacetime element is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-2M/r}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{1}$$

where the components $x^{\mu,\nu}$ denote spacetime coordinates and indices range over $0,1,2,3$. These hypersurfaces of constant universal time t define an absolute three-dimensional space described by the metric

$$ds^2 = g_{jk}dx^j dx^k = \frac{1}{1-2M/r}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{2}$$

where the indices i,j,k refer to coordinates in absolute space and range over $1,2,3$.

Consider now a set of observers at rest with respect to this absolute space. Such observers are known as fiducial observers (FIDO's). The FIDO's measure their proper time τ using clocks that they carry with them and make local measurements of physical quantities. Hence, in what follows, all quantities such as velocities $\mathbf{v}$ and fields $\mathbf{B}$ and $\mathbf{E}$ are defined as FIDO locally measured quantities and all rates as measured by the FIDO's are measured using FIDO proper time. When making these local measurements the FIDO's use a local Cartesian coordinate system that has basis vectors of unit length tangent to the coordinate lines:

$$\mathbf{e}_{\hat{r}} = \left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\frac{\partial}{\partial r} , \quad \mathbf{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial\theta} , \quad \mathbf{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{3}$$

The ratio of the rate of FIDO proper time to that of universal time is defined in terms of a redshift factor known as the *lapse* function,

$$\alpha(r) \equiv \frac{d\tau}{dt} = \left(1-\frac{2M}{r}\right)^{\frac{1}{2}} , \tag{4}$$

so called because it measures the amount of FIDO proper time which has elapsed during the passage of a unit amount of universal time.

If one considers a spacetime viewpoint rather than a $3+1$ split of spacetime, the set of orthonormal vectors also includes the basis vectors for the time coordinate which is therefore given by

$$\mathbf{e}_{\hat{0}} = \frac{d}{d\tau} = \frac{1}{\alpha}\frac{\partial}{\partial t}\,. \tag{5}$$

It is important to note that the FIDO proper time τ functions as a local laboratory time, where the FIDO's have the role of "local laboratories." It is not a global coordinate and does not provide a slicing of spacetime. The Schwarzschild time coordinate t is the logical choice to satisfy this role and in fact slices spacetime in the way that the FIDO's would do physically. For this reason, all subsequent equations are expressed in terms of the universal time coordinate t rather than the FIDO proper time τ.

The lapse function α plays the role of a gravitational potential as well as governing the ticking rates of clocks and redshifts. From the lapse function one can compute the gravitational acceleration felt by a FIDO:

$$\mathbf{a} = -\nabla \ln\alpha = -\frac{1}{\alpha}\frac{M}{r^2}\mathbf{e}_{\hat{r}}\,. \tag{6}$$

For a derivation of the gravitational acceleration see TPM [1] or for a generalized derivation not restricted to the Schwarzschild metric see TPM [6]. It is clear from Eq. (6) that far from the horizon the gravitational acceleration is weak and approaches the Newtonian value familiar from flat spacetime. Close to the horizon, however, the acceleration approaches infinity as $\alpha \to 0$. Note that the FIDO's are not unaccelerated observers. Their motion may become pathological near the event horizon (or any singularity) as seen by observers keeping universal time. A set of observers whose motion is always nonpathological with respect to observers keeping universal time (observers situated far from the horizon or other singularity) are the freely falling observers (FFO's), as denoted by TPM.

Finally, the rate of change of any scalar physical quantity or any three-dimensional vector or tensor, as measured by a FIDO, is defined by the convective derivative

$$\frac{D}{D\tau} \equiv \left(\frac{1}{\alpha}\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\,, \tag{7}$$

where $\mathbf{v}$ is the velocity of a fluid as measured locally by a FIDO.

For the purposes of the present work the other aspects of the membrane paradigm, as detailed in TPM [1] and [6] are not required. For a more detailed and general treatment of the $3+1$ split of spacetime and the concept of a set of fiducial observers (FIDO's) see the membrane paradigm book [1] and York [10]. The notation adopted throughout will be that used by TPM. In general, $G = c = k_B = 1$, cgs units will be used and all equations are valid in a FIDO rest frame (at rest with respect to the Schwarzschild coordinates).

III. TWO FLUID EQUATIONS

The derivations of the equations for continuity and the conservation of energy and momentum and Maxwell's equations in $3+1$ notation will not be detailed here. The reader is referred to TPM for this material. There follows a derivation of the equations required for all wave disturbances in the plasma including the effects of gravity. Only the Alfvén and high frequency transverse electromagnetic waves will be treated in the present paper, however, with the detailed analysis of the longitudinal waves and of the two-stream instability being reserved for the following paper, paper II.

Consider now a two-component plasma consisting of electrons and either positrons or ions. The dispersion relations that result from the following investigation are valid for either an electron-positron plasma or an electron-ion plasma since no assumptions are made regarding the mass, number density, pressure, or temperature of the fluids. Finally, due account must be taken of the variations of equilibrium fields and fluid quantities with respect to the radial coordinate since these will necessarily vary as the fluids move in toward the horizon. In the $3+1$ formalism, the continuity equation for each of the fluid species may be written

$$\frac{\partial}{\partial t}(\gamma_s n_s) + \nabla\cdot(\alpha\gamma_s n_s \mathbf{v}_s) = 0\,, \tag{8}$$

where s is the species index, 1 for electrons and 2 for positrons (or ions). The energy density ϵ_s and the components of both the three-dimensional momentum density, $\mathbf{S}_s$, and stress-energy tensor, $W_s^{j,k}$, for a perfect relativistic fluid, of species s, are given by

$$\epsilon_s = \gamma_s^2(\varepsilon_s + P_s\mathbf{v}_s{}^2)\,, \quad \mathbf{S}_s = \gamma_s^2(\varepsilon_s + P_s)\mathbf{v}_s\,, \quad W_s^{jk} = \gamma_s^2(\varepsilon_s + P_s)v_s^j v_s^k + P_s g^{jk}\,. \tag{9}$$

In the above equations $\mathbf{v}_s$ is the fluid velocity, n_s is the number density, P_s is the pressure, and ε_s is the total energy density defined by

$$\varepsilon_s = m_s n_s + P_s/(\gamma_g - 1)\,. \tag{10}$$

The gas constant γ_g takes the value 4/3 as $T\to\infty$ and 5/3 as $T\to 0$. The adiabatic equation of state is the simplest for a relativistic fluid. It is valid for both the high and low temperature regimes and so is a fairly good approximation for the plasma under consideration.

From studies of two temperature models of spherical accretion onto black holes (see Colpi [11], for example), it has been shown that the ion temperature profile is closely adiabatic and that ion temperatures approach 10^{12} K near the horizon. Far from the horizon electron (positron) temperatures are essentially equal to the ion temperatures; however, closer to the horizon the electrons are progressively cooled by mechanisms such as multiple Compton scattering and synchrotron radiation. These processes limit electron temperatures near the horizon to about 10^8–10^9 K. Again, the equation of state can be expressed in terms of the conservation of entropy:

$$\frac{D}{D\tau}(P_s/n_s^{\gamma_g}) = 0, \tag{11}$$

where $D/D\tau = (1/\alpha)\,\partial/\partial t + \mathbf{v}_s \cdot \nabla$. The full equation of state for a relativistic fluid, as measured in the fluid's rest frame, is given by Eq. (34) of Harris [12] or, alternatively, Eq. (52) of Jüttner [13]:

$$\varepsilon_s = m_s n_s + m_s n_s \left[\frac{P_s}{m_s n_s} - \frac{i H_2^{(1)'}(i m_s n_s/P_s)}{H_2^{(1)}(i m_s n_s/P_s)} \right] , \tag{12}$$

where the $H_2^{(1)}(x)$ are Hankel functions. This expression is obviously complicated and would make the following analysis unmanageable.

The use of the perfect fluid energy-momentum tensor guarantees that adiabaticity is automatically contained in the fluid equations. This means, in essence, that the energy equation can be set aside in favor of utilizing the adiabatic equation of state. The energy equation will be derived, below, however, for the sake of completeness. The quantities corresponding to the fluid ones of Eq. (9) but now for the electromagnetic field are

$$\epsilon_s = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) , \quad \mathbf{S}_s = \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B}) ,$$
$$W_s^{jk} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) g^{jk} - \frac{1}{4\pi}(E^j E^k + B^j B^k) . \tag{13}$$

The equations for the conservation of energy and momentum, as derived by TPM [1], are written, respectively, as

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\epsilon_s = -\nabla \cdot \mathbf{S}_s + 2\mathbf{a} \cdot \mathbf{S}_s, \tag{14}$$

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\mathbf{S}_s = \epsilon_s \mathbf{a} - \frac{1}{\alpha}\nabla \cdot (\alpha \overleftrightarrow{\mathbf{W}}_s) . \tag{15}$$

Maxwell's equations, coupling the two-fluid plasma to the electromagnetic fields, take the following 3 + 1 form:

$$\nabla \cdot \mathbf{B} = 0, \tag{16}$$

$$\nabla \cdot \mathbf{E} = 4\pi\sigma, \tag{17}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha \mathbf{E}), \tag{18}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla \times (\alpha \mathbf{B}) - 4\pi\alpha \mathbf{J}, \tag{19}$$

where the charge and current densities are defined as

$$\sigma = \sum_s \gamma_s q_s n_s \ , \quad \mathbf{J} = \sum_s \gamma_s q_s n_s \mathbf{v}_s \ . \tag{20}$$

Note that the fluid velocities and fields are all FIDO measured quantities whereas the fluid densities and pressures are measured in the rest frame of the fluid. Equations (9) and (13) can now be substituted directly into Eqs. (14) and (15). Using Maxwell's equations, coupling each single fluid of species s to the electromagnetic fields, and the expression for the total energy density, Eq. (10), the energy and momentum conservation equations may be rewritten for each species s as

$$\frac{1}{\alpha}\frac{\partial}{\partial t}P_s - \frac{1}{\alpha}\frac{\partial}{\partial t}\left[\gamma_s^2 (\varepsilon_s + P_s)\right] - \nabla \cdot \left[\gamma_s^2 (\varepsilon_s + P_s)\, \mathbf{v}_s\right]$$
$$+ \gamma_s q_s n_s \mathbf{E} \cdot \mathbf{v}_s + 2\gamma_s^2 (\varepsilon_s + P_s)\, \mathbf{a} \cdot \mathbf{v}_s = 0 \tag{21}$$

and

$$\gamma_s^2 (\varepsilon_s + P_s) \left(\frac{1}{\alpha}\frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla \right) \mathbf{v}_s + \nabla P_s - \gamma_s q_s n_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B})$$
$$+ \mathbf{v}_s \left(\gamma_s q_s n_s \mathbf{E} \cdot \mathbf{v}_s + \frac{1}{\alpha}\frac{\partial}{\partial t}P_s \right) + \gamma_s^2 (\varepsilon_s + P_s) \left[\mathbf{v}_s (\mathbf{v}_s \cdot \mathbf{a}) - \mathbf{a}\right] = 0 \ . \tag{22}$$

If, now, one sets the lapse function α to unity so that the acceleration goes to zero (the limit of zero gravity), these equations reduce to the corresponding special relativistic fluid equations given by SK [7], even though the equations in the case considered here are valid in a FIDO frame and the special relativistic equations of SK [7] are valid in a frame in which both fluids are at rest. A transformation from the FIDO frame to the comoving (fluid) frame involves a boost velocity which, in this case, is simply the freefall velocity onto the black hole. The corresponding relativistic Lorentz factor γ_{boost} depends on the lapse function. Since the fluid velocity is equal to the freefall velocity the freefall velocity is given by

$$v_{\text{ff}} = (1 - \alpha^2)^{\frac{1}{2}}. \tag{23}$$

Substitution into $\gamma_{\text{boost}} = (1 - v_{\text{ff}}^2)^{-\frac{1}{2}}$ yields $\gamma_{\text{boost}} = 1/\alpha$.

The two fluid equations in Schwarzschild coordinates cannot be evaluated analytically, although they form the basis of the numerical procedure for solving the linear two fluid equations as will be reported in a subsequent paper. This will lead to an understanding of how the energy density in the plasma waves varies with distance from the horizon. The Rindler coordinate system, however, in which space is locally Cartesian, provides a good approximation to the Schwarzschild metric close to the black hole horizon. The essential features of the horizon and the 3 + 1 split are retained without the complication of explicitly curved spatial three-geometries. The Schwarzschild metric is approximated in Rindler coordinates by

$$ds^2 = -\alpha^2 dt^2 + dx^2 + dy^2 + dz^2 , \tag{24}$$

where

$$x = 2M\left(\theta - \frac{\pi}{2}\right) , \quad y = 2M\phi , \quad z = 4M\left(1 - \frac{2M}{r}\right) . \tag{25}$$

A discussion of the transformation from the Schwarzschild metric to the Rindler metric can be found in TPM [6]. The standard lapse function $(1 - r_H/r)^{1/2}$ is again denoted by α and simplifies in Rindler coordinates to $z/2r_H$, where r_H is the Schwarzschild radius.

IV. ONE-DIMENSIONAL WAVE PROPAGATION

Consider now a restriction to one-dimensional wave propagation in the radial z direction. Equations (8) and (16)–(22) can be expressed in a form analogous to that used by SK. Introduce the complex variables

$$\begin{aligned} v_{sz}(z,t) &= u_s(z,t) , \quad v_s(z,t) = v_{sx}(z,t) + iv_{sy}(z,t), \\ B(z,t) &= B_x(z,t) + iB_y(z,t) , \quad E(z,t) = E_x(z,t) + iE_y(z,t) . \end{aligned} \tag{26}$$

Using these definitions for the transverse fields and velocities, it is possible to write Eqs. (8), (16)–(19), (21), and (22) in a more convenient form. Then set

$$v_{sx} = \frac{1}{2}(v_s + v_s^*), \quad v_{sy} = \frac{1}{2i}(v_s - v_s^*) \tag{27}$$

and

$$B_x = \frac{1}{2}(B + B^*), \quad B_y = \frac{1}{2i}(B - B^*) \tag{28}$$

so that

$$v_{sx}B_y - v_{sy}B_x = \frac{i}{2}(v_s B^* - v_s^* B) \tag{29}$$

and similarly for the transverse electric field components, where the $*$ denotes the complex conjugate.

The continuity equation, Eq. (8), becomes

$$\frac{\partial}{\partial t}(\gamma_s n_s) + \frac{\partial}{\partial z}(\alpha\gamma_s n_s u_s) = 0 , \tag{30}$$

while Poisson's equation, Eq. (17), takes the form

$$\frac{\partial E_z}{\partial z} = 4\pi\left(q_1 n_1 \gamma_1 + q_2 n_2 \gamma_2\right) . \tag{31}$$

Adding the $\mathbf{e}_{\hat{y}}$ component of Eq. (18), multiplied by i, to the $\mathbf{e}_{\hat{x}}$ component leads to

$$\frac{1}{\alpha}\frac{\partial B}{\partial t} = -i\left(\frac{\partial}{\partial z} - a\right)E . \tag{32}$$

Equation (19) can be simplified in much the same way. Differentiating this equation with respect to t and substituting from Eq. (32) yields

$$\left(\alpha^2\frac{\partial^2}{\partial z^2} + \frac{3\alpha}{2r_H}\frac{\partial}{\partial z} - \frac{\partial^2}{\partial t^2} + \frac{1}{(2r_H)^2}\right)E = 4\pi e\alpha\frac{\partial}{\partial t}\left(n_2\gamma_2 v_2 - n_1\gamma_1 v_1\right) . \tag{33}$$

In order to investigate longitudinal waves and transverse electromagnetic waves, it is more convenient to work from a combination of the continuity equation, Eq. (30), Maxwell's equations, (31), (32), and (33), and Eq. (22), the force, or conservation of momentum, equation. Having already dealt with both the continuity and Maxwell's equations, it remains to derive the force equation in terms of the new complex transverse fields. The force equation can be split up into its three vector components and this allows the longitudinal component to be separated out. The two transverse components of the force equation may then be combined into a single transverse equation for the newly defined transverse fields and velocities by adding the $\mathbf{e}_{\hat{x}}$ component to the $\mathbf{e}_{\hat{y}}$ component, multiplied by i. The resultant longitudinal and transverse components of the momentum conservation equation are then respectively given by

$$\begin{aligned} \rho_s\frac{Du_s}{D\tau} &= q_s n_s\gamma_s\left(E_z + \frac{i}{2}(v_s B^* - v_s^* B)\right) + (1 - u_s^2)\rho_s a \\ &\quad - u_s\left(q_s n_s\gamma_s \mathbf{E}\cdot\mathbf{v}_s + \frac{1}{\alpha}\frac{\partial P_s}{\partial t}\right) - \frac{\partial P_s}{\partial z} \end{aligned} \tag{34}$$

and

$$\begin{aligned} \rho_s\frac{Dv_s}{D\tau} &= q_s n_s\gamma_s\left(E - iv_s B_z + iu_s B\right) - u_s v_s\rho_s a \\ &\quad - v_s\left(q_s n_s\gamma_s \mathbf{E}\cdot\mathbf{v}_s + \frac{1}{\alpha}\frac{\partial P_s}{\partial t}\right) , \end{aligned} \tag{35}$$

where

$$\mathbf{E}\cdot\mathbf{v}_s = \frac{1}{2}\left(E v_s^* + E^* v_s\right) + E_z u_s$$

and the total energy density is defined as

$$\rho_s = \gamma_s^2\left(\varepsilon_s + P_s\right) = \gamma_s^2\left(m_s n_s + \Gamma_g P_s\right) \tag{36}$$

with $\Gamma_g = \gamma_g/(\gamma_g - 1)$.

V. LINEARIZATION

A. Perturbations

The above equations are linearized by introducing the quantities

$$u_s(z,t) = u_{0s}(z) + \delta u_s(z,t)\,, \quad v_s(z,t) = \delta v_s(z,t)$$
$$n_s(z,t) = n_{0s}(z) + \delta n_s(z,t)\,, \quad P_s(z,t) = P_{0s}(z) + \delta P_s(z,t)\,, \tag{37}$$

$$\rho_s(z,t) = \rho_{0s}(z) + \delta\rho_s(z,t)\,, \quad \mathbf{E}(z,t) = \delta\mathbf{E}(z,t)\,. \tag{38}$$

Choosing an applied magnetic field to lie in the radial $\mathbf{e}_{\hat{z}}$ direction, the radial and transverse components of the magnetic field are therefore respectively given by

$$\mathbf{B}_z(z,t) = \mathbf{B}_0(z) + \delta\mathbf{B}_z(z,t)\,, \quad \mathbf{B}(z,t) = \delta\mathbf{B}(z,t)\,. \tag{39}$$

The relativistic Lorentz factor is, therefore, also linearized such that

$$\gamma_s = \gamma_{0s} + \delta\gamma_s\,,$$

where

$$\gamma_{0s} = (1 - \mathbf{u}_{0s}^2)^{-\frac{1}{2}}\,, \quad \delta\gamma_s = \gamma_{0s}^3 \mathbf{u}_{0s}\cdot\delta\mathbf{u}_s\,. \tag{40}$$

B. Dependence of unperturbed values on z

The dependence of the unperturbed equilibrium quantities on z is determined in the following manner. Because the fluid elements are in a region close to the black hole horizon, the unperturbed radial velocity for each species as measured by a FIDO along $\mathbf{e}_{\hat{z}}$ is assumed to be the freefall velocity so that

$$u_{0s}(z) = v_{\rm ff}(z) = [1 - \alpha^2(z)]^{\frac{1}{2}}\,. \tag{41}$$

The unperturbed number density can be determined directly from the conservation law for the rest mass (continuity equation). From Eq. (30) one can deduce that

$$r^2 \alpha \gamma_{0s} n_{0s} u_{0s} = \text{const}\,,$$

where r is the Schwarzschild radial coordinate. It is then true that $r_H^2 \alpha_H \gamma_H n_H u_H = r^2 \alpha \gamma_{0s} n_{0s} u_{0s}$, where the values with a subscript H are the limiting values at the horizon. At the horizon, the freefall velocity becomes unity so that $u_H = 1$. Also, because $u_{0s} = v_{\rm ff}$, $\gamma_{0s} = 1/\alpha$ and so $\alpha\gamma_{0s} = \alpha_H \gamma_H = 1$. Since $v_{\rm ff} = (r_H/r)^{\frac{1}{2}}$, the number density for each species can be written as

$$n_{0s}(z) = n_{Hs} v_{\rm ff}^3(z)\,. \tag{42}$$

The unperturbed pressure follows from the equation of state in terms of the number density for each fluid species, as given in Eq. (42) above. That is, since

$$\frac{P_s}{n_s^{\gamma_g}} = \text{const}\,,$$

then

$$P_{0s} = P_{Hs}\left(\frac{n_{0s}}{n_{Hs}}\right)^{\gamma_g}\,.$$

So, from Eq. (42), the pressure for each fluid species may be written in terms of the freefall velocity as

$$P_{0s}(z) = P_{Hs} v_{\rm ff}^{3\gamma_g}(z)\,. \tag{43}$$

From this it is possible to derive the temperature profile as well. Since $P_{0s} = k_B n_{0s} T_{0s}$ then it follows that, with $k_B = 1$,

$$T_{0s}(z) = T_{Hs} v_{\rm ff}^{3(\gamma_g - 1)}(z)\,. \tag{44}$$

Since the unperturbed magnetic field has been chosen to be purely in the radial z direction, it is also parallel to the infall fluid velocity $u_{0s}(z)\mathbf{e}_{\hat{z}}$ for each fluid. Along with the infall fluid velocity, it thus does not experience effects of spatial curvature. The dependence of the magnetic field on the radial coordinate r is due entirely to flux conservation. Since $\nabla\cdot\mathbf{B}_0 = 0$ it follows that, in Schwarzschild coordinates,

$$r^2 B_0(r) = \text{const}\,.$$

From this, it is clear that

$$B_0(r) = B_H\,(r_H/r)^2$$

and so the unperturbed magnetic field may be written in terms of the freefall velocity as

$$B_0(z) = B_H v_{\rm ff}^4(z), \tag{45}$$

where $v_{\rm ff}(z) = [1 - \alpha^2(z)]^{\frac{1}{2}}$.

The derivatives, with respect to z, of the unperturbed field and fluid quantities are clearly directly proportional to the derivative of the freefall velocity with respect to z. Hence, since

$$\frac{dv_{\rm ff}}{dz} = -\frac{\alpha}{2r_H}\frac{1}{v_{\rm ff}}\,, \tag{46}$$

it follows that

$$\frac{du_{0s}}{dz} = -\frac{\alpha}{2r_H}\frac{1}{v_{\rm ff}}\,, \quad \frac{dB_0}{dz} = -\frac{4\alpha}{2r_H}\frac{B_0}{v_{\rm ff}^2}$$
$$\frac{dn_{0s}}{dz} = -\frac{3\alpha}{2r_H}\frac{n_{0s}}{v_{\rm ff}^2}\,, \quad \frac{dP_{0s}}{dz} = -\frac{3\alpha}{2r_H}\frac{\gamma_g P_{0s}}{v_{\rm ff}^2}\,. \tag{47}$$

C. Linearized equations

In the following linearization, products of perturbation terms are neglected. Substituting the linearized variables from Eqs. (38)–(40) directly into the continuity equation leads to

$$\gamma_{0s}\left(\frac{\partial}{\partial t}+u_{0s}\alpha\frac{\partial}{\partial z}+\frac{u_{0s}}{2r_H}+\gamma_{0s}^2\alpha\frac{du_{0s}}{dz}\right)\delta n_s+\left(\alpha\frac{\partial}{\partial z}+\frac{1}{2r_H}\right)(n_{0s}\gamma_{0s}u_{0s})$$

$$+n_{0s}\gamma_{0s}^3\left[u_{0s}\frac{\partial}{\partial t}+\alpha\frac{\partial}{\partial z}+\frac{1}{2r_H}+\alpha\left(\frac{1}{n_{0s}}\frac{dn_{0s}}{dz}+3\gamma_{0s}^2u_{0s}\frac{du_{0s}}{dz}\right)\right]\delta u_s \;=\; 0\,. \quad (48)$$

The same is done for the longitudinal and transverse components of the momentum conservation equation. Here, it is useful to eliminate the pressure variable δP_s using the equation of state or, equivalently, the condition for the conservation of entropy, Eq. (11), which, when linearized, gives

$$\delta P_s=\frac{\gamma_g P_{0s}}{n_{0s}}\delta n_s\,. \quad (49)$$

Hence, it is also possible to write

$$\delta\rho_s=\frac{\rho_{0s}}{n_{0s}}\left(1+\frac{\gamma_{0s}^2\gamma_g P_{0s}}{\rho_{0s}}\right)\delta n_s+2u_{0s}\gamma_{0s}^2\rho_{0s}\delta u_s\,, \quad (50)$$

where $\rho_{0s}=\gamma_{0s}^2\left(m_s n_{0s}+\Gamma_g P_{0s}\right)$. Therefore, the longitudinal part of the momentum equation becomes

$$\left\{\frac{\partial}{\partial t}+\alpha u_{0s}\frac{\partial}{\partial z}+\alpha\gamma_{0s}^2\left(1+u_{0s}^2\right)\frac{du_{0s}}{dz}\right\}\delta u_s-\frac{\alpha q_s n_{0s}}{\rho_{0s}\gamma_{0s}}\delta E_z+\frac{1}{\gamma_{0s}^2 n_{0s}}\left\{\frac{\gamma_{0s}^2\gamma_g P_{0s}}{\rho_{0s}}\left(\alpha\frac{\partial}{\partial z}+u_{0s}\frac{\partial}{\partial t}\right)\right.$$

$$\left.+\alpha\gamma_{0s}^2\frac{\gamma_g P_{0s}}{\rho_{0s}}\left(\frac{1}{P_{0s}}\frac{dP_{0s}}{dz}-\frac{1}{n_{0s}}\frac{dn_{0s}}{dz}\right)+\left(1+\frac{\gamma_{0s}^2\gamma_g P_{0s}}{\rho_{0s}}\right)\left(u_{0s}\gamma_{0s}^2\alpha\frac{du_{0s}}{dz}+\frac{1}{2r_H}\right)\right\}\delta n_s$$

$$+\left(u_{0s}\alpha\frac{du_{0s}}{dz}+\frac{\alpha}{\rho_{0s}}\frac{dP_{0s}}{dz}+\frac{1}{\gamma_{0s}^2 2r_H}\right)\;=\;0\,. \quad (51)$$

The transverse part of the momentum conservation equation is linearized and differentiated with respect to t. Substituting for the magnetic field using Eq. (32), it becomes

$$\left(\alpha u_{0s}\frac{\partial}{\partial z}+\frac{\partial}{\partial t}-\frac{u_{0s}}{2r_H}+\frac{i\alpha q_s\gamma_{0s}n_{0s}B_0}{\rho_{0s}}\right)\frac{\partial\delta v_s}{\partial t}-\frac{\alpha q_s\gamma_{0s}n_{0s}}{\rho_{0s}}\left(\alpha u_{0s}\frac{\partial}{\partial z}+\frac{\partial}{\partial t}+\frac{u_{0s}}{2r_H}\right)\delta E=0\,. \quad (52)$$

Poisson's equation is linearized to give

$$\frac{\partial\delta E_z}{\partial z}=4\pi e(n_{02}\gamma_{02}-n_{01}\gamma_{01})+4\pi e(\gamma_{02}\delta n_2-\gamma_{01}\delta n_1)+4\pi e(n_{02}u_{02}\gamma_{02}^3\delta u_2-n_{01}u_{01}\gamma_{01}^3\delta u_1) \quad (53)$$

and Eq. (33) is linearized such that

$$\left(\alpha^2\frac{\partial^2}{\partial z^2}+\frac{3\alpha}{2r_H}\frac{\partial}{\partial z}-\frac{\partial^2}{\partial t^2}+\frac{1}{(2r_H)^2}\right)\delta E=4\pi e\alpha\left(n_{02}\gamma_{02}\frac{\partial\delta v_2}{\partial t}-n_{01}\gamma_{01}\frac{\partial\delta v_1}{\partial t}\right). \quad (54)$$

VI. LOCAL APPROXIMATION

It is now possible to investigate the information which may be obtained from the linearized equations above by restricting consideration to effects on a local scale for which the distance from the horizon does not vary significantly. A local (or mean-field) approximation will be used here for the lapse function and therefore also for the equilibrium fields and fluid quantities. It then becomes possible to derive analytic expressions for the resulting dispersion relations which give a preliminary indication of the impact of gravitational effects on wave propagation.

This is not an unreasonable approximation even if it is assumed that the plasma is situated relatively close to the horizon (as in the present work using the Rindler metric). This means, in effect, that $\alpha^2 \ll 1$ so that even a relatively small change in distance z will make a significant difference to the magnitude of α. Thus it is important to choose a sufficiently small range in z for which α does not vary much. Note also that the terms due to the derivatives of these quantities in the above equations are $O(\alpha^2)$ and so are relatively small. This approach then amounts to considering thin layers in the $\mathbf{e}_{\hat{z}}$ direction, each layer with its own α_0, where α_0 is some mean value of α within a particular layer. A more complete picture can then be built up by considering a large number of layers within a chosen range of α_0 values.

Such a local approximation does, however, impose a restriction on the magnitude of the wavelength and, therefore, the wave number k. That is, it is assumed that the wavelength is small compared with the range over which the equilibrium quantities change significantly. Hence, the wavelength must be smaller in magnitude than the scale of the gradient of the lapse function, α. That is,

$$\lambda < \left(\frac{\partial \alpha}{\partial z}\right)^{-1} = 2r_H \simeq 5.896 \times 10^5 \text{ cm}$$

or, alternatively,

$$k > \frac{2\pi}{2r_H} \simeq 1.067 \times 10^{-5} \text{ cm}^{-1}$$

for a black hole of mass $\sim 1M_\odot$.

One of the drawbacks associated with the hydrodynamical approach used in the present work (and therefore in the special relativistic case, SK [7], as well) is that it is essentially a bulk, fluid approach and so the microscopic behavior of the two-fluid plasma is treated in a somewhat approximate manner via the equation of state. This means that the results are really only strictly valid in the long wavelength limit. The restriction on the wavelength, imposed by the local approximation, however, is not too severe and still permits the consideration of intermediate to long wavelengths so that the small k limit is still valid in the present work.

The unperturbed field and fluid quantities are not assumed to be constant with respect to α (and therefore z as well) so that the derivatives of these quantities are nonzero. The local approximation is then used for α so that the derivatives of the equilibrium quantities are evaluated at each layer for a given α_0. In the local approximation for α, $\alpha \simeq \alpha_0$ is valid within a particular layer, where α_0 is the "mean-field" value of α. Therefore, the unperturbed fields and fluid quantities and their derivatives, which are functions of α, take on their corresponding "mean-field" values for a given α_0. Hence, the coefficients in Eqs. (51), (48), and (53) become constant within each layer, evaluated at each fixed mean-field value, $\alpha = \alpha_0$. Because the coefficients in the linearized equations are no longer z dependent within each layer, it is possible to Fourier transform the equations with respect to z, assuming plane-wave-type solutions for the perturbations of the form $\sim e^{i(kz-\omega t)}$ for each α_0 layer.

VII. DISPERSION RELATION

Equations describing transverse Alfvén waves and high frequency electromagnetic waves propagating parallel to the constant magnetic field can be derived from Eqs. (21) and (22). In order to obtain the dispersion relation for the transverse electromagnetic wave modes, the transverse component of the momentum conservation equation, Eq. (52), and Maxwell's equation, Eq. (54), are Fourier transformed. First, Eq. (54), Fourier transformed, becomes

$$\delta E = \frac{i4\pi e\alpha_0\omega\left(n_{02}\gamma_{02}\delta v_2 - n_{01}\gamma_{01}\delta v_1\right)}{\alpha_0 k\left(\alpha_0 k - i3/2r_H\right) - \omega^2 - 1/(2r_H)^2} \,. \tag{55}$$

Similarly, the transverse part of the conservation of momentum equation, Eq. (52), when Fourier transformed, becomes

$$\omega\left(\alpha_0 k u_{0s} - \omega + \frac{iu_{0s}}{2r_H} + \frac{\alpha_0 q_s\gamma_{0s}n_{0s}B_0}{\rho_{0s}}\right)\delta v_s - i\alpha_0\frac{q_s\gamma_{0s}n_{0s}}{\rho_{0s}}\left(\alpha_0 k u_{0s} - \omega - \frac{iu_{0s}}{2r_H}\right)\delta E = 0 \,. \tag{56}$$

The dispersion relation follows and may be written as

$$\left[K_\pm\left(K_\pm \pm \frac{i}{2r_H}\right) - \omega^2 + \frac{1}{(2r_H)^2}\right] = \alpha_0^2\left\{\frac{\omega_{p1}^2\left(\omega - u_{01}K_\pm\right)}{\left(u_{01}K_\mp - \omega - \alpha_0\omega_{c1}\right)} + \frac{\omega_{p2}^2\left(\omega - u_{02}K_\pm\right)}{\left(u_{02}K_\mp - \omega + \alpha_0\omega_{c2}\right)}\right\} \tag{57}$$

for either the electron-positron or electron-ion plasma, where $\omega_{cs} = e\gamma_{0s}n_{0s}B_0/\rho_{0s}$ and $K_\pm = \alpha_0 k \pm i/2r_H$. As for the plasma frequency, the cyclotron frequency, ω_{cs}, is also frame independent. The factors of γ_{0s} do not cancel out explicity however because, although the fluid quantities are measured in the fluid frame, the field B_0 is measured in the FIDO frame. A boost to the fluid frame involves the transformation $B_0 \to \gamma_{0s}B_0$ for either fluid thereby canceling the γ_{0s} factors. The $+$ and $-$ denote the left and right modes, respectively (left and right circularly polarized) where the dispersion relation for the left mode is obtained by taking the complex conjugate of the dispersion relation for the right mode. As opposed to the special relativistic case, where both the R and L modes have the same dispersion relation, the terms due to gravitational acceleration in the present work contribute to make the dispersion relations for the R and L modes different.

VIII. NUMERICAL SOLUTION OF MODES

To determine all the physically meaningful modes for both the longitudinal and transverse waves it is necessary to solve the dispersion relations given above numerically. Even in the simplest cases for the electron-positron plasma where both species are assumed to have the same equilibrium parameters, the dispersion relations are complicated enough to make any attempt at an analytical solution cumbersome and unprofitable. The numerical analysis has been carried out using the well-known EISPACK routines based on the standard eigenvalue method [14]. The EISPACK routines require the equations to be supplied in the form of a matrix equation as follows:

$$(A - kI)\,X = 0\,, \tag{58}$$

where the eigenvalue is chosen here to be the wave number k, the eigenvector X is given by the relevant set of perturbations, and I is the identity matrix. The vector A is the sum of two matrices A_R and A_I. The elements of these are, respectively, the real and imaginary terms in the coefficients of the equations for the perturbations.

The perturbation equations for both the longitudinal and transverse waves must therefore be written in an appropriate form. The equations are then expressed in terms of the following set of dimensionless variables:

$$\begin{gathered}\tilde{\omega} = \frac{\omega}{\alpha_0\omega_*}\,,\quad \tilde{k} = \frac{kc}{\omega_*}\,,\quad k_H = \frac{1}{2r_H\omega_*}\,,\\ \delta\tilde{u}_s = \frac{\delta u_s}{u_{0s}}\,,\quad \delta\tilde{v}_s = \frac{\delta v_s}{u_{0s}}\,,\quad \delta\tilde{n}_s = \frac{\delta n_s}{n_{0s}}\,,\\ \delta\tilde{B} = \frac{\delta B}{B_0}\,,\quad \delta\tilde{E} = \frac{\delta E}{B_0}\,,\quad \delta\tilde{E}_z = \frac{\delta E_z}{B_0}\,.\end{gathered} \tag{59}$$

Although δu_s, δv_s, and u_{0s} are already dimensionless, it is convenient to define $\delta\tilde{u}_s$ and $\delta\tilde{v}_s$ as above for consistency.

For the case of an electron-positron plasma $\omega_{p1} = \omega_{p2}$ and $\omega_{c1} = \omega_{c2}$ (as the choice of input parameters is the same for each fluid) so that ω_* is defined as

$$\omega_* = \begin{cases} \omega_c \ \text{ Alfvén modes,} \\ (2\omega_p^2 + \omega_c^2)^{\frac{1}{2}} \ \text{ high frequency modes}\,, \end{cases} \tag{60}$$

where $\omega_p = \sqrt{\omega_{p1}\omega_{p2}}$ and $\omega_c = \sqrt{\omega_{c1}\omega_{c2}}$.

For the electron-ion plasma the choice of ω_* is a more complicated matter because both the plasma frequency and the cyclotron frequency are different for each fluid and so again it is not clear from the dispersion relations what the natural choice of ω_* should be. For simplicity, therefore, it has been assumed that

$$\omega_* = \begin{cases} \frac{1}{\sqrt{2}}\left(\omega_{c1}^2 + \omega_{c2}^2\right)^{\frac{1}{2}} \ \text{ Alfvén modes,} \\ (\omega_{*1}^2 + \omega_{*2}^2)^{\frac{1}{2}} \ \text{ high frequency modes}\,, \end{cases} \tag{61}$$

where $\omega_{*s}^2 = \left(2\omega_{ps}^2 + \omega_{cs}^2\right)$. These values have been chosen for ω_* because they reduce to the special relativistic cutoffs in the zero gravity limit for an electron-positron plasma. In the special relativistic case the cutoffs are determined by the dispersion relation in that the solutions to the dispersion relation are physical [that is, $\mathrm{Re}(k) > 0$] only for certain frequency regimes. Because the dispersion relations cannot be handled analytically it is difficult to determine what the cutoffs should be in the present case including gravity. Other similar combinations for ω_* should not make any real difference to the form of the results as ω_* is really only a scale factor.

The dimensionless eigenvector for the transverse set of equations is given by

$$\tilde{X}_{\text{transverse}} = \begin{bmatrix} \delta\tilde{v}_1 \\ \delta\tilde{v}_2 \\ \delta\tilde{B} \\ \delta\tilde{E} \end{bmatrix}. \tag{62}$$

Instead of using Eqs. (55) and (56), it is more convenient to begin with Eq. (56) in combination with Eq. (32) and the precursor to Eq. (33):

$$i\frac{\partial E}{\partial t} = -\alpha\left(\frac{\partial}{\partial z} - a\right)B - i4\pi e\alpha\left(\gamma_2 n_2 v_2 - \gamma_1 n_1 v_1\right)\,. \tag{63}$$

When Eqs. (32) and (63) are linearized and Fourier transformed they become

$$\left(k - \frac{i}{2r_H\alpha_0}\right)\delta E + \frac{i\omega}{\alpha_0}\delta B = 0 \tag{64}$$

and

$$\begin{aligned}\frac{i\omega}{\alpha_0}\delta E = {} & \left(k - \frac{i}{2r_H\alpha_0}\right)\delta B \\ & + 4\pi e\left(\gamma_{02}n_{02}\delta v_2 - \gamma_{01}n_{01}\delta v_1\right)\,.\end{aligned} \tag{65}$$

Using Eq. (59), Eqs. (56), (64), and (65) are then written in dimensionless form:

$$\begin{aligned}\tilde{k}\delta\tilde{v}_s = {} & \left(\frac{\tilde{\omega}}{u_{0s}} - \left(\frac{q_s}{e}\right)\frac{\omega_{cs}}{u_{0s}\omega_*} - \frac{ik_H}{\alpha_0}\right)\delta\tilde{v}_s \\ & + \left(\frac{q_s}{e}\right)\frac{\omega_{cs}}{u_{0s}\omega_*}\delta\tilde{B} - i\left(\frac{q_s}{e}\right)\frac{\omega_{cs}}{u_{0s}^2\omega_*}\delta\tilde{E}\,,\end{aligned} \tag{66}$$

$$\tilde{k}\delta\tilde{E} = -i\tilde{\omega}\delta\tilde{B} + \frac{ik_H}{\alpha_0}\delta\tilde{E}, \tag{67}$$

and

$$\tilde{k}\delta\tilde{B} = u_{01}\frac{\omega_{p1}^2}{\omega_{c1}\omega_*}\delta\tilde{v}_1 - u_{02}\frac{\omega_{p2}^2}{\omega_{c2}\omega_*}\delta\tilde{v}_2 + \frac{ik_H}{\alpha_0}\delta\tilde{B} + i\tilde{\omega}\delta\tilde{E}\,. \tag{68}$$

These equations are now in the required form to be used as input to Eq. (58).

IX. RESULTS

The cases considered here comprise both the electron-positron plasma and the electron-ion plasma, for completeness. The limiting horizon values for the magnetic field and the fluid parameters are as follows. For the electron-positron plasma the horizon values are chosen to be

$$n_{Hs} = 10^{18}\ \text{cm}^{-3},\quad T_{Hs} = 10^{10}\ \text{K},\quad B_H = 3\times 10^6\ \text{G},\quad \text{and } \gamma_g = \frac{4}{3}\,. \tag{69}$$

For the electron-ion plasma, where the ions can be assumed to be essentially nonrelativistic, the limiting values are taken to be

$$n_{H1} = 10^{18}\ \text{cm}^{-3}, \quad T_{H1} = 10^{10}\ \text{K},$$
$$n_{H2} = 10^{15}\ \text{cm}^{-3}, \quad T_{H2} = 10^{12}\ \text{K}.$$

The limiting horizon value for the equilibrium magnetic field takes on the same value as it does for the electron-positron case above. The choice of limiting horizon temperature for each species is derived from studies of two temperature models of spherical accretion onto black holes by Colpi *et al.* [11]. Values for the limiting horizon densities and the limiting horizon field are simply arbitrarily chosen values which appear to be not inconsistent with current ideas. The gas constant and the mass of the black hole have been chosen to be

$$\gamma_g = \frac{4}{3} \text{ and } M = 5M_\odot . \tag{70}$$

A. Alfvén modes

1. *Electron-positron plasma*

In the special relativistic case investigated by SK for the ultrarelativistic electron-positron plasma, only one, purely real Alfvén mode was found to exist. This is because both the left and right circularly polarized modes were described by the same dispersion relation thereby leading to the same mode. This is illustrated in Fig. 1 for similar field and fluid parameters as outlined for the case including gravity.

The Alfvén modes in the presence of the black hole's gravitational field are interesting in that there exist two Alfvén modes for the electron-positron plasma compared with four modes for the electron-ion plasma. These two modes for the electron-positron plasma coalesce into a single mode on taking the special relativistic limit so yielding the SK result. The two modes for the electron-positron plasma, shown in Fig. 2 and Fig. 3, show that the growth and damping effects are relatively small. The second mode is obtained from the complex conjugate dispersion relation to that which leads to the first mode. The modes are not, however, complex conjugates of each other. Here, damping corresponds to $\text{Im}(\tilde{k}) > 0$ and growth therefore corresponds to $\text{Im}(\tilde{k}) < 0$. This is because the convention used is $e^{ikz} = e^{i[\text{Re}(k)+i\text{Im}(k)]z}$.

2. *Electron-ion plasma*

In the case of the electron-ion plasma, however, there exist four modes, two of which are damped and two of which show growth. The first two modes, shown in Fig. 4, are a complex conjugate pair and are significantly damped and growing, respectively, whereas the other two modes, shown in Fig. 5 and Fig. 6, display only marginal damping and growth, respectively. It is these two modes which are equivalent to the electron-positron modes discussed above. For the remaining two

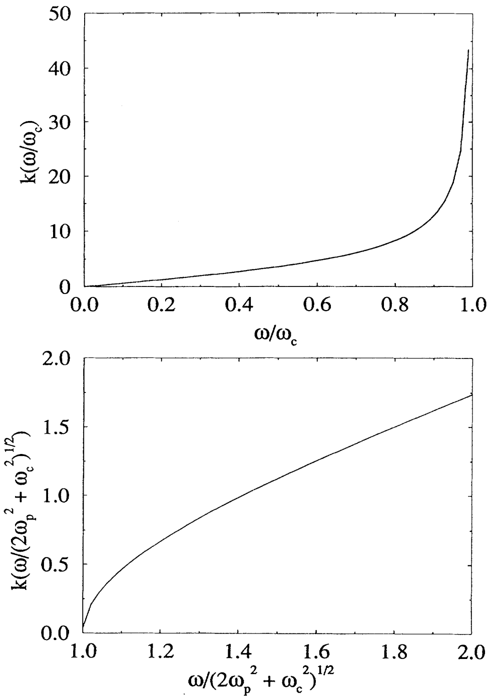

FIG. 1. Special relativistic results for the electron-positron plasma. The fluid and field parameters were chosen to be $n_0 = 10^{15}$ cm^{-3}, $T_0 = 10^9$ K, $B_0 = 3 \times 10^4$ G, and $\gamma_g = 4/3$. Top, Alfvén mode; and bottom, high frequency electromagnetic mode.

modes the differences in the magnitudes of the ω_{c1} and ω_{c2} are apparently sufficient to take the frequencies from their negative (and therefore unphysical) values for the electron-positron case to positive physical values for the electron-ion case. These changes are therefore due to the difference in mass and density factors as between the

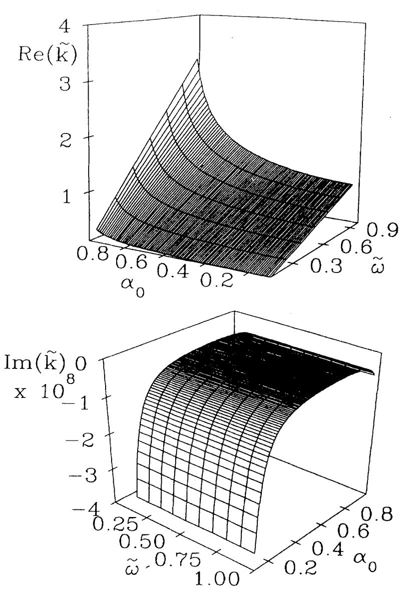

FIG. 2. Top: Real part of Alfvén growth mode for the electron-positron plasma. Bottom: Imaginary part of Alfvén growth mode.

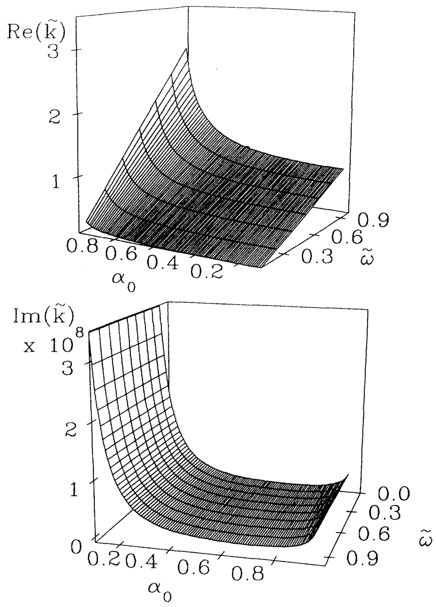

FIG. 3. Top: Real part of damped Alfvén mode for electron-positron plasma. Bottom: Imaginary part of damped Alfvén mode.

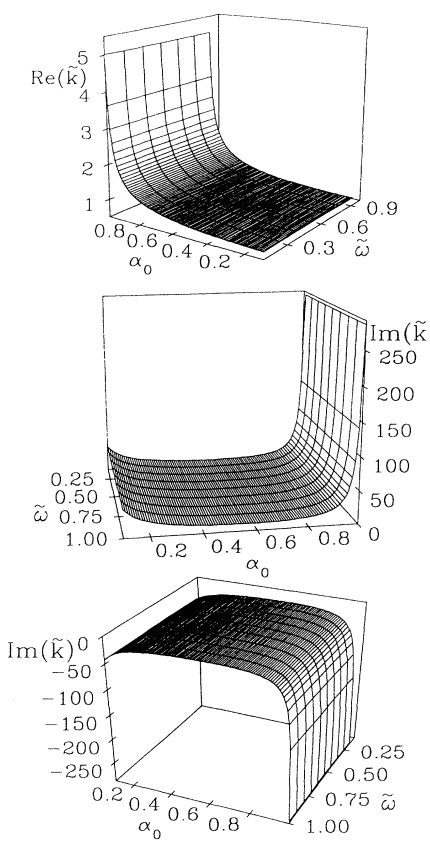

FIG. 4. Top: Real part of the complex conjugate pair of Alfvén modes for the electron-ion plasma. Center: Imaginary part of the damped mode. Bottom: Imaginary part of the growth mode.

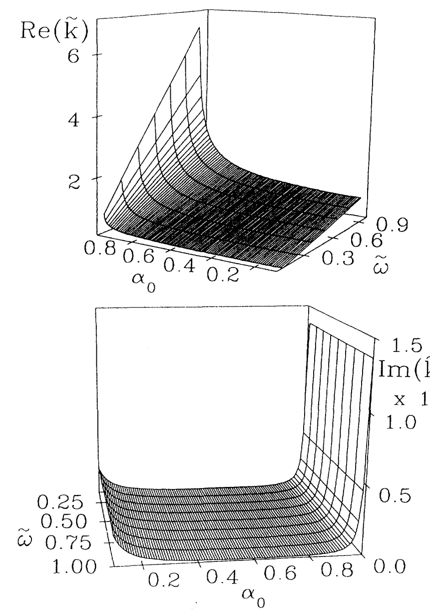

FIG. 5. Damped Alfvén mode for the electron-ion plasma.

positrons and ions. Note that the growth and damping rates are independent of frequency and only depend on the distance from the black hole horizon through α_0.

B. High frequency transverse modes

1. Electron-positron plasma

There exist three high frequency electromagnetic modes for the electron-positron plasma. These are illustrated in Figs. 7–9. Figures 7 and 8 show two modes

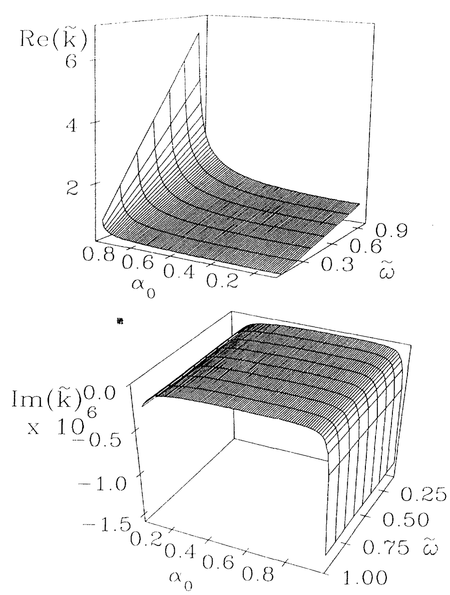

FIG. 6. Alfvén growth mode for the electron-ion plasma.

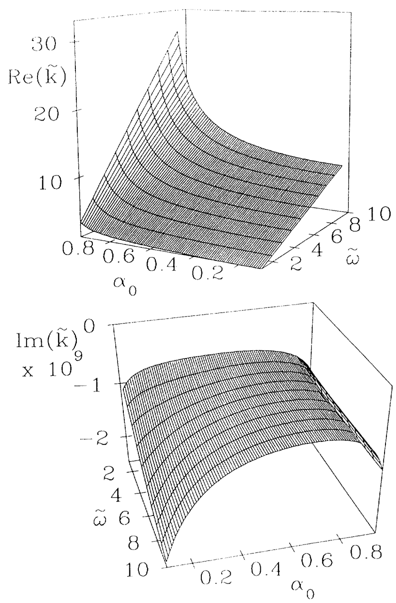

FIG. 7. High frequency transverse growth mode for the electron-positron plasma.

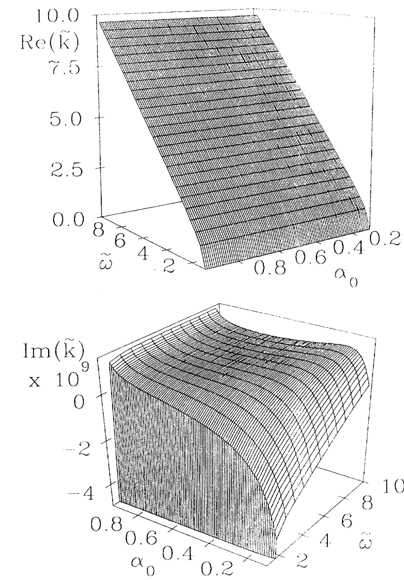

FIG. 9. High frequency transverse mode for the electron-positron plasma showing both damping and growth.

which are similar and although both show a small amount of growth, the mode in Fig. 8 is damped very close to the horizon. Thus at a distance from the horizon corresponding to $\alpha_0 < 0.2$, it appears that energy is no longer fed into the wave mode by the gravitational field but begins to be drained from the waves. The third mode, Fig. 9, is damped for most of the frequency domain but shows growth for lower frequencies as $\tilde{\omega} \to 1$ and $\alpha_0 \to 0$. Close to the horizon, below about $\alpha_0 \sim 0.3$, it becomes a growth mode for all frequencies. Again, in the special relativistic case investigated by SK, there exists only one purely real high frequency mode for the ultrarelativistic

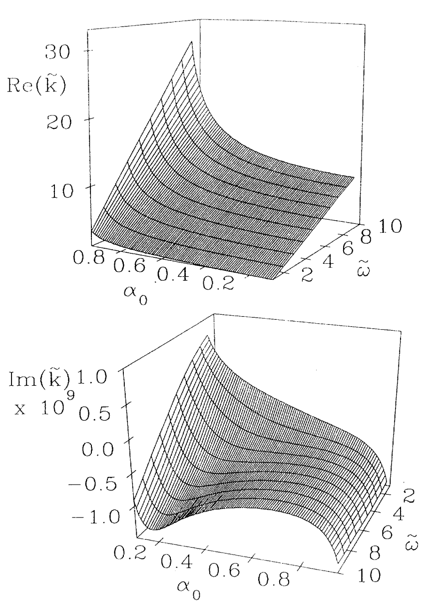

FIG. 8. High frequency transverse mode for the electron-positron plasma showing both damping and growth.

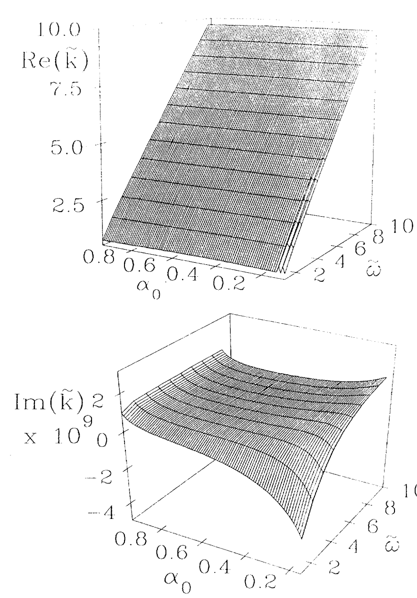

FIG. 10. High frequency transverse mode for the electron-ion plasma showing both damping and growth.

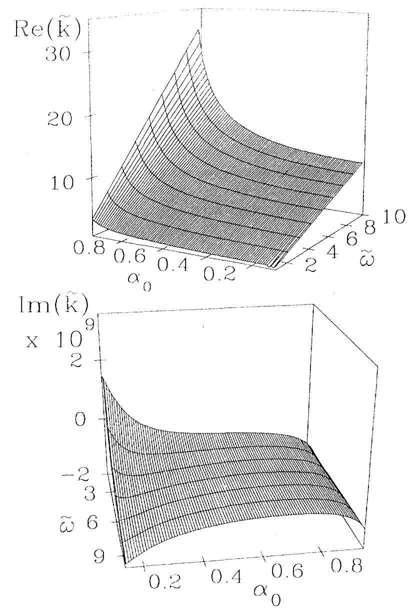

FIG. 11. High frequency transverse mode for the electron-ion plasma showing both damping and growth.

electron-positron plasma. This is shown in Fig. 1 and corresponds again to the coalescence of the modes illustrated in Fig. 7 and Fig. 8 as $\alpha_0 \to 1$.

2. Electron-ion plasma

As for the electron-positron plasma, the electron-ion plasma has three high frequency modes. The first two, Fig. 10 and Fig. 11, show damping and growth and for both of these modes, the solution becomes unstable as $\alpha_0 \to 0$ and $\tilde{\omega} \to 1$. This may simply be because the solution is too close to a resonant frequency. Figure 12 is purely a growth mode. Unlike the first two, this mode is stable for all frequencies and at all distances from the horizon. The dependence of the growth and decay rates on frequency is clearly evident, unlike the corresponding Alfvén modes.

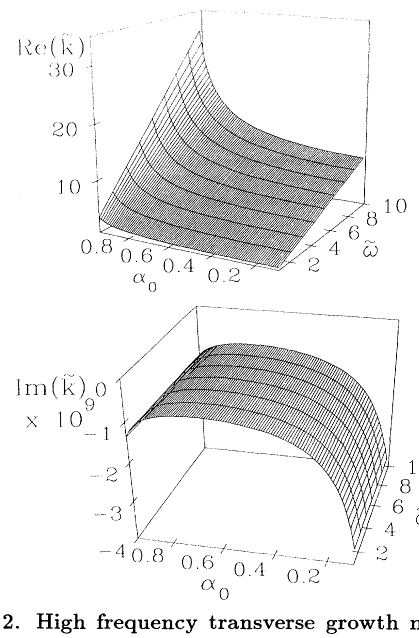

FIG. 12. High frequency transverse growth mode for the electron-ion plasma.

X. CONCLUSION

Using a local approximation, the dispersion relations for the Alfvén and high frequency electromagnetic waves have been derived. In the limit of zero gravity these results reduce in each case to the special relativistic results obtained by Sakai and Kawata [7].

Unlike the special relativistic SK work where only one purely real mode was found for both the Alfvén and high frequency electromagnetic waves, new modes, all of which are either damped or growing, arise in the present work because of the black hole's gravitational field. In general, for the electron-positron plasma, the damping and growth rates are smaller, by several orders of magnitude, compared with the real components of the wave number. On the other hand, for the electron-ion plasma, modes exist for which the damping and growth rates are significant. This is particularly true for the Alfvén waves. The damping and growth rates are clearly frequency independent for the Alfvén waves, being solely dependent on the radial distance from the horizon as denoted by the mean value of the lapse function α_0. This is not the case for the high frequency waves where the rate of damping or growth is dependent on both frequency and radial distance from the horizon.

The fact that some modes are damped demonstrates that, at least in this approximation, energy is being drained from some of the waves by the gravitational field. On the other hand, the majority of the modes show growth rates indicating that the gravitational field is, in fact, feeding energy into the waves.

The present paper has been exclusively devoted to the investigation, within the local approximation, of Alfvén and high frequency electromagnetic waves in a two-fluid plasma surrounding a Schwarzschild black hole. The next paper in the present series (paper II) will be concerned with the investigation of the longitudinal waves and the two-stream instability in this environment. These two papers conclude the material based on the local approximation discussed above.

Further papers will be based on the linear two fluid equations in Schwarzschild coordinates obtained from the fundamental nonlinear equations, Eqs. (8) and (16)–(22), introduced here. These will contain numerical solutions to the ordinary differential equations for the perturbations. In this way more detailed physical characteristics can be obtained about the longitudinal and transverse waves together with the two-stream instability. In particular the energy density in the waves will be calculated as a function of radial distance from the black hole horizon.

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