Hot string soup: Thermodynamics of strings near the Hagedorn transition

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Above the Hagedorn energy density closed fundamental strings form a long string phase. The dynamics of weakly interacting long strings is described by a simple Boltzmann equation which can be solved explicitly for equilibrium distributions. The average total number of long strings grows logarithmically with total energy in the microcanonical ensemble. This is consistent with calculations of the free single string density of states provided the thermodynamic limit is carefully defined. If the theory contains open strings the long string phase is suppressed.

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I. INTRODUCTION

The statistical mechanics of strings at high temperature differs significantly from that of pointlike objects. The exponential growth in the single-string density of states as a function of mass [1–4] makes the canonical partition function,

$$Z = \text{tr}\,[e^{-\beta H}], \tag{1}$$

of a free string gas ill defined as one approaches the so-called Hagedorn temperature. Different physical interpretations of this fact have been offered, including that the Hagedorn temperature defines an absolute limiting temperature in physics [1,2] or that it signals a transition to an unknown high-temperature phase where strings may be replaced by more fundamental degrees of freedom [5].

The high-energy limit of the single-string density of states for free closed strings in D noncompact space dimensions is found to be [6]

$$\omega(\varepsilon) \approx \frac{V \exp\left(\beta_H \varepsilon\right)}{\varepsilon^{D/2+1}}, \tag{2}$$

where V is the D-volume occupied by the system and β_H is the inverse Hagedorn temperature. Given this density of states one can compute the energy distribution function $d(\varepsilon, E)$ in the microcanonical ensemble, which gives the average number of strings carrying energy between ε and $\varepsilon + \delta\varepsilon$ in a system of total energy E, and at high-energy density it is found to favor the formation of a single long string which carries most of the available energy [3,4,7–10].

This physical picture is suspect because it neglects the effect of interactions. A long string, which carries a finite

fraction of the total energy in a system at high-energy density, will extend further than the size of the system itself and as the thermodynamic limit is approached it traverses the entire volume of the system many times over. In this limit the string has numerous opportunities to intersect itself and it is natural to ask how string interactions affect the equilibrium configuration. We address this question by writing down a Boltzmann equation appropriate for very long closed strings and obtaining self-consistent solutions. In the high-energy limit the density of states takes the form

$$\omega(\varepsilon) \approx \frac{\exp\left(\beta_H \varepsilon\right)}{\varepsilon}, \tag{3}$$

instead of (2) and this has important consequences for the energy distribution function and the thermodynamic behavior of the system. There is still a long string phase but it is dominated by a large number of long strings which may split and join. Similar conclusions were reached by Salomonson and Skagerstam [11] in the context of a discrete model of strings, but these seem to have been overlooked in much of the subsequent literature on the subject.

The discrepancy between (2) and (3) can be understood as follows. The microcanonical ensemble is defined only for finite volume systems with finite total energy and some care is required in defining the thermodynamic limit for extended objects such as strings. At present the only consistent description of strings in finite volume is to consider a compact target space. The density of states in the thermodynamic limit is then obtained by performing a calculation on a finite sized target space and then letting the size tend to infinity at the end of the day.

There is a further subtlety which needs to be considered. The assumption of equipartition, which is needed to make contact between the density of states and equilibrium distribution functions, requires the existence of interactions, however weak they may be. Since string interactions always include gravity the thermal ensemble is only defined for length scales satisfying

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$$R^2 < \frac{1}{g^2 \rho}, \tag{4}$$

due to the Jeans instability. In the thermodynamic limit we wish to consider fixed energy density, ρ, as the volume becomes large and from (4) we see that the limit can only be taken if at the same time the string coupling g is scaled to zero sufficiently rapidly. Alternatively, we can at the outset fix the value of the string coupling to be extremely small and then restrict our considerations to systems of large, but finite, volume consistent with (4).

Salomonson and Skagerstam [11] showed that when the extended dimensions of space are taken to be compact-ified on a D-torus the free string density of states for closed strings indeed takes the form (3) rather than (2), which was obtained for strings in noncompact space, and this was later confirmed by other groups [9, 10]. One might worry that this result depends on the choice of topology that is used to implement string theory in fi-nite volume. For example, one could consider instead of a torus a group manifold of some simple group where strings have no winding modes and the counting of high-energy states is naively quite different from the toroidal case. It turns out, however, that the global topology plays no role. A general argument based on modular in-variance, given by Brandenberger and Vafa [12], shows that the free density of states for closed strings takes the form (3) for sufficiently high energy on any compact tar-get space.

Thus the very existence of interactions has a subtle but important effect on the density of states of strings at high energy. This effect is entirely due to the extended nature of strings and can be seen in two different ways. On the one hand the Jeans instability forces one to consider only strings on a compact target space and this affects the counting of free single-string states. The other method will be discussed in the present paper and involves solving a set of transport equations for weakly interacting strings. These equations are quite simple in the long string limit and relatively little work is required to get at the density of single string states using our approach.

The plan of this paper is as follows. In Sec. II we intro-duce the Boltzmann equation for long closed strings and obtain self-consistent equilibrium solutions. These solu-tions agree with the discrete string model results of Sa-lomonson and Skagerstam [11]. The Boltzmann equation also allows one to consider time-dependent distributions of strings. As an example we compute the rate of decay of a small initial perturbation to an equilibrium string distribution. In Sec. III we compare our results with cal-culations based on the free string density of states and discuss further the physical picture that emerges. Sec-tion IV sets up similar Boltzmann equations for long open strings. In this case, however, no long string phase forms as the energy density is increased beyond the Hagedorn energy density. The probability for string decay increases as the length of string increases, while the probability for rejoining decreases inversely with the volume, so that in the thermodynamic limit long strings are suppressed in open string theory. Section V summarizes our conclu-sions.

II. BOLTZMANN EQUATION FOR LONG CLOSED STRINGS

In conventional statistical mechanics there are several ways to obtain equilibrium distribution functions. One is to start from the single object density of states computed in the noninteracting theory and using the equipartition theorem. This is the route that has been taken in much of the previous work on high-temperature string the-ory. Another method is to consider transport equations, which describe how energy and other conserved quanti-ties are redistributed in collisions among constituents of the ensemble, and impose equilibrium conditions.

Consider, for example, the Boltzmann equation for a gas of interacting massive particles:

$$\frac{\partial f(\mathbf{v}_1, t)}{\partial t} = \int d^D v_2 \int d\Omega \sigma(\Omega) |\mathbf{v}_1 - \mathbf{v}_2| (f'_2 f'_1 - f_2 f_1), \tag{5}$$

where $f(\mathbf{v}, t) d^D v$ is the number of particles per unit vol-ume lying in the velocity volume $d^D v$ about $\mathbf{v}$, and $\sigma(\Omega)$ is the differential scattering cross section. This equation describes the change in time of the distribution function due to binary collisions of the form $\{\mathbf{v}_1, \mathbf{v}_2\} \to \{\mathbf{v}_1', \mathbf{v}_2'\}$. The derivation of this equation makes the crucial assump-tion that the velocity of a particle is uncorrelated with its position, which is usually valid for sufficiently dilute sys-tems. The equilibrium solution of the Boltzmann equa-tion (5) is the Maxwell-Boltzmann distribution

$$f(\mathbf{v}) \sim e^{-\beta m v^2 / 2}, \tag{6}$$

which describes a gas of particles in the canonical ensem-ble, i.e., fixed volume and temperature.

We want to obtain an analogous equation for a gas of interacting string loops. In the limit of very small loops the extended nature of strings becomes irrelevant and the result for massless particles must be recovered. At intermediate scales we expect the loop equations of string theory to be complicated and we will not attempt to de-rive them in full generality here. Instead we focus our attention on very long string loops, which satisfy rather simple equations and whose distribution function can be explicitly computed. The limit of long strings is of con-siderable interest since previous studies of the free string density of states suggest that the microcanonical ensem-ble becomes dominated by long strings, as the Hagedorn energy density is approached [3, 4, 7–12].

A key observation is that when the string length be-comes large compared with the spatial size of the system the string will traverse the entire volume many times over and, assuming a generic parametrization of the string, two points on the string, which are separated by a finite parameter distance, are found at uncorrelated positions in the embedding space. The Boltzmann equation for such strings will only involve intrinsic properties of the strings, such as their length, but not any details of their embedding.

A further simplification for long strings is that their en-ergy is dominated by the string tension so we can simply

characterize the single-string states by their loop length. It follows from condition (4) that the interaction energy is a small fraction of the total energy of the system and at equilibrium that fraction will not change with time. Thus we only include terms in the Boltzmann equation that conserve total length, so that to a good approximation the interactions will conserve energy.

The analogue of the Boltzmann equation for long oriented closed strings takes the form

$$\frac{\partial n(\ell)}{\partial t} = \frac{\kappa}{V}\left\{-\frac{1}{2}\ell^2 n(\ell) - \int_0^\infty d\ell'\,\ell'n(\ell')\ell n(\ell)\right.$$
$$+\frac{1}{2}\int_0^\ell d\ell'\,\ell'(\ell-\ell')n(\ell')n(\ell-\ell')$$
$$\left.+\int_\ell^\infty d\ell'\,\ell'n(\ell')\right\}, \qquad (7)$$

where κ is some positive constant which depends on the string coupling, and for convenience we have set $\alpha' = 1$. Here $n(\ell)$ is the average number of strings of length ℓ. The first term on the right-hand side of (7) represents the effect of a loop of length ℓ self-intersecting and splitting into two smaller strings. The factor of ℓ^2/V reflects the probability of finding two bits of the same long string at the same point in the embedding space. The second term comes from two strings of length ℓ and ℓ' joining into a single string of length $\ell+\ell'$. The third term describes two smaller strings joining to form a single string of length ℓ. The fourth term describes a long string of length ℓ' self-intersecting and splitting to form a string of length ℓ and another string of length $\ell' - \ell$.

All four terms on the right-hand side of (7) thus involve a three string interaction and we are ignoring contributions from interactions of four or more strings, which are suppressed when the string coupling is weak. In general, the four terms carry different phase space factors but in the long string limit these factors are the same for all the terms, and are absorbed into the constant κ. This is because the basic interaction is the same in all cases: two short segments of string cut across each other and exchange ends at the intersection point. The interaction rate involves an average over the relative orientation and momentum of the segments when they meet but in the long string limit this average is not sensitive to the overall string length nor to whether or not the segments belong to the same long string before the interaction.

A similar Boltzmann equation holds for unoriented closed strings, with extra factors of $1/2$ appearing in front of the first and fourth terms on the right-hand side of (7). For simplicity, we will restrict our considerations to oriented closed strings in the following, and note that qualitatively similar conclusions hold for unoriented closed strings.

It should be noted that the Boltzmann equation (5) is a truncation of an exact set of equations known as the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy (see [13] for details). An analogous hierarchy governs the exact statistical dynamics of strings but the string Boltzmann equation (7) is sufficient for the purposes of this paper.

At equilibrium the average number of strings of a given length remains constant,

$$\frac{\partial n(\ell)}{\partial t} = 0, \qquad (8)$$

and we therefore consider static solutions to the Boltzmann equation. The Laplace transform of (7) then gives

$$\frac{1}{2}\partial_z \tilde{w}(z) - \bar{L}\tilde{w}(z) + \frac{1}{2}\tilde{w}(z)^2 + \frac{\bar{L} - \tilde{w}(z)}{z} = 0, \qquad (9)$$

where we have defined $w(\ell) = \ell n(\ell)$ and $\bar{L}$ is the average total length of string in the ensemble, given by $\bar{L} = \int_0^\infty d\ell\,w(\ell) = \tilde{w}(0)$. The Laplace transformed equation is solved by

$$\tilde{w}(z) = \frac{1}{z + 1/\bar{L}}, \qquad (10)$$

and taking the inverse Laplace transform gives

$$w(\ell) = e^{-\ell/\bar{L}}. \qquad (11)$$

The average number of strings as a function of length is then

$$n(\ell) = \frac{e^{-\ell/\bar{L}}}{\ell}. \qquad (12)$$

This is the analogue of the Maxwell-Boltzmann distribution for long fundamental strings, describing strings at fixed temperature in the canonical ensemble. Large energy fluctuations appear in the canonical ensemble as the Hagedorn temperature is approached and at that point the canonical and microcanonical ensembles are no longer equivalent. It is preferable to use the more fundamental microcanonical ensemble to describe the physics. In order to do that, we first obtain the single-string density of states from the canonical ensemble distribution (12) using

$$n(\ell) = \omega(\sigma\ell)\sigma e^{-\beta\sigma\ell}, \qquad (13)$$

where σ is the string tension and the energy of a single string is to a good approximation given by $\varepsilon = \sigma\ell$. The resulting density of states is

$$\omega(\varepsilon) = \frac{e^{\beta_H \varepsilon}}{\varepsilon}, \qquad (14)$$

where

$$\beta_H = \beta - \frac{1}{\sigma\bar{L}} \qquad (15)$$

is to be identified with the inverse Hagedorn temperature. Note there is no volume factor in (14) so the density of long string states is not an extensive quantity. We will return to this point in the following section.

Given the single-string density of states (14) one may obtain the multi-string density of states:

$$\Omega(E) \approx e^{\beta_H E}. \qquad (16)$$

The single string distribution function for fixed total energy E is then approximately

$$d(\varepsilon, E) \approx \frac{\omega(\varepsilon)\Omega(E - \varepsilon)}{\Omega(E)} \approx \frac{1}{\varepsilon}. \qquad (17)$$

This approximation is valid for $c < \varepsilon < E - c$ where c is some constant independent of E [10].

The number of long strings of a given length in the microcanonical ensemble thus depends inversely on the length and the average total number of long strings is $\ln E$. If we think of each string as a collection of small bits, whose number is proportional to the string length, then $\varepsilon d(\varepsilon, E)$ determines the relative probability that a given string bit finds itself a part of a string of length $\ell = \varepsilon/\sigma$. Since $\varepsilon d(\varepsilon, E)$ is independent of ε the string bit could belong to a string of any length with equal probability. This is reasonable since a long string traverses the entire volume of the system and its length can at any moment be modified by interactions which occur far away from the given string bit.

One advantage of using a Boltzmann equation to describe hot strings is that we can compute time dependent properties of the system. As an example, consider introducing a small perturbation $\delta n(\ell)$ to the equilibrium distribution (12) localized around some length $\ell = \ell_0$. Solving the linearized Boltzmann equation about the solution (12) one finds an exponential rate of decay of the perturbation,

$$\delta n(\ell, t) = \delta n(\ell, 0) e^{-\kappa(\bar{L} + \frac{1}{2}\ell_0)\ell_0 t / V}, \qquad (18)$$

confirming that the equilibrium distribution found above is stable.

III. COMPARISON TO FREE STRING CALCULATIONS

The above results can be compared to those obtained by direct counting of free string states but that requires some care. Weak string interactions must be introduced to allow the system to explore phase space, so that the equipartition theorem holds. As emphasized above, the Jeans instability then forces one to consider systems with finite size. In order for the calculation based on the free density of states to be fully consistent the target space should be taken to be compact. A thermodynamic limit may be taken at the end of the day provided the string coupling is scaled to zero sufficiently rapidly.

It has been shown by Brandenberger and Vafa [12] that, remarkably, the free string density of states in an arbitrary compact target space takes a universal form in the high-energy limit. This may be seen by considering the one-loop partition function for strings on a compact target space at finite temperature $T = 1/\beta$. The time is now regarded as a compact Euclidean direction with period β. As the Hagedorn temperature is approached the partition function begins to diverge as the state which winds once around the time direction becomes massless [14–16]. One finds that the expectation value of the energy diverges as

$$E \approx -\frac{\partial}{\partial \beta} \int^\infty \frac{d\tau}{\tau} e^{-(\beta - \beta_H)\tau} = \frac{1}{\beta - \beta_H} + \cdots, \qquad (19)$$

from which one may deduce that the asymptotic high-energy density of states is given by (3) [12]. Thus free

string calculations agree with results obtained from the Boltzmann equation, provided the thermodynamic limit is taken in the correct way, i.e., taking the limit of infinite volume only at the very end.

The single-string density of states for closed strings in a large but finite volume V has the form (2) as long as the string length remains small compared to the size of the system. It then crosses over to (3) in the high-energy limit once the string traverses the entire volume. The two expressions for the density of states differ by factor of $V/\varepsilon^{\frac{D}{2}}$, which can be understood by the following heuristic argument due to Polchinski [17]. The shape of a highly energetic string is approximated by a random walk. If the string is not too long the random walk occupies a volume of order $\ell^{\frac{D}{2}} \sim (\varepsilon/\sigma)^{\frac{D}{2}}$, and the density of states contains a factor of $V/\ell^{\frac{D}{2}}$ from the translation zero mode. If, on the other hand, the string is so long that $\ell > V$ then the random walk fills the entire system volume so that this zero mode factor is absent and the density of states takes its high-energy form.

The two expressions (2) and (3) for the free string density of states lead to very different thermodynamic behavior. If one uses the density of states in a noncompact space (2) one is led to conclude that there is a long string phase dominated by a single string (or very few) [3, 4, 7–10], and further that the specific heat derived from (2) is negative, indicating an instability. The calculation which leads to the single-string dominance is, however, inherently inconsistent, since, as stated above, once one allows arbitrarily weak interactions then the Jeans instability sets in and one has to consider a finite volume target space.

In order to determine the sign of the specific heat in the long string phase dictated by (3) it is necessary to calculate higher-order corrections to the density of states. Brandenberger and Vafa [12] showed that these corrections lead to an entropy $S(E)$ in the microcanonical ensemble of the form

$$S = \beta_H E - \frac{C}{E} + \text{const}, \qquad (20)$$

where C is a positive constant. This implies

$$\beta = \left(\frac{\partial S}{\partial E}\right)_V = \beta_H + \frac{C}{E^2}, \qquad (21)$$

and one finds that the specific heat is positive. At this point one might be surprised at the apparent discrepancy between Eqs. (21) and (15). It is a consequence of the inequivalence of the canonical and microcanonical ensembles as the Hagedorn temperature is approached as discussed above.

To sum up, the single long string phase is unstable when interactions are included. Instead, the stable long string phase involves a distribution of long strings which may split and join as described in the previous section.

IV. BOLTZMANN EQUATION FOR LONG OPEN STRINGS

One may apply similar arguments to set up a Boltzmann equation describing long open strings. In this case

we assume a string may split with a constant probability per unit length [18]. Unlike in the closed string case, this does not require self-intersection. When open strings join, the ends of two strings must collide, the likelihood of which falls inversely with the volume of the system [11]. The Boltzmann equation for long oriented open strings is then

$$\frac{1}{\kappa}\frac{\partial n(\ell)}{\partial t} = -\ell n(\ell) - \frac{a}{V}\int_0^\infty d\ell'\, n(\ell')n(\ell)$$
$$+\frac{a}{2V}\int_0^\ell d\ell'\, n(\ell')n(\ell-\ell') + 2\int_\ell^\infty d\ell'\, n(\ell').$$

(22)

Here κ is a positive constant that depends on the string coupling and a is a positive constant related to the string joining probability.

To find the static solutions, it is again convenient to take a Laplace transform to get

$$\tilde{n}'(z) - \frac{a}{V}\bar{N}\tilde{n}(z) + \frac{a}{2V}\tilde{n}(z)^2 + \frac{2}{z}[\bar{N} - \tilde{n}(z)] = 0.$$

(23)

Here $\bar{N}$ is the average total number of open strings. The solution is

$$\tilde{n}(z) = \frac{2V}{a(z + 2V/\bar{N}a)},$$

(24)

and taking the inverse Laplace transform gives

$$n(\ell) = \frac{2V}{a}e^{-2V\ell/\bar{N}a}.$$

(25)

This then is the analogue of the Maxwell-Boltzmann distribution for long open strings in the canonical ensemble. Reading off the density of states, one obtains

$$\omega(\varepsilon) \approx \frac{2V}{a\sigma}e^{\beta_H\varepsilon}.$$

(26)

Unlike for closed strings, the density of states of open strings is an extensive quantity.

Now consider the open string distribution function in the microcanonical ensemble. The multistring density of states, in a saddle point approximation valid in the limit, $V \to \infty$ with E/V fixed, is

$$\Omega(E) \approx \exp\left\{\sqrt{\frac{8EV}{a\sigma}} + \beta_H E\right\},$$

(27)

and for open strings carrying a finite fraction of the total energy of the system the string distribution function

$$d(\varepsilon, E) = \frac{\omega(\varepsilon)\Omega(E-\varepsilon)}{\Omega(E)},$$
$$\sim \exp\left\{-\sqrt{\frac{8EV}{a\sigma}}\left(1 - \sqrt{1 - \varepsilon/E}\right)\right\}$$

(28)

is strongly suppressed in this limit. Thus we conclude that no long open string phase exists in the thermodynamic limit in agreement with the discrete model results of Salomonson and Skagerstam [11]. Similarly, if closed strings are allowed to decay into open strings, no long closed string phase will form either.

V. DISCUSSION

In this paper we presented a Boltzmann equation describing the evolution of long fundamental strings. The equilibrium solution is easily found and describes a distribution where the available energy E is shared evenly between strings of different length and the total number of long strings is of order $\log E$. Our results agree with those of Salomonson and Skagerstam [11] who studied a discrete model of weakly interacting strings. This is not surprising since for the most part they made the same physical assumptions as we have. The advantage of our continuum approach is that it neatly summarizes the combinatorics of the discrete model and yields the equilibrium distribution with minimal effort. The Boltzmann equation can also be utilized to discuss nonequilibrium behavior.

The equilibrium solution of the Boltzmann equation corresponds to a single string density of states which agrees with that obtained by direct computation, provided care is taken to work on a compact target space [11, 12]. It is reassuring that these two different approaches to the statistical mechanics of strings are mutually consistent.

The physical picture that emerges from these considerations is as follows. Imagine a gas of weakly interacting strings in a space of large but finite volume as the energy density is slowly increased. At first the strings will predominantly be small and their behavior will be governed by some low-energy effective field theory. As the Hagedorn energy density is approached long strings begin to form. Eventually the ensemble will be dominated by the long string configurations described in this paper although there presumably remains a small component of short strings which behaves as a thermal gas of particles in equilibrium with the long strings.

According to Eq. (21) the Hagedorn temperature is a limiting temperature in the microcanonical ensemble. As further energy is pumped into the system most of it is spent on forming long strings rather than increasing the temperature. This behavior is reminiscent of a first-order phase transition with a large latent heat. Our approximations break down when the energy density becomes too large. At $\rho \sim 1/g^2\alpha'$ the string theory becomes strongly coupled but even before that the Jeans instability undermines the thermal ensemble. Once the inequality in (4) is violated the system becomes unstable to gravitational collapse.

The Boltzmann equation provides a starting point for

the study of nonequilibrium thermodynamics of fundamental strings. An important problem for future work is to generalize to nontrivial metric and dilaton backgrounds in order to get a handle on the gravitational collapse of string distributions. Such a generalization would also be relevant to the description of cosmic strings in the early Universe.

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[1] R. Hagedorn, Nuovo Cimento Suppl. **3**, 147 (1965).

[2] K. Huang and S. Weinberg, Phys. Rev. Lett. **25**, 895 (1970); S. Fubini and G. Veneziano, Nuovo Cimento A **64**, 1640 (1969).

[3] S. Frautschi, Phys. Rev. D **3**, 2821 (1971).

[4] R.D. Carlitz, Phys. Rev. D **5**, 3231 (1972).

[5] J.J. Atick and E. Witten, Nucl. Phys. **B310**, 291 (1988).

[6] See, e.g., M. Green, J. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987), Vol. 1.

[7] B. Sundborg, Nucl. Phys. **B254**, 538 (1985).

[8] D. Mitchell and N. Turok, Phys. Rev. Lett. **58**, 1577 (1987); Nucl. Phys. **B294**, 1138 (1987).

[9] M.J. Bowick and L.C.R. Wijewardhana, Phys. Rev. Lett. **54**, 2485 (1985); M.J. Bowick and S.B. Giddings, Nucl. Phys. **B325**, 631 (1989); S.B. Giddings, Phys. Lett. B **226**, 55 (1989).

[10] N. Deo, S. Jain, and C-I. Tan, Phys. Rev. D **40**, 2626 (1989); Phys. Lett. B **220**, 125 (1989); N. Deo, S. Jain, O. Narayan, and C-I. Tan, Phys. Rev. D **45**, 3641 (1992).

[11] P. Salomonson and B. Skagerstam, Nucl. Phys. **B268**, 349 (1986); Physica A **158**, 499 (1989).

[12] R. Brandenberger and C. Vafa, Nucl. Phys. **B316**, 391 (1989).

[13] E. Lifshitz and L. Pitaevskii, *Physical Kinetics* (Pergamon Press, New York, 1981).

[14] B. Sathiapalan, Phys. Rev. D **35**, 3277 (1987).

[15] Ya.I. Kogan, JETP Lett. **45**, 709 (1987).

[16] K.H. O'Brien and C-I. Tan, Phys. Rev. D **36**, 1184 (1987).

[17] J. Polchinski (private communication).

[18] J. Dai and J. Polchinski, Phys. Lett. B **220**, 387 (1989).