Exact solution (by algebraic methods) of the lattice Schwinger model in the strong-coupling regime

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Using the monomer-dimer representation of the lattice Schwinger model, with $N_f = 1$ Wilson fermions in the strong-coupling regime ($\beta = 0$), we evaluate its partition function Z exactly on finite lattices. By studying the zeros of $Z(k)$ in the complex plane (Re(k),Im(k)) for a large number of small lattices, we find the zeros closest to the real axis for infinite strips in the temporal direction and spatial extent $S = 2$ and 3. We find evidence for the existence of a critical value for the hopping parameter in the thermodynamic limit $S \to \infty$ on the real axis at about $k_c \simeq 0.39$. By looking at the behavior of quantities such as the chiral condensate, the chiral susceptibility, and the third derivative of Z with respect to $1/2k$, close to the critical point k_c, we find some indications for a continuous phase transition.

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I. INTRODUCTION

The Schwinger model [1], that is, two-dimensional QED (QED_2) with massless electrons, has always attracted the interest of theoreticians not only because it is a rather simple model, which can be solved analytically, but also (and mainly) because many of its properties are quite similar to those of four-dimensional QCD. Already in the basic version with $N_f = 1$ fermion flavors, one recovers a lot of QCD-like properties such as confinement for fermions, chiral symmetry breaking, due to the anomaly in the U(1) axial current, and charge screening [1–3].

By virtue of this similarity, one is also tempted to consider the lattice version of the Schwinger model as a test model for lattice four-dimensional QCD (QCD_4). Also here one faces the problem of the choice of a lattice scheme for fermions: the two most common choices are the well-known Wilson fermions [4] and the staggered (or Kogut-Susskind [KS]) fermions [5,6]. Most of the lattice calculations done up to now for the Schwinger model used the staggered fermion formulation [7–9], in which case the chiral limit is obtained by simply setting the bare fermion mass parameter m appearing in the lattice Lagrangian to zero: $m \to 0$. All these lattice calculations seem to reproduce well the expected properties of the continuum massless Schwinger model, known from analytical results.

On the contrary, very little is known about the lattice Schwinger model with Wilson fermions. Our interest in the Wilson formulation of QED_2 comes from the recently made observation [10] that the critical point in the hopping parameter, at which the chiral limit is reached, may not agree with the *naïve* expectation (see below, for a more detailed discussion of this point). This may also be of relevance for an understanding of the complicated phase diagram found in the Wilson formulation of lattice QCD_4 with large numbers of flavors [11].

In the lattice action with Wilson fermions the bare fermion mass m does not appear explicitly as in the lattice action with KS fermions. It contains as parameters the coupling constant β and the hopping parameter k, which is related to the bare fermion mass. Therefore in the Wilson fermion formulation there is the problem of defining a chiral limit, corresponding to $m \to 0$. In lattice QCD_4 with Wilson fermions the chiral limit, at a given value of β, is reached when the hopping parameter k approaches a certain critical value $k_c(\beta)$, often defined as the value of k for which the pion mass M_π vanishes. In fact, in the chiral limit of QCD_4 the pion becomes the Goldstone boson of the spontaneously broken chiral symmetry, and its mass is expected to vanish as $\sqrt{m}$ when $m \to 0$. It is tacitly assumed that this defines a critical point which coincides with a critical point (zero) of the partition function.

It is expected that the situation is similar for the lattice Schwinger model, in the sense that there will be a critical point for each value of the gauge coupling, $k = k_c(\beta)$, which defines the chiral limit. The continuum chiral limit will be reached following this line up to $k_c(\beta = \infty) = 1/2d = 1/4$. However, we cannot determine $k_c(\beta)$ in the same way as in QCD_4, since in the Schwinger model with $N_f = 1$ we have no Goldstone boson in the chiral limit $m \to 0$: the U(1) chiral symmetry is broken by the anomaly, and one is left with a massive pseudoscalar, similar to the η' in QCD_4. A determination of $k_c(\beta)$ thus has to proceed through the direct investigation of the singularities of $Z(k)$.

A quite common attitude is to assume that $k_c(\beta)$ coincides with the convergence radius $\bar{k}(\beta)$ for the joint expansion in the hopping parameter k and the inverse gauge coupling β. However, there is no proof that this is

correct: for example, in the lattice Schwinger model with KS fermions we have that $\bar{k} \leq 1/2$ [10], while $k_c = \infty$. For Wilson fermions, one does not know the precise values of $\bar{k}(\beta)$ and $k_c(\beta)$. Our aim is to compare these two values at least in the strong coupling regime $\beta = 0$. It was already found in Ref. [10] that $\bar{k}(0) \leq 1/2$ and some indications from Monte Carlo simulations were reported, indicating that $k_c(0) \neq \bar{k}(0)$.

In this work we will determine $k_c(0)$ by deriving analytically (using algebraic methods) the Lee-Yang zeros [12] of the partition function $Z(k)$ for the lattice Schwinger model, with $N_f = 1$ Wilson fermions, in the strong-coupling regime ($\beta = 0$). For a finite lattice $S \times T$ these zeros have a nonvanishing imaginary part, in the complex plane (Re(k),Im(k)), indicating that there is no real critical point for a finite lattice. This remains true for $T \to \infty$ with finite S. Yet, enlarging the lattice, they show a tendency to move towards the real k axis. By studying the zeros in the complex plane of the partition function $Z(k)$ for a large number of small lattices, and then extrapolating to the thermodynamic limit $\infty \times \infty$, we will find evidence for the existence of a real critical value for the hopping parameter, at about $k_c \simeq 0.39$. We will also study some relevant quantities, such as the chiral condensate $\langle\bar{\psi}\psi\rangle$, the chiral susceptibility, and the third derivative, with respect to $1/2k$, of the partition function, in order to get some information about the question of the order of the phase transition.

II. THE METHOD

The action for the lattice Schwinger model is written as the sum of a gauge action $S_G[U]$ and of a fermion action $S_F[\psi, \bar{\psi}, U]$:

$$S = S_G[U] + S_F[\psi, \bar{\psi}, U] \ . \tag{1}$$

The gauge part $S_G[U]$ is given by

$$S_G[U] = \beta \sum_P \left[1 - \frac{1}{2}(U_P + U_P^\dagger)\right] \ , \tag{2}$$

where $\beta \equiv 1/e^2$, e being the usual electromagnetic coupling constant. Since the gauge group is U(1), the basic lattice gauge variable $U_\mu(n)$, corresponding to the link connecting the sites n and $n + \hat{\mu}$, can be written in the form of a phase:

$$U_\mu(n) = \exp[i\phi_\mu(n)] \ . \tag{3}$$

In Eq. (2) U_P stands for the usual 1×1 Wilson plaquette, constructed using the link variables $U_\mu(n)$.

In the strong-coupling limit ($e^2 \to \infty$), the coefficient β in front of $S_G[U]$ goes to zero and the total action S reduces simply to the fermion action $S_F[\psi, \bar{\psi}, U]$. The action S_F for Wilson fermions (and only one flavor) can be written in the form

$$S_F = \frac{1}{2k} \sum_{n,m} \bar{\psi}(n) K_{nm}[U] \psi(m) \ , \tag{4}$$

where k is the so-called hopping parameter, and each K_{nm}, for a given couple of lattice sites n and m, is a matrix in Dirac space:

$$K_{nm}[U] = \delta_{nm}\mathbf{I} - k \sum_\mu [(r - \gamma_\mu) U_\mu(n) \delta_{n+\hat{\mu},m} + (r + \gamma_\mu) U_\mu^\dagger(n - \hat{\mu}) \delta_{n-\hat{\mu},m}] \ . \tag{5}$$

Therefore K_{nm} is of the form

$$K_{nm} = \delta_{nm}\mathbf{I} - k M_{nm}[U] \ , \tag{6}$$

where the only nonvanishing matrices M_{nm} are those connecting neighboring lattice sites:

$$M_{n,n+\hat{\mu}}[U] = (r - \gamma_\mu) U_\mu(n) \ ,$$

$$M_{n,n-\hat{\mu}}[U] = (r + \gamma_\mu) U_\mu^\dagger(n - \hat{\mu}) \ . \tag{7}$$

The parameter r, which satisfies $|r| \leq 1$, is called the Wilson parameter. In the following we will consider only the case $r = 1$. The matrices γ_μ, with $\mu = 1, 2$ are the 2×2 Euclidean Dirac matrices, corresponding to 1+1 space-time dimensions (in particular we will consider the index $\mu = 1$ as corresponding to the time dimension and the index $\mu = 2$ as corresponding to the space dimension): they satisfy the anticommutation relation $\{\gamma_i, \gamma_k\} = 2\delta_{i,k}\mathbf{I}_2$, with $\mathbf{I}_2$ being the 2×2 identity matrix. For our algebraic manipulations, we have chosen the following representation for the γ matrices:

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{8}$$

Finally, the partition function, in the strong coupling regime, $\beta = 0$, is given by the expression:

$$\begin{aligned} Z(k) &= \int [D\bar{\psi}D\psi] \int [DU] e^{-S_F[\psi,\bar{\psi},U]} \\ &= \int [D\bar{\psi}\psi] \int [DU] \\ &\quad \times \exp\left(-\frac{1}{2k} \sum_{n,m} \bar{\psi}(n) K_{nm}[U] \psi(m)\right) \ . \end{aligned} \tag{9}$$

Following the standard normalization convention, we also eliminate the factor $1/2k$ appearing in the exponent in Eq. (9) by rescaling the fermion fields with $\sqrt{2k}$:

$$\psi = \sqrt{2k}\bar{\psi}, \quad \bar{\psi} = \sqrt{2k}\bar{\bar{\psi}} \ . \tag{10}$$

When evaluating a matrix element of the form $\langle \prod_{n=1}^N \bar{\psi}_{A_n}(x_n) \bar{\bar{\psi}}_{B_n}(y_n) \rangle$, in terms of the rescaled fields, we must, however, keep in mind that the original correlation function is obtained by multiplying with $(1/2k)^N$. When considering a lattice with S lattice sites in the space direction and T lattice sites in the time direction, for a total of $N = S \times T$ lattice sites, the partition function (9) becomes

$$Z(k) = (2k)^{-2N} \tilde{Z}(k) \ , \tag{11}$$

where we have defined

$$\tilde{Z}(k) \equiv \int [D\bar{\tilde{\psi}} D\tilde{\psi}] \int [DU] \times \exp\left(-\sum_{n,m} \bar{\tilde{\psi}}(n) K_{nm}[U] \tilde{\psi}(m) \right) . \quad (12)$$

It is exactly $\tilde{Z}(k)$ that we have evaluated with algebraic methods for a large series of small lattices. First, we have put $\tilde{Z}(k)$ in a comfortable form for subsequent algebraic manipulations. Making use of the explicit expression (3) for the link variables U and remembering the Grassmann properties of the fermion fields, one finds the monomer-dimer representation [13] for the partition function $\tilde{Z}(k)$:

$$\tilde{Z}(k) \equiv \int [D\bar{\tilde{\psi}} D\tilde{\psi}] \prod_n F(n) L_1(n) L_2(n) , \quad (13)$$

where $F(n)$ is the monomer term at lattice site n, coming from the mass term in the action (i.e., the bilinear diagonal term not containing the gauge fields U):

$$F(n) = 1 - \bar{\tilde{\psi}}_1(n)\tilde{\psi}_1(n) - \bar{\tilde{\psi}}_2(n)\tilde{\psi}_2(n) + \bar{\tilde{\psi}}_1(n)\tilde{\psi}_1(n)\bar{\tilde{\psi}}_2(n)\tilde{\psi}_2(n) . \quad (14)$$

(The indices 1 and 2 are the Dirac indices). The quantities $L_\mu(n)$, with $\mu = 1, 2$, are the dimer terms defined on links of the lattice. They result from the direct integration over the gauge field $U_\mu(n) = \exp[i\phi_\mu(n)]$, corresponding to the link $n \to n + \hat{\mu}$. Explicitly:

$$L_\mu(n) = \int_{-\pi}^{\pi} \frac{d\phi_\mu(n)}{2\pi} e^{\{M_\mu(n)\exp[i\phi_\mu(n)] + N_\mu(n)\exp[-i\phi_\mu(n)]\}} , \quad (15)$$

where $M_\mu(n)$ and $N_\mu(n)$ are given by

$$M_\mu(n) = k\bar{\tilde{\psi}}(n)(1 - \gamma_\mu)\tilde{\psi}(n + \hat{\mu}) ,$$

$$N_\mu(n) = k\bar{\tilde{\psi}}(n + \hat{\mu})(1 + \gamma_\mu)\tilde{\psi}(n) . \quad (16)$$

Thanks to the particularly simple form (3) of the gauge variables U in the case of a U(1) gauge group, the integration in (15) can be performed in an elementary way, after having expanded as a power series the first exponential. Making use of the explicit representation (8) for the γ matrices and of the Grassmann properties of the fermion fields, one finds the following rather simple expressions for the one-link integrals $L_1(n)$ and $L_2(n)$:

$$\begin{aligned} L_1(n) &= 1 + M_1(n)N_1(n) \\ &= 1 + 4k^2 \bar{\tilde{\psi}}_2(n)\tilde{\psi}_2(n + \hat{1})\bar{\tilde{\psi}}_1(n + \hat{1})\tilde{\psi}_1(n) , \\ L_2(n) &= 1 + M_2(n)N_2(n) \\ &= 1 + k^2 \bar{\tilde{\psi}}(n) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tilde{\psi}(n + \hat{2}) \cdot \bar{\tilde{\psi}}(n + \hat{2}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tilde{\psi}(n) . \end{aligned} \quad (17)$$

After inserting expressions (14) and (17) in (13), one is left with an integral over the fermionic variables, which must be evaluated according to the integration rules for the Grassmann variables. Doing this "manually" turns out to be extremely boring and time-consuming, even for the very small 2×2 lattice. For this reason we have developed an algebraic method for evaluating the integral (13). We have used the algebraic computer language MATHEMATICA and have implemented the basis rules for the Grassmann algebra:

$$\{\eta_i, \eta_j\} = 0, \quad \int d\eta_i = 0, \quad \int d\eta_i \eta_k = \delta_{ik} . \quad (18)$$

In this way we were able to write computer programs (for MATHEMATICA) for evaluating products of polynomials of arbitrary strings of Grassmann variables [like those appearing in Eq. (13)] and for integrating them. In practice we have used the following strategy for calculating the partition function (13) for a given lattice of size $S \times T$. First, we have computed the *transfer matrix*, i.e., the products of all one-link terms, $L_1(n)$ and $L_2(n)$, and all mass terms $F(n)$ belonging to a given spacelike line (x, t) with $x = 1, 2, \ldots, S$. We call this object a "line." It is a function of all Grassmann fields belonging to the sites of the line under consideration:

$$\text{line}[t, 0] = \prod_{x=1}^{S} F(x, t) L_1(x, t) L_2(x, t) . \quad (19)$$

In evaluating this product we have already taken into account the toruslike topology of the lattice, in the form of periodic boundary conditions for the Grassmann fields along spacelike lines. The partition function is represented in terms of line$[t, 0]$ as

$$\tilde{Z}(k) = \int [D\bar{\tilde{\psi}} D\tilde{\psi}] \prod_{t=1}^{T} \text{line}[t, 0] . \quad (20)$$

Starting with the object line$[t, 0]$ one can now evaluate composite objects, such as the product of two adjacent lines: line$[t, 0]$line$[t + 1, 0]$. By virtue of Eqs. (19) and (17), all other lines, different from line$[t, 0]$ or line$[t + 1, 0]$, do not depend on the Grassmann fields on the sites $x, t + 1)$, $x = 1, 2, \ldots, S$. Therefore we can integrate the product line$[t, 0]$line$[t + 1, 0]$ with respect to these Grassmann fields and obtain

$$\text{line}[t,1] = \int \prod_{x=1}^{S} d\bar{\tilde{\psi}}(x,t+1) d\tilde{\psi}(x,t+1) \times \text{line}[t,0]\text{line}[t+1,0] \ . \quad (21)$$

We can then proceed in the same way and construct more extended objects. For example, we can multiply line$[t,1]$ with line$[t+2,0]$, or even with line$[t+2,1]$, and integrate over the Grassmann fields lying on the line $(x,t+2)$, $x = 1,2,\ldots,S$: In general we obtain line$[t,l]$ by performing the integration over the (l) lines of intermediate Grassmann fields at $x = t+1,\ldots,t+l$. Finally, for $l = T-1$ the resulting object covers just the entire lattice $S \times T$ and we must again take into account the toruslike topology of the lattice, i.e., impose antiperiodic boundary conditions for the Grassmann fields along timelike lines. (In fact, it turns out to be irrelevant, for the final result, if we impose periodic or antiperiodic boundary conditions for these Grassmann fields.) In the last step, we can integrate over the remaining Grassmann fields and obtain the final result for the partition function (13).

It turns out that the CPU time required in this approach is entirely controlled by the initial spatial extent S of the lattice as this determines the number of Grassmann fields one can combine in a given string of fields. Doubling the length in the time direction does not lead to a drastic increase of the computer time as the resulting object, line$[t,2l]$ contains exactly the same number and types of strings of Grassmann fields as line$[t,l]$: the only additional complication results from the more complex structure of the coefficients of these strings, which become higher-order polynomials in k for increasing T. As a curiosity, if we try to follow the strategy of "extending" in the space direction, instead of extending in the time direction, we need (for our programs) CPU times which are orders of magnitude larger than in the previous case. This is simply due to the space-time asymmetry in the representation (8): but of course the partition function for a lattice $S \times T$, namely $\tilde{Z}(k,S,T)$, is exactly equal to the partition function $\tilde{Z}(k,T,S)$ for a lattice $T \times S$. (One can always choose a representation for the γ matrices in which $\tilde{\gamma}_1 = \gamma_2$ and $\tilde{\gamma}_2 = \gamma_1$.)

III. RESULTS

Following the computational strategy discussed in the previous section, we have evaluated the partition function

TABLE I. The coefficients of the polynomial $\tilde{Z}(k,2,32) = \sum_{n=0}^{32} a_{4n} k^{4n}$.

n	a_{4n} (for the 2×32 lattice)
0	1
1	512
2	129 024
3	21 331 968
4	2 602 369 024
5	249 774 997 504
6	19 638 234 644 480
7	1 300 255 043 747 840
8	73 961 277 759 684 608
9	3 668 969 473 236 271 104
10	160 569 828 529 865 228 288
11	6 255 905 737 448 198 504 448
12	218 535 073 687 128 684 625 920
13	6 883 374 618 372 455 350 140 928
14	196 354 147 234 439 285 612 478 464
15	5 089 687 819 224 732 967 794 376 704
16	120 172 957 550 769 852 891 363 540 992
17	2 588 628 553 768 726 317 536 821 379 072
18	50 910 684 319 442 948 897 317 601 673 216
19	914 109 924 603 780 980 467 032 270 045 184
20	14 970 604 113 895 231 454 059 152 342 515 712
21	223 213 979 431 827 944 404 264 819 095 502 848
22	3 021 076 338 904 294 430 787 178 751 238 078 464
23	36 958 171 640 461 419 289 379 070 931 541 950 464
24	406 259 327 271 308 991 154 552 486 860 991 496 192
25	3 980 443 123 196 403 924 685 803 149 026 276 671 488
26	34 377 853 959 664 866 223 457 481 686 909 255 155 712
27	257 704 138 028 993 570 924 127 408 162 720 968 605 696
28	1 639 689 392 947 634 033 166 698 976 649 965 853 474 816
29	8 559 749 426 407 522 788 060 672 587 795 634 988 253 184
30	34 663 799 157 267 802 700 683 573 313 175 561 579 790 336
31	97 874 256 443 064 108 849 693 716 776 331 356 231 696 384
32	146 811 384 664 566 690 398 085 268 830 912 235 998 019 584

$\tilde{Z}(k)$, given by (13), for a large number of lattices of the form $S \times T$ with $S = 2$ and 3 ranging from 2×2 up to 2×32 and from 3×3 up to 3×16. These calculations could be performed on a workstation. For $S \geq 4$ considerably more computer time and memory would be required.

From the Grassmann properties of the fermion fields, it immediately follows that the function $\tilde{Z}(k, S, T)$, for a given lattice having $N = S \times T$ sites, is a polynomial of order $2N$ in the hopping parameter k,

$$\tilde{Z}(k, S, T) = \sum_{n=0}^{N} a_{2n} k^{2n} . \tag{22}$$

On a lattice with N lattice sites there are $4N$ different fermion fields (four fields for each site), so that, by virtue of the properties of the Grassmann algebra, one can construct strings of fermion fields with at most $4N$ fields. And since each power k^2 is always accompanied by four fermion fields, one can at most construct a string with $4N$ fields with a coefficient (proportional to) k^{2N} in front of it.

We have computed the partition functions $\tilde{Z}(k, S, T)$ for $(S = 2, T = 2, \ldots, 32)$ and for $(S = 3, T = 3, \ldots, 16)$: each of them is a polynomial of order $2N = (S \times T)$ in k, with $a_0 = 1$ ($\tilde{Z}(k = 0, S, T) = 1$) and $a_{2n} \geq 0$ for all n. The magnitude of these coefficients generally increases with the order of k, apart from deviations in the very first coefficients due the toruslike topology of the lattice which allows for special "trajectories," made up of chains of links wrapping around the lattice. In Tables I and II we report the list of these coefficients for the two lattices 2×32 and 3×16, respectively. Note that for S and T even the partition function $\tilde{Z}(k, S, T)$ is a polynomial of the form $\sum_{n=0}^{N/2} a_{4n} k^{4n}$. This is due to the larger set of symmetries on such lattices, as will be discussed below. Let us first discuss the distribution of

TABLE II. The (nonzero) coefficients of the polynomial $\tilde{Z}(k, 3, 16) = \sum_{n=0}^{48} a_{2n} k^{2n}$.

n	a_{2n} (for the 3×16 lattice)
0	1
3	1024
4	768
5	6144
6	516 096
7	811 008
8	6 334 464
9	174 620 672
10	431 947 776
11	3 203 923 968
12	45 588 021 248
13	153 911 033 856
14	1 063 856 898 048
15	9 906 042 175 488
16	40 917 798 223 872
17	261 009 935 695 872
18	1 857 033 179 496 448
19	8 585 692 506 488 832
20	50 608 039 715 143 680
21	303 834 862 874 263 552
22	1 470 365 700 697 620 480
23	8 023 810 955 173 429 248
24	42 904 506 735 217 606 656
25	205 822 945 618 110 185 472
26	1 036 084 831 241 631 694 848
27	5 045 499 989 985 857 634 304
28	23 116 972 970 746 438 483 968
29	106 903 512 053 003 436 687 360
30	476 889 028 736 929 390 133 248
31	2 038 448 818 744 602 390 429 696
32	8 604 050 273 287 356 083 601 408
33	34 923 088 125 822 911 234 703 360
34	136 172 615 876 993 812 147 470 336
35	515 227 117 640 733 139 176 259 584
36	1 857 978 729 092 258 641 800 593 408
37	6 360 979 177 485 248 012 381 847 552
38	20 656 762 024 719 021 784 826 904 576
39	62 433 587 250 829 914 862 254 030 848
40	174 406 412 859 094 327 618 009 300 992
41	446 095 973 409 032 980 536 257 150 976
42	1 025 256 020 010 205 168 904 800 567 296
43	2 088 088 022 191 744 886 698 175 102 976
44	3 738 100 459 121 645 858 804 962 689 024
45	5 610 148 109 882 451 164 480 363 560 960
46	7 107 243 167 143 739 985 776 879 861 760
47	6 755 399 440 834 383 071 115 485 380 608
48	5 629 499 534 323 800 464 442 257 309 696

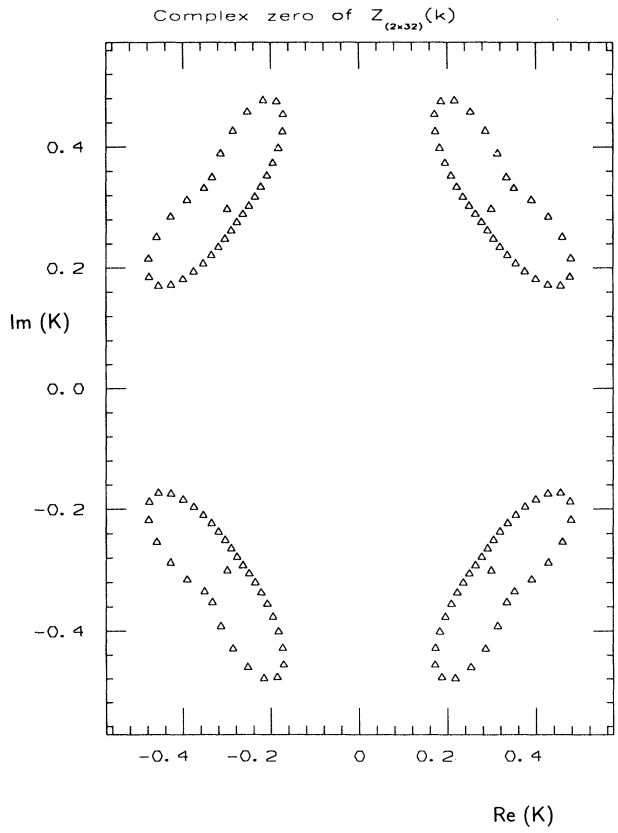

FIG. 1. The distribution of zeros of the partition function $\tilde{Z}(k, 2, 32)$ in the complex plane (Re(k),Im(k)).

zeros of the partition function. In Figs. 1 and 2 we show the distribution of the complex zeros, $(\mathrm{Re}(k), \mathrm{Im}(k))$, of the partition function $\tilde{Z}(k, S, T)$ for the 2×32 and 3×16 lattices, respectively.

By virtue of (22), $\tilde{Z}(k)$ is a polynomial in k^2 with real coefficients: so, if $\bar{k}$ is a complex zero of $\tilde{Z}(k)$, also $-\bar{k}$ and $\bar{k}^*$ (the complex conjugate of $\bar{k}$) will be zeros of $\tilde{Z}(k)$. As a consequence of this, the distribution of zeros (x, y) in the complex plane $(\mathrm{Re}(k), \mathrm{Im}(k))$ is invariant under the *parity* transformation $(x, y) \to (-x, -y)$ (*P symmetry*) and under the *complex-conjugate* transformation $(x, y) \to (x, -y)$ (*C symmetry*). In other words, the distribution of zeros turns out to be symmetric under reflections with respect to the real and/or the imaginary k axis: this is evident from Figs. 1 and 2. The distribution of zeros in Fig. 1, for the lattice 2×32, has an additional symmetry under reflections with respect to the axis $\mathrm{Re}(k) - \mathrm{Im}(k) = 0$ and/or to the axis $\mathrm{Re}(k) + \mathrm{Im}(k) = 0$. This additional symmetry is typical for lattices of size $S \times T$, where both S and T are even numbers. In fact, it turns out that for this class of lattices the partition function $\tilde{Z}(k)$ may be written in the following form:

$$\tilde{Z}(k) = \int \prod_{l \text{ odd}} d\bar{\tilde{\psi}}(l) d\tilde{\psi}(l) \prod_{m \text{ odd}} F(m) \prod_{n \text{ even}} \mathrm{cross}(n) , \tag{23}$$

where a given site (s, t) (with $s = 1, \ldots, S$ and $t = 1, \ldots, T$) is said to be even or odd if the integer number $s + t$ is, respectively, even or odd. While $F(m)$ is the usual monomer term which we have introduced before in Eq. (14), cross(n) is a new object obtained by multiplying the monomer term in the site n with the four dimer terms starting from or ending at the site n, and finally integrating with respect to the fermion fields in n:

$$\mathrm{cross}(n) = \int d\bar{\tilde{\psi}}(n) d\tilde{\psi}(n) F(n) L_1(n) L_2(n) \times L_1(n - \hat{1}) L_2(n - \hat{2}) . \tag{24}$$

By explicitly evaluating this expression, one finds that cross(n) may be written as $1 + k^4 \alpha(\bar{\tilde{\psi}}\tilde{\psi})$, where $\alpha(\bar{\tilde{\psi}}\tilde{\psi})$ is a sum of products of four fermion fields defined in the neighboring sites of n (i.e., $n + \hat{1}$, $n - \hat{1}$, $n + \hat{2}$, and $n - \hat{2}$). Therefore the partition function $\tilde{Z}(k)$ will be a polynomial in k^4, with real (and positive) coefficients. As a consequence, if $\bar{k}$ is a solution of $\tilde{Z}(\bar{k}) = 0$, also $i\bar{k}$ will be a solution. In other words, the distribution of zeros of $\tilde{Z}(k)$ in the complex k plane will be invariant under the transformation $(x, y) \to (-y, x)$: we will call this an *I symmetry*. After combining this I symmetry with the

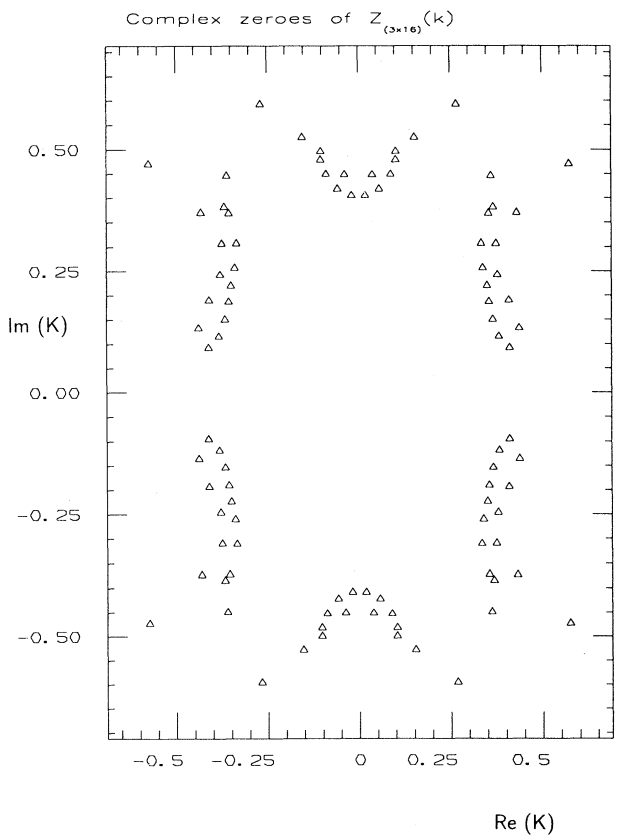

FIG. 2. The distribution of zeros of the partition function $\tilde{Z}(k, 3, 16)$ in the complex plane $(\mathrm{Re}(k), \mathrm{Im}(k))$.

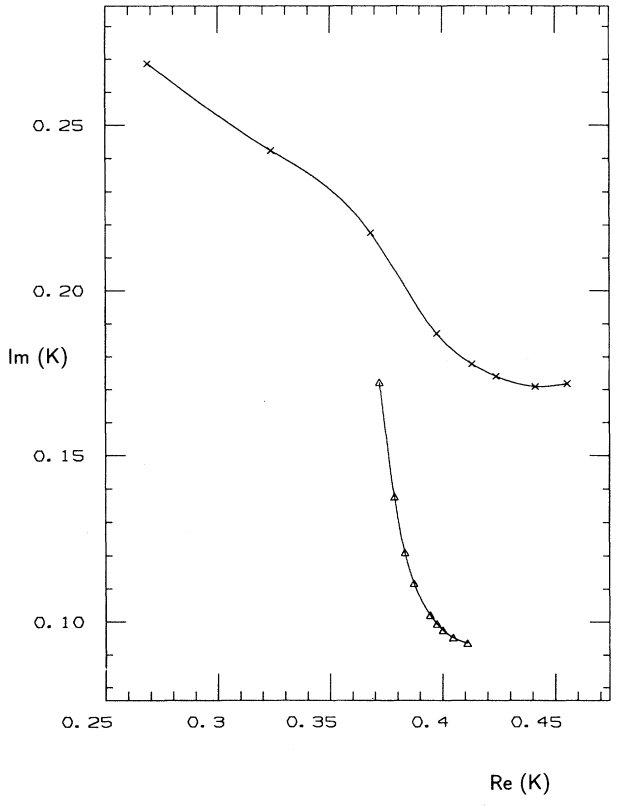

FIG. 3. The zeros of $\tilde{Z}(k, S, T)$ closest to the real k axis for various lattices of size $S \times T$: crosses refer to lattices with $S = 2$ and $T = 2, 3, 4, 6, 8, 10, 16, 32$ (from left to right), while triangles refer to lattices with $S = 3$ and $T = 3, 4, 5, 6, 8, 9, 10, 12, 16$ (from left to right).

other P and C symmetries, one immediately recognizes that the distribution of zeros is invariant under reflections not only with respect to the real or the imaginary k axes, but also with respect to the axes $\mathrm{Re}(k) - \mathrm{Im}(k) = 0$ and $\mathrm{Re}(k) + \mathrm{Im}(k) = 0$: this is in fact what one observes in Fig. 1.

With increasing lattice size the zero with the smallest imaginary part comes closer to the real axis. This zero of the partition function has been plotted in Fig. 3 for various lattices of size $S \times T$. As can be seen, for increasing T and fixed S the imaginary part of the zeros approaches a finite, nonzero value, indicating that there is no phase transition on infinite strips of extent $S < \infty$. We have then used a polynomial fit to derive, from the two previous series of points, the asymptotic values of the roots nearest to the real k axis corresponding to the lattices $2 \times \infty$ and $3 \times \infty$. In practice, we have performed separate fits of real and imaginary parts of the sequence of roots for given S, using the polynomial ansatz, $P(T) = a_0 + a_1/T + a_2/T^2 + \cdots$. The number of coefficients in the polynomials was chosen so as to minimize the χ^2 value. It turns out that for the lattices with $S = 2$ good fits are obtained if one uses a polynomial with six coefficients for the real part of the roots and a polynomial with five coefficients for the imaginary part. Instead, for the lattices with $S = 3$ good fits are obtained using polynomials with only three coefficients for both the real and imaginary part of the roots.

In this way we have found that the root nearest the real k axis is approximately $(0.464 \pm 0.003, 0.172 \pm 0.004)$ for the $2 \times \infty$ lattice and approximately $(0.43 \pm 0.01, 0.09 \pm 0.01)$ for the $3 \times \infty$ lattice.

If one tries to extrapolate these two points linearly, assuming the existence of a real critical value k_c in the thermodynamic limit $S \to \infty$ and $T \to \infty$ (see also below), one finds approximately (within the accuracy of our linear extrapolation)

$$k_c \simeq 0.39 \ . \tag{25}$$

Clearly our present analysis is restricted to small values of S. Any further quantitative investigation of the critical behavior, in particular a reliable determination of critical exponents, will require a sequence of larger values of S. Still it is interesting to analyze the present results for the complex zeros of infinite strips with $S = 2$ and 3 under the assumption of a second-order phase transition in the thermodynamical limit $S \to \infty$. The real and the imaginary parts of the zero closest to the real axis are then expected to scale like

$$\mathrm{Re}(k(S)) = k_c + \frac{a}{S^{1/\nu}} \ ,$$
$$\mathrm{Im}(k(S)) = \frac{b}{S^{1/\nu}} \ . \tag{26}$$

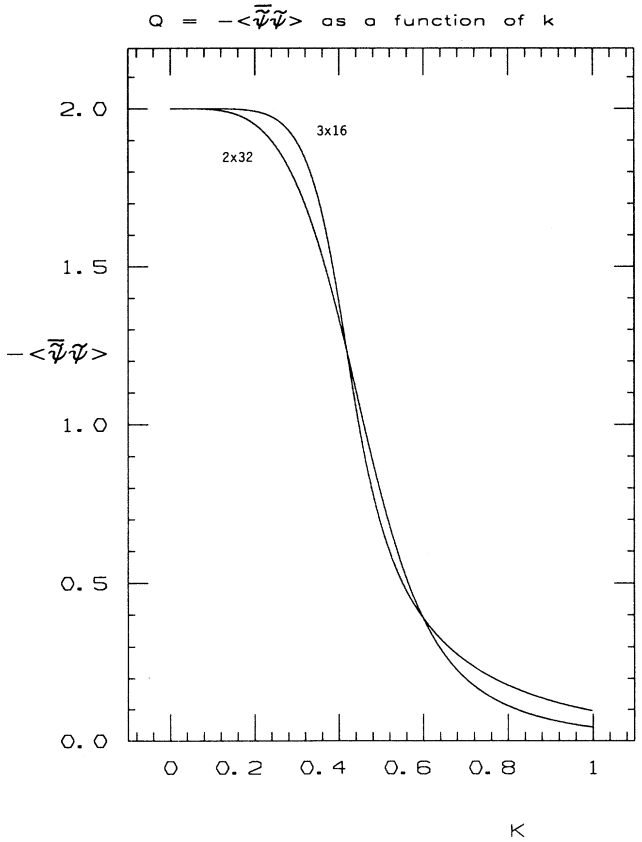

FIG. 4. The rescaled chiral condensate, $\tilde{Q} = -\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle$, as a function of k, for the two lattices 2×32 and 3×16.

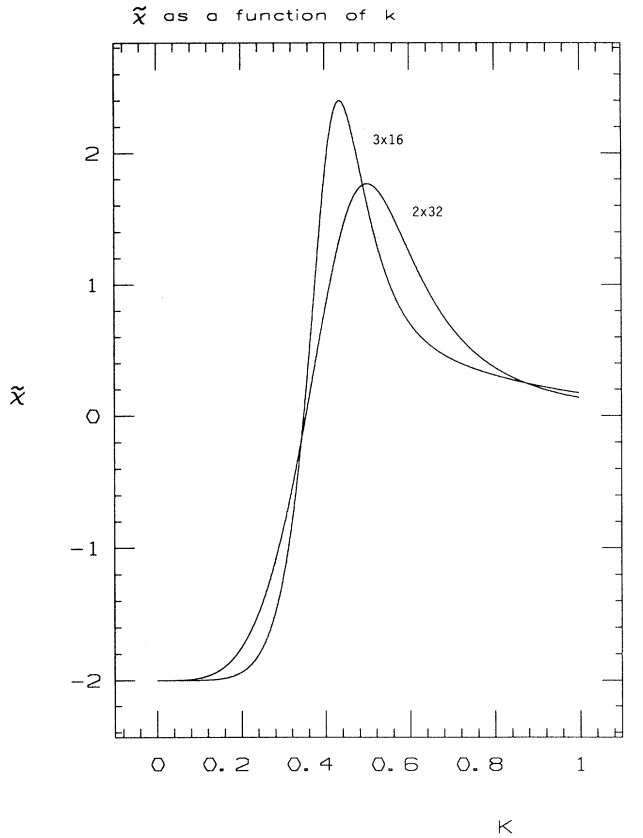

FIG. 5. The rescaled chiral susceptibility $\tilde{\chi}$ [defined in the text by Eq. (28)] as a function of k, for the two lattices 2×32 and 3×16.

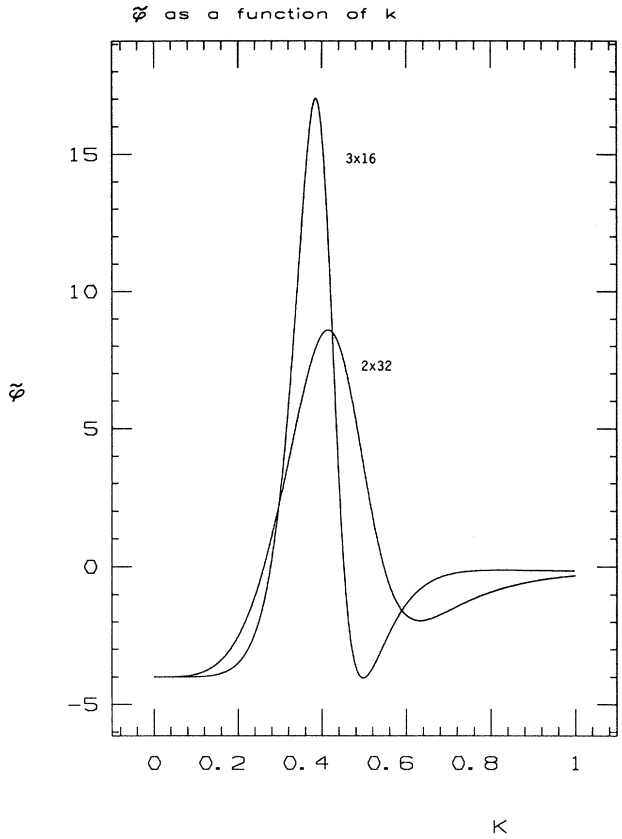

FIG. 6. The rescaled third derivative $\tilde{\varphi}(k)$ of $\ln(Z)$ with respect to $1/2k$ [defined in the text by Eq. (29)], for the two lattices 2×32 and 3×16.

Using this *Ansatz* for $S = 2$ and $S = 3$ results we find from the S dependence of $\mathrm{Im}(k)$ the critical exponent $\nu \simeq 0.63$ and using this to extract k_c from $\mathrm{Re}(k)$ we again find $k_c \simeq 0.39$. Of course, in order to get a realistic estimate of the errors on these numbers, one needs to consider also lattices with much larger S.

This is surely an extremely interesting result: the value we have found for k_c is considerably smaller than the value $\bar{k} = 1/2$ we would have expected from *naïve* argumentations. Since for $\beta = \infty$ (theory with free fermions, since $e^2 = 1/\beta \to 0$) it is well known that $k_c = 1/4$ (in general it will be $k_c = 1/2d$, with d space-time dimensions), we are led by the result (25) to believe in the existence of a line of phase transition from $(\beta = 0, k_c \simeq 0.39)$ to $(\beta = \infty, k_c = 1/4)$. (See also Ref. [14], where this same result was found making use of the so-called eight-vertex model [15] for the analytical computations.)

In order to get some further information about the question of the order of the phase transition, we have studied the behavior of the (rescaled) chiral condensate $\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle$, the (rescaled) chiral susceptibility $\tilde{\chi}$, and the (rescaled) third derivative, with respect to $1/2k$, of the logarithm of the partition function Z in the vicinity of the critical point k_c. The rescaled chiral condensate $\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle$ is given by the following expression in terms of the partition function $\tilde{Z}(k)$ [see Eqs. (5), (9), and (11)]:

$$\begin{aligned}\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle &= -\frac{1}{(2k)N}\frac{d}{d(1/2k)}\ln(Z) \\ &= -2 + \frac{k}{N}\left(\frac{1}{\tilde{Z}(k)}\frac{d\tilde{Z}(k)}{dk}\right) ,\end{aligned} \tag{27}$$

where $N = S \times T$ is, as usual, the number of sites of the lattice. In Fig. 4 we report the behavior of $\tilde{Q} \equiv -\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle$ as a function of k, for the two lattices 2×32 and 3×16. The rescaled chiral susceptibility $\tilde{\chi}$ is defined as

$$\begin{aligned}\tilde{\chi} &\equiv \frac{1}{(2k)^2 N}\frac{d^2}{d(1/2k)^2}\ln(Z) \\ &= \sum_n \langle\bar{\tilde{\psi}}(n)\tilde{\psi}(n)\bar{\tilde{\psi}}(0)\tilde{\psi}(0)\rangle - N\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle^2 \\ &= -2 + \frac{k}{N}\left[k\left(\frac{1}{\tilde{Z}}\frac{d^2\tilde{Z}}{dk^2}\right) - k\left(\frac{1}{\tilde{Z}}\frac{d\tilde{Z}}{dk}\right)^2 \right. \\ &\quad \left. +2\left(\frac{1}{\tilde{Z}}\frac{d\tilde{Z}}{dk}\right)\right] .\end{aligned} \tag{28}$$

In Fig. 5 we report the behavior of $\tilde{\chi}$ as a function of k, for the same two lattices. It is easy to verify that $\tilde{\chi} = -k(d\tilde{Q}/dk) - \tilde{Q}$, so that $\tilde{\chi}$ behaves as $\tilde{\chi} \simeq -\tilde{Q} \to -2$ for $k \to 0$. Finally, we also consider the third derivative, with respect to $1/2k$, of the logarithm of the partition function Z:

$$\begin{aligned}\tilde{\varphi} &\equiv -\frac{1}{(2k)^3 N}\frac{d^3}{d(1/2k)^3}\ln(Z) \\ &= \sum_{m,n}\langle\bar{\tilde{\psi}}(m)\tilde{\psi}(m)\bar{\tilde{\psi}}(n)\tilde{\psi}(n)\bar{\tilde{\psi}}(0)\tilde{\psi}(0)\rangle - 3N\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle\tilde{\chi} - N^2\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle^3 \\ &= -4 + \frac{k}{N}\left[k^2\left(\frac{1}{\tilde{Z}}\frac{d^3\tilde{Z}}{dk^3}\right) + 6k\left(\frac{1}{\tilde{Z}}\frac{d^2\tilde{Z}}{dk^2}\right) - 3k^2\left(\frac{1}{\tilde{Z}}\frac{d\tilde{Z}}{dk}\right)\left(\frac{1}{\tilde{Z}}\frac{d^2\tilde{Z}}{dk^2}\right)\right] \\ &\quad + \frac{k}{N}\left[2k^2\left(\frac{1}{\tilde{Z}}\frac{d\tilde{Z}}{dk}\right)^2 - 6k\left(\frac{1}{\tilde{Z}}\frac{d\tilde{Z}}{dk}\right)^2 + 6\left(\frac{1}{\tilde{Z}}\frac{d\tilde{Z}}{dk}\right)\right] .\end{aligned} \tag{29}$$

The behavior of $\tilde{\varphi}$ as a function of k is shown in Fig. 6, for the two lattices 2×32 and 3×16. It is easy to demonstrate that $\tilde{\varphi} = k(d\tilde{\chi}/dk) + 2\tilde{\chi}$: from this one can immediately derive that $\tilde{\varphi}$ behaves as $\tilde{\varphi} \simeq 2\tilde{\chi} \to -4$ for $k \to 0$.

Clearly the three quantities are closely related. The sharpening of the crossover in $\langle\bar{\tilde{\psi}}\tilde{\psi}\rangle$ is reflected in the rise of the peak in $\tilde{\chi}$ with increasing S, and the narrowing of the peak in $\tilde{\chi}$ is expressed in terms of the rapidly rising peaks with opposite signature in $\tilde{\varphi}$. Certainly the behavior of these three quantities is consistent with that expected for a continuous phase transition, i.e., a second- or third-order phase transition.

IV. CONCLUSIONS

We have evaluated analytically, using algebraic methods, the partition function Z for the lattice Schwinger model, with $N_f = 1$ Wilson fermions, in the strong-coupling regime ($\beta = 0$). For a given lattice $S \times T$, the partition function is of the form $Z(k,S,T) = (2k)^{-2N}\tilde{Z}(k,S,T)$, where $N = S \times T$ is the total number of lattice sites and $\tilde{Z}(k,S,T)$ is a polynomial in k of order $O(2N)$. By studying the zeros in the complex plane $(\mathrm{Re}(k), \mathrm{Im}(k))$ of the partition function $\tilde{Z}(k,S,T)$ for a large series of small lattices $S \times T$, we have found evidence for the existence of a critical value for the hopping parameter, which in the thermodynamic limit, $S,T \to \infty$ lies on the real axis at about $k_c \simeq 0.39$. We are led by this result to believe in the existence of a line of phase transition from $(\beta = 0, k_c \simeq 0.39)$ to $(\beta = \infty, k_c = 1/4)$. In order to determine the order of the transition, it is clearly important to study in more detail the density of the zeros near k_c. This requires larger values of S. We have analyzed the chiral condensate $\langle\bar{\psi}\psi\rangle$, the chiral susceptibility, and the third derivative, with respect to $1/2k$, of the partition function, in order to get some insights into the question of the order of the phase transition. Even though the present analysis does not yet allow drawing a definite conclusion on the order of the transition we have found some indications that the phase transition might be third order or even second order.

ACKNOWLEDGMENTS

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