

Aspects of black hole quantum mechanics and thermodynamics in 2+1 dimensions

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We discuss the quantum mechanics and thermodynamics of the (2+1)-dimensional black hole, using both minisuperspace methods and exact results from Chern-Simons theory. In particular, we evaluate the first quantum correction to the black hole entropy. We show that the dynamical variables of the black hole arise from the possibility of a deficit angle at the (Euclidean) horizon, and briefly speculate as to how they may provide a basis for a statistical picture of black hole thermodynamics.

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I. INTRODUCTION

In the 20 years since Bekenstein's proposal that black holes have an entropy [1] and Hawking's discovery that they can evaporate [2], a great deal has been learned about the thermodynamics of black holes. Nevertheless, some key questions remain unanswered.

(1) Despite considerable work over the past few years, the "information loss paradox," the apparently nonunitary transition from pure particle states to thermal Hawking radiation, remains an open problem (see, for example, [3]).

(2) Standard approaches to black hole thermodynamics involve semiclassical approximations of one kind or another, and can tell us little about the final stages of the process of evaporation, where the effects of quantum gravity are sure to be important.

(3) Although a number of interesting suggestions have been made, we do not yet have a generally accepted model of the microscopic statistical mechanics that should presumably underlie black hole thermodynamics.

The recent discovery of black hole solutions in (2+1)-dimensional gravity offers a promising new arena for investigating such problems. In contrast with (3+1)-dimensional general relativity, the (2+1)-dimensional model has only finitely many physical degrees of freedom. As a result, questions about quantum gravity can be explored in considerable detail, and we can be reasonably confident that our conclusions are at least self-consistent. The purpose of this paper is to begin that exploration.

The plan of the paper is the following. In Sec. II, we discuss the geometry of the Euclidean black hole and its relation to hyperbolic three-space \mathbf{H}^3 , the complete

Riemannian space of constant negative curvature, with appropriate identifications. We explain how to generalize these identifications to allow for a conical singularity (and a helical twist) at the horizon. The holonomies of this generalized geometry become the dynamical variables of the black hole, and are assigned Poisson brackets from previously available results for the Chern-Simons formulation. In Sec. III we discuss the black hole from the Hamiltonian point of view, concentrating on a minisuperspace model. We again find that the parameters describing a singularity at the horizon become the dynamical variables of the black hole, with Poisson brackets equivalent to those derived from the Chern-Simons approach. We show that the partition function arises from a sum over these parameters, and we calculate it in the classical approximation, obtaining the standard black hole entropy. We then evaluate the first quantum correction, again using known results from the Chern-Simons formulation. For large black holes, we find that this correction does not involve \hbar , and merely renormalizes the gravitational constant, a typical occurrence in Chern-Simons theory. Finally, we devote Sec. IV to speculation on the possible microscopic origin of black hole entropy as it emerges from the descriptions obtained in Secs. II and III.

II. EUCLIDEAN BLACK HOLE

We shall investigate black hole thermodynamics in terms of the "Wick-rotated" Euclidean black hole. One may take the point of view that the Euclidean geometry emerges as a complex stationary point of the Lorentzian action (see, for example, [4]), or one may argue that the Euclidean description is the more fundamental one. The analysis that follows applies in either case.

A. Geometry and identifications

We start with the static Lorentzian black hole metric [5]

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$$ds_{\text{Lor}}^2 = - \left(\frac{r^2}{\ell^2} - M \right) dt^2 + \left(\frac{r^2}{\ell^2} - M \right)^{-1} dr^2 + r^2 d\phi^2, \quad (2.1)$$

which is a solution of the vacuum Einstein equations in 2+1 dimensions with a cosmological constant $\Lambda = -1/\ell^2$. This metric and its spinning counterpart [Eq. (2.14) below] are discussed in detail in [6]. For us, a key feature is the geometric significance of the time coordinate t as the “Killing time,” the displacement along the timelike Killing vector at spatial infinity. Since it is this Killing vector that determines the static nature of the solution and the applicability of equilibrium thermodynamics, t is the appropriate time for thermodynamic considerations.

The metric (2.1) has an obvious analytic continuation to

$$ds^2 = \left(\frac{r^2}{\ell^2} - M \right) d\tau^2 + \left(\frac{r^2}{\ell^2} - M \right)^{-1} dr^2 + r^2 d\phi^2, \quad (2.2)$$

which satisfies the Euclidean field equations. To analyze the resulting geometry, we exploit the simplicity of (2+1)-dimensional gravity: The full curvature tensor in 2+1 dimensions depends linearly on the Ricci tensor, and any solution of the empty space field equations is a space of constant curvature. We can exhibit this characteristic explicitly for the metric (2.2) by changing to coordinates

$$\begin{aligned} x &= \left(1 - \frac{M\ell^2}{r^2} \right)^{1/2} \cos \frac{\sqrt{M}\tau}{\ell} e^{\sqrt{M}\phi}, \\ y &= \left(1 - \frac{M\ell^2}{r^2} \right)^{1/2} \sin \frac{\sqrt{M}\tau}{\ell} e^{\sqrt{M}\phi}, \\ z &= \frac{\sqrt{M}\ell}{r} e^{\sqrt{M}\phi}. \end{aligned} \quad (2.3)$$

The metric becomes

$$ds^2 = \frac{\ell^2}{z^2} (dx^2 + dy^2 + dz^2), \quad z > 0, \quad (2.4)$$

which may be recognized as the standard metric for the upper half-space model of hyperbolic three-space \mathbf{H}^3 . To account for the periodicity of the Schwarzschild angular coordinate ϕ , we must make the (isometric) identifications

$$(x, y, z) \sim (e^{2\pi\sqrt{M}}x, e^{2\pi\sqrt{M}}y, e^{2\pi\sqrt{M}}z). \quad (2.5)$$

The (2+1)-dimensional Euclidean black hole may thus be described as the quotient of hyperbolic space \mathbf{H}^3 by the isometry (2.5). As in four dimensions, the Euclidean black hole corresponds to the region outside the event horizon of the Lorentzian solution; the event horizon itself is mapped to the circle $x = y = 0, 1 \leq z \leq e^{2\pi\sqrt{M}}$.

It is straightforward to find a fundamental region for the identifications (2.5). The result is most easily expressed in “spherical” coordinates

$$\begin{aligned} x &= R \cos \theta \cos \chi, \\ y &= R \sin \theta \cos \chi, \\ z &= R \sin \chi, \end{aligned} \quad (2.6)$$

with θ periodic, i.e.,

$$(R, \theta, \chi) \sim (R, \theta + 2\pi, \chi). \quad (2.7)$$

The identifications (2.5) then become

$$(R, \theta, \chi) \sim (e^{2\pi\sqrt{M}}R, \theta, \chi). \quad (2.8)$$

We may therefore choose as a fundamental region the space between the hemispheres $R = 1$ and $R = e^{2\pi\sqrt{M}}$, with points on the boundaries identified along radial lines as in Fig. 1.

Topologically, the resulting manifold is a solid torus. For $\chi \neq \pi/2$, each slice of fixed χ is an ordinary two-torus, with circumferences parametrized by the periodic coordinates $\ln R$ and θ ; the degenerate surface $\chi = \pi/2$ is a circle at the core of the solid torus. Physically, R is an angular coordinate, equal to $e^{\sqrt{M}\phi}$ in the original Schwarzschild coordinates; the azimuthal angle θ measures time; and χ is a radial coordinate. Note that with the metric (2.4), the x - y plane $\chi = 0$ is infinitely far from the interior of the manifold, while $\chi = \pi/2$ is the horizon.

(We caution the reader that it is important to distinguish between the angular coordinate θ in the upper half-space representation, which is physically a time coordinate, and the Schwarzschild angular coordinate ϕ . In the discussion that follows, we shall move freely between these two useful coordinate systems.)

The metric (2.4)–(2.5) already gives us important information about (2+1)-dimensional black hole thermodynamics. The periodicity in θ reflects a periodicity in Killing time, and as usual in Euclidean quantum field theory, we can interpret the period as an inverse temperature. Indeed, comparing (2.3) and (2.6), we see that

$$\theta = \frac{\sqrt{M}}{\ell} \tau, \quad (2.9)$$

and so a period of 2π in θ corresponds to a period of

$$\beta = \frac{2\pi\ell}{\sqrt{M}} \quad (2.10)$$

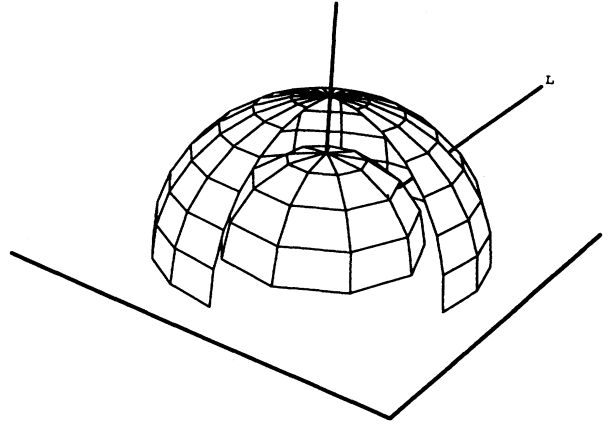


FIG. 1. The Euclidean black hole is obtained by identifying the inner and outer hemispheres of this figure along radial lines such as L . The circle γ_A is the segment of L between the two hemispheres, whose end points are identified. (The outer hemisphere in this figure has been cut open to show the inner hemisphere.)

in the Killing time.

As we shall see in the next section, the off-shell extension of the black hole solution requires a generalization of this periodicity. If θ has a period Θ , that is, if (2.7) is replaced by

$$(R, \theta, \chi) \sim (R, \theta + \Theta, \chi), \quad (2.11)$$

then the metric (2.4) acquires a conical singularity along the z axis with deficit angle $2\pi - \Theta$, and will not solve the equations of motion there. This mildly singular geometry can also be expressed in the Schwarzschild coordinates (2.2), where Θ is now determined by the condition

$$(\tau_2 - \tau_1)(N^\perp)^2 = 2\Theta(r - r_+) + O((r - r_+)^2), \quad (2.12)$$

with

$$(N^\perp)^2 = \frac{r^2}{\ell^2} - M, \quad r_+ = \sqrt{M}\ell. \quad (2.13)$$

Equation (2.12) holds for the off-shell black hole in any spacetime dimension [7].

The extension of this analysis to the spinning black hole is fairly straightforward. The Lorentzian metric is now

$$ds_{\text{Lor}}^2 = -(N_{\text{Lor}}^\perp)^2 dt^2 + f_{\text{Lor}}^{-2} dr^2 + r^2 (d\phi + N_{\text{Lor}}^\phi dt)^2, \quad (2.14)$$

with

$$N_{\text{Lor}}^\perp = f_{\text{Lor}} = \left(-M_{\text{Lor}} + \frac{r^2}{\ell^2} + \frac{J_{\text{Lor}}^2}{4r^2} \right)^{1/2}, \quad (2.15)$$

$$N_{\text{Lor}}^\phi = -\frac{J_{\text{Lor}}}{2r^2}.$$

The corresponding Euclidean solution is obtained by letting $t = -i\tau$, $M_{\text{Lor}} = M$, and $J_{\text{Lor}} = iJ$, and reads

$$ds^2 = (N^\perp)^2 d\tau^2 + f^{-2} dr^2 + r^2 (d\phi + N^\phi d\tau)^2, \quad (2.16)$$

with

$$N^\perp = f = \left(-M + \frac{r^2}{\ell^2} - \frac{J^2}{4r^2} \right)^{1/2}, \quad (2.17)$$

$$N^\phi = -iN_{\text{Lor}}^\phi = -\frac{J}{2r^2}.$$

If we set

$$r_\pm^2 = \frac{M\ell^2}{2} \left[1 \pm \left(1 + \frac{J^2}{M^2\ell^2} \right)^{1/2} \right], \quad (2.18)$$

it is not hard to check that the coordinate transformation

$$x = \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{1/2} \cos \left(\frac{r_+}{\ell^2} \tau + \frac{|r_-|}{\ell} \phi \right) \\ \times \exp \left\{ \frac{r_+}{\ell} \phi - \frac{|r_-|}{\ell^2} \tau \right\},$$

$$y = \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{1/2} \sin \left(\frac{r_+}{\ell^2} \tau + \frac{|r_-|}{\ell} \phi \right) \\ \times \exp \left\{ \frac{r_+}{\ell} \phi - \frac{|r_-|}{\ell^2} \tau \right\}, \quad (2.19)$$

$$z = \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2} \right)^{1/2} \exp \left\{ \frac{r_+}{\ell} \phi - \frac{|r_-|}{\ell^2} \tau \right\}$$

again takes the metric to the form (2.4). Here we have set

$$|r_-| = ir_- = \frac{J\ell}{2r_+}. \quad (2.20)$$

The identifications analogous to those of Eq. (2.5) or (2.8), in “spherical” coordinates (2.6), are now

$$(R, \theta, \chi) \sim \left(Re^{2\pi r_+/\ell}, \theta + \frac{2\pi|r_-|}{\ell}, \chi \right), \quad (2.21)$$

reducing to (2.8) when $J = 0$. The fundamental region is again the region between two hemispheres, as in Fig. 1, but the boundaries are now identified with a twist around the z axis.

In terms of the Schwarzschild coordinates ϕ and τ , the identification (2.21) simply reads

$$(\phi, \tau) \sim (\phi + 2\pi, \tau), \quad (2.22)$$

whereas the demand that θ should have a period 2π , that is, the absence of a conical singularity in the Euclidean black hole metric, translates into

$$(\phi, \tau) \sim (\phi + \Phi, \tau + \beta), \quad (2.23)$$

with

$$\beta = \frac{2\pi r_+ \ell^2}{r_+^2 - r_-^2}, \quad \Phi = \frac{2\pi|r_-|\ell}{r_+^2 - r_-^2}. \quad (2.24)$$

Equation (2.24) shows that the angle ϕ appearing in (2.16) is not the usual Schwarzschild azimuthal angle, for which the identification (2.23) would read

$$(\phi', \tau) \sim (\phi', \tau + \beta). \quad (2.25)$$

Rather, the relationship between ϕ and ϕ' is evidently

$$\phi' = \phi - \frac{\Phi}{\beta} \tau = \phi - \frac{|r_-|}{\ell r_+} \tau. \quad (2.26)$$

It follows that

$$d\phi' + N^{\phi'} d\tau = d\phi + N^\phi d\tau, \quad (2.27)$$

with

$$N^{\phi'} = N^\phi + \frac{|r_-|}{\ell r_+} = \frac{|r_-|}{\ell r_+} - \frac{r_+ |r_-|}{\ell r^2}, \quad (2.28)$$

and, in particular,

$$N^{\phi'}(r_+) = 0. \quad (2.29)$$

The identifications (2.21), expressed in terms of the “usual Schwarzschild angle” ϕ' , thus lead to a vanishing shift at the horizon.

B. Conical singularity with a helical twist

As in the static case, it is straightforward to extend the rotating black hole solution to one admitting a conical singularity along the z axis. A slight generalization is now natural. Recall that for the static black hole, the periodic coordinates $\ln R$ and θ parametrized the two circumferences of the torus $\chi = \text{const}$. The natural generalization of the periodicity (2.11) in θ therefore includes an associated twist in the $\ln R$ direction; that is,

$$(R, \theta, \chi) \sim (e^{\Sigma} R, \theta + \Theta, \chi). \quad (2.30)$$

As before, the nonstandard periodicity in θ indicates the presence of a conical singularity along the z axis, while the shift in $\ln R$ represents a simultaneous “radial” twist, resembling the time helical structure of the spacetime around a spinning particle in 2+1 dimensions [8].

Note that the identifications (2.21) are not altered by the introduction of this singularity, and so Eq. (2.22) is maintained. The Schwarzschild coordinate periods Φ and β appearing in (2.23), on the other hand, now become

$$\begin{aligned} \beta &= \frac{\ell^2}{r_+^2 - r_-^2} (-|r_-| \Sigma + r_+ \Theta), \\ \Phi &= \frac{\ell}{r_+^2 - r_-^2} (r_+ \Sigma + |r_-| \Theta). \end{aligned} \quad (2.31)$$

Correspondingly, it is now necessary to replace (2.26) by

$$\phi' = \phi + \frac{r_+ \Sigma + |r_-| \Theta}{|r_-| \Sigma - r_+ \Theta} \frac{\tau}{\ell}, \quad (2.32)$$

which generates a shift

$$N^{\phi'}(r) = -\frac{r_+ \Sigma + |r_-| \Theta}{\ell(|r_-| \Sigma - r_+ \Theta)} - \frac{r_+ |r_-|}{\ell r^2}. \quad (2.33)$$

In particular, it follows that

$$\Sigma = 0 \Leftrightarrow N^{\phi'}(r_+) = 0, \quad (2.34)$$

and, once this is taken into account,

$$\Theta = 2\pi \Leftrightarrow \beta = \frac{2\pi r_+ \ell^2}{r_+^2 - r_-^2}. \quad (2.35)$$

We shall see in Sec. III how the conditions (2.34) and (2.35) reappear in a very different approach.

C. Isometries and holonomies

It is well known that the full isometry group of the hyperbolic metric (2.4) is the group $\text{SL}(2, \mathbb{C})$, the universal covering group of the Lorentz group. It is useful to express the isometry (2.21) explicitly as an element of this group. The action of $\text{SL}(2, \mathbb{C})$ is most easily expressed in the language of quaternions [9]: If we write the coordinates x , y , and z as a quaternion

$$q = x + yi + zj, \quad (2.36)$$

then a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}), \quad a, b, c, d \in \mathbb{C} \quad (2.37)$$

acts by

$$q \mapsto (aq + b)(cq + d)^{-1}. \quad (2.38)$$

It is easy to show that the identifications (2.21) are represented by the matrix

$$H = \begin{pmatrix} e^{\pi(r_+ + i|r_-|)/\ell} & 0 \\ 0 & e^{-\pi(r_+ + i|r_-|)/\ell} \end{pmatrix}. \quad (2.39)$$

H itself is coordinate dependent, but its trace gives an invariant characterization of the geometry.

Experience from elsewhere in (2+1)-dimensional gravity [10, 11] has taught us that it is often useful to look at the theory in first-order form, with variables consisting of a triad $e^a = e^a_\mu dx^\mu$ and a spin connection $\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{bc\mu} dx^\mu$. For a Euclidean spacetime with a negative cosmological constant, these two one-forms can be combined to form a single $\text{SL}(2, \mathbb{C})$ connection

$$A^a = \omega^a + \frac{i}{\ell} e^a, \quad (2.40)$$

and Witten has shown [10] that the ordinary Einstein action then reduces to the Chern-Simons action

$$\begin{aligned} I &= \frac{i\ell}{64\pi G} \int_M d^3x \epsilon^{ijk} \left[A_i^a (\partial_j A_{ka} - \partial_k A_{ja}) \right. \\ &\quad \left. + \frac{2}{3} \epsilon_{abc} A_i^a A_j^b A_k^c \right] + \text{c.c.} \end{aligned} \quad (2.41)$$

In particular, if we start with the hyperbolic metric (2.4) in the coordinates (2.6), the triad can be chosen to be

$$e^1 = \frac{\ell}{\sin \chi} \frac{dR}{R}, \quad e^2 = \frac{\ell}{\sin \chi} d\chi, \quad e^3 = \ell \cot \chi d\theta, \quad (2.42)$$

with a corresponding spin connection

$$\omega^1 = -\frac{1}{\sin \chi} d\theta, \quad \omega^2 = 0, \quad \omega^3 = \cot \chi \frac{dR}{R}. \quad (2.43)$$

It is then straightforward to show that A^a is flat, as required by the Chern-Simons field equations. We stress that this flatness does not imply triviality—although its curvature vanishes, A^a has a nonvanishing holonomy around the noncontractible closed curve

$$\gamma_A : s \mapsto \left(e^{2\pi r_+ s/\ell}, \theta_0 + \frac{2\pi|r_-|}{\ell} s, \chi_0 \right), \quad s \in [0, 1] \quad (2.44)$$

connecting the inner and outer hemispheres of Fig. 1. The value of this holonomy can be computed directly from the connection, but it is more easily understood geometrically: Up to conjugation by a rather complicated coordinate-dependent matrix, it is precisely the matrix (2.39) describing the identifications that characterize the black hole. Cangemi *et al.* [12] have found an analogous result for the Lorentzian black hole, but a poor coordinate choice made their computations rather difficult; see also [6] and [13].

It is perhaps worth emphasizing this relationship between identifications and holonomies. The direct solution of the Einstein field equations leads to the Euclidean black hole metric (2.16). On the other hand, we know

that any vacuum solution in three dimensions with $\Lambda < 0$ must have constant negative curvature, and ought therefore to be obtainable from \mathbf{H}^3 by suitable identifications. The Chern-Simons formulation provides the bridge between these two pictures: The required identifications of \mathbf{H}^3 are precisely the holonomies of the connection (2.40). Mathematically, this correspondence comes from the relationship between flat connections and geometric structures, a subject of considerable research in the past few years [14].

We conclude this section with a brief discussion of the large diffeomorphisms. Since the Euclidean black hole is topologically a solid torus, we should expect to find a one-parameter family of large diffeomorphisms, corresponding to Dehn twists (twists by multiples of 2π) around the contractible circumference [15]. These diffeomorphisms are evident in (2.21): the identifications that determine the geometry are unchanged by the replacement

$$r_+ \rightarrow r_+, \quad |r_-| \rightarrow |r_-| + n\ell \quad (2.45)$$

for any integer n . We thus find a rather strange symmetry, under which solutions with different masses and angular momenta are related by large diffeomorphisms. We do not yet understand the physical significance of this invariance.

D. Holonomies as canonical variables

We now turn to a discussion of the dynamical variables of the system and the corresponding Poisson brackets. In the Chern-Simons description of (2+1)-dimensional gravity, the fundamental physical observables are the holonomies of the connection A^a , defined by (2.40), around curves γ in M . It is almost possible to express these holonomies as functions of homotopy classes $[\gamma]$ alone: Under a smooth deformation of γ , the holonomy $H(\gamma)$ of a flat connection is invariant up to overall conjugation. In particular, the traces $\text{Tr}H(\gamma)$ are homotopy invariant, and if γ is contractible, the entire holonomy matrix is the identity.

The Poisson algebra of the holonomies $H(\gamma)$ has been analyzed in detail by Nelson *et al.* [16, 17]. The holonomies around two curves γ_1 and γ_2 have nonvanishing Poisson brackets only when the projections of γ_1 and γ_2 onto a surface of constant time intersect and cannot be separated by a smooth deformation. The resulting algebra is simplest when spacetime has the topology $\mathbf{R} \times \Sigma$, with Σ a closed genus g surface. In that case, the homotopy classes $[\gamma]$ automatically come in $2g$ inter-

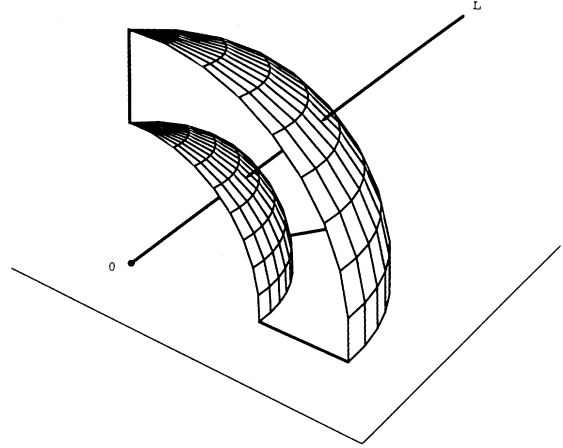


FIG. 2. Transition amplitudes are obtained from a section of the black hole between two constant τ surfaces. Each boundary of constant τ is an annulus, with an inner circumference at r_+ (the z axis) and an outer circumference at infinity (the intersection with the x - y plane).

secting pairs, the archetype being the two independent circumferences of a torus. It may be shown that the resulting Poisson brackets give rise to a natural symplectic structure on the space of observables, thus permitting a simple Hamiltonian formulation of the theory [18].

For the Euclidean black hole, on the other hand, there is only one nontrivial homotopy class, $[\gamma_A]$, and only one nontrivial holonomy. It would thus seem that there is no classical symplectic structure, and no starting point for this approach to quantization.

Fortunately, this is not quite true. Consider first a section of the geometry (2.2) lying between times τ_1 and τ_2 , as illustrated in upper half-space coordinates in Fig. 2. (The figure shows the $J = 0$ case, but the generalization to nonzero spin is straightforward.) In addition to the noncontractible path γ_A discussed above, there is now a nontrivial path γ_B joining the constant time surfaces $\tau = \tau_1$ and $\tau = \tau_2$. The curves γ_A and γ_B are linked, and their holonomies therefore have nontrivial Poisson brackets.

More specifically, one can evaluate the Poisson brackets by “radial quantization,” treating the radial Schwarzschild coordinate r , or the corresponding coordinate χ in the upper half-space representation, as time. It is easy to check that for a curve $(R(s), \theta(s))$ on a surface of constant χ , the holonomy of the connection formed from (2.42) and (2.43) is

$$H = \begin{pmatrix} \cos w + i \cot \chi \sin w & -\csc \chi \sin w \\ -(\cos^2 \chi - \sin^2 \chi) \csc \chi \sin w & \cos w - i \cot \chi \sin w \end{pmatrix}, \quad (2.46)$$

with $w = \theta - i \ln R$. Using the identifications (2.21), we obtain, for the curve γ_A ,

$$\text{Tr}H(\gamma_A) = 2 \cosh \frac{2\pi}{\ell} (r_+ + i|r_-|), \quad (2.47)$$

while, for a curve γ_B connecting (R_1, θ_1) and $(R_2 = e^{2\Sigma} R_1, \theta_2 = \theta_1 + \Theta)$,

$$\text{Tr}H(\gamma_B) = 2 \cosh (\Sigma + i\Theta). \quad (2.48)$$

The Poisson brackets can then be read off from Eq. (5.2) of [17]; with factors of G restored, we obtain

$$\begin{aligned} \{r_+, \Theta\} &= \{|r_-|, \Sigma\} = 4G, \\ \{|r_-|, \Theta\} &= \{r_+, \Sigma\} = 0. \end{aligned} \quad (2.49)$$

If we now return to the complete, periodic Euclidean black hole (Fig. 1 rather than Fig. 2) we see that Θ and Σ are precisely the deficit angle at the horizon and the associated helical twist described in Sec. II B. It was shown in [19] that the horizon area is canonically conjugate to the opening angle for a black hole in any number of dimensions. Since the horizon area in 2+1 dimensions is proportional to r_+ , this agrees with (2.49).

Of course, the complete, classical vacuum solution requires that there be no conical singularity. But the Poisson brackets (2.49) show that such a requirement is inconsistent in quantum theory, where the deficit angle and the horizon radius are complementary observables. We shall see in the next section that the same formulation arises naturally in a minisuperspace approach to the black hole.

The proper Lorentzian interpretation of the deficit angle Θ remains an interesting open question. As a variable conjugate to the horizon size, Θ may be viewed as a kind of pressure. Indeed, it is proportional to the “canonical pressure” of Brown *et al.* [20], although the observational significance of such a pressure in a Euclidean context is not clear. Perhaps more to the point, recall that the circle $r = r_+$ is the Euclidean analog of the two-dimensional horizon of the Lorentzian black hole. We might therefore view the deficit angle as a kind of “compressed” description of the dynamics of the horizon (see Sec. IV A). Indeed, there is some evidence that a detailed analysis of the horizon dynamics of the Lorentzian black hole can reproduce the entropy that we shall derive in the Euclidean context in Sec. III C [21].

III. MINISUPERSPACE AND RADIAL QUANTIZATION

We now turn to a Hamiltonian approach to the Euclidean black hole, and develop a minisuperspace model that will provide further insight into the system. We shall see that many of the conclusions of Sec. II, including the role of the horizon radius and the deficit angle as conjugate variables, may be duplicated in this approach, in a context that is directly generalizable to 3+1 dimensions.

A. Action and boundary terms

As discussed in Ref. [19], the Euclidean action for a black hole may be taken to be

$$I = \frac{1}{4G}(\text{area of horizon}) + I_{\text{can}} + B_{\infty}, \quad (3.1)$$

where I_{can} is the canonical Arnowitt-Deser-Misner (ADM) action and B_{∞} is a local boundary term at large spatial distances whose form depends on what is held fixed at infinity. (The sign is such that one path integrates e^{+I} .) For the complete black hole spacetime, the action (3.1) differs from the Hilbert action¹

$$I_H = \frac{1}{8\pi G} \left[\frac{1}{2} \int \sqrt{g} (R + 2\ell^{-2}) d^3x - \int_{\partial M} \sqrt{h} K d^2x \right] \quad (3.2)$$

by another boundary term at infinity. In order to agree with the conventions used in [6], we shall set $G = 1/8$ in this section. We now specialize (3.1) to a class of fields that includes our rotating black hole, but in which all fluctuations in azimuthal angle and time are frozen. Thus we admit all metrics of the form

$$ds^2 = \beta^2(r) f^2(r) d\tilde{\tau}^2 + f^{-2}(r) dr^2 + r^2 [d\phi' + \tilde{N}^{\phi'}(r) d\tilde{\tau}]^2, \quad r_+ \leq r < \infty, \quad 0 \leq \tilde{\tau} \leq 1, \quad (3.3)$$

with

$$f^2(r_+) = 0 \quad (3.4)$$

and

$$f^2(r) - \left(\frac{r}{\ell}\right)^2 \rightarrow -M \quad \text{as } r \rightarrow \infty. \quad (3.5)$$

We have replaced the lapse function $N^\perp(r)$ with the “Killing lapse”

$$\beta(r) = f^{-1}(r) N^\perp(r) \quad (3.6)$$

for later convenience, and have denoted the azimuthal angle by ϕ' to agree with the notation of Sec. II.

The canonical Euclidean action has the form

$$I_{\text{can}} = \int d\tilde{\tau} d^2x \left(\pi^{ij} \frac{\partial g_{ij}}{\partial \tilde{\tau}} - N^\perp \mathcal{H}_\perp - N^i \mathcal{H}_i \right), \quad (3.7)$$

and specialized to the minisuperspace (3.3), it becomes [6]

$$I_{\text{can}} = - \int_{r_+}^{\infty} dr \left\{ \beta(r) \left[(f^2)'(r) - \frac{p^2(r)}{2r^3} - \frac{2r}{\ell^2} \right] + \tilde{N}^{\phi'}(r) p'(r) \right\}, \quad (3.8)$$

where $p(r)$ is the r - ϕ component of the gravitational momentum, related to the extrinsic curvature, and the prime denotes differentiation with respect to r . We shall be interested in the action principle based on (3.1) with boundary conditions that permit the existence of a unique classical solution. This is because we wish to investigate the classical and semiclassical approximations to the path integral, which start with the classical action \bar{I} for an extremum with specified boundary conditions. We therefore fix the functions $\beta(r)$ and $\tilde{N}^{\phi'}(r)$ at spatial infinity:

$$\beta(r) \rightarrow \beta(\infty), \quad \tilde{N}^{\phi'}(r) \rightarrow \tilde{N}^{\phi'}(\infty) \quad \text{as } r \rightarrow \infty, \quad (3.9)$$

and we also fix

$$p(r_+) = p_+ \quad (3.10)$$

to supplement (3.4). Equations (3.4), (3.9), and (3.10) provide a complete set of boundary conditions for the

¹Similar results have been obtained in Refs. [20, 22, 23] in slightly different contexts.

action principle, and the four numbers r_+ , p_+ , $\beta(\infty)$, and $\tilde{N}^{\phi'}(\infty)$ fully determine the classical solution. Since the “momenta” β and $\tilde{N}^{\phi'}$ are fixed at the upper end point, one must add to the action (3.7) a boundary term

$$-\beta(\infty)M - \tilde{N}^{\phi'}(\infty)J,$$

where neither the parameter M defined in (3.5) nor $J = -p(\infty)$ are held fixed.

When varied with respect to f^2 , p , β , and $\tilde{N}^{\phi'}$ with these boundary conditions, the full action

$$I = 4\pi r_+ + I_{\text{can}} - \beta(\infty)M - \tilde{N}^{\phi'}(\infty)J \quad (3.11)$$

has an extremum when the following equations hold:

$$\begin{aligned} (f^2)' - \frac{p^2}{2r^3} - \frac{2r}{\ell^2} &= 0, \\ p' &= 0, \\ \beta' &= 0, \\ (\tilde{N}^{\phi'})' + \frac{p\beta}{r^3} &= 0. \end{aligned} \quad (3.12)$$

The solution is given by

$$\begin{aligned} f^2(r) &= -\left(\frac{r_+^2}{\ell^2} - \frac{p_+^2}{4r_+^2}\right) + \frac{r^2}{\ell^2} - \frac{p_+^2}{4r^2}, \\ p(r) &= p_+, \\ \beta(r) &= \beta(\infty), \\ \tilde{N}^{\phi'}(r) &= \tilde{N}^{\phi'}(\infty) + \frac{p_+\beta(\infty)}{2r^2}. \end{aligned} \quad (3.13)$$

B. Canonical variables

The action (3.8) is in canonical form, with the radial coordinate r playing the role of time² (“radial quanti-

zation”). It contains two canonical pairs (β, f^2) and $(\tilde{N}^{\phi'}, p)$, with equal r brackets given by

$$\{\beta, f^2\} = \{\tilde{N}^{\phi'}, p\} = 1. \quad (3.14)$$

In Sec. II, on the other hand, we described the black hole in terms of two different pairs of canonically conjugate variables, (r_+, Θ) and $(|r_-|, \Sigma)$. We now further investigate the connection between these pairs.

Let us begin by comparing the solution (3.13) to the metric (2.16) and (2.17), permitting the periodicities (2.31) that describe a general conical singularity. It is easy to check that the two metrics are identical if we set

$$\tilde{\tau} = \tau/\beta, \quad \tilde{N}^{\phi'} = \beta N^{\phi'}, \quad (3.15)$$

with

$$\begin{aligned} p_+ &= -J = -\frac{2r_+|r_-|}{\ell}, \\ M &= \frac{r_+^2 + r_-^2}{\ell^2}, \\ \beta(\infty) &= \beta = \frac{\ell^2}{r_+^2 - r_-^2} (-|r_-|\Sigma + r_+\Theta), \\ \beta^{-1}\tilde{N}^{\phi'}(\infty) &= N^{\phi'}(\infty) = -\frac{r_+\Sigma + |r_-|\Theta}{\ell(|r_-|\Sigma - r_+\Theta)}. \end{aligned} \quad (3.16)$$

We can now compute the Poisson brackets $\{\beta, f^2\}$ and $\{\tilde{N}^{\phi'}, p\}$ directly from the brackets (2.49). We find that the only nonvanishing equal r brackets are

$$\{\beta, f^2\} = \left\{ \frac{\ell^2}{r_+^2 - r_-^2} (-|r_-|\Sigma + r_+\Theta), \frac{|r_-|^2 - r_+^2}{\ell^2} - \frac{r_+^2|r_-|^2}{\ell^2 r^2} \right\} = 1 \quad (3.17)$$

and

$$\{\tilde{N}^{\phi'}, p_+\} = \left\{ \frac{\ell r_+|r_-|}{r^2(r_+^2 - r_-^2)} (|r_-|\Sigma - r_+\Theta) + \frac{\ell}{r_+^2 - r_-^2} (r_+\Sigma + |r_-|\Theta), -\frac{2r_+|r_-|}{\ell} \right\} = 1, \quad (3.18)$$

in agreement with (3.14).

We can thus view the Chern-Simons quantization of Sec. II as a form of “covariant canonical quantization,” that is, quantization of the space of classical solutions [24, 25]. The space of classical solutions, which is isomorphic to the reduced phase space, is parametrized by constants of motion $\{r_+, r_-, \Sigma, \Theta\}$, and the canonical commutation relations of the full theory, or, in this case, the minisuper-space model, are equivalent to the commutation relations among these parameters.

An alternative derivation of the conjugacy of r_+ and Θ may offer further insight. In the canonical action (3.8), r clearly plays the role of a “time” variable, with a corresponding “Hamiltonian”

$$H_r(r) = -\beta(r) \left[\frac{p^2(r)}{2r^3} + \frac{2r}{\ell^2} \right]. \quad (3.19)$$

It is then a standard result that the variable conjugate to r_+ is $-H_r(r_+)$, which by Eq. (3.13) is equal to $(\beta f^2)'(r_+)$. But from (2.12), this is just the opening angle, or rather 2Θ . The r_+ dependence of the remaining terms in (3.11) gives an additional contribution of -4π to the momentum conjugate to r_+ ; combining the two contributions, we find

²A similar radial foliation has been considered by Brown *et al.* [20] in the context of black hole thermodynamics.

$$P_{r_+} = -2(2\pi - \Theta), \quad (3.20)$$

in accordance with the general discussion of [19] and in agreement with (2.49). The extra 4π does not contribute to the Poisson brackets, but it will be important in the determination of the partition function in the next section.

C. Partition function

We now turn to the computation of the thermal partition function of the (2+1)-dimensional black hole. The partition function may be obtained as a trace of the Euclidean propagation amplitude, which is derived in turn from the path integral for the wedge-shaped manifold $\tau_1 \leq \tau \leq \tau_2$ of Fig. 2. In this section, we shall concentrate on the classical approximation, in which the propagation amplitude is expressed as the exponential of the classical action \bar{I} ; the first quantum correction will be described in the next section.

The Euclidean propagation amplitude

$$K[(^{(2)}\mathcal{G}_2, (^{(2)}\mathcal{G}_1; r_+, p_+; \beta(\infty), \tilde{N}^{\phi'}(\infty))]$$

depends on the two-geometries of the slices $\tau = \tau_1$ and $\tau = \tau_2$; the horizon geometry, which is determined by r_+ and p_+ ; and the asymptotic geometry, characterized by $\beta(\infty)$ and $\tilde{N}^{\phi'}(\infty)$. Upon performing a suitable Laplace transform, the latter two parameters may be replaced by the mass M and the angular momentum J . It is to be emphasized that the asymptotic constants M and J are to be treated as independent parameters; they are determined in terms of r_+ and p_+ only “on shell.”

To obtain the partition function, we must take the trace of K over the initial and final geometries, including the horizon geometries. The metric in our minisuper-space model is τ independent; so $(^{(2)}\mathcal{G}_1$ and $(^{(2)}\mathcal{G}_2$ are automatically equal, and the trace reduces to an integral over r_+ and p_+ . For the full path integral, one must be careful about the contour of integration and the measure [see [26, 27] for a related discussion in a (3+1)-dimensional setting], but for the classical approximation, this integration amounts to extremizing the action with respect to r_+ and p_+ , to obtain the grand canonical partition function with the temperature $\beta(\infty)$ and the rotational chemical potential

$$\mu = (\beta^{-1} \tilde{N}_{\text{Lor}}^{\phi'})(\infty) \quad (3.21)$$

held fixed. (Note that $\tilde{N}^{\phi'}(\infty)J$ is equal to $[-i\tilde{N}_{\text{Lor}}^{\phi'}(\infty)](-iJ_{\text{Lor}}) = -\beta\mu J_{\text{Lor}}.$)

Extremizing with respect to r_+ and p_+ amounts to setting their canonical conjugates equal to zero. The variable conjugate to p_+ is $\tilde{N}^{\phi'}(r_+)$, and so we find

$$\tilde{N}^{\phi'}(r_+) = 0, \quad (3.22)$$

in agreement with (2.34). The variable conjugate to r_+ is $P_{r_+} = -2(2\pi - \Theta)$, and so $\Theta = 2\pi$ at the extremum, again in accordance with the results of Sec. II. Moreover, the canonical action I_{can} vanishes when the equations

of motion (3.12) are satisfied. We therefore obtain an extremal action of

$$\bar{I}[\beta, \mu] = 4\pi r_+ - \beta M - \tilde{N}^{\phi'}(\infty)J, \quad (3.23)$$

with M , J , and r_+ expressed in terms of β and $\tilde{N}^{\phi'}(\infty)$ through (3.17) with $\Sigma = 0$ and $\Theta = 2\pi$. The classical approximation to the partition function is therefore

$$Z(\beta, \mu) = e^{\bar{I}}, \quad (3.24)$$

and the entropy in this approximation is

$$S = 4\pi r_+, \quad (3.25)$$

or, reinstating the universal constants,

$$S = \frac{2\pi r_+}{4\hbar G}. \quad (3.26)$$

Note that in conventional units ($\hbar = G = 1$), the entropy is just a quarter of the horizon size, as expected for Einstein’s theory in any spacetime dimension [19].

D. First quantum correction

The results of the previous section are essentially those of a tree-level approximation. In particular, the exponent appearing in Eq. (3.24) is the classical action of the Euclidean black hole. By returning to the Chern-Simons formulation, however, we can easily obtain the first quantum (“one-loop”) correction.

As we observed in Sec. II A, the Euclidean black hole is topologically a solid torus. For such a topology, the one-loop contribution to the Chern-Simons path integral has already been worked out, albeit in a rather different context, in Appendix 3 of Ref. [28]. This contribution, essentially the Van Vleck–Morette determinant, takes a very simple form, depending only on the holonomy (2.39): The prefactor is simply

$$\Delta = 4\pi \left(\cosh \frac{2\pi r_+}{\ell} - \cos \frac{2\pi |r_-|}{\ell} \right). \quad (3.27)$$

For r_+/ℓ large, this becomes

$$\Delta \approx 2\pi e^{2\pi r_+/\ell}, \quad (3.28)$$

and so the entropy (3.26) is corrected to read

$$S = \frac{2\pi r_+}{\ell} \left(\frac{\ell}{4\hbar G} + 1 \right). \quad (3.29)$$

Note that while the “classical” term involves the area expressed in Planck units, the first quantum correction depends instead on the scale ℓ set by the cosmological constant, and involves neither \hbar nor G .

In some approaches to quantum gravity, the partition function also receives contributions from other topologies. These contributions may be quite large, and in some cases they dominate the path integral [28, 29]. For the Euclidean black hole, however, this is not the case. It will be shown elsewhere [30] that the only complete extrema of the Euclidean action with the *asymptotic* geometry of the black hole are the black hole itself and the “hot empty space” solution obtained by identifying

$$(x, y, z) \sim (x + 1, y, z) \sim (x + \tau_1, y + \tau_2, z) \quad (3.30)$$

in the metric (2.4).

IV. STEPS TOWARDS BLACK HOLE STATISTICAL MECHANICS

We have now seen that the thermal properties of the $(2+1)$ -dimensional black hole are intimately related to the existence of new degrees of freedom associated with a conical singularity at the Euclidean event horizon. It is shown in Ref. [7] that a similar conclusion holds for the black hole in any number of spacetime dimensions. The off-shell Euclidean black hole in d dimensions has the topology $\mathbf{R}^2 \times S^{d-2}$, and just as in the three-dimensional case, one must generically allow a conical singularity in the \mathbf{R}^2 plane. As in the previous section, such a singularity leads to a term $(2\pi - \Theta)\delta A$ in the variation of the Hilbert action, where A is now the area (volume if $d > 4$) of the $(d - 2)$ -sphere at the horizon; it is shown in [19] that this term has its geometrical origin in the dimensional continuation of the two-dimensional Gauss-Bonnet theorem. This boundary variation implies that the opening angle Θ is canonically conjugate to the area of the horizon, in agreement with (2.49) for $2+1$ dimensions.

This is an attractive picture, but it does not yet provide a “statistical mechanical” explanation of the black hole entropy (3.26). Ideally, one would like to explain this entropy as a logarithm of the number of macroscopically indistinguishable states. Our analysis suggests that these states ought to be associated with the conical singularity at $r = r_+$, but a more detailed microscopic description seems to require a full quantization of the black hole.

We do not yet know how to complete such a program. We can, however, point to several suggestive results.

A. Counting Euclidean states

One starting point for black hole statistical mechanics is the Chern-Simons approach to three-dimensional gravity. As discussed in Sec. II B, three-dimensional Euclidean gravity can be described by the Chern-Simons action (2.41) with gauge group $\text{SL}(2, \mathbf{C})$. The quantization of such an action has been discussed by Witten [31] and Hayashi [32]. In the special case that the coupling constant $\ell/8\hbar G$ is an integer, Hayashi argues that the space of states on a torus is isomorphic to two copies of the Hilbert space of an $\text{SU}(2)$ Chern-Simons theory with coupling constant $k = \ell/8\hbar G$. The group $\text{SU}(2)$ appears because $\text{SL}(2, \mathbf{C})$ is the complexification of $\text{SU}(2)$; roughly speaking, the two $\text{SU}(2)$ gauge fields are the connection A^a of (2.40) and its complex conjugate, treated as independent fields.

The states of an $\text{SU}(2)$ Chern-Simons theory can be created by inserting a Wilson line carrying an $\text{SU}(2)$ representation of spin $j/2$ ($j = 1, \dots, k$) at the core of a solid torus. Since such Wilson lines are the Chern-Simons analogue of conical singularities [10, 33], this description closely resembles the analysis of Sec. II. In this relatively

simple approach to quantization, however, there appears to be no evidence for the exponentially rising density of states needed to explain black hole entropy; it may be shown that one only obtains, roughly speaking, a set of states evenly spaced in r_+ .

This straightforward version of the Chern-Simons quantization can be replaced, however, by one in which the role of the horizon becomes more fundamental. For the Lorentzian black hole, the horizon is not a single line, but is rather a boundary between the interior and exterior regions. To mimic this feature in the Euclidean solution, we can remove a small cylinder around the horizon $r = r_+$, obtaining a Euclidean version of the “stretched horizon” [34, 35].

The introduction of such a boundary dramatically changes the $\text{SL}(2, \mathbf{C})$ Chern-Simons theory: The Chern-Simons action now induces a dynamical chiral Wess-Zumino-Witten action on the stretched horizon, whose states represent genuine horizon degrees of freedom. For a noncompact group such as $\text{SL}(2, \mathbf{C})$, the structure of the resulting Wess-Zumino-Witten (WZW) Hilbert space is poorly understood. But we can again look for analogies with the $\text{SU}(2)$ theory, as suggested by Witten in the last section of Ref. [31].

An $\text{SU}(2)$ WZW model has an infinite number of states, but only finitely many of these occur for any given eigenvalue of the Virasoro operator L_0 . Now, L_0 can be interpreted as a generator of time translations, and so we should expect its eigenvalues to be proportional to the mass of the black hole. For large values of L_0 , it is known that the number of states increases exponentially with $L_0^{1/2}$ [36]. Since the mass of the three-dimensional black hole is proportional to r_+^2 , this gives at least the right qualitative dependence of the number of states on the horizon size. The exponential dependence of the density of states on $L_0^{1/2}$ is a generic feature of Kač-Moody algebras, at least for those based on compact Lie groups [36], and so this result may not be too sensitive to our use of $\text{SU}(2)$ as a stand-in for the correct gauge group.

B. Topological field theory and four-dimensional black holes

The results of the last section appear to be peculiar to the three-dimensional black hole. A generalization to four dimensions lies beyond the scope of this paper. We would, however, like to make two brief observations.

First, the geometry of the Euclidean black hole is largely independent of the dimension of spacetime. The black hole in d dimensions has the topology $\mathbf{R}^2 \times S^{d-2}$, and the entropy comes quite generically from the existence of a possible conical singularity in the \mathbf{R}^2 plane [7, 19]. The exponential form of the entropy can be interpreted as the existence of “one degree of freedom on \mathbf{R}^2 per unit horizon area.” An understanding of the role of \mathbf{R}^2 in any dimension, essentially a question of topological field theory on a cone, might therefore lead to a microscopic interpretation of black hole entropy in all dimensions.

Second, although the Chern-Simons formulation is unique to three dimensions, the meaning of the WZW degrees of freedom can be generalized. The $SL(2, \mathbb{C})$ Wess-Zumino-Witten theory on the horizon arises because it is not consistent to gauge-fix all of the “gauge” degrees of freedom; the horizon dynamics is that of would-be gauge transformations that are forced to become dynamical [37, 38]. But $SL(2, \mathbb{C})$ gauge transformations in the Chern-Simons formulation are equivalent to diffeomorphisms and local Lorentz transformations in the metric formulation. We therefore suggest that an understanding of black hole entropy is likely to require a careful analysis of the process of gauge fixing at the horizon. We hope to return to these questions elsewhere.

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- [1] J. D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973).
 - [2] S. W. Hawking, *Nature (London)* **248**, 30 (1974).
 - [3] D. Page, in *Proceedings of the Fifth Canadian Conference on General Relativity and Relativistic Astrophysics*, edited by R. B. Mann and R. G. McLenaghan (World Scientific, Singapore, 1994).
 - [4] J. D. Brown and J. W. York, *Phys. Rev. D* **47**, 1420 (1993).
 - [5] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
 - [6] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **48**, 1506 (1992).
 - [7] S. Carlip and C. Teitelboim, “The Off-Shell Black Hole,” IAS Report No. IASSNS-HEP-93/84 and Davis Report No. UCD-93-34, 1993 (unpublished).
 - [8] S. Deser, R. Jackiw, and G. 't Hooft, *Ann. Phys. (N.Y.)* **152**, 220 (1984).
 - [9] See, for example, A. F. Beardon, in *Discrete Groups and Automorphic Functions*, edited by W. J. Harvey (Academic Press, New York, 1977).
 - [10] E. Witten, *Nucl. Phys.* **B323**, 113 (1989).
 - [11] S. Carlip, in *Proceedings of the Fifth Canadian Conference on General Relativity and Relativistic Astrophysics* [3].
 - [12] D. Cangemi, M. Leblanc, and R. B. Mann, *Phys. Rev. D* **48**, 3606 (1993).
 - [13] C. Vaz and L. Witten, *Phys. Lett. B* **327**, 29 (1994).
 - [14] See W. M. Goldman, in *Geometry of Group Representations*, edited by W. M. Goldman and A. R. Magid (American Mathematical Society, Providence, 1988).
 - [15] See, for example, J. S. Birman, in *Discrete Groups and Automorphic Functions* [9].
 - [16] J. E. Nelson and T. Regge, *Nucl. Phys.* **B328**, 190 (1989).
 - [17] J. E. Nelson, T. Regge, and F. Zertuche, *Nucl. Phys.* **B339**, 516 (1990).
 - [18] The mathematics of this symplectic structure is discussed by W. M. Goldman, *Invent. Math.* **85**, 263 (1986).
 - [19] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **72**, 957 (1994).
 - [20] J. D. Brown *et al.*, *Class. Quantum Grav.* **7**, 1433 (1990).
 - [21] S. Carlip, this issue, *Phys. Rev. D* **51**, 632 (1995).
 - [22] G. Hayward, *Phys. Rev. D* **47**, 3275 (1993).
 - [23] G. Hayward and J. Louko, *Phys. Rev. D* **42**, 4032 (1990).
 - [24] A. Ashtekar and A. Magnon, *Proc. R. Soc. London* **A346**, 375 (1975).
 - [25] C. Crnkovic and E. Witten, in *Three Hundred Years of Gravity*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987).
 - [26] J. Louko and B. F. Whiting, *Class. Quantum Grav.* **9**, 457 (1992).
 - [27] J. Melmed and B. F. Whiting, *Phys. Rev. D* **49**, 907 (1994).
 - [28] S. Carlip, *Phys. Rev. D* **46**, 4387 (1992).
 - [29] S. Carlip, *Class. Quantum Grav.* **10**, 207 (1993).
 - [30] S. Carlip (in preparation).
 - [31] E. Witten, *Commun. Math. Phys.* **137**, 29 (1991).
 - [32] N. Hayashi, *Prog. Theor. Phys. Suppl.* **114**, 125 (1993).
 - [33] S. Carlip, *Nucl. Phys.* **B324**, 106 (1989).
 - [34] K. S. Thorne, R. H. Price, and D. A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University Press, New Haven, CT, 1986).
 - [35] L. Susskind, L. Thorlacius, and J. Uglum, *Phys. Rev. D* **48**, 3743 (1993).
 - [36] V. G. Kač and D. H. Peterson, *Adv. Math.* **53**, 125 (1984), Sec. 4.7.
 - [37] S. Elitzur *et al.*, *Nucl. Phys.* **B326**, 108 (1989).
 - [38] S. Carlip, *Nucl. Phys.* **B362**, 111 (1991).