"Nongeometric" contribution to the entropy of a black hole due to quantum corrections

Sergey N. Solodukhin*
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Head Post Office, P.O. Box 79, Moscow, Russia

(Received 12 August 1994; revised manuscript received 28 November 1994)

The quantum corrections to the entropy of charged black holes are calculated. The Reissner-Nordström and dilaton black holes are considered. The appearance of logarithmically divergent terms not proportional to the horizon area is demonstrated.

PACS number(s): 04.70.Dy, 04.62.+v

The classical Bekenstein-Hawking entropy of a four-dimensional black hole is known to be proportional to the area of the horizon:

$$S_{\rm BH} = \frac{1}{4}\frac{A_h}{\kappa}, \tag{1}$$

where κ is the gravitational constant [1]. Roughly speaking, the horizon is a two-dimensional surface which separates the whole space into two different regions, the free exchange of information between which is impossible. Thus, an outside observer does not have information about states of quantum fields in the region inside the horizon and therefore must trace over all such states. The entropy, characterizing this lack of knowledge, turns out to be determined only by the geometry of the surface separating these two regions; namely, it is proportional to the area of the surface. This fact occurs to be a feature of not just gravitational objects but is rather typical [2].

It is reasonable to ask whether this geometric character of hole entropy remains valid when quantum corrections (say, due to quantum fluctuations of matter fields in the black hole background) are taken into account.

Approximating the metric of a black hole of infinitely large mass by a more simple Rindler metric, it was shown [3] that the quantum correction to (1) again takes the geometric character

$$S^q = \frac{1}{48\pi}\frac{A_h}{\epsilon^2}\ , \tag{2}$$

though it is divergent when the ultraviolet cutoff ϵ tends to zero. This divergence was related with information loss in the black hole [4].

However, recently [5] we showed that the Rindler metric is not a good model for black hole space-time. The reason is that the horizon surfaces of a black hole (sphere) and Rindler space (plane) are topologically different. For a black hole of finite mass at the same time with (2) one also observes the logarithmically divergent mass-independent term,

$$S^q = \frac{1}{48\pi}\frac{A_h}{\epsilon^2} + \frac{1}{45}\ln\frac{\Lambda}{\epsilon}, \tag{3}$$

where Λ is the infrared cutoff. This term does not have geometric character and resembles the quantum correction to entropy of a two-dimensional black hole [6]. Generally, entropy is defined up to an arbitrary additive constant. Hence, one could assume that this term is not essential and does not influence the physics. In this paper we show (with the example of charged black holes) that the appearance of such nongeometric, logarithmically divergent terms is typical in four dimensions. In the general case, these terms depend on the characteristics of the black hole (charge, mass, etc.) and therefore cannot be neglected as nonessential additive constants. We use the path integral method of Gibbons and Hawking [7] to calculate the corrections to the entropy of the black hole. The basic formulas can be found in [5].

In the Euclidean path integral approach to a statistical field system taken with a temperature $T=(2\pi\beta)^{-1}$ one considers the fields which are periodic with respect to imaginary time τ with a period $2\pi\beta$. For arbitrary β the classical black hole metric is known to have a conical singularity which disappears only for a special Hawking inverse temperature β_H.

Let matter be described by the action

$$I_{\rm mat} = \frac{1}{2}\int (\nabla\Phi)^2\sqrt{g}d^4x\ . \tag{4}$$

Then the contribution to the energy and entropy due to matter fluctuations is given by

$$E^q = \frac{1}{2\pi}\partial_\beta I_{\rm eff}(\beta,\Delta)|_{\beta=\beta_H}\ ,$$

$$S^q = (\beta\partial_\beta - 1)I_{\rm eff}(\beta,\Delta)|_{\beta=\beta_H}, \tag{5}$$

where $\Delta = \nabla_\mu\nabla^\mu$ is the Laplace operator; $I_{\rm eff}(\beta,\Delta) = \frac{1}{2}\ln\det\Delta_{g_\beta}$ is the one-loop effective action calculated in the classical black hole background with conical singularity at the horizon. In order to take the derivative ∂_β in (5) we assume that β is slightly different from β_H.

The logarithm of the determinant in the DeWitt-Schwinger proper time representation is

$$\ln\det\Delta = -\int_{\epsilon^2}^{\infty} s^{-1}{\rm Tr}(e^{-s\Delta}), \tag{6}$$

where the integral over s is cut on the lower limit under

*Electronic address: solod@thsun1.jinr.dubna.su

$\epsilon^2 = L^{-2}$, and L and is maximal impulse.

In four dimensions we have the asymptotic expansion

$$\mathrm{Tr}(e^{-s\Delta}) = \frac{1}{(4\pi s)^2}\sum_{n=0}^{\infty} a_n s^n \ . \quad (7)$$

The divergent part of the effective action is given by

$$I_{\mathrm{eff}} = -\frac{1}{32\pi^2}\left[\frac{1}{2}a_0\epsilon^{-4} + a_1\epsilon^{-2} + a_2 \ln\left(\frac{\Lambda}{\epsilon}\right)^2\right] . \quad (8)$$

The black hole metric in the vicinity of the horizon ($\rho = 0$) has the form

$$ds^2 = \alpha^2(\rho^2 + C\rho^4)d\phi^2 + d\rho^2 + [\gamma_{ij}(\theta) + h_{ij}(\theta)\rho^2]d\theta^i d\theta^j \ , \quad (9)$$

where C = const, $\alpha = \beta/\beta_H$, and we introduce the new coordinate $\phi = \beta^{-1}\tau$ which has a period 2π. Near the horizon ($\rho = 0$) this Euclidean space looks topologically like a direct product $M_\alpha = C_\alpha \otimes \Sigma$. C_α is a two-dimensional cone with the metric $ds^2 = \alpha^2\rho^2 d\phi^2 + d\rho^2$, and Σ is the horizon surface with the metric $\gamma_{ij}(\theta)$. It was shown recently [8] that for a background such as this the coefficients in the expression (7) take the form

$$a_n = a_n^{\mathrm{reg}} + a_{\alpha,n} \ ,$$

where a_n^{reg} are standard coefficients $a_n = \int_{M_\alpha} a_n(x, x)d\Omega(x)$, given by the integrals over the smooth domain of M_α; the coefficients $a_{\alpha,n}$ are surface terms determined by integrals over the horizon Σ:[1]

$$a_{\alpha,0} = 0, \quad a_{\alpha,1} = \frac{\pi}{3}\frac{(1-\alpha)(1+\alpha)}{\alpha}\int_\Sigma \sqrt{\gamma}d^2\theta \ ,$$

$$a_{\alpha,2} = \frac{\pi}{18}\frac{(1-\alpha)(1+\alpha)}{\alpha}\int_\Sigma R\sqrt{\gamma}d^2\theta - \frac{\pi}{180}\frac{(1-\alpha)(1+\alpha)(1+\alpha^2)}{\alpha^3} \times \int_\Sigma (R_{\mu\nu}n_i^\mu n_i^\nu - 2R_{\mu\nu\alpha\beta}n_i^\mu n_i^\alpha n_j^\nu n_j^\beta)\sqrt{\gamma}d^2\theta \ , \quad (10)$$

where n^i are two-vectors orthogonal to the surface Σ ($n_i^\mu n_j^\nu g_{\mu\nu} = \delta_{ij}$).

Inserting (8) and (10) into (5) we obtain for the correction to the entropy.

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \frac{1}{144\pi}\left[\int_\Sigma R\sqrt{\gamma}d^2\theta - \frac{1}{5} \times \int_\Sigma (R_{\mu\nu}n_i^\mu n_i^\nu - 2R_{\mu\nu\alpha\beta}n_i^\mu n_i^\alpha n_j^\nu n_j^\beta)\sqrt{\gamma}d^2\theta\right]\ln\frac{\Lambda}{\epsilon} \quad (11)$$

For the metric (9) we may take $n_1^\mu = ((\alpha\rho)^{-1}, 0, 0, 0)$, $n_2^\mu = (0, 1, 0, 0)$.

For the metric (9) we obtain, at $\rho = 0$,

$$R = R_\Sigma - 6C - 4\gamma^{ij}h_{ij} \ ,$$
$$R_{\mu\nu}n_i^\mu n_i^\nu - 2R_{\mu\nu\alpha\beta}n_i^\mu n_i^\alpha n_j^\nu n_j^\beta = 6C - 2\gamma^{ij}h_{ij}, \quad (12)$$

where R_Σ is a scalar curvature determined with respect to the two-dimensional metric γ_{ij}.

The correction to the entropy (11) then reads

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left[\frac{1}{18} - \frac{1}{16\pi}\int_\Sigma \left(\frac{4}{5}C + \frac{2}{5}\gamma^{ij}h_{ij}\right)\sqrt{\gamma}d^2\theta\right] \times \ln\frac{\Lambda}{\epsilon} \ , \quad (13)$$

where we used the fact that the horizon surface Σ is a sphere and hence $\frac{1}{4\pi}\int_\Sigma R_\Sigma\sqrt{\gamma}d^2\theta = 2$; $A_\Sigma = \int_\Sigma \sqrt{\gamma}d^2\theta$ is the horizon area.

If we start with a black hole metric written in the Schwarzschild-like form

$$ds^2 = \beta^2 g(r)d\phi^2 + \frac{1}{g(r)}dr^2 + r^2\tilde{g}_{ij}(\theta)d\theta^i d\theta^j \ , \quad (14)$$

where $\tilde{g}_{ij}(\theta)$ is a metric of the two-dimensional (2D) sphere, then introducing a new radial coordinate $\rho = \int g^{-1/2}dr$ we obtain in the vicinity of the horizon [which is determined as simple zero of $g(r)$] the metric in the form (9) where $\gamma_{ij} = r_h^2\tilde{g}_{ij}$, $h_{ij} = \frac{r_h}{\beta_H}\tilde{g}_{ij}$, and $C = \frac{1}{6}g_r''|_{r_h}$; r_h is the radius of the horizon sphere. The corresponding Hawking temperature is $\beta_H = 2[g_r'(r_h)]^{-1}$.

Finally we get, for the quantum correction to the entropy,

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left[\frac{1}{18} - \frac{A_\Sigma}{20\pi}\left(\frac{1}{6}g_r''|_{r_h} + \frac{1}{r_h\beta_H}\right)\right]\ln\frac{\Lambda}{\epsilon} \ . \quad (15)$$

We see that the logarithmic term in (15) is formally proportional to the horizon area A_Σ. However, the coefficient of proportionality depends on the background black hole geometry and therefore the whole expression does not take the form (1).

Let us consider some particular examples.

Example 1: Reissner-Nordström black hole. The charged black hole is described by the metric (14) with

$$g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad M \geq Q. \quad (16)$$

The largest horizon is located at

$$r_h = M + \sqrt{M^2 - Q^2} \ . \quad (17)$$

The corresponding Hawking inverse temperature is

$$\beta_H = \frac{r_h^2}{\sqrt{M^2 - Q^2}} \ , \quad (18)$$

and for the second derivative g'' we have

$$g''|_{r_h} = 2r_h^{-4}(3Q^2 - 2M^2 - 2M\sqrt{M^2 - Q^2}) \ . \quad (19)$$

Inserting (17)–(19) into (15) we obtain, for the quantum correction to the entropy,

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left(\frac{1}{18} - \frac{M}{15r_h}\right)\ln\frac{\Lambda}{\epsilon} \ . \quad (20)$$

When the mass M becomes infinitely large, $r_h \to 2M$

[1]Our convention for the curvature and Ricci tensor is $R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha{}_{\nu\beta} - \cdots$, and $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$.

and (20) coincides with (3). It is interesting to note that for an extreme black hole ($M = Q$, $\beta_H = \infty$, $r_h = M$) the second term in (20) becomes negative:

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} - \frac{1}{90}\ln\frac{\Lambda}{\epsilon} \ . \quad (21)$$

However, this does not mean that the whole expression (21) is negative since in the limit $\epsilon \to 0$ the first positive term is dominant.

Example 2: Dilaton charged black hole. The metric of a dilaton black hole having an electric charge Q and magnetic charge P takes the form [9]

$$ds^2 = gdt^2 + g^{-1}dr^2 + R^2 d\Omega \ , \quad (22)$$

with the metric function

$$g(r) = \frac{(r-r_+)(r-r_-)}{R^2}, \quad R^2 = r^2 - D^2 \ , \quad (23)$$

where D is the dilaton charge: $D = (P^2 - Q^2)/2M$. The outer and the inner horizons are defined as

$$r_\pm = M \pm r_0 \ , \quad r_0^2 = M^2 + D^2 - P^2 - Q^2 \ . \quad (24)$$

Near the outer horizon we have

$$R^2 = R_+^2 + \frac{r_+}{\beta_h}\rho^2, \quad R_+^2 = r_+^2 - D^2 \ ,$$

and the metric (22) takes the form (9). The Hawking temperature β_H is

$$\beta_H = \frac{2(r_+^2 - D^2)}{r_+ - r_-} \ , \quad (25)$$

and for the second derivative g_r'' we have

$$g_r''(r_h) = \frac{2}{(r_+^2 - D^2)^2}[r_+^2 - D^2 - 2(r_+ - r_-)r_+] \ . \quad (26)$$

From the general expression (15) we get, for this type of black hole,

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left(-\frac{1}{90} + \frac{2}{15}\frac{r_+(r_+ - r_-)}{(r_+^2 - D^2)} + \frac{1}{10}\frac{(r_+ - r_-)}{r_+}\right) \times \ln\frac{\Lambda}{\epsilon} \ , \quad (27)$$

where $A_\Sigma = 4\pi(r_+^2 - D^2)$.

It is instructive to consider the black hole with only electric charge ($P = 0$). Then $r_0 = M - \frac{Q^2}{2M}$ ($2M^2 > Q^2$), $\frac{r_+(r_+ - r_-)}{(r_+^2 - D^2)} = 1 - \frac{Q^2}{4M^2}$, and expression (27) takes the form

$$S^q = \frac{A_\Sigma}{48\pi\epsilon^2} + \left[\frac{1}{18} + \frac{1}{15}\frac{2M^2 - Q^2}{2M^2} - \frac{1}{5}\left(\frac{2M^2 - Q^2}{4M^2 - Q^2}\right)\right] \times \ln\frac{\Lambda}{\epsilon} \ . \quad (28)$$

For large M we again obtain the result (3).

In the case of a dilaton extreme black hole, $2M^2 = Q^2$, the horizon area vanishes, $A_\Sigma = 0$, and the whole black hole entropy is determined only by the logarithmically divergent term

$$S^q_{\rm extr} = \frac{1}{18}\ln\frac{\Lambda}{\epsilon} \ . \quad (29)$$

Notice that (29) is positive. Expression (29) is very similar to the entropy of a two-dimensional black hole [6]. This can be considered as an additional justification of the point that the dilaton extreme black hole is effectively two dimensional that has been widely exploited recently [10].

Thus, we demonstrated with a number of examples the appearance of logarithmically divergent terms in a quantum correction to the entropy which are not proportional to the horizon area. One could conclude from this that the classical law (1) is broken due to the quantum corrections. However, one can show that the complete black hole entropy,

$$S = S_{\rm BH} + S^q \ , \quad (30)$$

which is the sum of classical Bekenstein-Hawking entropy (1) and the quantum correction, again takes the form (1), being defined with respect to the renormalized quantities. The renormalized gravitational constant $\kappa_{\rm ren}$ is determined as [5]

$$\frac{1}{\kappa_{\rm ren}} = \frac{1}{\kappa} + \frac{1}{12\pi\epsilon^2} \ . \quad (31)$$

Then (30) can be written in a form similar to (1),

$$S = \frac{1}{4\kappa_{\rm ren}}A_{\Sigma,\rm ren} \ , \quad (32)$$

if we define the quantum-corrected radius of horizon $r_{h,\rm ren}$

$$4\pi r_{h,\rm ren}^2 = 4\pi r_h^2 + \eta l_{\rm pl}^2 \ , \quad (33)$$

where $l_{\rm pl}^2 = \kappa_{\rm ren}$ is the Planck length; the quantity $\eta = \eta(M,Q)\ln\frac{\Lambda}{\epsilon}$ absorbs the logarithmic divergence of (30) and in general depends on the bare black hole characteristics M, Q, etc. For the Schwarzschild black hole η is a positive constant. An expression such as (33) appears in the work of York [11] describing the quantum fluctuations of the horizon and recently in [12] as a result of the quantum deformation of the Schwarzschild solution. On the other hand, for the charged Reissner-Nordström black hole we have $\eta = (\frac{2}{9} - \frac{4M}{15r_h})\ln\frac{\Lambda}{\epsilon}$ and for the extreme black hole ($Q = M$) η is negative.

Expression (33) means that quantum corrections result in shifting the horizon radius by the Planck distance. For the charged black hole with $Q < M < \sqrt{\frac{25}{24}}Q$ the quantum corrections decrease the horizon radius while for $M > \sqrt{\frac{25}{24}}Q$ it is increasing. The quantum-corrected entropy is determined then with respect to this quantum-corrected horizon in such a way that the law (1) remains valid. For a massive black hole ($M \gg M_{\rm pl}$) this shifting of the horizon is negligible. However, it becomes essential and important for a black hole of the Planck mass.

One of the reasons for (33) to hold could be the renormalization of mass of the black hole that can be calculated in principle from (5). One must take into account the boundary terms in the effective action which contribute to the energy [in (8) we neglected such a boundary term]. On the other hand, (33) can be considered as a result of deformation on small distances of the Schwarzschild solution due to quantum corrections (see [12]).

On the other hand, the expresssion (11) can be interpreted in the sense that the quantum entropy of the black hole does not depend just on the *intrinsic* geometry of a horizon (i.e., of the horizon area $A_{\sum}$) [13]. Generically, the entropy (11) depends also on the *extrinsic* geometry of the horizon, namely on the way the horizon surface is embedded in the larger four-dimensional manifold. Such a general possibility was recently discussed in [14] for the higher derivative gravity. In this regard, it is interesting to compare the result (11) with that of Ref. [14]. This will be consdiered elsewhere.

Note added in proof. A result similar to (29) was also derived by a different method in Ref. [15].

This work was supported in part by Grant No. RFL000 of the International Science Foundation.

[1] J.D. Bekenstein, Nuovo Cimento Lett. **4**, 737 (1972); Phys. Rev. D **7**, 2333 (1973); S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).
[3] G. 't Hooft, Nucl. Phys. **B256**, 727 (1985).
[4] L. Susskind and J. Uglum, Phys. Rev. D **50**, 2700 (1994).
[5] S.N. Solodukhin, Phys. Rev. D **51**, 609 (1995).
[6] C. Callan and F. Wilczek, Phys. Lett. B **333**, 55 (1994); D. Kabat and M.J. Strassler, *ibid.* **329**, 46 (1994); C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. **B424**, 443 (1994); J.S. Dowker, Class. Quantum Grav. **11**, L55 (1994).
[7] G.W. Gibbons and S.W. Hawking, Phys. Rev. D **15** 2752 (1977); S.W. Hawking, in *General Relativity*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[8] D.V. Fursaev, Phys.Lett. B **334**, 53 (1994).
[9] G.W. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).
[10] D. Garfinkle, G. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991); G. Horowitz and A. Strominger, Nucl. Phys. **B360**, 197 (1991); S.B. Giddings and A. Strominger, Phys. Rev. Lett. **67**, 2930 (1991).
[11] J.W. York, Jr., Phys. Rev. D **28**, 2929 (1983).
[12] D. I. Kazakov and S. N. Solodukhin, Nucl. Phys. **B429**, 153 (1994).
[13] A discussion of this point is also in D. V. Fursaev, *Black-hole Thermodynamics and Renormalization,* Report No. DSF-32/94, 1994 (unpublished).
[14] T. Jacobson, G. Kang, and R. C. Myers, Phys. Rev. D **49**, 6587 (1994).
[15] A. Ghosh and P. Mitra, Phys. Rev. Lett. **73**, 2521 (1994).