Conical singularity and quantum corrections to the entropy of a black hole

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For a general finite temperature different from the Hawking one there appears a well known conical singularity in the Euclidean classical solution of gravitational equations. The method of regularizing the cone by a regular surface is used to determine the curvature tensors for such a metric. This allows one to calculate the one-loop matter effective action and the corresponding one-loop quantum corrections to the entropy in the framework of the path integral approach of Gibbons and Hawking. The two-dimensional (2D) and four-dimensional cases are considered. The entropy of Rindler space is shown to be divergent logarithmically in two dimensions and quadratically in four dimensions that coincides with results obtained earlier. For the eternal 2D black hole we observe a finite, dependent on the mass, correction to the entropy. The entropy of the 4D Schwarzschild black hole is shown to possess an additional (in comparison with the 4D Rindler space) logarithmically divergent correction which does not vanish in the limit of infinite mass of the black hole. We argue that infinities of the entropy in four dimensions are renormalized by the renormalization of the gravitational coupling.

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One of the interesting problems of black hole physics is a microscopic explanation as the state counting of the Bekenstein-Hawking entropy of the black hole which is in four dimensions proportional to the area of the horizon. In quantum field theory one can define the "geometric entropy" associated with a pure state and a geometrical region by considering the pure state density matrix, tracing over the field variables inside the region to form the density matrix which describes the state of the field outside the region [1–3]. Taking this region to be a sphere in flat space-time, a recent numerical study [3] shows that the corresponding entropy scales as the surface area of a sphere. Thus, no gravity is present but the entropy thus defined behaves typically for a black hole. In [3] it has also been observed that this quantity is quadratically divergent when the ultraviolet cutoff (the size of the lattice) $\epsilon \to 0$. In two dimensions [3,5–8], the geometric entropy is logarithmically divergent, $S = \frac{1}{6}\ln\frac{\Sigma}{\epsilon}$, where Σ is the size of a box to eliminate infrared problems.

The somewhat related definition of the *black hole* entropy by tracing states outside the horizon has been suggested in [4].

It has been argued in [5] that quantum mechanical geometric entropy is the first quantum correction to the thermodynamical entropy. In flat space, the appropriately defined geometric entropy of a free field is just the quantum correction to the Bekenstein-Hawking entropy of Rindler space [5]. In the case of a black hole, the fields propagating in the region just outside the horizon give the main contribution to the entropy [1]. For very massive black holes this region is approximated as a flat Rindler space. Therefore, one can expect that for a black hole the corresponding quantum correction to the entropy is essentially the same as for Rindler space.

In [8] it has been shown that all these results can be obtained by using the known finite-temperature expression for the renormalized $\langle T^0_0\rangle$ in Rindler space.

The main goal of this paper is to calculate the quantum correction to Rindler space and black hole entropy by means of the path integral approach of Gibbons and Hawking [9]. We consider at first the two-dimensional case and extend the results obtained to four dimensions. The one-loop effective action for matter in $D = 2$ and its divergent part in $D = 4$ are well known and expressed in terms of geometrical invariants constructed from curvature. For a general finite temperature different from the Hawking one there appears the well-known conical singularity in the classical solution of gravitational equations. Therefore, we are faced with the problem of describing the geometrical invariants for manifolds with conical singularities. Fortunately, for the cases under consideration this can be performed. The extension of gravitational action to the conical geometries can also be found in [10].

Let us consider the canonical ensemble for the system of gravitational field ($g_{\mu\nu}$) and matter (φ) under temperature $T = 1/(2\pi\beta)$. Then, the partition function of the system is given as the Euclidean functional integral

$$Z(\beta) = \int [\mathcal{D}\varphi][\mathcal{D}g_{\mu\nu}] \exp[-I(\varphi, g_{\mu\nu})] \ , \tag{1}$$

where the integration is taken over all fields $(\varphi, g_{\mu\nu})$ which are real in the Euclidean sector and periodic with respect to the imaginary time coordinate τ with a period $2\pi\beta$. The action integral in (1) is a sum of pure gravitational action and action of matter fields:

$$I(\varphi, g) = I_{\rm gr}(g) + I_{\rm mat}(\varphi, g) \ . \tag{2}$$

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In the stationary phase approximation, neglecting the contribution of the thermal gravitons, one quantizes only matter fields considering the metric as classical. Then, in the one-loop approximation one obtains

$$Z(\beta) = \exp[-I_{gr}(g,\beta) - \tfrac{1}{2}\ln\det\Delta_g] \ , \tag{3}$$

where the leading contribution is given by the metric which is a classical solution and periodic with a period $2\pi\beta$ with respect to imaginary time τ. For arbitrary β this metric is known to have a conical singularity.[1] Therefore, in general, the integration in (1) must include the metrics with conical singularities as well [11]. As is well known, there exists a special Hawking inverse temperature β_H for which the conical singularity disappears. It seems reasonable to calculate thermodynamical quantities such as energy and entropy for arbitrary β and then take the limit $\beta \to \beta_H$.

With respect to the partition function (1) one can define the average of the energy,

$$\langle E\rangle = -\frac{1}{2\pi}\partial_\beta \ln Z(\beta) \ , \tag{4}$$

and entropy,

$$S = (-\beta\partial_\beta + 1)\ln Z(\beta) \ . \tag{5}$$

Hence, for the Hawking temperature we have

$$S_{\rm BH} = (-\beta\partial_\beta + 1)\ln Z(\beta)|_{\beta=\beta_H} \ . \tag{6}$$

As follows from (3), in the stationary phase approximation $Z(\beta)$ can be represented by the effective action

$$\begin{aligned} &\ln Z(\beta) = -\mathcal{G}_{\rm eff}(g), \\ &\mathcal{G}_{\rm eff}(g) = I_{gr}(g) + \mathcal{G}_{\rm eff}(\Delta), \end{aligned} \tag{7}$$

where $\mathcal{G}_{\rm eff}(\Delta) = \frac{1}{2}\ln\det\Delta_g$ is the one-loop contribution to the effective action due to matter fields. Then, from (6) we obtain

$$S_{\rm BH} = (\beta\partial_\beta - 1)\mathcal{G}_{\rm eff}(g)|_{\beta=\beta_H} \ , \tag{8}$$

where $\mathcal{G}_{\rm eff}(\Delta)$ is considered in the background classical metric with conical singularity. Inserting (7) into (8) we obtain that the total entropy

$$S_{\rm BH} = S_{\rm BH}^{\rm clas} + S_{\rm BH}^{q} \tag{9}$$

is a sum of the classical entropy,

$$S_{\rm BH}^{\rm clas} = (\beta\partial_\beta - 1)I_{gr} \ , \tag{10}$$

and quantum corrections,

$$S_{\rm BH}^{q} = (\beta\partial_\beta - 1)\mathcal{G}_{\rm eff}(\Delta) \ . \tag{11}$$

[1] In the strict sense, this metric is no longer the solution of the Einstein equations. Nevertheless, it still gives the main contribution to the functional integral (1) in the class of metrics with conical singularity.

Using these general formulas, let us consider the case of two-dimensional (2D) gravity interacting with scalar conformal matter:

$$I_{\rm mat} = \int \frac{1}{2}(\nabla\varphi)^2\sqrt{g}d^2z \ . \tag{12}$$

Then, we get that

$$\mathcal{G}_{\rm eff}(\Delta) = \frac{1}{2}\ln\,\det(-\Box) \ , \tag{13}$$

where $\Box = \nabla_\mu\nabla^\mu$ is the Laplace operator defined with respect to a metric with a conical singularity.

In two dimensions the description of manifolds with conical singularity is essentially simplified. We begin with the consideration of the simplest example of a surface such as that described by the metric

$$ds^2 = d\rho^2 + \left(\frac{1}{\beta_H}\right)^2\rho^2 d\tau^2 \ . \tag{14}$$

This space can be considered as a Euclidean variant of 2D Rindler space-time. Assuming that τ is periodical with period $2\pi\beta$, let us consider the new variable $\phi = \beta^{-1}\tau$ which has a period 2π. Then, (14) reads

$$ds^2 = d\rho^2 + \alpha^2\rho^2 d\phi^2, \tag{15}$$

where $\alpha = (\frac{\beta}{\beta_h})$. It is the standard cone with a singularity at $\rho = 0$. When $\alpha = 1$ ($\beta = \beta_H$) the conical singularity in (15) disappears.

Having calculated the scalar curvature $R_{\rm con}$ for the conical metric (15) let us approximate [12] the cone by a regular surface determined in 3D Euclidean space by the equation: $x = \alpha\rho\cos\phi$, $y = \alpha\rho\sin\phi$, $z = \sqrt{1-\alpha^2}\sqrt{\rho^2+a^2}$, and with the metric

$$ds^2 = \frac{\rho^2 + a^2\alpha^2}{\rho^2 + a^2}d\rho^2 + \alpha^2\rho^2 d\phi^2 \ . \tag{16}$$

In the limit $a \to 0$, the metric (16) coincides with the metric of the cone (15). Calculating the curvature for (16) and taking off the regularization ($a \to 0$) for the scalar curvature of the cone (15) we obtain (see also [13]):

$$R_{\rm con} = \frac{2(\alpha - 1)}{\alpha}\delta(\rho), \tag{17}$$

where $\delta(\rho)$ is the delta function defined with respect to the measure

$$\int_0^{+\infty}\delta(\rho)\rho d\rho = 1 \ .$$

This regularization of the cone allows us to use the results obtained for determinants of elliptic operators on regular surfaces and then, taking the approximation (16), to obtain the relevant expressions for surfaces with conical singularities.

On general grounds, the one-loop effective action (13) contains the divergent and finite parts [14],

$$\mathcal{G}_{\rm eff}(\Box) = \mathcal{G}_{\rm inf}(\Box) + \mathcal{G}_{\rm fin}(\Box), \tag{18}$$

for which, neglecting boundary terms, we have[2]

$$\mathcal{G}_{\rm inf}(\Box) = \frac{1}{48\pi}\int R\sqrt{g}d^2z \ln\left(\frac{L}{\mu}\right)^2 , \quad (19)$$

$$\mathcal{G}_{\rm fin}(\Box) = \frac{1}{96\pi}\int R\Box^{-1}R\sqrt{g}d^2z , \quad (20)$$

where L is an ultraviolet cutoff. Let us proceed with the conical metric (15). As follows from (17), the Euler number for the cone

$$\chi = \frac{1}{4\pi}\int R_{\rm con}\sqrt{g}d^2z = (\alpha - 1) \quad (21)$$

and, consequently, for the divergent part of the effective action we get

$$\mathcal{G}_{\rm inf}(\Box) = \frac{1}{12}(\alpha - 1)\ln\left(\frac{L}{\mu}\right)^2 . \quad (22)$$

In order to calculate the finite part (20), let us consider the function $\psi_{\rm con}$ which is the solution of the equation

$$\Box\psi_{\rm con} = R_{\rm con}, \quad R_{\rm con} = \frac{2(\alpha-1)}{\alpha}\delta(\rho) . \quad (23)$$

Equation (23) can be rewritten as

$$\left(\rho^{-1}\partial_\rho(\rho\partial_\rho) + \frac{1}{\alpha^2\rho^2}\partial^2_\phi\right)\psi = \frac{4\pi(\alpha-1)}{\alpha}\delta^2(x,x') , \quad (24)$$

where $\delta^2(x,x')$ is the two-dimensional delta function satisfying the condition.

$$\int_0^{2\pi} d\phi \int_0^{+\infty} \delta^2(x,x')\rho d\rho = 1.$$

One can see that (24) is just the equation for the Green function. The solution is well known:

$$\psi_{\rm con} = \frac{2(\alpha-1)}{\alpha}\ln\rho + w, \quad (25)$$

where w is a harmonical function, $\Box w = 0$. Taking into account (23), (25) we obtain the following for the finite part (20):

$$\mathcal{G}_{\rm fin}(\Box) = \frac{1}{96\pi}\int R_{\rm con}\psi_{\rm con}\sqrt{g}d^2z = \frac{1}{12}\frac{(\alpha-1)^2}{\alpha}\ln\epsilon + \frac{1}{24}(\alpha-1)w(0) , \quad (26)$$

where $w(0)$ is the value of the function w at $\rho = 0$. One can see that the "finite" part of the effective action is really divergent in the low limit of the integration over ρ. Therefore we introduced the regularization of the distance ϵ to the top of the cone. In the limit $\epsilon \to 0$ (26) is logarithmically divergent. Thus, the complete one-loop effective action on the cone takes the form

$$\mathcal{G}_{\rm eff}(\Box) = \frac{(\alpha-1)}{12}\ln\left(\frac{L}{\mu}\right)^2 + \frac{1}{12}\frac{(\alpha-1)^2}{\alpha}\ln\epsilon + \frac{(\alpha-1)}{24}w(0) . \quad (27)$$

We see that there are two types of divergences on the cone. The first is the ultraviolet one related to the cutoff of Feynman diagrams on the energy L (what is equivalent to the introduction of some minimal distance L^{-1}). The other divergences arise when the distance to the top of the cone goes to zero. One can expect that in the self-consistent renormalization procedure all the distances cannot be smaller than the fixed ultraviolet scale L^{-1}. From this point of view, the identification $\epsilon = L^{-1}$ is fairly natural. Moreover, this identification turns out to be necessary when we compare (27) with results obtained earlier for determinants on the cone by means of the ζ function [5] (see also [15]). Indeed, assuming $\epsilon = L^{-1}$ in (27) we obtain

$$\mathcal{G}_\infty(\Box) = \frac{1}{12}\left(\alpha - \frac{1}{\alpha}\right)\ln\frac{L}{\mu} , \quad (28)$$

which coincides with that obtained in [5].

Now we can calculate the corresponding correction to the entropy. Inserting (27) into (11) we observe that the second term in (27) does not contribute to the entropy (for $\beta = \beta_H$) and the first term in (27) leads to

$$S^q_{\rm BH} = \frac{1}{12}\ln\left(\frac{L}{\mu}\right)^2 = \frac{1}{6}\ln\frac{\Sigma}{\epsilon}, \quad (29)$$

where we assumed that $w(0) = 0$. This coincides with a result previously obtained for the quantum correction to the entropy of 2D Rindler space-time.

Some remarks concerning the ultraviolet divergences of entropy (29) are in order. According to general recipes of renormalization in quantum field theory, one must add the relevant counterterms to the "bare" action in order to cancel the divergences. In our case one must add the following term to the classical action:

$$\frac{s}{4\pi}\int R\sqrt{g}d^2z ,$$

which does not affect the classical equations of motion but contributes to the effective action $\Delta\mathcal{G}_{\rm eff} = s(\alpha - 1)$ and entropy $\Delta S = s$. The total entropy is finite but undefined. This procedure seems to be reasonable. The entropy is determined up to an arbitrary constant. The above cancellation of the divergences means only the renormalization of this additive constant which does not influence the physics and cannot be determined from the

[2]Our convention for the curvature and Ricci tensor is $R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha{}_{\nu\beta} - \cdots$, and $R_{\mu\nu} = R^\alpha{}_{\nu\mu\alpha}$.

experiment.

On the other hand, introducing the ultraviolet cutoff L in statistical description of the system we introduce a grain scale L^{-1}. This means that we define an elementary state of the system characterized by the size $\epsilon = L^{-1}$. The situation looks similar to the one that we have in classical statistical physics [16]. Quasiclassically, one can define the number of states in the region of phase space (p,q) as $\Delta\Gamma = \frac{\Delta p \Delta q}{(2\pi\hbar)^s}$ (s is number of freedoms of the system) where $\epsilon = (2\pi\hbar)$ is a scale characterizing the elementary state of the system. Then the entropy $S = \ln \Delta\Gamma$ is divergent when the Planck constant $\hbar \to 0$ ($\epsilon \to 0$). Thus, the result (29) can be interpreted as an indication that there must exist a fundamental scale which plays the role of ultraviolet regulator and naturally characterizes (like the Planck constant $\hbar$ in standard statistical physics) the size of the elementary state of the quantum gravitational system in phase space (concerning this, see also [17]).

Let us now consider the 2D black hole with the metric written in the Schwarzschild-like gauge:

$$ds^2 = g(x)d\tau^2 + \frac{1}{g(x)}dx^2 \ . \tag{30}$$

The metric is supposed to be asymptotically flat: $g(x) \to 0$ if $x \to +\infty$. We assume that τ in (30) is periodical with period $2\pi\beta$. Consider the new angle coordinate $\phi = \tau/\beta$ which has period 2π. If one introduces a new radial coordinate ρ,

$$\rho = \int \frac{dx}{\sqrt{g(x)}}, \tag{31}$$

the metric (30) takes the form

$$ds^2 = \beta^2 g(\rho) d\phi^2 + d\rho^2 \ . \tag{32}$$

Let the metrical function $g(x)$ have zero of the first order at the point $x = x_h$. In Minkowski space this point is the event horizon. Near the horizon we have $g(x) = g'|_{x_h}(x - x_h)$ for the metric function. For ρ [Eq. (31)] we obtain

$$\rho = \frac{2}{\sqrt{g'_{x_h}}}(x - x_h)^{1/2} \ , \tag{33}$$

in which it is assumed that the horizon is located at $\rho = 0$. For the function $g(\rho)$ in the vicinity of this point we get

$$g(\rho) \approx \frac{\rho^2}{\beta_H^2} \ , \tag{34}$$

where $\beta_H = 2/g'_{x_h}$. The metric (32) can be rewritten in the form

$$ds^2 = \left[\left(\frac{\beta}{\beta_H} \right)^2 \rho^2 d\phi^2 + d\rho^2 \right] + f(\rho) d\phi^2 \ , \tag{35}$$

where we directly extract the cone part of the metric with singularity at $\rho = 0$. The function $f(\rho) = \beta^2 g(\rho) - (\frac{\beta}{\beta_H})^2 \rho^2$ near the point $\rho = 0$ behaves as $f(\rho) \sim \rho^4$. Now we may regularize the cone part of the metric (35) as before [Eq. (16)], calculate the scalar curvature, and then remove the regularization ($a \to 0$).[3] At the end, we obtain the following result for the curvature:

$$R = \frac{2(\alpha - 1)}{\alpha}\delta(\rho) + R_{\mathrm{reg}}, \quad \alpha = \frac{\beta}{\beta_H} \ , \tag{36}$$

where the first term is the contribution due to the conical singularity, while the second, regular, term takes the form

$$R_{\mathrm{reg}} = \frac{g''_\rho}{g} - \frac{1}{2}\left(\frac{g'_\rho}{g} \right)^2 \ . \tag{37}$$

One can see that R_{reg} [Eq. (37)] has at $\rho = 0$ the finite value determined by the term of fourth order in the expansion of $g(\rho)$ [or $f(\rho)$].

In order to find one-loop quantum corrections in the background metric (30), (32), we have to find the function ψ satisfying the equation

$$\Box\psi = \frac{2(\alpha - 1)}{\alpha}\delta(\rho) + R_{\mathrm{reg}} \ , \tag{38}$$

where the Laplacian $\Box$ for the metric (32) reads

$$\Box = \partial_\rho^2 + \frac{g'}{2g}\partial_\rho + \frac{1}{\beta^2 g}\partial_\phi^2 \ . \tag{39}$$

Assuming that ψ is independent of ϕ we get that outside the point $\rho = 0$ the general solution of (38) is

$$\psi = \ln g + b\int_\rho^{\Lambda'} \frac{d\rho}{\sqrt{g}}, \tag{40}$$

where b and $\Lambda' > 0$ are still arbitrary constants. In the limit $\rho \to 0$ the Laplacian (39) coincides with the Laplacian for the cone [Eq. (24)]. Hence, in order to obtain the δ singularity on the right-hand side (RHS) of Eq. (38) the ψ [Eq. (40)] has to coincide with the corresponding solution for the cone, ψ_{con}, in the limit $\rho \to 0$:

$$\psi \to \psi_{\mathrm{con}} = \frac{2(\alpha - 1)}{\alpha}\ln\rho \quad \text{if} \quad \rho \to 0 \ . \tag{41}$$

Because of (34), we get, for the leading terms of (40),

$$\psi = (2 - b\beta_H)\ln\rho \ . \tag{42}$$

The condition (41) gives value of the constant $b = 2/\beta$. Finally, the solution of (38) reads

$$\psi = \ln g(\rho) + \frac{2}{\beta}\int_\rho^{\Lambda'} \frac{d\rho}{\sqrt{g(\rho)}} \tag{43}$$

or, equivalently, in terms of the coordinate x,

$$\psi = \ln g + \frac{2}{\beta}\int_x^{\Lambda} \frac{dx}{g(x)} \ . \tag{44}$$

[3] At this last stage, the important point is the behavior of the function $f(\rho) \sim \rho^4$. Because of this, cross terms such as $\delta(\rho)f(\rho)$ and $\delta(\rho)f'(\rho)$ do not contribute.

It is worth observing that the renormalized energy density of the scalar field in the space-time [Eq. (30)], as follows from (20), is

$$\langle T_0^0\rangle_{\rm ren} = \frac{1}{48\pi}\left(2g''_x - g'_x\psi'_x + \frac{g}{2}(\psi'_x)^2\right) . \tag{45}$$

For $\psi(x)$ [Eq. (40)] this reads

$$\langle T_0^0\rangle_{\rm ren} = \frac{1}{48\pi}\left(2g''_x - \frac{1}{2g}(g'^2_x - b^2)\right) . \tag{46}$$

This expression can be obtained by integrating the conformal anomaly [18]. For $b = 2/\beta$ this energy density at the space infinity ($x \to \infty$),

$$\langle T_0^0\rangle_{\rm ren} \to \frac{1}{24\pi}\left(\frac{1}{\beta}\right)^2 , \tag{47}$$

coincides with the energy density of massless bosons with temperature $T = \frac{1}{2\pi\beta}$.

It should be noted that the choice of constant b in (40) means the choice of the quantum state of the scalar field in the space-time of black hole. Therefore, the fact that this constant is related to the temperature β of the gravitational system seems to be natural: The thermal states of the black hole [4] and quantum field in the black hole space-time are the same.

On the other hand, we can see that (46) is divergent at the horizon ($x = x_h$) for general β and becomes regular only if $\beta = \beta_H$ (see also [8]). Thus, the Hawking temperature β_H is distinguished also in the sense that only for this temperature does the renormalized energy density of the quantum field, being in thermal equilibrium with the black hole, turn out to be finite at the horizon. Really, the infinite energy density means that something singular can happen at the horizon when back reaction is taken into account. Therefore, for $\beta \neq \beta_H$ the back reaction must be essential for justifying the semiclassical approximation [when we consider (3) instead of the functional integral (1)].

Before calculating the quantum corrections to the entropy of the 2D black hole, one would like to have some concrete description of 2D gravity. The simplest way is to use string-inspired dilaton gravity with the action [19]

$$I_{\rm gr} = -\int d^2z\sqrt{g}\{e^{-2\Phi}[-R + 4(\nabla\Phi)^2 + Q^2] + 4\nabla^\mu(e^{-2\Phi}\nabla_\mu\Phi)\} , \tag{48}$$

where the last boundary term is added [20] to the on-shell action (48) in order for the flat space-time to satisfy the condition $I_{gr}(g = 1)|_{\rm on\ shell} = 0$. Then, from the field equations we obtain

$$ds^2 = g(x)d\tau^2 + \frac{1}{g(x)}dx^2, \quad g(x) = 1 - 2me^{-Qx} ,$$
$$\Phi = -\frac{Q}{2}x . \tag{49}$$

The action (48), considered in the solution (49), takes the form [20]

$$I_{\rm gr} = \int d\tau dx[\partial_x(e^{-2\Phi}\partial_x g)] . \tag{50}$$

Assuming that τ is periodic with period $2\pi\beta$, for (50) we obtain

$$I_{\rm gr} = 2\pi\beta[e^{-2\Phi}\partial_x g]_{+\infty} - 4\pi\frac{\beta}{\beta_H}[e^{-2\Phi}]_{x_h}, \tag{51}$$

where x_h is the point of the horizon, $g(x_h) = 0$, and $2/\beta_H = [\partial_x g]_{x_h} = Q$. However, this naive calculation of the action does not take into account that for $\beta \neq \beta_H$ there exists a conical singularity at $x = x_h$ with contribution (36) to the curvature. This leads to an additional term in the action:

$$4\pi\left(\frac{\beta}{\beta_H} - 1\right)[e^{-2\Phi}]_{x_h} .$$

Therefore, the action (48) being considered on the classical metric (49) with $\beta \neq \beta_H$ is

$$I_{\rm gr} = 2\pi\beta[e^{-2\Phi}\partial_x g]_{+\infty} - 4\pi[e^{-2\Phi}]_{x_h} . \tag{52}$$

Thus, the β-dependent terms, calculable on the horizon, are mutually canceled in (52) and for the classical entropy of the black hole [20,21] we get

$$S^{\rm clas} = 4\pi[e^{-2\Phi}]_{x_h} = 8\pi m . \tag{53}$$

On the other hand, we obtain $M = 2mQ$ for the mass of the black hole. Hence, the entropy (53) can be written $S^{\rm clas} = 2\pi\beta_H M$.

Now let us calculate quantum corrections to (53) according to the above considered procedure.

From (36) we obtain that the Euler number for the metric (30), (32), when $\beta \neq \beta_H$, is the sum

$$\frac{1}{4\pi}\int R\sqrt{g}d^2z = \chi_{\rm con} + \chi_{\rm reg} , \tag{54}$$

where the first term on the RHS of (54) is the contribution due to the conical singularity while the second term is a regular contribution. As before, we have that

$$\chi_{\rm con} = \left(\frac{\beta}{\beta_H} - 1\right) . \tag{55}$$

To calculate the regular part $\chi_{\rm reg}$, it is convenient to use metric in the form (30). Then, we obtain $R_{\rm reg} = g''$ for the curvature. Consequently, for the regular term in (54) one gets

[4] This state is fixed when the integration in (1) is performed over the Euclidean manifolds with the cyclic Killing vector ∂_τ with period $2\pi\beta$.

$$\chi_{\rm reg} = \frac{1}{4\pi}\int g'' dx d\tau = -\frac{\beta}{\beta_H} \quad (56)$$

and the total Euler number (54) turns out to be independent of β: $\chi = -1$. The divergent part of the one-loop effective action (19) is

$$\mathcal{G}_{\rm inf}(\Box) = -\frac{1}{12}\ln\left(\frac{L}{\mu}\right)^2 . \quad (57)$$

For the finite part of the effective action (20) we get

$$\mathcal{G}_{\rm fin} = \frac{1}{96\pi}\int R\Box^{-1}R\sqrt{g}d^2z = \frac{1}{96\pi}\int R\psi\sqrt{g}d^2z = \mathcal{G}_{\rm con} + \mathcal{G}_{\rm reg} , \quad (58)$$

where $\mathcal{G}_{\rm con}$ is

$$\mathcal{G}_{\rm con} = \frac{\beta}{24}\left(\frac{\alpha-1}{\alpha}\right)\int_0^{+\infty}\delta(\rho)\psi\sqrt{g(\rho)}d\rho . \quad (59)$$

The integrand in (59) is nonzero only at $\rho = 0$ where $\psi = \psi_{\rm con}$ and $\sqrt{g} = \beta_H^{-1}\rho$. Hence, $\mathcal{G}_{\rm con}$ coincides with one we had for the cone [Eq. (22)]:

$$\mathcal{G}_{\rm con} = \frac{1}{12}\frac{(\alpha-1)^2}{\alpha}\ln\epsilon , \quad (60)$$

where the regularization, the distance ϵ from the horizon ($\rho = 0$), was introduced.

The regular part of (58),

$$\mathcal{G}_{\rm reg} = \frac{\beta}{48}\int_{x_h}^{+\infty} R_{\rm reg}\psi dx , \quad (61)$$

does not contain divergences in the low limit of the integration (the terms such as $\epsilon\ln\epsilon$ vanish in the limit $\epsilon \to 0$). In (61) we use the black hole metric in the form (30). For the concrete metric (49) we get that

$$\int_x^{\Lambda}\frac{dx}{g(x)} = -\frac{\beta_H}{2}\ln g(x) - x + \frac{1}{Q}\ln(e^{Q\Lambda} - 2m)$$

and $\psi(x)$ [Eq. (44)] takes the form

$$\psi = \left(1 - \frac{\beta_H}{\beta}\right)\ln g - \frac{2x}{\beta} + \frac{2}{\beta Q}\ln(e^{Q\Lambda} - 2m) . \quad (62)$$

Inserting this into (61), after the calculations we obtain

$$\mathcal{G}_{\rm reg} = \frac{\alpha}{12} - \frac{1}{12}\ln\left(\frac{e^{Q\Lambda} - 2m}{2m}\right) . \quad (63)$$

Collecting (60), (63), and (57), for the effective action we finally obtain

$$\mathcal{G}_{\rm eff}(\Box) = -\frac{1}{12}\ln\left(\frac{L}{\mu}\right)^2 + \frac{1}{12}\frac{(\alpha-1)^2}{\alpha}\times\ln\epsilon + \frac{\alpha}{12} - \frac{1}{12}\ln\left(\frac{e^{Q\Lambda} - 2m}{2m}\right) , \quad (64)$$

where $\alpha = \frac{\beta}{\beta_H}$. Identifying $\epsilon = L^{-1}$, for the infinite part of the effective action we obtain

$$\mathcal{G}_{\infty}(\Box) = -\frac{1}{12}\frac{(\alpha^2+1)}{\alpha}\ln\frac{L}{\mu} . \quad (65)$$

As one can see, the infinite part does not depend on the concrete form of the black hole solution. Probably, the result (65) is worth checking by means of the alternative calculation, for example, with the help of the ζ function. For the quantum correction to the entropy [Eq. (53)] for $\beta = \beta_H$ we obtain

$$S^q_{\rm BH} = \frac{1}{6}\ln\frac{\Sigma}{\epsilon} + \frac{1}{12}\ln\left(\frac{e^{Q\Lambda} - 2m}{2m}\right) . \quad (66)$$

The divergent part of (66) coincides with the quantum correction to the entropy in the case of the Rindler space-time [Eq. (29)]. Obviously, this justifies the approximation of the black hole space-time near the horizon by Rindler space, which was considered earlier [1,5–8].

In terms of the classical mass M and the Hawking temperature β_H the total entropy (9) can be written as

$$S_{\rm BH} = 2\pi\beta_H M + \frac{1}{12}\ln\left[\frac{(\frac{2}{\beta_H})e^{\frac{2\Sigma}{\beta_H}} - M}{M}\right] + \frac{1}{6}\ln\frac{\Sigma}{\epsilon} , \quad (67)$$

where we identified $\Lambda = \Sigma$.

In comparison with Rindler space, for the black hole case we observe the finite correction to the entropy [Eq. (67)] which logarithmically depends on the black hole mass M. This means, in particular, that the temperature of the system defined as $T^{-1} = \partial_M S$ is no longer T_H but possesses some corrections. This can be considered as an indication that back reaction must be taken into account. Indeed, the classical black hole solution does not give the extremum of the semiclassical statistical sum [Eq. (7)]. The configuration, which is the minimum of the one-loop effective action $\mathcal{G}_{\rm eff}$, must be considered. Generally, this quantum-corrected configuration may essentially differ from the classical one [22–24]. In any case, such thermodynamical quantities as temperature β_H, mass M, and entropy must be recalculated. Unfortunately, in general the quantum-corrected field equations are not exactly solvable. Recently [25–27], this was considered for the Russo-Susskind-Thorlacius (RST) model where the exact solution is known. In particular, paper [26] derived by a different method the correction term $-\frac{1}{12}\ln M$ in the entropy which is similar to our result (67).

Let us now apply our method to the 4D case. Assume that the gravitational field in four dimensions is described by the standard Einstein-Hilbert action:

$$I_{\rm gr} = \frac{1}{16\pi\kappa} \int d^4x \sqrt{g} R^{(4)} + \text{ boundary terms} , \qquad (68)$$

where the gravitational constant κ has dimensionality of length squared $[l^2]$.

Rindler space in four dimensions is described by the metric

$$ds^2 = \frac{\beta^2}{\beta_H^2} d\phi^2 + d\rho^2 + dx^2 + dy^2 , \qquad (69)$$

which for $\beta \neq \beta_H$ can be represented as a direct product of the two-dimensional cone [Eq. (15)] on the 2D plane: $C^2 \otimes R^2$. Applying the regularization procedure (16) to the cone part of the metric (69), we obtain that the 4D scalar curvature for (69) in the limit $a \to 0$ coincides with the curvature of the 2D cone [Eq. (17)]:

$$R^{(4)} = \frac{2(\alpha - 1)}{\alpha} \delta(\rho), \quad \alpha = \frac{\beta}{\beta_H} . \qquad (70)$$

We are also interested in the spherically symmetric metric describing the 4D black hole,

$$ds^2 = \beta^2 g(\rho) d\phi^2 + d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2\theta d\varphi^2) . \qquad (71)$$

Near the horizon we have $g(\rho) = \frac{\rho^2}{\beta_H^2}$ and $r(\rho) = r_h + \frac{\rho^2}{2\beta_H}$, where r_h is the value of the radius r at the horizon. For $\beta \neq \beta_H$ there again exists a conical singularity at the horizon ($\rho = 0$). The part of the metric (71) in the plane (ϕ, ρ) coincides with the 2D metric (32), (35). Regularizing the conical singularity at $\rho = 0$ as before we obtain that the complete Riemann tensor is a sum of the regular part (which is nonsingular in the limit $a \to 0$) and the part coming from the cone:

$$R^{\mu}_{\ \nu\alpha\beta} = R^{\mu}_{\rm con\ \nu\alpha\beta} + R^{\mu}_{\rm reg\ \nu\alpha\beta} . \qquad (72)$$

The only nontrivial component of the contribution from the cone in (72) is (for finite a)

$$R^{\phi}_{\rm con\ \rho\phi\rho} = \frac{a^2(1-\alpha^2)}{(\rho^2+a^2)(\rho^2+a^2\alpha^2)} . \qquad (73)$$

Though the whole consideration can be generalized, we study here only the case when the regular part of the metric is Ricci flat ($R^{\rm reg}_{\mu\nu} = 0$); i.e., it is the solution of the Einstein equation in vacuum. Then, we obtain from (72), (73) that the scalar curvature for the metric (71) in the limit $a \to 0$ is also given by expression (70).

The divergent part of the one-loop effective action for scalar matter described by the action

$$I_{\rm mat} = \frac{1}{2} \int (\nabla\varphi)^2 \sqrt{g} d^4x$$

in four dimensions (neglecting the boundary terms) takes the form (see, for example, [28])

$$\mathcal{G}_{\rm inf} = \frac{1}{2} (\ln \det \Box)_\infty = -\frac{1}{32\pi^2} \left[\frac{1}{2} B_0 L^4 + B_2 L^2 + B_4 \ln \left(\frac{L}{\mu} \right)^2 \right] , \qquad (74)$$

where L is the ultraviolet cutoff. The coefficients B_k in (74) take the form (we omit the overall irrelevant coefficients dependent on the type of matter)

$$B_0 = \frac{1}{2} \int \sqrt{g} d^4x ,$$

$$B_2 = -\frac{1}{6} \int R^{(4)} \sqrt{g} d^4x ,$$

$$B_4 = \int \left(\frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R^{\mu\nu}{}_{\alpha\beta} R_{\mu\nu}{}^{\alpha\beta} - \frac{1}{30} \Box R \right) \sqrt{g} d^4x . \qquad (75)$$

Considering (74) on the Rindler background (69), and using (70) we obtain

$$\mathcal{G}_{\rm inf} = \frac{1}{48\pi} (\alpha - 1) A_h L^2 - \frac{\alpha}{64\pi^2} V L^4 + (\alpha-1)^2 A_h T(a,\alpha) \ln \frac{L}{\mu} , \qquad (76)$$

where $A_h = \int dx dy$ is the area of the Rindler horizon; V is the volume of the space (69) if $\beta = \beta_H$. As one can see, B_4 is quadratic in curvature. For finite a it gives the last term in the effective action (76) with the function $T(a, \alpha)$ having the form

$$T(a,\alpha) = \frac{1}{a^2} T(\alpha) , \qquad (77)$$

where $T(\alpha)$ is a nonsingular function which takes finite value at $\alpha = 1$. Thus, we obtain an additional (to the ultraviolet) divergence when we take limit $a \to 0$. Fortunately, the last term in (76) is proportional to $(\alpha - 1)^2$ and does not contribute to the energy (4) or entropy (6) calculable at $\beta = \beta_H$.

From (76) we get the quantum correction to the entropy:

$$S^q_{\rm BH} = \frac{1}{48\pi} \frac{A_h}{\epsilon^2} , \qquad (78)$$

where the ultraviolet distance $\epsilon = L^{-1}$ was introduced.

The result (78) is exactly the one obtained for the geometrical entropy [1,3]. One can give this the following interpretation. Though we start from a flat space-time, considering the system at the finite temperature β, we obtain the statistical system in effective 4D Euclidean space with a conical singularity. Therefore, the induced gravitational effects of the curvature play a role leading to the nontrivial effective action (74) and entropy (78).

Notice that only the contribution coming from the scalar curvature $R^{(4)}$ to the first power in the effective action (74) leads to the quantum correction to the entropy of Rindler space. This term takes the same form as the classical ("bare") gravitational action (68) but with the L^2-divergent coefficient. The two-dimensional example teaches us that an extra ultraviolet divergence (in the limit $\epsilon \to 0$) can come from the "finite" nonlocal terms in the complete effective action which are omitted in (74).

These terms are not exactly known in four dimensions. By means of methods different from ours, there recently appeared results on the heat kernel asymptotic expansion on the curved cone [29]. They allow one to obtain *all* divergences due to the conical singularity, which could come both from the infinite and finite parts of the complete effective action. Comparing our result for Rindler space [Eq. (76)] with that of [29] we observe that divergence coming from the finite part is proportional to $(\alpha-1)^2L^2$. Renormalizing the infinities of (74), (76) we introduce the same counterterms as for the manifolds without conical singularities. Thus, to renormalize the L^2 divergence of (74), (76) it is enough to renormalize the gravitational constant [30]:

$$\kappa^{-1}=\kappa_B^{-1}+\frac{L^2}{12\pi}\ . \tag{79}$$

On the other hand, we must add new (absent in the regular case) local counterterms (cf. [31]), determined on the horizon surface, in order to absorb the additional divergences coming from the finite terms in the effective action. However, this divergence is proportional to $(\alpha-1)^2$ and hence does not contribute to the entropy. Therefore, the renormalization of the gravitational constant [Eq. (79)] is enough to renormalize the ultraviolet divergence of the quantum correction to the entropy [Eq. (78)].

In a recent interesting report Susskind and Uglum [32] have also calculated the quantum correction to the entropy of Rindler space which they consider as an infinite mass limit of black hole space-time. In particular, it has also been observed that the quantum correction to the entropy is equivalent to the quantum correction to the gravitational constant (for the discussion of this point, see also [33]).

The metric (69) for $\alpha\neq 1$ is similar to the metric of a cosmic string. In the cosmic string interpretation of the metric (69) our procedure of regularizing the conical singularity has a natural physical justification. It means that we consider the string with finite radius "a" of the kernel. This description is more realistic while the infinitely thin cosmic string (in the limit $a\to 0$) is an idealization. Therefore, we could consider the parameter "a" in our above consideration as a "phenomenological" one which is small but finite. This assumption allows us to avoid an additional divergence in the effective action related with the limit $a\to 0$.

Consider now the black hole described by the Schwarzschild solution. The metric takes the form (71). Near the horizon ($\rho=0$) we have

$$g(\rho)=\frac{\rho^2}{\beta_H^2}-\frac{4\rho^4}{3\beta_H^4}, \tag{80}$$

where $\beta_H=4M$; M is the mass of the black hole.

The classical Bekenstein-Hawking entropy of the black hole is well known:

$$S_{\rm BH}^{\rm clas}=\frac{A_h}{4\kappa}\ , \tag{81}$$

where $A_h=4\pi r_h^2$ is the area of the horizon sphere; for the Schwarzschild solution one has $r_h=2M$.

Calculating the quantum correction to this entropy we observe a new point in comparison with the Rindler case. Though the regular part of the metric is Ricci flat, the Riemann tensor $R^{\mu\nu}_{\rm reg\ \alpha\beta}$ is nonzero. From (72) we obtain that the term

$$R^{\mu\nu}_{\rm reg\ \alpha\beta}R^{\alpha\beta}_{\rm con\ \mu\nu} \tag{82}$$

contributes nontrivially to B_4 and to the effective action. The conical Riemann tensor in (82) is proportional to $(\alpha-1)$ and hence (82) leads to an additional correction to the entropy of the black hole.

In the limit $a\to 0$, the conical Riemann tensor $R^{\mu\nu}_{\rm con\ \alpha\beta}$ is proportional to the delta function $\delta(\rho)$. Hence, only the value of the regular Riemann tensor at the horizon is essential when we integrate (82). From (71) and (80) we obtain

$$R^{\phi}_{\rm reg\ \rho\phi\rho}(\rho=0)=\frac{4}{\beta_H^2}\ . \tag{83}$$

Substituting (72), (73), and (83) into the expression for B_4 [Eq. (75)] in the limit $a\to 0$ we have

$$B_4=\frac{32\pi}{270}\frac{A_h}{\beta_H^2}\frac{(1-\alpha)(\alpha^2+\alpha+1)}{\alpha^2}+\frac{(\alpha-1)^2}{a^2}T(\alpha)+\alpha B_4^0\ , \tag{84}$$

where B_4^0 is the coefficient B_4 [Eq. (75)] calculated for the Schwarzschild solution if $\beta=\beta_H$.

The infinite part of the one-loop effective action (74) then takes the form

$$\mathcal{G}_{\rm inf}=\frac{(\alpha-1)}{48\pi}A_hL^2-\frac{\alpha VL^4}{64\pi^2}+\frac{(\alpha-1)(\alpha^2+\alpha+1)}{2160\pi\alpha^2}\frac{A_h}{M^2}\ln\frac{L}{\mu}+\frac{(\alpha-1)^2}{a^2}T(\alpha)\ln\frac{L}{\mu}-\frac{\alpha}{16\pi^2}B_4^0\ln\frac{L}{\mu}\ . \tag{85}$$

Finally, for the quantum correction to the entropy we get

$$S^q_{\rm BH}=\frac{A_h}{4}\left(\frac{1}{12\pi\epsilon^2}+\frac{1}{180\pi M^2}\ln\frac{\Sigma}{\epsilon}\right), \tag{86}$$

where the ultraviolet distance $\epsilon=L^{-1}$ was introduced. The entropy (86) is proportional to the horizon area as before. However, in comparison with the Rindler case we observe an additional logarithmically divergent term in (86) which is dependent on the mass of the black hole.

Considering the entropy per horizon area in the limit of the infinite black hole mass ($M\to\infty$), we obtain the entropy for Rindler space. This probably could justify the approximation of the infinite mass black hole by Rindler space [1,32]. However, since the horizon area for the Schwarzschild solution is $A_h=16\pi^2M^2$, we observe that the logarithmically divergent term in the complete entropy,

$$S^q_{\rm BH}=\frac{A_h}{48\pi\epsilon^2}+\frac{1}{45}\ln\frac{\Sigma}{\epsilon}\ , \tag{87}$$

is independent of the mass. It takes a form which is very similar to that we had in the two-dimensional case [see (67)]. The reason for the different results for Rindler space and the black hole lies obviously in the different topologies of these manifolds. The topological numbers (like the Euler one) vanish for flat Rindler space while they are nonzero for the black hole and independent of the black hole mass.

To renormalize the L^2 and $\ln L$ divergences, in (85) we must add to the bare gravitational action not only the Einstein-like term but also the term $\kappa_1 B_4$ quadratic in curvature with new coupling constant κ_1. A comparison with the exact results [29] shows that divergences (both L^2 and $\log L$), additional to (85) and coming from the "finite" terms in the complete effective action, are again proportional to $(\alpha - 1)^2$ and they do not contribute to the entropy. Thus, we again obtain that the infinities of entropy (86) (but not of effective action) are renormalized by the renormalization of only the coupling constants κ and κ_1.

Finally, several remarks are in order of discussion. As has been noted in [32], only the quantum corrections but not the classical entropy have a clear interpretation in terms of counting the states. To overcome this, we may start from a zero bare gravitational action, assuming that the whole gravitational dynamics is determined by an induced matter effective action. Then, roughly speaking, the whole entropy of the black hole is a quantum correction. An interesting example of the induced gravity is given by superstring theory (see also [34]) which is probably free from ultraviolet divergences. In string theory, the space-time metric is not a primary object. It appears in the low-energy approximation as a "quantum condensate" of string excitations at energies $E << (\alpha')^{-\frac{1}{2}}$ (see, for example, [35]). Therefore, considering the low-energy effective action of the string, we obtain that already the "classical" entropy can be identified with the logarithm of an appropriately counted number of such string states. However, this speculation needs further detailed investigation.

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