Exact solution of two-dimensional Poincaré gravity coupled to fermion matter

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(Received 3 June 1994)

The two-dimensional model of gravity with zweibeins e^a and the Lorentz connection one-form $\omega^a{}_b$ as independent gravitational variables coupled to 2D massless Dirac matter is considered. It is shown that the classical equations of motion are exactly integrated and the general solution is found in the case of chiral fermions.

PACS number(s): 04.60.Kz, 04.20.Jb

The numerous recent attempts to formulate the theory of gravity in the framework of a consistent gauge approach resulted in constructing the gauge gravity models for the de Sitter and Poincaré groups (for a review see, e.g., [1]). The independent variables are now vielbeins $e^a = e^a_\mu dx^\mu$ and the Lorentz connection one-form $\omega^a{}_b = \omega^a{}_{b,\mu} dx^\mu$. These methods being applied in two dimensions (2D), give us a dynamical description of 2D gravity. It was argued that the investigation of simple two-dimensional model leads to a better understanding of four-dimensional gravity and its quantization [2]. It was shown in [2] that the Lagrangian $L = \gamma R^2 + \beta T^2 + \lambda$ is the most general one quadratic in curvature R and torsion T, and containing a cosmological constant λ. The classical equations of motion for this type of two-dimensional gravity were analyzed in the conformal gauge [3] and the light cone gauge [4] and their exact integrability was demonstrated. The various aspects of the quantization of the model were recently considered in [5]. In Ref. [6] it was shown that the formulation of the model on the language of differential forms is very useful. This allows one to find exactly the solution of vacuum gravitational equations using appropriate (and rather natural) coordinates on the 2D space-time. The resulting metric can be written in the Schwarzschild-like form and describes an asymptotically de Sitter black hole configuration [6]. Using this method in [7] one proves the integrability of the general 2D Poincaré gauge gravity with the Lagrangian being an arbitrary (not necessarily quadratic) function of curvature and torsion and demonstrates that the solution of the field equations is again of the black hole type.

One of the motivations for the study of two-dimensional gravity is that it can be considered as a "toy model" for the investigation of old problems of black hole formation and evaporation [8]. Therefore, the interaction of 2D Poincaré gravity with matter is worth studying. However, the coupling with matter in the general case breaks the above exact integrability. One exceptional case noted in [6,7] is the 2D Yang-Mills field. In this note we consider the coupling of 2D Poincaré gauge gravity with 2D massless Dirac fermions. The particular solutions of this system were earlier studied in [7,9]. Here we find exactly the general solution of the field equations using the method of Ref. [6].

We begin with a brief description of the Poincaré gauge gravity and Dirac spinors in two dimensions.[1] In this paper we follow the notation of paper [6]. The 2D gauge gravity is described in terms of zweibeins $e^a = e^a_\mu dz^\mu$, $a = 0,1$ [the 2D metric on the surface M^2 has the form $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, $\eta_{ab} = \mathrm{diag}(+1,-1)$] and the Lorentz connection one-form $\omega^a{}_b = \omega \varepsilon^a{}_b$, $\omega = \omega_\mu dz^\mu$ ($\varepsilon_{ab} = -\varepsilon_{ba}$, $\varepsilon_{01} = 1$). The curvature and torsion two-forms are

$$R = d\omega, \quad T^a = de^a + \varepsilon^a{}_b \omega \wedge e^b. \tag{1}$$

With respect to the Lorentz connection ω one can define the covariant derivative ∇ which acts on the Lorentz vector A^a as

$$\nabla A^a := dA^a + \varepsilon^a{}_b \omega \wedge A^b.$$

The Dirac matrices γ^a, $a = 0,1$ in two dimensions satisfy the relations

$$\gamma^a \gamma^b = \eta^{ab} - \varepsilon^{ab} \gamma_5, \tag{2}$$

where $\gamma_5 = \gamma^0 \gamma^1$, $(\gamma_5)^2 = 1$. The following identities are also useful:

$$\gamma^a \gamma_5 + \gamma_5 \gamma^a = 0 \tag{3}$$

and

$$\gamma^a \gamma_5 = \varepsilon^a{}_b \gamma^b. \tag{4}$$

In further consideration we use an explicit realization of γ matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

The Dirac spinors in two dimensions have two complex components,

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[1] The exhausted introduction to 2D Dirac spinors one can find in [7].

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{6}$$

and under local Lorentz rotation (on angle Ω) transform as

$$\Psi \to \Psi' = S\Psi, \quad \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}S^{-1}, \tag{7}$$

where the Dirac conjugated spinor is defined as $\bar{\Psi} = \Psi^{\dagger}\gamma^0$. The matrix S realizing the spinor representation of 2D Lorentz group is given by

$$S = \cosh\left(\frac{\Omega}{2}\right) - \gamma_5 \sinh\left(\frac{\Omega}{2}\right). \tag{8}$$

One can see that the components ψ_1 and ψ_2 transform independently:

$$\psi_1' = e^{\frac{\Omega}{2}}\psi_1, \quad \psi_2' = e^{-\frac{\Omega}{2}}\psi_2. \tag{9}$$

This means that left- (right-) chiral spinors defined as

$$\gamma_5\Psi = \mp\Psi \tag{10}$$

give us the irreducible representations of the Lorentz group.

It is useful to define the covariant spinor derivative ∇ as a differential operator acting on the field Ψ considered as a zero-form with values in the two-dimensional complex spinor space:

$$\nabla\Psi := d\Psi + \frac{1}{2}\omega\gamma_5\Psi, \quad \nabla\bar{\Psi} := d\bar{\Psi} - \frac{1}{2}\omega\bar{\Psi}\gamma_5. \tag{11}$$

This definition means that operator ∇ acts on spinor bilinear combinations, such as $\bar{\Psi}\Psi$, $\bar{\Psi}\gamma^a\Psi$, $\bar{\Psi}\gamma^{[a}\gamma^{b]}\Psi$, as the usual covariant derivative on the Lorentz scalar, vector, and bivector, correspondingly. One can see from (11) that the spinor covariant derivative ∇ acts on the components of the spinor field (6) as follows:

$$\nabla\psi_1 = d\psi_1 - \frac{1}{2}\omega\psi_1 \ ; \ \nabla\psi_2 = d\psi_2 + \frac{1}{2}\omega\psi_2.$$

The dynamics of the 2D gravitational (e^a, ω) and fermion (Ψ) variables is determined by the action

$$S = S_{\rm gr} + S_{\rm fer}, \tag{12}$$

where

$$S_{\rm gr} = \int_{M^2} \frac{\alpha}{2} * T^a \wedge T^a + \frac{1}{2} * R \wedge R - \frac{\lambda}{4}\varepsilon_{ab}e^a \wedge e^b \tag{13}$$

is the standard action of 2D Poincaré gauge gravity quadratic in curvature and torsion; $*$ is the Hodge dualization, and α, λ are arbitrary constants.

The action for 2D Dirac fermions in terms of differential forms can be written as

$$S_{\rm fer} = \int \frac{\imath}{2}\varepsilon_{ab}e^a \wedge (\bar{\Psi}\gamma^b\nabla\Psi - \nabla\bar{\Psi}\gamma^b\Psi). \tag{14}$$

Notice that here we consider only the massless fermions. One can see that due to the identity (3) the Lorentz connection ω is dropped out from expression (14) and really one can use the usual external derivative d instead of ∇ in (14).

Instead the curvature R and torsion T^a two-forms let us consider the dual zero-forms $\rho = *R$, $q^a = *T^a$.

The variation of (12) with respect to the Lorentz connection ω and zweibeins e^a gives the equations

$$d\rho = -\alpha q^a \varepsilon_{ab} e^b, \tag{15}$$

$$\nabla q^a = -\frac{1}{2\alpha}\Phi(\rho, q^2)\varepsilon^a{}_b e^b + J^a, \tag{16}$$

where $q^2 = q^a q^b \eta_{ab}$. In (16) the following notation was introduced: $\Phi(q^2, \rho) = \rho^2 + \alpha q^2 - \lambda$. The matter one-form J^a takes the form

$$J^a = -\frac{\imath}{2}\varepsilon^a{}_b(\bar{\Psi}\gamma^b\nabla\Psi - \nabla\bar{\Psi}\gamma^b\Psi). \tag{17}$$

It should be noted that $J^a = J^a_\mu dx^\mu$ is related with matter energy-momentum tensor: $T_{\mu\nu} = \frac{1}{2}(\varepsilon_\mu{}^\alpha J^a_\alpha e^a_\nu + \varepsilon_\nu{}^\alpha J^a_\alpha e^a_\mu)$.

Variation of action (12) with respect to the fermion field Ψ gives

$$(e^a\varepsilon_{ab}\gamma^b) \wedge (\nabla\Psi) = \frac{1}{2}T^a\varepsilon_{ab}\gamma^b\Psi. \tag{18}$$

From (1) we obtain

$$\omega = \tilde{\omega} - q_a e^a, \tag{19}$$

where $\tilde{\omega}$ is the torsionless part of the Lorentz connection:

$$de^a + \varepsilon^a{}_b\tilde{\omega} \wedge e^b = 0. \tag{20}$$

Using (19) and identity (4), Eq. (18) can be rewritten as

$$(e^a\varepsilon_{ab}\gamma^b) \wedge (d\Psi + \tfrac{1}{2}\tilde{\omega}\gamma_5\Psi) = 0; \tag{21}$$

i.e., the torsion is dropped in the Dirac equation. Taking the Hodge dualization of (21) one can transform (21) to a more standard form of the Dirac equation:

$$\gamma^\mu(\partial_\mu + \tfrac{1}{2}\tilde{\omega}_\mu\gamma_5)\Psi = 0,$$

where $\gamma^\mu = e^\mu_a\gamma^a$.

Using the Dirac equation (21) one can show that one-forms J^a (17) satisfy the identities

$$J_a \wedge e^a = 0, \quad \varepsilon_{ab}J^a \wedge e^b = 0. \tag{22}$$

Really the identities (22) are consequences of invariance action (14) under the local Lorentz and conformal transformations correspondingly [10].

The components of the spinor field (6) can be written as $\psi_i = e^{\chi_i}$, $i = 1, 2$, where $\chi_1 = \beta + \imath v$, $\chi_2 = \gamma + \imath u$ are complex fields. Then the one-forms J^a (17) take the form

$$J^0 = [e^{2\gamma}du - e^{2\beta}dv], \quad J^1 = [e^{2\gamma}du + e^{2\beta}dv], \tag{23}$$

while the Dirac equation (18) reads

$$(e^0 - e^1) \wedge (\imath du + d\gamma + \tfrac{1}{2}\omega)e^{\gamma} = \tfrac{1}{2}(T^0 - T^1)e^{\gamma},$$
$$(e^0 + e^1) \wedge (\imath du + d\gamma - \tfrac{1}{2}\omega)e^{\beta} = \tfrac{1}{2}(T^0 + T^1)e^{\beta}. \quad (24)$$

Assume that the orthonormal basis $\{e^a\}$ takes the conformal-Lorentz form

$$e^a = e^{\sigma}(n^a d\tau - \varepsilon^a{}_b n^b dx), \quad (25)$$

where $n^a, a = 0,1$ is a unit Lorentz vector, $n^2 = n^a n_a = \pm 1$. By means of diffeomorphism transformations in two dimensions, the arbitrary basis $\{e^a\}$ always can be transformed to the form (25).

The corresponding metric $ds^2 = \eta_{ab} e^a_\mu e^b_\nu dx^\mu dx^\nu$ takes the conformally flat form

$$ds^2 = n^2 e^{2\sigma}(d\tau^2 - dx^2).$$

By means of the identity

$$A \wedge e^a = \varepsilon^a{}_b e^b \wedge (*A), \quad (26)$$

where A is arbitrary one-form, we get, for the differential of (25),

$$de^a = \varepsilon^a{}_b[- * (d\sigma) + n^{\alpha}\varepsilon_{\alpha\beta} dn^{\beta}] \wedge e^b. \quad (27)$$

Inserting (27) into (20) we obtain, for $\breve{\omega}$,

$$\breve{\omega} = *(d\sigma) - n^a \varepsilon_{ab} dn^b. \quad (28)$$

Assuming for definiteness that $n^2 = 1$, components n^a can be written as $n^0 = \cosh\theta,\ n^1 = \sinh\theta$, so we have $n^a \varepsilon_{ab} dn^b = d\theta$. Under local Lorentz rotation on the angle Ω, the variable θ transforms as $\theta \to \theta - \Omega$. So the last term in (28) is a pure gauge part of the Lorentz connection.

Substituting the expression (28) into the Dirac equation (21) and using identities (4) and (26) we get

$$(e^a \varepsilon_{ab} \gamma^b) \wedge (d + \tfrac{1}{2} d\sigma - \tfrac{1}{2} d\theta \gamma_5)\Psi \quad (29)$$

or, in the spinor components (6),

$$(e^0 + e^1) \wedge (d + \tfrac{1}{2} d\sigma\psi_1 + \tfrac{1}{2} d\theta)\psi_1 = 0,$$
$$(e^0 - e^1) \wedge (d + \tfrac{1}{2} d\sigma\psi_1 - \tfrac{1}{2} d\theta)\psi_2 = 0. \quad (30)$$

For basis (25) we have

$$(e^0 \mp e^1) = e^{(\sigma \mp \theta)}(d\tau \pm dx).$$

Taking this into account, Eqs. (30) are easily solved and we obtain, for the spinor field,

$$\Psi = e^{-\frac{\sigma}{2}} \begin{pmatrix} e^{-\frac{\theta}{2}} e^{\imath v(x^-)} p(x^-) \\ e^{\frac{\theta}{2}} e^{\imath u(x^+)} f(x^+) \end{pmatrix}, \quad (31)$$

where v, p and u, f are arbitrary functions of the light-cone coordinates $x^- = \tau - x$ and $x^+ = \tau + x$, correspondingly.

Thus the Dirac equation (18),(21), taken separately, is exactly solved in the conformal-Lorentz gauge (25) and the general solution takes the form (31). However, now one must insert (31) in the gravitational equations (15),(16) and find the consistent solutions of the coupled gravity-Dirac system.

As in the vacuum case [6], there are two types of solutions of Eqs. (15)–(18). The first one is characterized by the torsion squared is zero, $q^2 \equiv 0$. One can see from Eqs. (15)–(18) that it is possible only in the case when torsion is identically zero: $q^a \equiv 0,\ a = 0,1$, the space-time has constant curvature[2] : $\rho^2 = \lambda$, and the one-forms (17) vanish: $J^a \equiv 0,\ a = 0,1$.

If zweibeins are taken in the form (25) the constant curvature condition $*(d\omega) = \rho = \text{const}$ gives us the equation for conformal factor σ: $*d * (d\sigma) = \rho = \text{const}$, which is equivalent to the Liouville equation

$$2\partial_- \partial_+ \sigma = \frac{\rho}{2} e^{2\sigma}, \quad (32)$$

where $\rho = \pm\sqrt{\lambda}$. The general solution of the Liouville equation is well known. By means of the coordinate changing, it can be transformed to the form

$$2\sigma = -\ln\left(1 - \frac{\rho}{4} x^+ x^-\right)^2.$$

Correspondingly, we have, for the metric,

$$ds^2 = \frac{dx^+ dx^-}{(1 - \frac{\rho}{4} x^+ x^-)^2}$$

and, for the Lorentz connection (28),

$$\omega = \frac{\frac{\rho}{4}}{1 - \frac{\rho}{4} x^+ x^-}(x^+ dx^- - x^- dx^+) - d\theta.$$

The one possible solution for the Dirac field is trivial: $\Psi = 0$ ($e^{\gamma} = e^{\beta} = 0$). The nontrivial Ψ with vanishing forms J^a (23) is given by (31) where u and v are constant functions:

$$\Psi = \left(1 - \frac{\rho}{4} x^+ x^-\right)^{\frac{1}{2}} \begin{pmatrix} e^{-\frac{\theta}{2}} e^{\imath v} p(x^-) \\ e^{\frac{\theta}{2}} e^{\imath u} f(x^+) \end{pmatrix}. \quad (33)$$

Let us now assume that $q^2 \neq 0$ identically on 2D space-time. We begin the analysis with the case when $J^a = 0,\ a = 0,1$. Then the gravitational field equations (15),(16) completely decouple from the Dirac equation (18). One sees from (23) that J^a vanish if e^{γ}, e^{β} are zero and/or the imaginary parts of χ_i, u, and v are constant functions. The gravitational equations reduce to the vacuum case. The general vacuum solution was obtained in [6] (for more accurate definitions see [11]). It is essential that one uses the variable ρ as one of the space-time coordinates. Introducing ϕ as additional, orthogonal to ρ, coordinate, we can write the vacuum solution for the zweibeins,

[2]Note that only if $\lambda \geq 0$ there exists the constant curvature solution.

$$e^a = q^a e^{-\frac{\rho}{\alpha}} d\phi - \frac{1}{\alpha q^2} \varepsilon^a{}_b q^b d\rho \qquad (34)$$

and, for the Lorentz connection,

$$\omega = -\frac{1}{q^2} q^a \varepsilon_{ab} dq^b - \frac{\alpha}{2} (q^2)'_\rho e^{-\frac{\rho}{\alpha}} d\phi, \qquad (35)$$

where q^2 is a known function of ρ:

$$q^2(\rho) = -\frac{1}{\alpha}(\rho + \alpha)^2 + \Lambda + \epsilon e^{\frac{\rho}{\alpha}}, \qquad (36)$$

where $\Lambda = \lambda/\alpha - \alpha$, ϵ is the integrating constant.

The corresponding metric

$$ds^2 = q^2 e^{\frac{-2\rho}{\alpha}} d\phi^2 - \frac{1}{\alpha^2 q^2} d\rho^2 \qquad (37)$$

was shown to describe the asymptotically de Sitter black hole configuration with Arnowitt-Deser-Misner (ADM) mass proportional to ϵ. The zeros of q^2 are points of the horizons [6].

It is worth observing that (34) takes the form (25) if we identify $n^a = \frac{q^a}{q}$, $e^\sigma = qe^{-\frac{\rho}{\alpha}}$, $\tau = \phi$, $x = \int^\rho \frac{e^{\frac{\rho'}{\alpha}}}{\alpha q^2(\rho')} d\rho'$. For definiteness we assume that $q^2 > 0$, then $q \equiv \sqrt{q^2}$.

Indeed, in coordinates (ϕ, x) the metric (37) is conformally flat:

$$ds^2 = q^2(\rho) e^{-2\frac{\rho}{\alpha}} (d\phi^2 - dx^2), \qquad (38)$$

where ρ can be, in principle, expressed as function of x. Note again that the first term in (35) is pure gauge: $\frac{1}{q^2} q^a \varepsilon_{ab} dq^b = d\theta$.

Since the solution of the Dirac equations (18), (21) for zweibeins taken in the form (25) is already known (31), we obtain the following expression for the fermion field:

$$\Psi = q^{-1/2} e^{\frac{\rho}{2\alpha}} \begin{pmatrix} e^{-\frac{\theta}{2}} e^{\imath v} p(x^-) \\ e^{\frac{\theta}{2}} e^{\imath u} f(x^+) \end{pmatrix}, \qquad (39)$$

where u and v are constants, and $x^\mp = \phi \mp x$. We see that Ψ (39) diverges at points where q^2 has zeros. Remember that these points are regular horizons of the vacuum metric (37), (38). Nevertheless, nothing singular happens at these points since the energy-momentum tensor for the spinor configuration (39) is identically zero. The fermion field Ψ also diverges at the point $e^{-\frac{\rho}{\alpha}} = 0$, where the black hole singularity is located (see [6]), while it tends to zero, $\Psi \to 0$, if $e^{-\frac{\rho}{\alpha}} \to \infty$.

Let us assume that $q^2 \neq 0$ identically on 2D spacetime. The fermion action (14) is invariant under (global) chiral (γ_5) transformations: $\Psi \to \Psi' = \exp[\mu\gamma_5]\Psi$. Therefore, for simplicity we may restrict ourselves by considering only the fermions of fixed chirality:

$$\gamma_5 \Psi = \Psi.$$

In this case the fermion field has only one nonzero component: $\psi_1 = 0$, $\psi_2 = e^\chi$, where $\chi = \gamma + \imath u$ is a complex field.

Then only the first of Eqs. (24) is nontrivial. It gives us, in particular, that $du \sim (e^0 - e^1)$. In the Lorentz invariant form it can be written as

$$q_a e^a - q^a \varepsilon_{ab} e^b = B du, \qquad (40)$$

where B is still an unknown scalar function. As seen from (9), only the real part of χ transforms under the Lorentz group: $\gamma \to \gamma - \frac{\Omega}{2}$. So the imaginary part u is the Lorentz scalar.

One can see from (15) and (40) that variables ρ and u can be naturally chosen as coordinates on 2D spacetime. Then basis of one-forms e^a is expressed in terms of $(d\rho,\ du)$:

$$e^a = \frac{q^a}{q^2}\left(-\frac{d\rho}{\alpha} + B du\right) - \frac{1}{\alpha q^2} \varepsilon^a{}_b q^b d\rho. \qquad (41)$$

The metric $ds^2 = \eta_{ab} e^a_\mu e^b_\nu dx^\mu dx^\nu$ correspondingly takes the form

$$ds^2 = \frac{1}{q^2}\left(B du - \frac{d\rho}{\alpha}\right)^2 - \frac{1}{\alpha^2 q^2} d\rho^2. \qquad (42)$$

In terms of the field $\chi = \gamma + \imath u$ the one-form J^a (23) has the components

$$J^0 = J^1 = e^{2\gamma} du.$$

It is convenient to introduce the one-form

$$J = \frac{2}{q^0 + q^1} e^{2\gamma} du.$$

Assuming for definiteness that $q^2 > 0$, let us introduce variable θ: $q^0 = q\cosh\theta$, $q^1 = q\sinh\theta$, $q \equiv \sqrt{q^2}$. Then we have, for J,

$$J = \frac{2}{q} e^{2\gamma - \theta} du.$$

Under local Lorentz rotation on angle Ω variable θ transforms as $\theta \to \theta - \Omega$. So the combination $(2\gamma - \theta)$ is really Lorentz invariant.

Multiplying Eq. (16) on q^a and $q^b \varepsilon_{ba}$ separately, we obtain

$$dq^2 = \frac{\Phi}{\alpha^2} d\rho + q^2 J \qquad (43)$$

$$\omega + d\theta = -\frac{\Phi}{2\alpha q^2} q_a e^a + \frac{1}{2} J, \qquad (44)$$

where we used $\frac{1}{q^2} q^a \varepsilon_{ab} dq^b = d\theta$. The Lorentz connection ω with respect to Lorentz rotations transforms as $\omega \to \omega + d\Omega$. So that $(\omega + d\theta)$ is again the Lorentz invariant. Equation (43) gives us q^2 as a function of ρ and u, while (44) is the equation on the Lorentz connection ω. Equation (43) is equivalent to

$$\partial_\rho q^2 = \frac{\Phi}{\alpha^2}(\rho, q^2), \quad \partial_u q^2 = 2q e^{2\gamma - \theta}. \qquad (45)$$

It follows from the first equation (45) that q^2 as a function of ρ has the same form as in the vacuum case [6] [see Eq. (36)]. However, ϵ now is a function of u, $\epsilon = \epsilon(u)$, which is found from the second equation (45). Taking into account that $\partial_u q^2 = \partial_u \epsilon e^{\rho/\alpha}$ we get

$$\partial_u \epsilon = 2qe^{-\rho/\alpha} e^{2\gamma-\theta}. \tag{46}$$

Since the left-hand side of Eq. (46) is a function of only variable u we obtain that $(2\gamma - \theta)$ must have the form

$$2\gamma - \theta = -\ln q + \frac{\rho}{\alpha} + 2\ln f(u), \tag{47}$$

where $f(u)$ is a function of variable u related with $\epsilon(u)$ by means of the equation

$$\partial_u \epsilon = 2f^2(u). \tag{48}$$

Acting now by external differential d on both sides of Eq. (40) we obtain

$$B'_\rho d\rho \wedge du = \left(\frac{\Phi}{\alpha} - q^2\right) V + J_a \wedge e^a - \varepsilon_{ab} J^a \wedge e^b. \tag{49}$$

From (41) we have $d\rho \wedge du = \frac{\alpha}{B} q^2 V$. Then using (22) and (45), the Eq. (49) gives us the equation on function B:

$$\frac{B'_\rho}{B} = \frac{1}{q^2}\partial_\rho q^2 - \frac{1}{\alpha}. \tag{50}$$

From this we finally find

$$B = B_0(u) q^2 e^{-\frac{\rho}{\alpha}}, \tag{51}$$

where B_0 is an arbitrary function of u. Now inserting (51) into Eq. (44) we obtain the expression for the Lorentz connection:

$$\omega + d\theta = -\frac{\alpha}{2}\partial_\rho q^2 e^{-\frac{\rho}{\alpha}} B_0(u) du + \frac{1}{2q^2}\partial_\rho q^2 d\rho + \frac{1}{q^2} e^{\frac{\rho}{\alpha}} f^2(u) du.$$

Taking into account Eq. (48) we finally obtain

$$\omega + d\theta = -\frac{\alpha}{2}\partial_\rho q^2 e^{-\frac{\rho}{\alpha}} B_0(u) du + \frac{1}{2} d(\ln q^2). \tag{52}$$

It should be noted that modulo exact forms this expression for ω takes the same form as in the vacuum case [6] [see (35)].

Now it is easy to check the self-consistency condition: $*(d\omega) = \rho$. Really this procedure is the same as in the vacuum case.

Let us again consider the Dirac equation (24). It is easy to see from (41) that

$$e^0 - e^1 = B_0 q e^{-\frac{\rho}{\alpha}} e^{-\theta} du.$$

Inserting this and Eq. (52) into the first equation (24) we obtain

$$B_0 e^{-\frac{\rho}{\alpha}} du \wedge \left(d\gamma - \frac{1}{2} d\theta + \frac{1}{2q}\partial_\rho q d\rho\right) = -\frac{1}{2} V. \tag{53}$$

Using the obtained expressions for γ (47) and Eq. (48) we obtain that (53) holds identically.

This completes the proof of exact integrability of equations (15)–(18). The complete solution is given by the expression

$$e^a = \frac{q^a}{q^2}\left(-\frac{d\rho}{\alpha} + q^2 e^{-\frac{\rho}{\alpha}} B_0 du\right) - \frac{1}{\alpha q^2}\varepsilon^a{}_b q^b d\rho$$

for zweibeins, expression (52) for the Lorentz connection ω, and

$$\Psi = q^{-1/2} e^{\frac{\theta}{2}} e^{\frac{\rho}{2\alpha}} \begin{pmatrix} 0 \\ e^{iu} f(u) \end{pmatrix} \tag{54}$$

for the chiral fermion field. q^2 is a known function of ρ and u:

$$q^2(\rho) = -\frac{1}{\alpha}(\rho + \alpha)^2 + \Lambda + \epsilon(u) e^{\frac{\rho}{\alpha}},$$

where

$$\epsilon(u) = 2\int^u f^2(u') du'.$$

Note that up to this moment everything was Lorentz invariant. As a result, the general solution depends on an arbitrary field θ that is a reflection of the underlying Lorentz symmetry. Now one can fix the gauge, say $\theta = 0$ (see [11]).

The solution also depends on arbitrary function $f(u)$ which is not determined from the field equations and is found from initial conditions for fermion field.

In the case when fermions of both chiralities present Eqs. (15)–(18) can be integrated in the same manner taking the imaginary parts, u and v, of the spinor components [see (31)] as light-cone coordinates. However, the solution takes a more complicated form.

The sense of a found solution becomes more transparent if we consider the δ-like impulse of fermion matter:

$$f^2(u) = \frac{E}{2}\delta(u - u_0), \ E > 0. \tag{55}$$

Then Eq. (48) is easily solved:

$$\epsilon(u) = \epsilon_0 + E\theta(u - u_0), \tag{56}$$

where $\theta(x)$ is a step function. In regions $u < u_0$ and $u > u_0$ taken separately, the function $\epsilon(u)$ is constant and one can consider here a new variable v:

$$v = u - \int^\rho \frac{2e^{\frac{\rho'}{\alpha}}}{\alpha q^2(\rho')} d\rho'. \tag{57}$$

Then in coordinates (u, v) the metric (42) takes the vacuum conformally flat form (38):

$$ds^2 = q^2 e^{\frac{-2\rho}{\alpha}} du dv. \tag{58}$$

For $u < u_0$ we have the vacuum black hole solution (34)–(38) with a mass $\epsilon = \epsilon_0$. The fermion impulse with energy E falls into this space-time at $u = u_0$ along the v direction. As a result, for $u > u_0$ we again obtain the vacuum black hole solution but with a mass $\epsilon = \epsilon_0 + E$.

It was shown in [6] that the space-time structure of the vacuum solution (34)–(38) essentially depends on the value of the constant ϵ. The falling of the fermion matter leads to the re-construction of the initial vacuum according to the new value of ϵ. It should be noted that in this aspect the found solution is similar to that of the 2D dilaton gravity coupled with scalar (conformal) matter [8]. However, there are some essential differences. The flat space-time is one of solutions in 2D dilaton gravity. The falling of the scalar matter into the flat space-time leads to the formation of the black hole. In the case under consideration there is no such solution describing the black hole formation from regular space-time (in our case it is the de Sitter one) due to fermion matter. The "bare" vacuum black hole configuration is necessary. The reason is that the vacuum constant curvature solution is not obtained from the black hole one (34)–(38) for a value of integrating constant ϵ, i.e., these solutions are not parametrically connected.[3] Instead, in 2D dilaton gravity [13] the flat space-time is obtained as zero mass black hole solution.

[3]Really this situation is typical for 2D gravity described by action polynomial in curvature [12].

In conclusion, we studied the 2D Poincaré gauge gravity coupled to 2D massless Dirac fermions and showed that the classical equations are exactly integrated. As in the vacuum case, there are two types of solutions. The solution of the first type is a space-time of constant curvature ($\rho^2 = \lambda$) and zero torsion, $q^a = 0$, $a = 0, 1$. The corresponding fermion field can take trivial ($\Psi = 0$) and nontrivial configurations. The solution of the second type is characterized by a torsion that is not identically zero. The space-time is of the black hole type with a mass dependent on the incoming fermion matter energy.

I would like to thank Professor F.W. Hehl and Dr. Yu.N. Obukhov for their kind hospitality at the University of Cologne. This work was supported in part by Grant No. 94-02-03665-a of the Russian Fund of the Fundamental Investigations.

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