Relativistic two-particle scattering resonances in the Tamm-Dancoff approximation

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The controversial problem of the appearance of resonances in relativistic charged particle scattering that has been reported in the literature is investigated for the case of scalar particles exchanging scalars and spin-1/2 particles exchanging photons. A systematic search for resonances is performed on the basis of a homogeneous eigenvalue equation for the coupling constant at scattering energies ($E > 2m$) as well as the scattering equation itself. The energy-independent kernels are shown to be free of any irregularities in the s wave. With energy-dependent Tamm-Dancoff kernels either resonances or shape resonances of finite width are found at strong coupling.

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I. INTRODUCTION

The appearance of spikes in correlated positron-electron spectra produced in heavy-ion collisions near the coulomb barrier [1–5] has sparked a flurry of activity in the theoretical study of Bhabha scattering based on approximations to quantum electrodynamics (QED) that result in effective interactions that are energy dependent. Four known findings for s- and p-wave resonances [6–9] at realistic couplings ($\alpha = \frac{1}{137}$) have been questioned in [10–16].

If resonances indeed were found for a wide class of interaction kernels this could have far-reaching implications. It is possible that the two-body kernels that do not produce resonances are simply incomplete. Most known quasipotential approaches derived from equal-time quantization fail to reproduce the positronium structure beyond α^4 due to the neglect of virtual pairs. This has been investigated in detail in the single-particle sector [17]. A contribution from virtual particle-antiparticle pairs is required, however, to cancel an α^5 contribution from the kinetic energy as well as the interaction.

As compared to the thorough investigation [10,11] of the coordinate-space models [6,7], the studies [12–16] for the models derived in the momentum representation [8,9] do not provide conclusions of comparable definiteness. The sole independent calculation for exactly the same model as discussed by Arbuzov *et al.* [8] was performed by Walet *et al.* in [12]. However, the authors note [12] that their procedure of the momentum variable rotation in the complex plane is not fully justified for the equation under study. The remark is found to be even more pertinent if one takes into account the nonanalytic behavior of the possible solution at a finite value of the momentum [15]. A recent proof [16] that the model discussed by Arbuzov *et al.* cannot support the proposed bound states in the continuum of Neumann-Wigner-type [18] for coupling constants $f < \frac{1}{2}$ refers to the coordinate-space analogue of the model [19], which is close but not identical to the momentum-space original [8]. To obtain definite final conclusions for the Tamm-Dancoff (TD) model from [8] and elucidate their relation to the approximations involved is the aim of the present paper.

Two technical difficulties are to be resolved. First, our search should account both for resonances and bound states in the continuum. To address this problem, a search for homogeneous s-wave solutions in the continuum to the TD equation from [8] is accompanied by phase-shift calculations. Another problem comes from the coincidence of the pole and logarithmic singularities at a finite momentum value arising from the free Green's function and interaction kernel, respectively. This we address by introduction of a small exchanged particle mass regulator μ, as well as by considering a modified equation for $\mu = 0$.

Although the model discussed in [8] was proposed for small couplings, we are considering it also in a wider scale of coupling constants f. This is motivated by the problem of electron-positron scattering in the background field of a high charge Z nucleus, $f = Z\alpha$, as well as by findings of two-particle scattering resonances at strong couplings in other models [11,20]. Investigating coordinate-space Breit-type equations from [7] Wong and Wong [11] have found bound $^1S_0$ states embedded in the continuum for $f \approx 1.2$. These zero-width resonances were related to the pole singularity in the effective interaction potential. Regularization of the potential developed them into either finite-width resonances or shape resonances, the so-called echoes [21], associated with a rapid falloff of the scattering phase shift through $\pi/2+n\pi$. The authors conjectured that it would be desirable to investigate instead of the Breit equation a Tamm-Dancoff-type equation based on the opinion that this transition will remove the pole singularity in the potential [22]. Finite-width resonances for similar values of f were found by Hirata and Munakata [20] from a completely different model based on a bosonization technique. These later findings are deemed to support the suggestions of a new phase in QED [20,23,24].

Note that the model from [8] is based on the Tamm-Dancoff or limited Fock space approximation with equal-time quantization. Equal-time Fock space methods, lacking covariance, are known to be problematic at strong coupling [25]. The proposed alternative light-front quantization method, when combined with the TD approximation, introduces other difficulties instead [26–28], com-

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pounded for the scattering problem by the loss of rotational symmetry [29]. When starting from the region of small coupling constants, the equal-time quantization limited Fock space approach seems to be preferable for practical calculations. It has been shown to work well for positronium fine structure [30(a),(b)] for which equivalent light-front calculations so far have experienced difficulties in demonstrating the correct $O(\alpha^4)$ energy spectrum [31]. Within equal-time quantization, in fact, one was able to go a step further, i.e., to show how positronium appears as a resonance in the photon-photon scattering channel [32].

However, one should be careful when drawing conclusions from results obtained in the equal-time approach at strong couplings. A proper account for self-energy terms and their counterterms is expected to be of utmost importance at strong coupling. These terms are deemed responsible for the possible phase transition in QED [24]. The effect of different treatment of these terms on some bound-state results has been shown recently in the Wick-Cutkosky model [26,27]. To account for this problem we consider in addition to the model from [8] a related model with an energy-independent effective interaction. No irregularities were found for small coupling with both models. At strong coupling either resonances or shape resonances were obtained for the energy-dependent case. The behavior of these resonances with varying energy and coupling strength have much in common with the ones reported in [11,20].

The plan of the paper is as follows. Theory and numerical procedures are outlined in Secs. II and III, respectively. The results for exchange quanta of finite and zero mass are described in Secs. IV and V. Section VI deals with particles with spin $\frac{1}{2}$ in the pseudoscalar ($J^{PC} = 0^{-+}$) channel. Section VII provides conclusions.

II. THEORY

From the Hamiltonian variational viewpoint [14,32,33] the derivation of the model equation discussed by Arbuzov *et al.* [8] is straightforward. The Fock space amplitudes for two scalars of mass m exchanging scalars of mass μ obey the equations:

$$[2\omega_m(p) - E]\phi_0(p) = \lambda \int \frac{d^3q}{\omega_\mu^2(\mathbf{p}-\mathbf{q})} \frac{m}{\sqrt{\omega_m(p)\omega_m(q)}} [\phi_1(\mathbf{q},-\mathbf{p}) + \phi_1(-\mathbf{q},\mathbf{p})] + \cdots , \tag{1}$$

$$[\omega_m(p_1) + \omega_m(p_2) + \omega_\mu(\mathbf{p_1}+\mathbf{p_2}) - E]\phi_1(\mathbf{p_1},\mathbf{p_2}) = \lambda \frac{m\omega_\mu(\mathbf{p_1}+\mathbf{p_2})}{\sqrt{\omega_m(p_1)\omega_m(p_2)}} [\phi_0(\mathbf{p_1}) + \phi_0(-\mathbf{p_2})] + \cdots , \tag{2}$$

where E is the energy in the c.m. system and $\omega_m(p) = \sqrt{m^2+p^2}$. The functions $\phi_0(\mathbf{p})$ and $\omega_\mu^{-3/2}(\mathbf{p_1}+\mathbf{p_2})\phi_1(\mathbf{p_1},\mathbf{p_2})$ describe the states with exchange of 0 and 1 quanta, respectively. The dots in Eqs. (1) and (2) denote mass renormalization terms and terms responsible for the coupling to the higher components $\phi_i(p)$, $i = 2,\ldots$. Ignoring both of them allows one to eliminate ϕ_1 yielding the following equation for the s-wave amplitude:

$$[2\omega_m(p) - E]\varphi(p) = \frac{f}{\pi}\int_0^\infty k(p,q,E)\varphi(q)dq , \tag{3}$$

$$k(p,q,E) = \frac{m^2}{\omega_m(p)\omega_m(q)} \ln \left| \frac{\omega_\mu(p+q) + R(p,q,E)}{\omega_\mu(p-q) + R(p,q,E)} \right| , \tag{4}$$

$$R(p,q,E) = \omega_m(p) + \omega_m(q) - E , \tag{5}$$

where $\varphi(p) = p\phi_0(p)$, $f = (2\pi\lambda)^2$. For energies $E > 2m$ the kernel $k(p,q,E)$ has, in fact, an imaginary part that is being ignored. The equation used in [8] follows from (3) when $\mu = 0$.

Alternative derivations of field-theoretic two-particle equations with some other nonlinear energy dependence can also be found in the literature [12]. Treating the terms omitted in Eqs. (1) and (2) in a different manner, using the perturbative decoupling approximation [14,32], leads to the energy-independent kernel (4) with $R(p,q,E) = 0$. We consider here both the model defined by Eqs. (3)–(5) and its energy-independent analogue determined by $R(p,q,E) = 0$ in order to have some indications concerning the sensitivity of the results obtained to the approximations involved.

Because of the energy dependence of the kernel $k(p,q,E)$ we solved instead of (3) the corresponding eigenvalue problem for π/f and fixed E:

$$\frac{\pi}{f}\psi(p) = \int_0^\infty \frac{k(p,q,E)}{2\omega_m(q) - E}\psi(q)dq \tag{6}$$

with $\psi(p) = [2\omega_m(p) - E]\varphi(p)$. In addition, the s-wave scattering phase shift $\delta(E)$ and cross section $\sigma(E)$ were calculated from the inhomogeneous equation

$$\xi(p) = \frac{f}{\pi}\int_0^\infty \frac{k(p,q,E)}{2\omega_m(q) - E}\xi(q)dq + \frac{2\pi f}{p_0}k(p,p_0,E) \tag{7}$$

with

$$\xi(p_0) = \frac{4\pi}{m}\tan\delta , \tag{8}$$

$$\sigma = \frac{4\pi}{p_0^2}\sin^2\delta . \tag{9}$$

The relevance of the solution of the homogeneous equa-

tion (6) to the scattering problem becomes obvious from the following. If one finds for a given scattering energy E eigenvalues for the coupling constant f_ν, one of three cases can occur.

(i) The wave function $\varphi_\nu(p)$ is square integrable, i.e., $\psi_\nu(p_0) = 0$: a bound state is embedded in the continuum. At the current level of approximation to the physics it is irrelevant for the scattering problem, but it might develop into a resonance of finite width, i.e., couple to the scattering problem once more physics is included.

(ii) The eigenfunction normalized as $\max_p[\psi_\nu(p)] = 1$ is small at the on-shell momentum $\psi_\nu(p_0) \ll 1$: a resonance in the scattering channel occurs provided that the phase shift $\delta(E)$ passes through $n\pi/2$ (n is assumed to be an odd integer) from below as the energy E increases through $E = E_0$. The resonance width Γ_ν is related to $\psi_\nu(p_0)$. For example, in the case of a separable kernel $k(p,q,E) = v(p)v(q)$ the relation is quadratic

$$\Gamma_\nu \approx c\psi_\nu^2(p_0) \ . \tag{10}$$

The amplitude of the asymptotic oscillatory part of the coordinate-space wave function is proportional to $\psi_\nu(p_0)$. If the passage through $n\pi/2$ is from above, then the width of the structure is limited by the Wigner causality condition [34].

(iii) The eigenfunction $\psi_\nu(p)$ is not small at the on-shell momentum p_0: passages of $\delta(E)$ through $\pi/2 + n\pi$ from above are slow and do not yield narrow structures in the partial cross section (9). Originally the name "echo" was given to this case [21], but recently it was also used for narrow structures [11] that correspond to the second possibility (ii).

The combination of the eigenvalue calculations in the continuum [Eq. (6)] with the scattering calculations [Eq. (7)] constitutes a procedure, which enables one to obtain reliable results concerning narrow scattering resonances.

III. NUMERICAL PROCEDURES

Equations (6) and (7) were solved independently with two somewhat different techniques, which are modifications of those used in [13–15]. First of all, we turn to a new variable θ, that is confined to a finite interval, using a mapping of the form

$$p = \alpha \tan \theta/2 \tag{11}$$

with $0 < \theta < \pi$ and a free parameter α. We derive the relevant equation for $\tilde{\psi}(\theta) = \psi(p)$ following from (6) or (7). Use of the Fourier-sine approximation

$$\tilde{\psi}(\theta) \approx \sum_{n=1}^{N} a_n \sin(n\theta) \tag{12}$$

converts the integral equation for $\tilde{\psi}(\theta)$ into a system of linear equations for a_n that is solved by standard methods. The coefficients of the system involve the calculation of one-dimensional integrals of known integrands.

An alternative approach uses a piecewise polynomial representation for $\tilde{\psi}(\theta)$. The whole interval $0 < \theta < \pi$ is divided into subintervals $x_m < \theta < x_{m+1}$, $m = 1, \ldots, M$ with $x_0 = 0$ and $x_{M+1} = \pi$. Each subinterval is covered by 10 nodes θ_{mi}, where $\theta_{m1} > x_m$ and $\theta_{m10} < x_{m+1}$. Then $\tilde{\psi}(\theta)$ is approximated as

$$\tilde{\psi}(\theta) = \sum_{m=1}^{M} \sum_{i=1}^{10} P_{mi}(\theta)\eta_m(\theta) \ , \tag{13}$$

where P_{mi} are Lagrange interpolation polynomials, $P_{mi}(\theta_{mj}) = \delta_{ij}$, and $\eta_m(\theta)$ equals one for $x_m < \theta < x_{m+1}$ and zero when θ is outside this interval.

Since we are seeking not only solutions in square-integrable functions $\varphi(p)$ for $E > 2m$, the integral in (6) or (7) is to be taken in the principal value (PV) sense at $q = p_0$. Also, for $p < p_0$ two logarithmically singular points $q_k(p)$ should be taken into account. As μ/m decreases, these tend towards $q_1 = (b - b^{-1})/2$ with $b = [E - p - \omega_m(p)]/m$, when the numerator in the logarithm (4) vanishes, and $q_2 = (b^{-1} - b)/2$ or $q_2 = (c - c^{-1})/2$ with $c = [E + p - \omega_m(p)]/m$ for vanishing denominator in the log argument.

Particular attention is given to the region $p \approx p_0$ where oscillations appear in $\psi(p)$ for small μ/m. Within the Fourier-sine method this is achieved by increasing N in Eq. (12) up to 200 or using more elaborate mappings rather than (11) designed to concentrate points near $p = p_0$, e.g., $p = \alpha \tan[a + b(\theta - \theta_1)^3]$ with appropriate α, a, b, and θ_1. Within the piecewise polynomial technique, M in Eq. (13) is increased up to 25 and additional intervals in the p_0 vicinity are allocated by hand.

IV. FINITE-μ RESULTS FOR SPINLESS PARTICLES

For the E-independent kernel (4) with $R(p,q,E) = 0$ and finite mass μ we were able to find solutions of type (iii) in those cases where the scattering phase shift $\delta(E)$ is dropping from $\delta(0) = k\pi$ to $\delta(\infty) = 0$ with k being the number of true bound states supported by the potential according to Levinson's theorem. These solutions $\psi(p)$ invariably have a maximum near p_0, and φ has a first-order pole there rendering it non-square integrable. For couplings that are near critical for the first true bound state to appear we obtain at very low energies a passage through $\pi/2$ in $\delta(E)$ at $E = E_1$ with positive slope and a fall through $\pi/2$ nearby at $E_2 > E_1$ in accordance with the Bethe-Salpeter results in the ladder approximation [35]. Tracing the eigenvalues f_ν as a function of E in (6) also permits one to draw conclusions which of the cases (ii) and (iii) occurs in the scattering problem.

For the E-dependent kernel (4) with $R(p,q,E) \neq 0$ and finite μ solutions of both types (ii) and (iii) were found. In Fig. 1 the smallest eigenvalues f_ν calculated from (6) are shown as a function of the exchanged particle mass μ for $E = 3m$. Two kinds of levels can be distinguished from the figure: those that stabilize with decreasing μ and those that do not. As counted from below at $\mu/m = 1 \times 10^{-4}$, the second level, $\nu = 2$, is

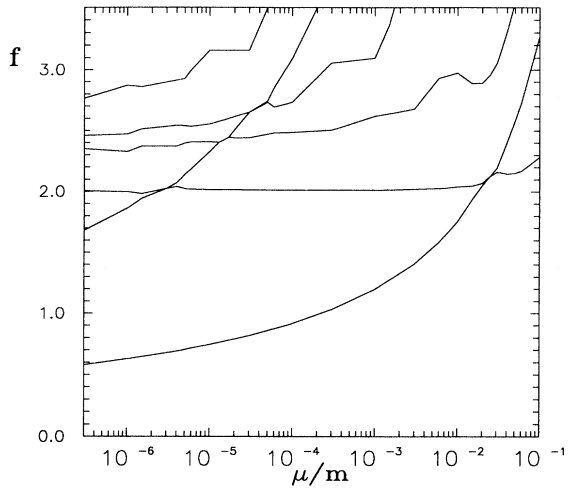

FIG. 1. The lowest coupling constants f_ν satisfying Eq. (6) as a function of the ratio $3\times10^{-7} < \mu/m < 10^{-1}$ for $E = 3m$. No true level crossing occurs: near the avoided crossings the accuracy of the solution is not sufficient to resolve the correct repulsive behavior of the real-valued eigenvalues.

definitely stable and some signs of stability show up for levels $\nu = 3$ and 4, while those with $\nu = 1$ and 5 are unstable. This feature correlates with the value of $\psi_\nu(p_0)$ (normalization $\max_p[\psi_\nu(p)] = 1$ is imposed), which is about 0.05–0.1 for $\nu = 2$, 3, and 4 and is about 0.5 to 1 for $\nu = 1$ and 5, suggesting the correspondence of these groups of levels to cases (ii) and (iii), respectively.

For several lower levels these suggestions were tested and confirmed by scattering calculations using Eq. (7). The cross-section results for $f = f_2 = 2.014$ with $\mu/m = 1\times10^{-3}$ and 1×10^{-4} and $f = f_3 = 2.006$ with $\mu/m = 3\times10^{-7}$ are represented by the solid, long-, and short-dashed lines in Fig. 2, respectively. The peak in the cross section is accompanied by a rapid increase in the scattering phase

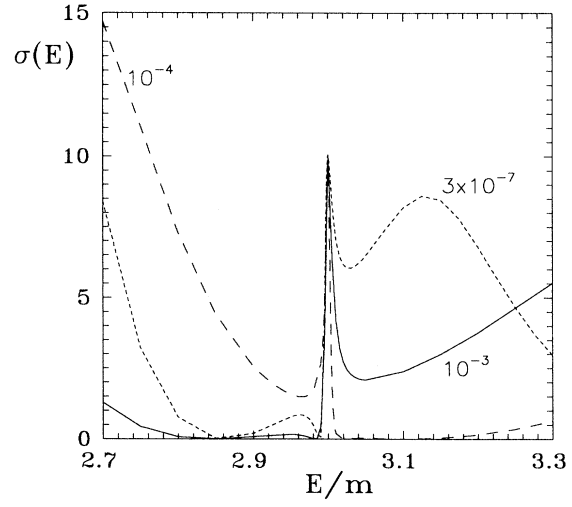

FIG. 2. Energy dependence of the s-wave cross section for $f \approx 2$ and different μ/m values indicated in the graph (see text).

shift as the energy increases through $E = 3m$, which confirms the finding of a resonance.

The fact that the lowest resonant level $\nu = 2$ at $\mu/m = 10^{-4}$ becomes $\nu = 3$ at $\mu/m \approx 10^{-6}$ is related to a rapid falloff of the higher-lying nonresonant level, which "crosses" the lower-lying ones, and becomes the second level at $\mu/m \approx 10^{-6}$. In the vicinity of the level crossing the eigenvalues become complex with small imaginary part. Two possible origins of the breakdown in our computational methods in this region were considered: an inadequate evaluation of eigenvalues from the nonsymmetric matrix that represents the integral operator in Eq. (6), and a significant decrease in the accuracy of representation of the integral equation itself. A straightforward symmetrization that is possible for simple quadratures [36] does not work in our case. We attempted to improve the accuracy by inverse iteration near the crossings, but were unsuccessful. For any initial choice of the eigenvalue the iterations yielded a divergent series of corrections, suggesting that the problem resides with the representation of the inverse operator in the chosen basis. Indeed, investigating the relation between the numerical errors in the calculated eigenvalues and wave functions we find that the relation changes from a quadratic dependence into a linear one, as one moves from the region of well-resolved eigenvalues into the region, where the eigenvalues are nearly degenerate. This result agrees well with the known quantum-mechanical effects of introducing a small perturbation on the behavior of energy levels. Special improvements in the representation of the function $\psi_i(p)$ enabled us to resolve the eigenvalues f_i and f_{i+1} up to about 10^{-2}. Inspection of the behavior of the functions $\psi_i(p)$ and $\psi_{i+1}(p)$ for $p \approx p_0$ provided us with an independent confirmation that no real crossing occurs: instead, level redistributions take place, where the lower level successively exchanges its characteristic with the one coming from above. The signature of this nonresonant level of the type (iii) can be seen from Fig. 2 in the form of a broad approach towards the unitarity limit in the cross section $\sigma(E)$ at $E \approx 3.14m$. This structure moves in the diagram as μ/m changes in correspondence with the behavior of the levels shown in Fig. 1.

The resonant coupling constants f_ν decrease with decreasing energy. For $\mu/m = 1\times10^{-4}$ the two lowest resonant levels f_ν are shown in Fig. 3. At $E = 3m$ these correspond to the levels $\nu = 2$ and 3 in Fig. 1, which become, respectively, $\nu = 7$ and 8 at $E = 2.02m$. $f_r \approx 1.34$ stands for the lowest coupling constant value for which a resonance occurs at finite μ/m.

The lower $E/m < 2$ part of Fig. 3 shows the energy of the two-particle ground state as a function of the coupling strength for both the E-dependent and E-independent kernels. With the TD kernel the energy shows a slower decrease with increasing f, such that the two-particle bound state is weakly relativistic in the range of strong coupling for which resonances appear.

The width of the resonance Γ_ν decreases rapidly with decreasing energy (Fig. 4). The same is true for the energy dependence of the eigenfunction value at the singular point, $\psi_\nu(p_0)$. In general, the linear relationship (10) between Γ_ν and $\psi^2_\nu(p_0)$ is found to hold with a viola-

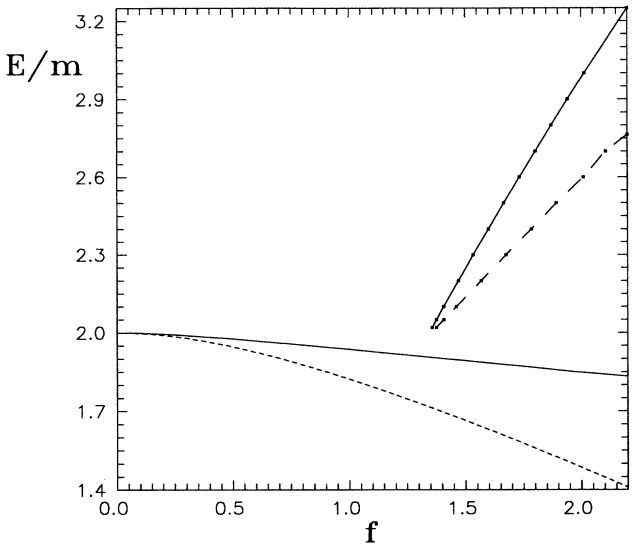

FIG. 3. $E/m>2$: Energy vs coupling constant plot for the smallest (solid line) and the second-smallest (dashed line) coupling constants f_ν for which a resonance occurs for $\mu/m = 1\times10^{-4}$. $E/m < 2$: ground-state energy as a function of the coupling constant for the energy-dependent (solid line) and the energy-independent (short-dashed line) cases.

tion in the neighborhood of the level "crossing," when the eigenfunction behavior undergoes significant changes with varying energy.

The Γ_ν-E dependence finds its reflection in the resonant wave function behavior in the coordinate representation

$$\phi_0(r) = \int e^{i\mathbf{pr}}\phi_0(p)d^3p = \frac{4\pi}{r}\int_0^\infty \frac{\sin pr\psi_\nu(p)}{2\omega_m(p)-E}dp\ . \quad (14)$$

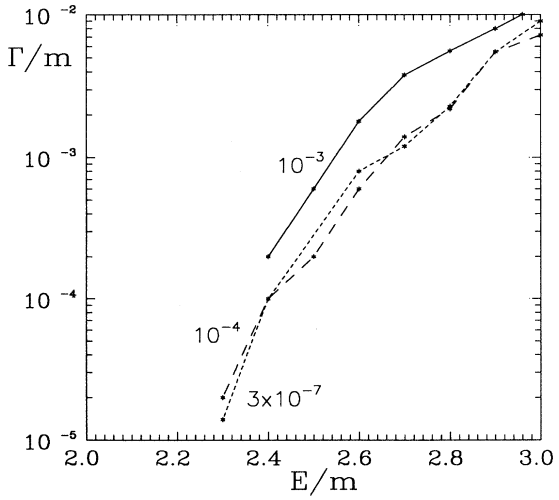

FIG. 4. Energy dependence of the resonance width for the resonance with the smallest coupling constant and different μ/m values indicated in the graph.

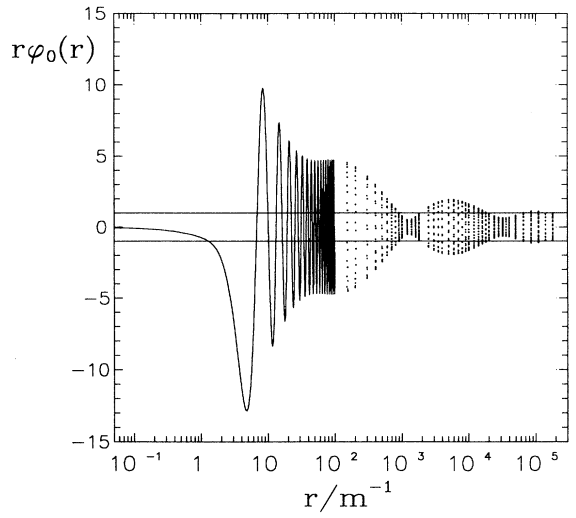

FIG. 5. The r dependence of the wave function $r\phi_0(r)$ with $\phi_0(r)$ from Eq. (14) for $E = 3m$ and $\mu/m = 1\times10^{-4}$. In contrast with the normalization $\max_p[\psi_\nu(p)] = 1$ used in the text, here we make use of the normalization $r\phi_0(r) \approx 1\times\cos p_0 r$, $r\to\infty$. Horizontal lines in the figure indicate the asymptotic envelopes, $+-1$. For $mr > 100$ the calculations were made along a period of $r\phi_0(r)$ only: these results are represented by the dots in the figure.

Figure 5 reveals a large $r\phi_0(r)$ amplitude at finite r compared to its asymptotic behavior

$$r\phi_0(r) \approx \pi^2 p_0^{-1}\omega_m(p_0)\psi_\nu(p_0)\cos p_0 r \quad \text{as } r\to\infty\ , \quad (15)$$

which is attained for both phase and amplitude at $r \approx 10/\mu$. The excess at finite r significantly increases with decreasing energy (Fig. 6), being in accord with the decrease in the width Γ_ν mentioned above. It is interesting to note that resonances of similar behavior, which show an increase in the resonance energy and in the resonance

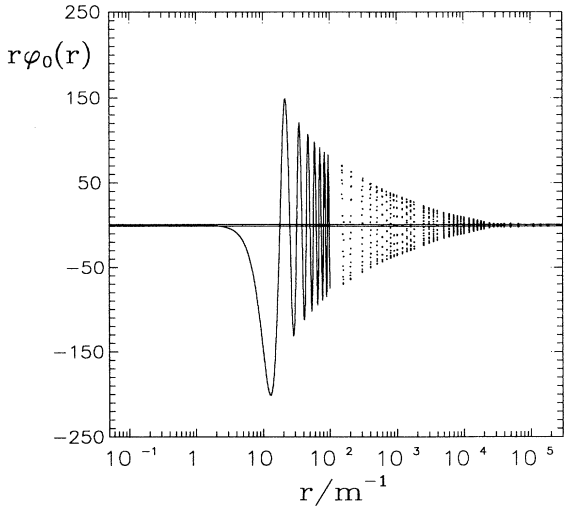

FIG. 6. Same as in Fig. 5 for $E = 2.3m$ and $\mu/m = 1\times10^{-4}$.

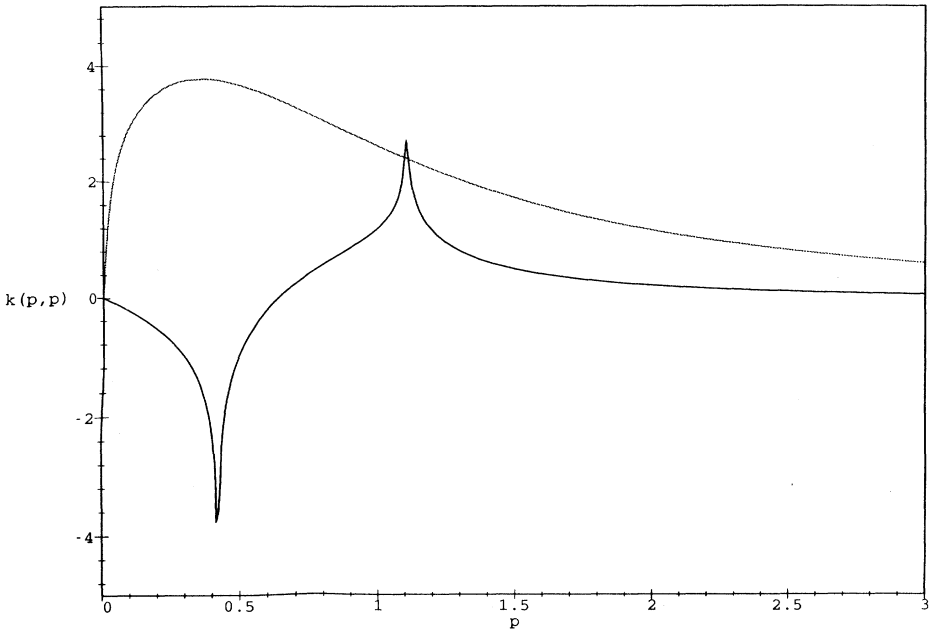

FIG. 7. The diagonal part of the full kernel (4) (solid line) and its energy-independent counterpart (dotted line) for $E = 3m$ and $\mu/m = 10^{-2}$.

width for an increase in the coupling strength, were found in other approaches [11,20].

A graph of the interaction kernels for both energy-dependent and -independent cases reveals a dramatically different behavior. Figure 7 shows the diagonal part $k(p,p,E)$ for the case $E = 3m$ in both approximations for the case of $\mu/m = 10^{-2}$. In the E-independent case we find the usual Yukawa-type ridge in $k(p,q)$ along $p = q$. In the TD case, however, for p and q below the on-shell momentum p_0 there is an appreciable structure away from the diagonal. In addition to the ridge resulting from the log argument in (4) approaching large values there appears a ridge of opposite sign from those locations in the (p,q) plane where the log argument approaches zero. It is this barrier in (p,q) space perpendicular to the diagonal that gives rise to a trapping of the wave function in momentum space. We emphasize that this trapping (and consequent resonant behavior) persists even if the vanishing of the log argument is prohibited by a regulator [37].

V. THE $\mu = 0$ CASE FOR SPINLESS PARTICLES

In Fig. 4, the resonance width Γ_ν was shown as determined from the cross section calculations at three different values of μ/m. The figure suggests that after a certain decrease accompanying the μ/m changes from 1×10^{-3} to 3×10^{-7}, the width stabilizes. Definitely, the difference between the widths for $\mu/m = 1 \times 10^{-4}$ and 3×10^{-7} does not exceed the numerical errors. The value of $\psi_\nu(p_0)$ also remains quite stable (excluding the neighborhood of the level crossing), varying between 0.04 and 0.05 for $E = 3m$, and between 0.0012 and 0.0017 for $E = 2.3m$ as μ/m changes from 1×10^{-4} to 3×10^{-7}.

At this point it is appropriate to recall the basic idea underlying a search for zero-width resonances, i.e., bound states in the continuum, in [8] for the $\mu = 0$ case. It was argued in [19] that performing the Fourier transform on Eq. (3) for $\mu = 0$ changes its right-hand side into a form, which at large distances $r \to \infty$ becomes simply $V(r)\phi_0(r)$ with $\phi_0(r)$ given by Eq. (14), and the local interaction potential $V(r)$ with a real part:

$$\mathrm{Re}V(r) \approx \frac{\cos(E-2m)r}{r} . \tag{16}$$

Comparison of this asymptotic behavior to that from the well-known nonrelativistic example [18] of a potential that supports bound states in the continuous spectrum led the authors of [19] to suggest that this situation arises in Eq. (3) with $\mu = 0$.

In our opinion such a conjecture is not fully justified. Indeed, following the line of reasoning by Arbuzov *et al.* [19] we find that the extraction of the local potential $V(r)$ from the right-hand side (RHS) of Eq. (3) can only be justified in the $E \to 2m$ limit. At fixed E greater than $2m$ neither the effective interaction potential nor the kinetic energy can be considered as local operators and one should expect at best the appearance of finite-width resonances with the width rapidly increasing with growing $E - 2m$. Figure 4 is found to be in accordance with this picture.

Yet, to obtain definite results for the $\mu = 0$ case (Wick-Cutkosky model), special calculations are needed. To avoid having to deal with cross-section calculations for this limiting case, we restrict ourselves to a search for bound states in the continuum. Thus, we seek solutions to Eq. (3) with $\mu = 0$, which satisfy the condition $\psi(p_0) = 0$, or

$$\int_0^\infty k(p_0,q,E)\varphi(q)dq = 0 . \tag{17}$$

Because of the overlap of the pole and logarithmic singu-

larities at $p = p_0$ arising from the free Green's function and interaction kernel, respectively, the kernel of Eq. (3) becomes too singular for a direct solution of Eq. (3) with the methods described in Sec. III. Note that the solutions, which obey the condition $\psi(p_0) = 0$, satisfy the following asymptotic behavior [15] as $p \to p_0$:

$$\varphi(p) \approx c|p - p_0|^{s-1} \tag{18}$$

with $1 < s < \frac{3}{2}$ for $f > 0$. It follows from Eq. (18) that $\varphi(p_0) = 0$ or

$$\lim_{p \to p_0} \int_0^\infty \frac{k(p,q,E) - k(p_0,q,E)}{2\omega_m(p) - E} \varphi(q) dq = 0 \ . \tag{19}$$

Combining Eqs. (3), (17), and (19) we arrive at the equation

$$\varphi(p) = \frac{f}{\pi} \int_0^\infty \left(\frac{k(p,q,E) - k(p_0,q,E)}{2\omega_m(p) - E} - \kappa(q) \right) \varphi(q) dq \ , \tag{20}$$

$$\kappa(q) = \frac{m^2 \omega_m(p_0)}{2p_0 \omega_m(p) \omega_m(q)} \left(\frac{1 + p_0/\omega_m(p_0)}{q + p_0 + \omega_m(q) + \omega_m(p_0) - E} + \frac{\mathrm{sgn}(q - p_0) - p_0/\omega_m(p_0)}{|q - p_0| + \omega_m(q) + \omega_m(p_0) - E} \right) ,$$

which accounts for the conditions $\psi(p_0) = 0$ and $\varphi(p_0) = 0$ explicitly.

We stress that although Eq. (20) incorporates all the solutions to Eq. (3) obeying (17) and (19), it may have additional solutions, which do not satisfy conditions (17) and (19), and, consequently, the original equation (3). For $E < 4m$ and $f < 2$ only solutions of this latter kind are found in numerical calculations performed on Eq. (20). From this we deduce that Eq. (3) with $\mu = 0$ does not have solutions that could be interpreted as bound states embedded in the continuous spectrum. This result supports the suggestion made upon inspection of Fig. 4 that the resonance remains to be of finite width even in the $\mu \to 0$ limit.

We come to the conclusion that the two-particle system described by Eq. (3) is capable of supporting resonances of finite width only, and that this occurs only for a very strong coupling, $f > f_r \approx 1.34$. The resonance coupling constant value f_r appears to be higher than the electromagnetic interaction coupling constant $\alpha = \frac{1}{137}$ by 2 orders of magnitude. With these findings we are not in conflict with either the negative resonance search of [12] for this model that was primarily concerned with small values of the coupling constant, or with the statement from [16] that bound states in the continuum are absent in the associated coordinate-space model for $f < 1/2$.

VI. PARTICLES WITH SPIN $\frac{1}{2}$

Let us now consider the fermion-antifermion system. In [30] a number of J^{PC} eigenstates of positronium were investigated in the perturbative decoupling approximation, $R(p,q,E) = 0$. One of those systems, namely, an electron-positron pair in the state $0^{-+}(^1S_0)$ is studied in the present paper in the TD approach. For this state irregularities were obtained in the Breit equation at strong coupling [11].

We first present the TD version of the relevant QED equation (see Eqs. (21) and (27) in [30(a)]) for the 0^{-+} state of the electron-positron system, which takes into account the exchange of transverse photons:

$$2[2\omega_m(p) - E]\phi_0(p) = \frac{f}{2\pi^2} \int d^3q \frac{\phi_0(q)}{|\mathbf{p} - \mathbf{q}|[|\mathbf{p} - \mathbf{q}| + R(p,q,E)]} K(\mathbf{p},\mathbf{q}) \ , \tag{21}$$

where

$$K(\mathbf{p},\mathbf{q}) = [3\omega_m(p)\omega_m(q) - 2m^2 + (\mathbf{p}\cdot\mathbf{q}) - (\mathbf{p}\cdot\mathbf{Q})(\mathbf{q}\cdot\mathbf{Q})]/\omega_m(p)\omega_m(q) \ , \tag{22}$$

with $\mathbf{Q} = (\mathbf{p} - \mathbf{q})/|\mathbf{p} - \mathbf{q}|$ and $R(p,q,E)$ given by Eq. (5). Performing the angular integration in Eq. (21) and introducing the regulator "photon mass" μ yields the following equation for the function $\varphi(p) = p\phi_0(p)$:

$$[2\omega_m(p) - E]\varphi(p) = \frac{f}{\pi} \int_0^\infty k(p,q,E)\varphi(q) dq \ , \tag{23}$$

with

$$k(p,q,E) = \frac{1}{4\omega_m(p)\omega_m(q)} \Bigg\{ (\omega_m^2(p) + \omega_m^2(q) + 6\omega_m(p)\omega_m(q) - 4m^2) \times \ln \left| \frac{\omega_\mu(p+q) + R(p,q,E)}{\omega_\mu(p-q) + R(p,q,E)} \right| - \frac{(p^2 - q^2)^2}{R(p,q,E)} \left[\frac{1}{\omega_\mu(p-q)} - \frac{1}{\omega_\mu(p+q)} + \frac{1}{R(p,q,E)} \times \left(\ln \left| \frac{\omega_\mu(p+q) + R(p,q,E)}{\omega_\mu(p-q) + R(p,q,E)} \right| - \ln \frac{\omega_\mu(p+q)}{\omega_\mu(p-q)} \right) \right] \Bigg\} \ . \tag{24}$$

In the limits $\mu \to 0$ and $R(p,q,E) \to 0$, Eqs. (23) and (24) reproduce the corresponding Eqs. (A2) – (A4) from [30(a)].

In our search for resonances within this model we followed the scheme described in Sec. IV. As compared to the scalar kernel given by Eq. (4), the kernel determined by Eq. (24) reveals stronger attraction at high momenta, introducing additional complications in the numerical procedure. Calculations of the eigenvalues f_ν of the homogeneous Eq. (6) with $k(p,q,E)$ from Eq. (24) for $E > 2m$ resulted in a $f_\nu - \mu$ dependence, which is qualitatively the same as the one given in Fig. 1 for the scalar case. As the photon regulator mass μ is removed, some of the eigenvalues f_ν remain stable. The energy dependence of the smallest stable coupling constant f_ν is shown in Fig. 8 for $\mu/m = 1 \times 10^{-4}$. Remarkably, the coupling constant decreases slowly with increasing energy in this case.

The eigenfunction $\psi_\nu(p)$ for this eigenvalue f_ν appears to be small at $p = p_0$, suggesting the resonance case (ii) from Sec. II. The scattering phase shift $\delta(E)$ calculated from Eq. (7) with $k(p,q,E)$ from Eq. (24) indeed shows a quick passage through $n\pi/2$ as E passes through the relevant value of the energy, but of the negative slope: $d\delta(E)/dE < 0$. The corresponding cross-section behavior is similar to the one presented in Fig. 2. The width of these "shape resonances" decreases with decreasing energy and the regulator mass μ. However, the width Γ remains larger than the inverse of the range of the potential μ, so the causality condition [34] is not violated. The wave function behavior in the coordinate representation qualitatively reproduces the ones displayed in Figs. 5 and 6. In contrast with these findings no irregularities are observed in the related energy-independent model determined by $R(p,q,E) = 0$.

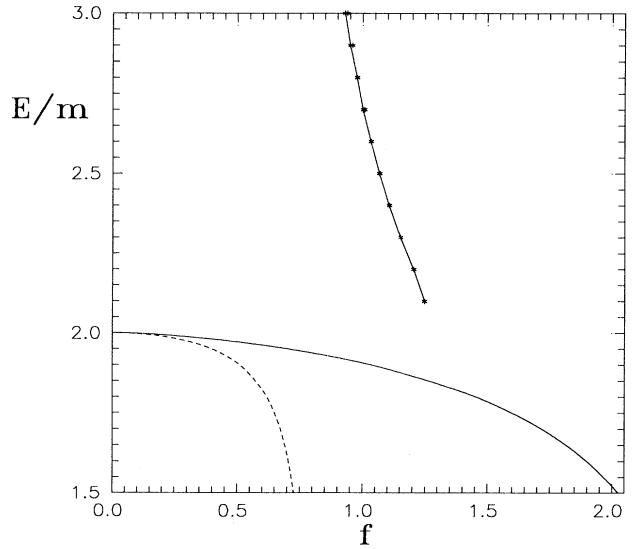

FIG. 8. $E/m>2$: The E-f plot for the smallest eigenvalue f_ν of Eq. (6) with $k(p,q,E)$ from Eq. (24) that stabilizes with decreasing μ for $\mu/m = 1 \times 10^{-4}$. $E/m < 2$: ground-state energy as a function of the coupling constant for the energy-dependent (slid line) and the energy-independent (short-dashed line) cases.

As follows from Fig. 7, in the energy interval considered the irregularities appear for strong-coupling constants $f > 0.9$. Quantitatively "strong coupling" may be characterized by the critical coupling constant value f_c. The discrete bound states of the system at energies $E < 2m$ exist only for $f < f_c$. For $f > f_c$ the "collapse to the center" phenomenon occurs and the energy spectrum is continuous and unbounded from below. For the present TD model the critical coupling constant is approximately $f_c \approx 2.2$ It is about three times higher than the critical value $f_c = 8\pi/(4 + 3\pi^2) \approx 0.7478$ obtained for the 0^{-+} state with the E-independent kernel [30,38]. Although the irregularities at energies higher than $2m$ appear in the physical region $f < f_c$, the coupling constant exceeds the critical value for the E-independent model.

An interpretation of the irregularities is not obvious. A graph of the rather complicated kernel (24) in comparison with the scalar case (4) reveals very similar behavior at small p and q, but a slow decrease along $p = q$ in the asymptotic region in the fermionic case. This strongly attractive behavior at large momenta, i.e., short distances, could induce very strong accelerations that lead to passages through $\pi/2 + n\pi$ from above over narrow energy intervals. Further investigations of this phenomenon are desirable.

We finish this section with a remark concerning the behavior of the ground-state energies presented in Fig. 8 for the energy-dependent and independent models. The energy-dependent TD kernel has a well-behaved ground-state energy in contrast with the one for the simpler kernel, as it contains less attraction which is obvious from the graph and the critical coupling f_c given above. In the weak-coupling limit $f \to 0$ the energy E of the ground state behaves as

$$E/m = 2 - \tfrac{1}{4}f^2 - \tfrac{21}{64}f^4 + \cdots \tag{25}$$

for the energy-independent case [30] and as

$$E/m = 2 - \frac{1}{4}f^2 + \frac{1}{\pi}f^3 \ln\frac{1}{f} + \cdots \tag{26}$$

for the energy-dependent case. Equation (25) represents the α^4 correct positronium fine structure [30]. The correction of the order of $f^3 \ln f$ to the nonrelativistic Coulomb energy in Eq. (26) comes from the first term in the curly brackets in the right-hand side of Eq. (24). It is qualitatively similar to the relevant kernel (4) for the scalar model for which the expansion (26) is also valid. This unusual behavior indicates that the TD versions of the kernels (24) or (4) could not serve as a reasonable starting approximation if the coupling is small. On the other hand, it is not clear *a priori* whether at strong coupling one should prefer a scattering wave equation that implies (25) vs (26) as its bound-state spectrum.

VII. CONCLUSIONS

We have studied equal-time TD wave equations for the appearance of resonances in the s-wave particle-

antiparticle scattering for scalars exchanging scalars as well as fermions exchanging photons. Also the related models with energy-independent effective interactions were analyzed. A demonstration as to how to perform a systematic investigation of a resonance problem in momentum space based both on the homogeneous eigenvalue integral equation for scattering energies $E > 2m$ and the scattering equation itself was presented.

We have shown that no resonances appear for weak coupling, if the problem is solved with finite exchanged particle mass regulator μ that is continuously decreased over several orders of magnitude. For the scalar model also the $\mu = 0$ case was investigated.

In contrast, resonances were found at strong coupling ($f \approx 1$) in the TD scalar Yukawa model and rapid decreases of the phase shift through $\pi/2 + n\pi$ for the $^1S_0$-channel electron-positron scattering problem. Two skeptical remarks about the nature of the resonances seem appropriate here: the effects are model dependent, since no irregularities were observed with the related models based on the perturbative decoupling Hamiltonian variational approach; the TD models do not provide the correct α^4 positronium structure. On the other hand, the parameters of the resonances show the dependence on the coupling strength similar to those obtained from different approaches [11,20].

Returning to the original motivation of this study, namely, the GSI peaks problem [1–5] we note that appearance of shape resonances in a strongly coupled truncated Fock space approach cannot be ruled out definitely. Whether strong external fields as present in heavy-ion collisions can induce peaks in e^+e^- scattering cross sections remains an unsolved problem.

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