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BRIEF REPORTS

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Improved limits on anisotropy and inhomogeneity from the cosmic background radiation

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Recently we presented limits on the deviation of the Universe from a Friedmann-Robertson-Walker form, which are implied directly by anisotropies in the cosmic background radiation, without assuming inflationary or other models of evolution. Here we weaken some of our previous assumptions, ensuring that all assumptions are in principle observational, and obtain improved limits.

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I. INTRODUCTION

In [1] we developed a covariant and gauge-invariant formalism for analyzing the direct relationship between temperature anisotropies of the cosmic background radiation (CBR), and deviations of the Universe from a Friedmann-Robertson-Walker (FRW) form. We presented a set of limits on the kinematic, dynamic, and geometric indicators of anisotropy and inhomogeneity. Since completing [1], we have been able to weaken some of the assumptions, ensuring that all assumptions are in principle observational, and then to derive a more complete and a sharper set of limits.

This paper is supplementary to [1], and we assume the discussion, basic equations, and notation of [1]. References to equations from [1] will be in the form (I:number).

II. AN IMPROVED SET OF OBSERVATIONAL ASSUMPTIONS

In [1], we introduced, in addition to observational assumptions (i.e., assumptions on the CBR anisotropies, in principle subject to observational testing), the following nonobservational assumption.

(I:C3): There exist constants α, β, γ of the order of 1 such that

$$|\dot{\sigma}_{ab}| < \alpha\Theta|\sigma_{ab}|, \quad |\hat{\nabla}_a\hat{\nabla}_b\mu| < \beta\Theta|\hat{\nabla}_a\mu|, \quad |\hat{\nabla}_a E_{bc}| < \gamma\Theta|E_{ab}| \ .$$

This assumption was introduced to bound certain derivatives, and then led to limits on the density inhomogeneity, vorticity, and electric Weyl tensor. No limits were found in [1] for the expansion inhomogeneity or the magnetic Weyl tensor.

We can avoid the disadvantage of the nonobservational assumption **(I:C3)**, and at the same time complete the set of limits, as follows. We retain assumption **(I:B1)**, which simply formalizes the observed bounds on the covariant temperature multipoles $\tau_{a_1\cdots a_L}$: i.e., there exist $O[1]$ constants ϵ_L such that

$$|\tau_{a_1\cdots a_L}| < \epsilon_L \ . \quad (1)$$

The assumptions **(I:B2)**, **(I:B3)** of [1], which express in general form that the first derivatives of the temperature multipoles are bounded, are simply extended up to third order derivatives (and thus remain observational in nature). Thus we replace **(I:B2)**, **(I:B3)** by

(B2′): There exist $O[1]$ constants $\epsilon_L^*, \epsilon_L^{**}, \epsilon_L^{***}$ such that

$$|\dot{\tau}_{a_1\cdots a_L}| < \epsilon_L^*\Theta, \quad |\ddot{\tau}_{a_1\cdots a_L}| < \epsilon_L^{**}\Theta^2, \quad |\dddot{\tau}_{a_1\cdots a_L}| < \epsilon_L^{***}\Theta^3 \ , \quad (2)$$

(**B3′**): There exist $O[1]$ constants $\epsilon_L^\dagger, \epsilon_L^{\dagger\dagger}, \epsilon_L^{\dagger\dagger\dagger}$ such that

$$|\hat\nabla_b \tau_{a_1\cdots a_L}| < \epsilon_L^\dagger \Theta, \quad |\hat\nabla_b \hat\nabla_c \tau_{a_1\cdots a_L}| < \epsilon_L^{\dagger\dagger}\Theta^2 , \quad |\hat\nabla_b \hat\nabla_c \hat\nabla_d \tau_{a_1\cdots a_L}| < \epsilon_L^{\dagger\dagger\dagger}\Theta^3 , \qquad (3)$$

and $O[1]$ constants $\epsilon_L^{\dagger *}, \epsilon_L^{\dagger **}, \epsilon_L^{\dagger\dagger *}$ such that

$$|(\hat\nabla_c \tau_{a_1\cdots a_L})^\cdot| < \epsilon_L^{\dagger *}\Theta^2, \quad |(\hat\nabla_c \tau_{a_1\cdots a_L})^{\cdot\cdot}| < \epsilon_L^{\dagger **}\Theta^3, \quad |(\hat\nabla_d \hat\nabla_c \tau_{a_1\cdots a_L})^\cdot| < \epsilon_L^{\dagger\dagger *}\Theta^3 . \qquad (4)$$

We have normalized the derivatives relative to the expansion of the Universe, and we will not need the higher derivative limits. Note that ϵ_L^* was denoted ϵ_L' in [1], and $\epsilon_L^\dagger$ was denoted ϵ_L'' in [1]. On the basis of (**B2′**), (**B3′**), we can find limits on all important quantities, including the expansion inhomogeneity and magnetic Weyl tensor (see Sec. III). Assumption (**I : C3**) may be discarded.

In practice $\epsilon_L^*, \epsilon_L^{**}$ and especially $\epsilon_L^\dagger, \epsilon_L^{\dagger\dagger}, \ldots$ are not known from observations. In order to produce more useful versions of the limits, we need a reasonable estimate of these quantities. We make a simple extension of assumptions (**I:C1**), (**I:C2**) to cover second and third derivatives:

(**C1′**): The spatial gradients of the temperature multipoles are not greater than their time derivatives:

$$\epsilon_L^\dagger \le \epsilon_L^*, \quad \epsilon_L^{\dagger\dagger}, \epsilon_L^{\dagger *} \le \epsilon_L^{**}, \quad \epsilon_L^{\dagger\dagger\dagger}, \epsilon_L^{\dagger **}, \quad \epsilon_L^{\dagger\dagger *} \le \epsilon_L^{***} . \qquad (5)$$

(**C2′**): The bounds on the time derivatives of the temperature harmonics are estimated by $\Theta\epsilon_L^* \simeq \epsilon_L/t_R$, $\Theta^2\epsilon_L^{**} \simeq \epsilon_L/t_R^2$, $\Theta^3\epsilon_L^{***} \simeq \epsilon_L/t_R^3$, so that

$$\epsilon_L^* \simeq \tfrac{1}{3}\epsilon_L, \quad \epsilon_L^{**} \simeq \tfrac{1}{9}\epsilon_L, \quad \epsilon_L^{***} \simeq \tfrac{1}{27}\epsilon_L . \qquad (6)$$

In [1] we employed the Copernican assumption as formulated in (**I:A1**). George Smoot (private communication) recently pointed out to us that the Copernican principle is ultimately partially testable. The Sunyaev-Zeldovich effect allows us to confirm to some degree that distant galaxies do see isotropic CBR, for otherwise the scattered radiation would have a significantly distorted blackbody spectrum. Thus, it may eventually be possible to determine some of the limits given in Eqs. (2) to (4) observationally.

III. IMPROVED AND EXTENDED LIMITS ON ANISTROPY AND INHOMOGENEITY

To begin with, we assume only (1)–(4), and not (5),(6). (1) leads immediately to the limits (I:46) on the radiation anisotropy tensors q_a, π_{ab}, ξ_{abc}. Limits on the derivatives of the radiation tensors arise from differentiating (I:41), using (1)–(4) and the evolution and constraint equations (I:11)–(I:24). A lengthy but straightforward calculation leads to the results

$$|\dot q_a| < \tfrac{4}{3}\mu\Theta(\tfrac{4}{3}\epsilon_1 + \epsilon_1^*) , \qquad (7)$$

$$|\hat\nabla_a q_b| < \tfrac{4}{3}\mu\Theta\epsilon_1^\dagger , \qquad (8)$$

$$|\ddot q_a| < \tfrac{4}{27}\mu\Theta^2[(20 + 4\Omega_R + 2\Omega_M)\epsilon_1 + 24\epsilon_1^* + 9\epsilon_1^{**}] , \qquad (9)$$

$$|\hat\nabla_a \hat\nabla_b q_c| < \tfrac{4}{3}\mu\Theta^2\epsilon_1^{\dagger\dagger} , \qquad (10)$$

$$|(\hat\nabla_a q_b)^\cdot| < \tfrac{4}{9}\mu\Theta^2(3\epsilon_1^{\dagger *} + 4\epsilon_1^\dagger) , \qquad (11)$$

$$|(\hat\nabla_a \hat\nabla_b q_c)^\cdot| < \tfrac{4}{9}\mu\Theta^3(3\epsilon_1^{\dagger\dagger *} + 4\epsilon_1^{\dagger\dagger}) , \qquad (12)$$

$$|\dot\pi_{ab}| < \tfrac{8}{15}\mu\Theta(\tfrac{4}{3}\epsilon_2 + \epsilon_2^*) , \qquad (13)$$

$$|\hat\nabla_a \pi_{bc}| < \tfrac{8}{15}\mu\Theta\epsilon_2^\dagger , \qquad (14)$$

$$|\ddot\pi_{ab}| < \tfrac{8}{135}\mu\Theta^2[(20 + 4\Omega_R + 2\Omega_M)\epsilon_2 + 24\epsilon_2^* + 9\epsilon_2^{**}] , \qquad (15)$$

$$|(\hat\nabla_a \pi_{bc})^\cdot| < \tfrac{8}{45}\mu\Theta^2(4\epsilon_2^\dagger + 3\epsilon_2^{\dagger *}) , \qquad (16)$$

$$|(\hat\nabla_a \pi_{bc})^{\cdot\cdot}| < \tfrac{8}{135}\mu\Theta^3[(20 + 4\Omega_R + 2\Omega_M)\epsilon_2^\dagger + 24\epsilon_2^{\dagger *} + 9\epsilon_2^{\dagger **}] , \qquad (17)$$

$$|\hat\nabla_a \hat\nabla_b \hat\nabla_c \pi_{de}| < \tfrac{8}{15}\mu\Theta^3\epsilon_2^{\dagger\dagger\dagger} , \qquad (18)$$

$$|\hat\nabla_a \xi_{bcd}| < \tfrac{8}{35}\mu\Theta\epsilon_3^\dagger , \qquad (19)$$

$$|(\hat\nabla_a \xi_{bcd})^\cdot| < \tfrac{8}{105}\mu\Theta^2(4\epsilon_3^\dagger + 3\epsilon_3^{\dagger *}) , \qquad (20)$$

$$|\hat\nabla_a \hat\nabla_b \xi_{cde}| < \tfrac{8}{35}\mu\Theta^2\epsilon_3^{\dagger\dagger} , \qquad (21)$$

$$|(\hat\nabla_a \hat\nabla_b \xi_{cde})^\cdot| < \tfrac{8}{105}\mu\Theta^3(4\epsilon_3^{\dagger\dagger} + 3\epsilon_3^{\dagger\dagger *}) . \qquad (22)$$

The density parameters are $\Omega_R = \mu/3H^2, \Omega_M = \rho/3H^2$. Limits on other derivatives are not needed.

Now (7)–(22) are used in the evolution and constraint equations (I:11)–(I:24) and their derivatives in order to find limits on the quantities that characterize the deviation of the Universe from FRW form. We emphasize that the following limits are gauge invariant, covariant and imposed directly by observational quantities, i.e., the temperature dipole, quadrupole, octopole, and their derivatives.

From (I:13) and (I:14) we get immediately

$$\frac{|\hat\nabla_a \mu|}{\mu} = 4\frac{|\hat\nabla_a T|}{T} < H(8\epsilon_1 + 12\epsilon_1^* + \tfrac{72}{5}\epsilon_2^\dagger) , \qquad (23)$$

$$\frac{|\sigma_{ab}|}{\Theta} < \tfrac{8}{3}\epsilon_2 + \epsilon_2^* + 5\epsilon_1^\dagger + \tfrac{9}{7}\epsilon_3^\dagger \ . \tag{24}$$

The remaining limits require more complicated manipulation of the evolution and constraint equations. From (I:27), using $\hat{\nabla}_b$ (I:13) to get $|\hat{\nabla}_a\hat{\nabla}_b\mu|$, we find

$$\frac{|\omega_{ab}|}{\Theta} < 9\epsilon_1^\dagger + 3\epsilon_1^{\dagger *} + \tfrac{6}{5}\epsilon_2^{\dagger\dagger} \ . \tag{25}$$

From (I:23), using $\hat{\nabla}_c$ (I:16), $\hat{\nabla}_c$ (I:14), and $[\hat{\nabla}_c$ (I:14)$]^{\cdot}$ to get limits on $|\hat{\nabla}_c E_{ab}|$, $|\hat{\nabla}_c\sigma_{ab}|$, and $|(\hat{\nabla}_c\sigma_{ab})^{\cdot}|$, we find

$$\frac{|\hat{\nabla}_a\rho|}{\Theta} < \tfrac{27}{2}H\epsilon_2^\dagger + \left(\frac{\Omega_R}{\Omega_M}\right)H[12\epsilon_1 + 12\epsilon_1^* + 61\epsilon_2^\dagger$$

$$+\left(\frac{3}{\Omega_M}\right)H[165\epsilon_1^{\dagger\dagger} + 45\epsilon_1^{\dagger\dagger *} + 110\epsilon_2^\dagger + 69\epsilon_2^{\dagger *} + 9\epsilon_2^{\dagger **} + 18\epsilon_3^{\dagger\dagger}] \ . \tag{26}$$

From $\hat{\nabla}_a$(I : 12), using (I : 13)$^{\cdot}$ and (I : 26) to get $|(\hat{\nabla}_a\mu)^{\cdot}|$, we find

$$\frac{|\hat{\nabla}_a\Theta|}{\Theta} < H[50\epsilon_1 + 51\epsilon_1^* + 9\epsilon_1^{**} + 3\epsilon_1^{\dagger\dagger} + \tfrac{24}{5}\epsilon_2^* + 18\epsilon_2^\dagger + \tfrac{18}{5}\epsilon_2^{\dagger *}] + 4(2\Omega_R + \Omega_M)H\epsilon_1 \ . \tag{27}$$

From (I : 16), using (I : 14)$^{\cdot}$ to get $|\dot{\sigma}_{ab}|$ and (I : 12), (I : 15), (I : 26), we find

$$\frac{|E_{ab}|}{\Theta} < H[50\epsilon_1^\dagger + 15\epsilon_1^{\dagger *} + \tfrac{88}{3}\epsilon_2 + 14\epsilon_2^* + 3\epsilon_2^{**} + \tfrac{66}{7}\epsilon_3^\dagger + \tfrac{9}{7}\epsilon_3^{\dagger *}] + \tfrac{4}{45}(11\Omega_R + 15\Omega_M)H\epsilon_2 \ . \tag{28}$$

From (I:22), using $\hat{\nabla}_c$(I : 27) (with $\psi = \mu$) and $\hat{\nabla}_c\hat{\nabla}_b$(I : 13) to get $|\hat{\nabla}_c\omega_{ab}|$, and $\hat{\nabla}_c$(I : 14) to get $|\hat{\nabla}_c\sigma_{ab}|$, we find

$$\frac{|H_{ab}|}{\Theta} < H[45\epsilon_1^{\dagger\dagger} + 9\epsilon_1^{\dagger\dagger *} + 9\epsilon_2^\dagger + 3\epsilon_2^{\dagger *} + \tfrac{18}{5}\epsilon_2^{\dagger\dagger\dagger} + \tfrac{9}{7}\epsilon_3^{\dagger\dagger\dagger}] \ . \tag{29}$$

If the dipole is assumed negligible, then all the ϵ_1's may be set to zero in (23)–(29). Equations (23)–(29) complete the limits (I:49), (I:50). They show explicitly the role of the dipole, quadrupole, and octopole in determining limits on all indicators of anisotropy and inhomogeneity; they are direct (i.e., model independent); and they are based on minimal observational assumptions (1)–(4).

We can now use the stronger observational assumptions (5),(6) to recast (23)–(29) in terms of the observationally realistic ϵ_L:

$$\frac{|\hat{\nabla}_a\mu|}{\mu} = 4\frac{|\hat{\nabla}_a T|}{T} < H(12\epsilon_1 + \tfrac{24}{5}\epsilon_2) \ , \tag{30}$$

$$\frac{|\sigma_{ab}|}{\Theta} < \tfrac{5}{3}\epsilon_1 + 3\epsilon_2 + \tfrac{3}{7}\epsilon_3 \ , \tag{31}$$

$$\frac{|\omega_{ab}|}{\Theta} < \tfrac{10}{3}\epsilon_1 + \tfrac{2}{15}\epsilon_2 \ , \tag{32}$$

$$\frac{|\hat{\nabla}_a\rho|}{\Theta} < \tfrac{9}{2}H\epsilon_2 + \left(\frac{H}{\Omega_M}\right)[60\epsilon_1 + 134\epsilon_2 + 6\epsilon_3] + \left(\frac{\Omega_R}{\Omega_M}\right)H[16\epsilon_1 + \tfrac{61}{3}\epsilon_2] \ , \tag{33}$$

$$\frac{|\hat{\nabla}_a\Theta|}{\Theta} < H(\tfrac{205}{3}\epsilon_1 + 8\epsilon_2) + 4(2\Omega_R + \Omega_M)H\epsilon_1 \ , \tag{34}$$

$$\frac{|E_{ab}|}{\Theta} < H(\tfrac{55}{3}\epsilon_1 + \tfrac{103}{3}\epsilon_2 + \tfrac{23}{7}\epsilon_3) + \tfrac{4}{45}(11\Omega_R + 15\Omega_M)H\epsilon_2 \ , \tag{35}$$

$$\frac{|H_{ab}|}{\Theta} < H(\tfrac{16}{3}\epsilon_1 + \tfrac{52}{15}\epsilon_2 + \tfrac{1}{21}\epsilon_3) \ . \tag{36}$$

If the dipole is neglected, then we can set $\epsilon_1 = 0$ in (30)–(36). The bounds (30)–(36) improve and extend the results (I:56), (I:57), (I:62)–(I:64). They are the main results of the quest for *a direct link from feasible observational limits on the CBR to limits on the deviations of the universe from FRW since the last scattering (within our past light cone).*

From these limits we can obtain conservative estimates of present-time bounds on the anisotropy and inhomogeneity of the universe, improving and extending the limit (I:65)–(I:67). Let

$$\epsilon \equiv \max(\epsilon_1, \epsilon_2, \epsilon_3)$$

denote the upper limit of currently observed anisotropy in the CBR temperature variation, and take $(\Omega_R)_0 \ll 1$. Then (30)–(36) imply

$$\left(\frac{|\hat{\nabla}_a \mu|}{\mu}\right)_0 < 17 H_0 \epsilon, \quad \left(\frac{|\sigma_{ab}|}{\Theta}\right)_0 < 6\epsilon, \quad \left(\frac{|\omega_{ab}|}{\Theta}\right)_0 < 4\epsilon \,. \tag{37}$$

$$\left(\frac{|E_{ab}|}{\Theta}\right)_0 < [\tfrac{4}{3}(\Omega_M)_0 + 56] H_0 \epsilon, \quad \left(\frac{|H_{ab}|}{\Theta}\right)_0 < 10 H_0 \epsilon \,, \tag{38}$$

$$\left(\frac{|\hat{\nabla}_a \Theta|}{\Theta}\right)_0 < [4(\Omega_M)_0 + 77] H_0 \epsilon, \quad \left(\frac{|\hat{\nabla}_a \rho|}{\rho}\right)_0 < C(\Omega) H_0 \epsilon \,, \tag{39}$$

where $C(\Omega) \equiv 5 + 200/(\Omega_M)_0$. The latter is stronger than the limit (I:67), but is still relatively weak, consistent with inhomogeneities in the large-scale matter distribution. The improved limit (39) leads to a table of values improving on that in [1]:

$(\Omega_M)_0$	0.02	0.1	0.3	1
$C(\Omega)$	10005	2005	672	205

If the dipole is neglected, then we can set $\epsilon_1 = 0$ in (30)–(36) to obtain "dipole-free" estimates improving on (I:65a)–(I:67a):

$$\left(\frac{|\hat{\nabla}_a \mu|}{\mu}\right)_0 < 5 H_0 \epsilon, \quad \left(\frac{|\sigma_{ab}|}{\Theta}\right)_0 < 4\epsilon \,,$$

$$\left(\frac{|\omega_{ab}|}{\Theta}\right)_0 < \epsilon \,, \tag{40}$$

$$\left(\frac{|E_{ab}|}{\Theta}\right)_0 < [\tfrac{4}{3}(\Omega_M)_0 + 38] H_0 \epsilon \,,$$

$$\left(\frac{|H_{ab}|}{\Theta}\right)_0 < 4 H_0 \epsilon \,, \tag{41}$$

$$\left(\frac{|\hat{\nabla}_a \Theta|}{\Theta}\right)_0 < 4[(\Omega_M)_0 + 2] H_0 \epsilon \,,$$

$$\left(\frac{|\hat{\nabla}_a \rho|}{\rho}\right)_0 < \left[\frac{140}{(\Omega_M)_0} + 5\right] H_0 \epsilon \,. \tag{42}$$

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[1] R. Maartens, G. F. R. Ellis, and W. R. Stoeger, Phys. Rev. D **51**, 1525 (1995).