Invariant fermion correlator in the Schwinger model on the torus

J.V. Steele, J.J.M. Verbaarschot, and I. Zahed

Department of Physics, State University of New York, Stony Brook, New York 11794-3800

(Received 25 July 1994)

We construct the gauge invariant fermion correlator in the Schwinger model on the torus. At zero temperature, this correlator falls off with a rate given by the Coulomb energy of an infinitely heavy charge. At high temperature, the screening mass approaches $\pi T/2$, and this in the presence of a mass gap. The fractional Matsubara frequency arises from the action of a pair of induced merons at high temperature that are localized over a range on the order of the meson Compton wavelength $1/m = \sqrt{\pi}/g$. We discuss the quenched approximation in this model, and comment on the possible relevance of some of these results to higher dimensions.

PACS number(s): 11.15.Tk, 11.30.Rd, 12.38.Aw

I. INTRODUCTION

A fundamental aspect of QCD is that chiral symmetry is spontaneously broken at low temperature and restored at high temperature. As the low temperature phase is also confining, the low temperature excitations are hadronic. The nature of the excitations above the phase transition is still debated. The phase is believed to be screened, but lattice simulations indicate strong spacelike correlations at *all* temperatures (see Ref. [1], and references therein).

Lattice simulations as well as analytical calculations have mainly focused on the behavior of the hadronic correlators in the spacelike (Euclidean) domain at low and high temperature [2, 3]. The mesonic and baryonic correlators were found to asymptote $2\pi T$ and $3\pi T$, respectively, with the exception of the π and the σ [2]. Lattice simulations for the quark correlator in the Landau gauge have also been performed [4].

In nongauge theories the fermionic propagator carries interesting information on the single as well as collective excitations of the system. In gauge theories the situation is somewhat unclear, since the naive fermion propagator is gauge dependent. In two-dimensional QCD, 't Hooft has shown that in the axial gauge the quark propagator is infrared sensitive and zero in the infrared limit [5]. Einhorn later noted that the infrared sensitivity was directly related to the gauge sensitivity of the quark propagator [6]. This feature was also noted by Casher, Kogut, and Susskind in the context of the Schwinger model [7]. The fermion propagator has been used to probe the high temperature phase of QCD [4, 8].

Recently, we have analyzed the mesonic correlation functions in the Schwinger model at finite temperature both in real and Euclidean time [9]. All correlators were found to fall off with the meson mass $m = g/\sqrt{\pi}$ in the spatial direction at high temperature (for $x \gg 1/m$). Similar correlators were also investigated by Abada and Schrock [10] and Smilga [11] with somewhat similar conclusions. In this paper, we would like to extend the analysis to the gauge-invariant fermion correlator in Euclidean space. By construction, this correlator is a combination of a fermion propagator and a path ordered exponential (string), and thus is gauge invariant and free from the ambiguities discussed above. The gauge-invariant correlator reduces to the *ordinary* fermion correlator for (timelike) axial gauges, barring the difficulties associated with these gauges in perturbation theory [12]. Also, the gauge-invariant correlator offers a suitable framework for probing both chiral symmetry breaking and confinement (screening) at zero and finite temperature through its short and large distance limits [12].

To illustrate these points, we will use the Schwinger model on the torus to calculate exactly this correlator both at zero and finite temperature. At zero temperature, the diagonal part of the gauge-invariant correlator is saturated by the fermion condensate at short distances, and falls off exponentially at large distances. The falloff rate is related to the screening length of the attached string, a direct consequence of the screening character of the QED ground state. This aside, the singularities of the gauge-invariant correlator reflect on a mass gap in the spectrum. At high temperature, this correlator asymptotes $\exp(-\pi T x/2)$ (for $x \gg 1/m$) in the spatial direction and has a free field behavior along the temporal direction *despite* the fact that the spectrum exhibits a mass gap at all temperatures.

We show that the deviation from πT in the spatial asymptote is due to a pair of induced merons. At high temperature, the merons are localized over a range on the order of the meson Compton wavelength $1/m = \sqrt{\pi}/g$. We discuss the quenched approximation, which means the fermion determinant is neglected in the average over the partition function, and suggest that it cannot be applied to this model. The possible relevance of some of these results to higher dimensions is discussed in our concluding remarks.

II. THE INVARIANT CORRELATOR

Consider the Euclidean gauge-invariant fermion correlator on a strip of temporal length $\beta = 1/T$ and spatial length L:

$$\mathcal{S}_F(x,\beta) = -\left\langle q(0)\exp\left(ig\int_0^x A_\mu d\xi_\mu\right)\overline{q}(x)\right\rangle_\beta . \tag{2.1}$$

Although this correlator is gauge invariant, it depends on the choice of the path between zero and x. Below, we will always choose the shortest path on the torus staying within the interval $[0,\beta]$ in the time direction and $[0,L]$ in the space direction. The expectation value is over the QED measure on the torus (see Sachs and Wipf [13] for more details). On the torus, the gauge field A obeys the general (Hodge) decomposition ($V=\beta L$) [13]

$$A_\mu(x)=\frac{2\pi}{g}\left(-\frac{kx^1}{V}+\frac{h_0}{\beta},\frac{h_1}{L}\right)_\mu-\varepsilon_{\mu\nu}\partial_\nu\phi. \qquad (2.2)$$

The first part refers to the instanton in two dimensions with a topological charge k, the second part refers to the constant modes associated with the Polyakov line, and the final part is just the transverse gauge field related to the electric polarization. The following calculations are understood to be in covariant gauges [13].

The correlator (2.1) receives contributions only from the $k=0,\pm1$ sectors. In higher fluxes, the zero modes, not being absorbed by the quark fields, will make the fermion determinant vanish. The $k=0$ sector will contribute to the off-diagonal part[1] (since the Green's function anticommutes with γ_5) and the diagonal elements will be from the $k=\pm1$ sectors. Inserting the explicit form of the gauge potential into (2.1) shows that the line integral gives a phase factor from the harmonic part (the h's) and an integral from the classical background field depending on the flux factor k. The latter integral vanishes for paths exclusively along either the time or space direction.[2] The ϕ-dependent term

$$ig\varepsilon_{\mu\nu}\int_0^x d\xi_\mu\frac{\partial}{\partial\xi^\nu}\phi(\xi) \qquad (2.3)$$

represents the quantum fluctuations around the background field and must be averaged in the path integral.

The spinor structure in two dimensions allows ϕ to be factored out of the Dirac operator $\not{D}\equiv\gamma_\mu(\partial_\mu-igA_\mu)$:

$$\not{D}=e^{g\gamma_5\phi}\tilde{\not{D}}_k e^{g\gamma_5\phi}, \qquad (2.4)$$

where $\tilde{\not{D}}_k$ is the Dirac operator in the instanton background in sector k [so A_μ is reduced to the k-dependent part of Eq. (2.2) in the covariant derivative]. With this in mind, the $k=1$ sector only contributes to the upper left entry of the (spinor) matrix, (2.1). The explicit form is (here the matrix indices are labeled by the eigenvalue from γ_5)

$$\mathcal{S}_{F,++}(x,\beta)=-\frac{\int d^2h\,\psi^\dagger{}_1(x)\psi_1(0)\det{}'(i\tilde{\not{D}}_1)\int D\phi\exp\left(-\frac12\int\phi(\Box^2-m^2\Box)\phi-2\pi/m^2V\right)\exp\left(ig\int_0^x Ad\xi\right)}{\int d^2h\det(i\tilde{\not{D}}_0)\int D\phi\exp\left(-\frac12\int\phi(\Box^2-m^2\Box)\phi\right)}. \qquad (2.5)$$

The zero modes are (for arbitrary positive flux k and p running from 1 to k)

$$\psi_{p,k}(x)=e^{-g\phi(x)}\tilde\psi_{p,k}(x)\ , \qquad (2.6)$$

$$\tilde\psi_{p,k}(x)=\left(\frac{2k}{\beta^2V}\right)^{1/4}U(x)e^{-\pi\tau\tilde h_0{}^2/k-2\pi i\tilde h_0(z+h_1/k)-\pi k(x^1)^2/V}\vartheta_3(kz+h_1-i\tau\tilde h_0|ik\tau)\ ,$$

with

$$U(x)=e^{2\pi i(h_0x^0/\beta+h_1x^1/L)},\tau=L/\beta, \qquad (2.7)$$

$$\tilde h_0=h_0-p+\frac12,\ \text{and}\ z=\frac{x^0+ix^1}{\beta}. \qquad (2.8)$$

For negative flux $-k$, the sign of ϕ must be changed and $\tilde\psi(x)\to\tilde\psi^*(-x)$. The determinants of $\tilde{\not{D}}$ have been evaluated by Sachs and Wipf [13] and may be concisely written as

$$\det(i\tilde{\not{D}}_0)=\left|\frac{\vartheta_4(h_1-i\tau h_0)}{\eta}\right|^2e^{-2\pi\tau h_0^2}\ \text{and}\ \det{}'(i\tilde{\not{D}}_1)=\sqrt{\frac{V}{2}}$$

with the prime denoting exclusion of the zero mode (since its contribution is taken into account in the field itself) and we use the four elliptic ϑ functions as defined in [14]. As in [13], η refers to Dedekind's η function evaluated at $i\tau$. The phase factor from the zero modes, $U(x)$, cancels with the phase factor from the gauge line integral. The Gaussian integrals can be calculated by completing squares (see Appendix A).

After integration over ϕ in both numerator and denominator, $\mathcal{S}_{F,++}(x,\beta)$ reduces to an integration over only the harmonic part of the potential:

$$-\frac{\sqrt{2\tau}|\eta|^2}{\beta}e^{2g^2K_{xx}}e^{-2\pi/m^2V-\pi(x^1)^2/V-\pi ix^0x^1/V}e^{I_3(x,\beta)}\int d^2he^{-2\pi\tilde h_0^2\tau+2\pi i\tilde h_0\bar z}\vartheta_3(\bar z+h_1+i\tau\tilde h_0)\,\vartheta_3(h_1-i\tau\tilde h_0) \qquad (2.9)$$

[1]We use Hermitian γ matrices with $\gamma_0=\sigma_1$, $\gamma_1=\sigma_2$, and $\gamma_5=\sigma_3$.

[2]For other choices of paths this term would vanish upon the limit $L\to\infty$ anyway.

with K_{xy} the bosonic Green's function defined by $(\Box^2 - m^2\Box)K(x,y) = \delta(x-y)$ (see Appendix A), and $\Box = \partial_0{}^2 + \partial_1{}^2$. The h_1 integration may be done after expressing the ϑ functions as Fourier series. This gives a Kronecker δ that combines the two sums into one. This one sum then extends the h_0 integration from the segment $[0,1]$ to $(-\infty,\infty)$ which allows a Gaussian integration to be done. Identifying the fermion condensate [13]

$$\langle \bar{q}q \rangle_\beta = -\frac{2|\eta|^2}{\beta} e^{2g^2 K_{xx}} e^{-2\pi/m^2 V}, \tag{2.10}$$

the answer for the $k = 1$ sector is

$$\mathcal{S}_{F,++}(x,\beta) = \left\langle \bar{q}(x) \exp\left(ig \int_0^x A d\xi\right) q(0) \right\rangle_\beta^{k=1} = \frac{\langle \bar{q}q \rangle_\beta}{2} e^{I_3(x,\beta) - \pi[(x^0)^2 + (x^1)^2]/2V}, \tag{2.11}$$

with I_3 given by

$$I_3(x,\beta) = \frac{g^2}{2} \int_0^x d\xi_\mu d\xi'_\mu \Box K_{\xi\xi'}. \tag{2.12}$$

Since $\Box K = (\Box - m^2)^{-1}$, the integration may be carried out by expanding in complete eigenfunctions of ϕ as done in Appendix B. The result, for $L \to \infty$, in the temporal direction is $(t > 0)$

$$I_3(t,\beta) = -\frac{\pi m t}{4} - \frac{m^2}{2} \int_0^\infty \frac{dk}{(k^2+m^2)^{\frac{3}{2}}} \left(e^{-t\sqrt{k^2+m^2}} - 1 + 2\frac{(\cosh t\sqrt{k^2+m^2} - 1)}{e^{\beta\sqrt{k^2+m^2}} - 1} \right) \tag{2.13}$$

and, in the spatial direction,

$$I_3(x^1,\beta) = -m^2 \int_0^\infty \frac{dk}{\sqrt{k^2+m^2}} \frac{\sin^2 \frac{kx^1}{2}}{k^2} \coth \frac{\beta}{2} \sqrt{k^2+m^2}. \tag{2.14}$$

Note that in the spatial direction, the high temperature limit of I_3 is $(x^1 > 0)$

$$-\frac{\pi x^1}{2\beta} + \frac{\pi}{2m\beta}(1 - e^{-mx^1}).$$

This results in a screening mass $\pi T/2$. We want to note that this factor is not from the Dirac string but from the K_{x0} term of the ϕ integration.

Next we proceed with the $k = 0$ sector. In this case only the off-diagonal matrix elements of the invariant correlator are nonzero. They can be expressed as

$$\mathcal{S}_{F,+-}(x,\beta) = -\frac{\int d^2h\, G_{+-}(0,x) \det(i\not{D}_0) \int D\phi \exp\left(-\frac{1}{2}\int \phi(\Box^2 - m^2\Box)\phi - 2\pi/m^2V\right) \exp\left(ig\int_0^x A d\xi\right)}{\int d^2h \det(i\not{D}_0) \int D\phi \exp\left(-\frac{1}{2}\int \phi(\Box^2 - m^2\Box)\phi\right)}, \tag{2.15}$$

where $G_{+-}(0,x)$ is the upper left off-diagonal element of the inverse of the Dirac operator. Its explicit representation is given by [9, 15]

$$G_{+-}(0,x) = e^{g[\phi(x)-\phi(0)]} U^\dagger(x) \frac{i|\eta|^3}{\beta} \frac{\vartheta_4(z-H)}{\vartheta_1(z)\vartheta_4(H)} e^{2\pi i h_0 z}, \tag{2.16}$$

with $H = h_1 - i\tau h_0$ and $U(x)$ defined in (2.7). The lower left matrix element of the gauge-invariant propagator is obtained from (2.15) by replacing G_{+-} by G_{-+} which is just obtained by taking the complex conjugate and the coordinates to their negative in G_{+-}. Substituting in for the determinant, the h integrations may be done and the square completed in the ϕ integration to give

$$\mathcal{S}_{F,+-}(x,\beta) = -\frac{i|\eta|^3}{\beta} \frac{e^{2g^2(K_{xx}-K_{x0})}}{\vartheta_1(z)} e^{I_3(x,\beta) - \pi z^2/2\tau}.$$

Using the form for the bosonic propagator

$$g^2 K_{xy} = \pi\Delta_m(x-y) + \ln\left|\frac{\eta}{\vartheta_1(z_1 - z_2)}\right|^{\frac{1}{2}} + \frac{\pi}{2V}(x^1 - y^1)^2 + \frac{\pi}{m^2V},$$

found elsewhere [9, 15], the result is

$$\mathcal{S}_{F,+-}(x,\beta) = i\frac{\langle\bar{q}q\rangle_\beta}{2}\frac{|\vartheta_1(z)|}{\vartheta_1(z)}e^{I_3(x,\beta)-2\pi\Delta_m(x)}e^{-\pi[(x^0)^2+(x^1)^2]/2V-\pi i x^0x^1/V}. \tag{2.17}$$

Here Δ_m is defined by $(\Box - m^2)\Delta_m(x-y) = \delta(x-y)$. In the thermodynamic limit ($V \to \infty$) this result can be simplified further. The volume-dependent exponential in (2.17) then becomes unity and the phase of the ϑ_1 function is given by

$$e^{-i\varphi_x} \equiv \frac{|\vartheta_1(z)|}{\vartheta_1(z)} = \frac{|\sin(\pi z)|}{\sin(\pi z)} \tag{2.18}$$

which is $\mathrm{sgn}(x^0)$ for purely temporal paths and $-i\,\mathrm{sgn}(x^1)$ for purely spatial ones. For $L \to \infty$, Δ_m can replaced by its Poisson resummation

$$-2\pi\Delta_m(x,\beta) = \sum_n K_0\left[m\sqrt{(x^0-\beta n)^2+(x^1)^2}\right].$$

The result for the lower left entry in the matrix is just the conjugate of Eq. (2.17). Putting together (2.11) and (2.17) we finally have, for the gauge-invariant fermion correlator in the Schwinger model at finite temperature in the infinite volume limit,

$$\mathcal{S}_F(x,\beta) = \frac{\langle\bar{q}q\rangle_\beta}{2}\,e^{I_3(x,\beta)}\left[\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} + ie^{-2\pi\Delta_m(x,\beta)}\begin{pmatrix} 0 & e^{-i\varphi_x} \\ e^{i\varphi_x} & 0\end{pmatrix}\right]. \tag{2.19}$$

In the final result (2.19), we have explicitly included a nonvanishing vacuum angle [16]. This leads to a weight of $\exp(ik\theta)$ for each topological sector k in (2.5) and (2.15). We recall that the zero modes only contribute to the diagonal parts of (2.19). The off-diagonal parts follow from (2.17) in the limit $V \to \infty$. For $x \to 0$, (2.19) is consistent with the operator product expansion. The explicit form for I_3 is given in (2.13) and (2.14) for both temporal and spatial directions. Note that we can rewrite those expressions in the equivalent form

$$I_3(t,\beta) = -\frac{\pi m t}{4} - \int\frac{d^2p}{2\pi}\,\delta(p^2-m^2)\frac{\pi m^2}{{p_0}^2}\,[\theta(p_0)+n_\beta]\,(e^{-p_0t}-1) \tag{2.20}$$

along the temporal direction, and

$$I_3(x^1,\beta) = +\int\frac{d^2p}{2\pi}\,\delta(p^2-m^2)\frac{\pi m^2}{{p_1}^2}\,[\theta(p_0)+n_\beta]\left(e^{ip_1x^1}-1\right) \tag{2.21}$$

along the spatial direction. Here $n_\beta = (e^{\beta\sqrt{k^2+m^2}}-1)^{-1}$ is the Bose distribution. The expressions (2.20) and (2.21) are amenable to a spectral analysis. At zero temperature, (2.21) can be rewritten in the form

$$I_3 = -\frac{\pi m x^1}{4} - \int\frac{d^2p}{2\pi}\,\delta(p^2-m^2)\frac{\pi m^2}{{p_0}^2}\,\theta(p_0)\left(e^{-p_0x^1}-1\right) \tag{2.22}$$

as expected from O(2) invariance. The result for zero temperature in the spatial direction has also been obtained in the gauge $A_0 = 0$ [17]. Although the Schwinger factor is absent in that case it seems to us that the present approach is more transparent.

III. DIMENSIONAL REDUCTION

At zero temperature, we note that the trace of the gauge-invariant correlator (2.19) reduces to $\langle\bar{q}q\rangle_\beta\cos\theta$, which is the expected fermion condensate at finite temperature and nonzero vacuum angle. At large Euclidean separations, it falls off as $e^{-m\pi|x|/4}$ in both the spatial and temporal directions. We note that

$$\frac{\pi m}{4} = \frac{g^2}{2}\int\frac{dp}{2\pi}\frac{1}{p^2+m^2} \tag{3.1}$$

is just the Coulomb energy of a screened infinitely heavy charge. The Coulomb energy follows from the screening of the line integral present in the gauge-invariant correlator in the QED vacuum. It is a purely "kinematical" term. This aside, the expressions (2.20) and (2.22) show that the singularities of the gauge-invariant correlator are related to the mass gap m with a form factor $\pi m^2/{p_0}^2$. The off-diagonal part of the gauge-invariant correlator reduces to the free fermion propagator at short distances $i\gamma\cdot x/|x|^2$. The large distance behavior is similar to the diagonal part. Our result (2.19) at zero temperature for large spatial range is

$$\mathrm{Tr}\,\mathcal{S}_F = \langle\bar{q}q\rangle\frac{\sqrt{e}}{2}\,e^{-\pi m x^1/4}\begin{pmatrix} e^{i\theta} & ie^{-i\varphi_x} \\ ie^{i\varphi_x} & e^{-i\theta}\end{pmatrix}$$

showing a mass gap, as expected, and hence in disagreement with the brief analysis by Sachs and Wipf of a related correlator ([13], Sec. 6). The asymptotics are dictated by I_3 which [as easily seen by Eq. (2.12)] is the amplitude of the massive boson of the theory being exchanged between any two points along the path. Since this comes from the gauge field line integral, one may think that this correlator is qualitatively the same in the quenched approximation. However, a detailed study shows that Eq. (2.19) results from the cancellation of factors between the fermions and gauge field contributions. The validity of the quenched approximation may then be questioned as is further discussed in the next section.

At finite temperature, we note that the diagonal part of the gauge-invariant correlator reduces to the temperature-dependent condensate at zero separation, while the off-diagonal part gives the usual free fermion propagator. At high temperature, the gauge-invariant correlator along the temporal direction exhibits a free field behavior. Along the spatial directions, the gauge-invariant correlator falls off at a rate which is given by $e^{-\pi T|x|/2}$ despite the fact that the spectrum has a mass gap $m = g/\sqrt{\pi}$.

To understand this behavior, we note that at high temperature the model dimensionally reduces to one dimension since ($\omega_n = 2\pi nT$)

$$\phi(t,x) = \sum_n e^{-i\omega_n t}\,\phi_n(x) \sim \phi_0(x)\ . \tag{3.2}$$

Therefore, in the thermodynamic limit, $A_1 \sim 0$ and

$$A_0(x) \sim \frac{2\pi h_0}{g\beta} - \partial_1\phi_0(x). \tag{3.3}$$

With this in mind, the diagonal part of the spatial gauge-invariant correlator reduces to

$$\left\langle \bar{q}(0,x)\exp\left(ig\int_0^x dx' A_1(0,x')\right) q(0,0)\right\rangle_\beta \sim \left\langle \bar{q}(0,x)\,q(0,0)\right\rangle_\beta. \tag{3.4}$$

Because of the trace over Dirac indices, only the zero modes contribute to (3.4). In the dimensionally reduced theory, the expectation value becomes

$$\left\langle \bar{q}(0,x)\,q(0,0)\right\rangle_\beta \sim \left\langle e^{-g[\phi_0(x)+\phi_0(0)]}\right\rangle_\beta. \tag{3.5}$$

The expectation value is with respect to the dimensionally reduced action

$$S_{\rm red} = \beta\int dx \frac{1}{2}\phi_0\left({\partial_x}^4 - m^2{\partial_x}^2\right)\phi_0. \tag{3.6}$$

The integration, being quadratic in ϕ_0, can be performed at once using the saddle point method. The saddle point equation is

$$\left({\partial_z}^4 - m^2{\partial_z}^2\right)\phi_0(z) = -gT\left[\delta(z-x)+\delta(z-0)\right], \tag{3.7}$$

where the source terms result from the exponents in (3.5).

The solution is an induced instanton (first introduced in the context of mesonic correlation functions [18]):

$$\bar{\phi}_0(z) = \frac{\pi T}{2g}\left(\frac{1}{m}e^{-m|z|} + |z| + \frac{1}{m}e^{-m|z-x|} + |z-x|\right)\ . \tag{3.8}$$

Inserting (3.8) into (3.5) and (3.6), and using (3.7) yields ($|x| \to \infty$)

$$\begin{aligned}\left\langle \bar{q}(0,x)\,q(0,0)\right\rangle_\beta &\sim e^{-\frac{\pi T}{2}\left(|x|+\frac{1}{m}+\frac{1}{m}e^{-m|x|}\right)} \\ &\sim e^{-\pi T|x|/2-\pi T/m}.\end{aligned} \tag{3.9}$$

For the off-diagonal part the induced instanton gives rise to a factor $\exp(\pi Tx/2)$ which together with a factor $\exp(-\pi Tx)$ from the free propagator results in the same asymptotics. As final answer we find

$$\left\langle q(0,0)\,\bar{q}(0,x)\right\rangle_\beta \sim -e^{-\pi T|x|/2-\pi T/m}\left(\frac{1}{2}\mathbf{1} - \frac{i\gamma^1}{2\beta}\right) \tag{3.10}$$

at high temperature.

We remark that for $x \gg 1/m$ the induced instanton (3.8) is a linear superposition of two merons. Indeed, the classical field

$$\bar{\phi}_0(z) = \frac{\pi T}{2g}\left(\frac{1}{m}e^{-m|z|} + |z|\right) \tag{3.11}$$

gives rise to the potential ($h_0 = -1/4g$)

$$A_0(z) = \frac{\pi T}{2g}{\rm sgn}(z)\left(e^{-m|z|}-1\right) - \frac{\pi T}{2g} \tag{3.12}$$

with $A_0(-\infty) = 0$ and $A_0(+\infty) = -\pi T/g$. The topological charge carried by (3.11) is

$$\nu = \frac{g\beta}{2\pi}\int dx\,E(x) = -\frac{g}{2\pi T}\left[A_0(+\infty) - A_0(-\infty)\right] = \frac{1}{2} \tag{3.13}$$

one-half of the instanton charge. In the Schwinger model the merons are induced [see Eq. (3.7)] and localized over the meson Compton wavelength $1/m$ since the electric field $E(x) \sim Te^{-m|x|}$. When x becomes on the order of $1/m$, the two merons merge into a single instanton. The solution for $x = 0$ was studied in [11]; its classical action is responsible for the temperature dependence of the condensate.

Mesonic correlation functions can also be analyzed along these lines [11]. In that case the action of the classical solution cancels the $\exp(-2\pi Tx)$ asymptotics of the free propagator resulting in an exponential fall off determined by the meson mass. In the present case both the classical solution and its action are a factor two smaller (as compared to the mesonic action in the $k = 2$ sector) and in the $k = \pm 1$ sectors there is no additional $\exp(-\pi Tx)$ dependence resulting in the asymptotic screening mass $\pi T/2$. In the $k = 0$ sector the action of the two meron configuration gives rise to a factor $\exp(\pi Tx/2)$ but in this case a factor $\exp(-\pi Tx)$ comes from the free propagator. Merons at high temperature therefore give rise to fractional Matsubara frequencies in the Schwinger model.

IV. THE QUENCHED APPROXIMATION

The quenched approximation for the Schwinger model has been discussed extensively in the literature [19] and a number of contradictory statements have been made. Part of the confusion will be discussed below. It should be stressed that at zero temperature the quenched approximation has no gauge degrees of freedom. The fermionic ground state is trivial, and as such $\langle\bar{q}q\rangle$ should be zero. In the absence of the Dirac sea, there is no (axial) anomaly in the quenched approximation in any dimension. At finite temperature, and in the Euclidean formulation, the Polyakov lines play the role of nonlocal gauge degrees of freedom. Dimensional reduction arguments suggest that the high temperature phase in the quenched approximation is still trivial.

Having said this, we may *formally* calculate $\langle\bar{q}q\rangle$ for fixed fermionic mass μ, number of flavors N_F and Euclidean volume V, and consider the limits $\mu \to 0$, $N_f \to 0$, and $V \to \infty$. Some of these calculations have already appeared in the literature, with some confusion related to the order of the limits. In general, each pair of these limits does not commute. Let us first discuss the mass (μ) dependence. The important parameter is $\xi \equiv \mu V\langle\bar{q}q\rangle$. For $\xi \ll 1$, the mass is much smaller than the smallest eigenvalue and, as a consequence, only the sectors with topological charge equal to ±1 contribute to the condensate. For $\xi \gg 1$ the mass is much larger than the smallest eigenvalue, and the condensate receives contributions from many different topological sectors [20]. Remarkably, in both cases the value of the condensate is the same. To be more precise, if $\xi \ll 1$, the primed sum in the expression for the condensate

$$\langle\bar{q}q\rangle = \frac{1}{V}\left\langle\left(\frac{1}{i\mu} + \sum{}' \frac{1}{\lambda_n + i\mu}\right) i\mu \prod_n (\lambda_n^2 + \mu^2)\right\rangle \tag{4.1}$$

is of order μ and vanishes with respect to the contribution from the zero eigenvalue. For $\xi \gg 1$ the second term becomes dominant.

In the quenched approximation, the gauge-invariant correlator receives contributions from all k sectors since the restriction from the zero modes in the path integral is no longer present. The same effect occurs by taking a sufficiently large quark mass, and we expect that in this limit the quenched approximation is valid. In the opposite case, $\xi \ll 1$, the condensate diverges in the chiral limit. This is clear from (4.1) because the factor $i\mu$ from the determinant is absent in this case. The extra weight in the bosonic integration $\frac{1}{2}m^2\,\phi\Box\phi$, which originally was from the fermionic determinant [13] is no longer there. This results in a different bosonic Green's function. With this in mind, the fermionic propagator reads

$$q(0)\bar{q}(x) = \sum_n \frac{\psi_n(0)\overline{\psi}_n(x)}{\lambda_n + i\mu}\ , \tag{4.2}$$

with μ being a regulator for the zero modes which will be taken to zero after $L \to \infty$. In the Schwinger model and also for QCD with one flavor the result for the condensate is independent of the order of the limits $\mu \to 0$ and $V \to \infty$. Assuming that this is also the case for the Schwinger model in the quenched approximation, we choose $\mu \ll 1/\sqrt{V}$ so that only the $k = \pm1$ sectors contribute to the condensate in the quenched approximation. Therefore, for $\xi \ll 1$, the fermion condensate becomes

$$\langle\bar{q}q\rangle_Q = \frac{2i}{\mu V}\frac{\sum_{k>0} k e^{-2\pi k^2/m^2 V}}{\sum_{k>0} e^{-2\pi k^2/m^2 V}}. \tag{4.3}$$

In the limit $V \to \infty$ this sum can be calculated:

$$\langle\bar{q}q\rangle_Q = \frac{i\sqrt{2}}{\pi}\frac{m}{\mu\sqrt{V}} \tag{4.4}$$

in agreement with an analysis using the proper time representation of the fermion propagator [21]. The fact that the condensate diverges for $\mu \to 0$ seems to be in contradiction with arguments presented at the beginning of this section. However, the condensate as calculated in (4.3) follows from the trace over *all* fermionic states and cannot be identified with the condensate that would be obtained in the absence of the Dirac sea (which is zero).

The gauge-invariant fermion correlator cannot be calculated exactly in the quenched approximation. The Wilson line, being different in the quenched approximation, shows an area law (quenched) as opposed to a perimeter law (unquenched). As such, it would be interesting to evaluate the ratio of the invariant correlator to the fermion condensate in the quenched approximation.

V. CONCLUSION

We have explicitly constructed the gauge-invariant fermion correlator for the Schwinger model on the torus. We have shown that at zero temperature, the result is O(2) invariant and asymptotes $e^{-\sigma_P|x|}$ at large separations with $\sigma_P = \pi m/4$ being the screening mass associated with the line integral. This screening mass is purely kinematical, and relates directly to the perimeter law of the Wilson loop at zero temperature

$$\langle \exp ig \oint_C A\cdot d\xi\rangle_\beta \sim e^{-\sigma_P\ P}\ , \tag{5.1}$$

where P is the perimeter spanned by the *large* loop C. At short distances, the diagonal part of the gauge-invariant correlator reduces to the fermion condensate, while the off-diagonal part reduces to the free fermion propagator. After proper subtractions, the gauge-invariant correlator displays singularities that are related to the mass gap in the theory. These singularities may be related to heavy-light systems, with the string playing the role of an infinitely heavy charge.

At finite temperature, we have shown that the diagonal part of the gauge-invariant correlator at short distances reduces to the fermion condensate and asymptotes $\exp(-\pi Tx/2)$ at large spatial separations, even though the spectrum exhibits a mass gap at all temperatures. The fractional Matsubara frequency results from the action of localized merons at high temperature. We have shown that the quenched approximation does not work in the Schwinger model. These results suggest that this might be the case for QCD with massless quarks. However, it may very well be that for nonzero quark masses the effect of the fermion determinant can be ignored for most observables. The relation of these results to the gauge-invariant correlator in real-time will be discussed elsewhere [22].

Are the above results relevant for four-dimensional QCD? We do not know. We suspect however, that the gauge-invariant correlator for QCD with one flavor should reflect on the fermion condensate along the diagonal. It should asymptotically approach $e^{-\sigma_P |x|}$ at zero temperature in all directions with σ_P as the perimeter-law coefficient in the Wilson loop. It would be interesting to check this point by lattice QCD simulations and relate the results to the (renormalized) Coulomb energy in four dimensions. This kinematical term aside, we suspect that the rest of the correlator should fall off at a rate determined by the heavy-light bound states of QCD.

At finite temperature, the diagonal part should be proportional to the finite temperature fermion condensate. In particular, it should vanish in the chirally symmetric phase. Lattice simulations of the mesonic screening masses together with dimensional reduction arguments [1, 3] suggest that at high temperature the screening masses asymptote $2\pi T$.

Repeating the dimensional reduction arguments, and barring induced merons at high temperature, we would conclude that the invariant fermion correlator asymptotes πT and not a fraction of πT in QCD. We recall, however, that merons in QCD have been proposed as candidates for understanding confinement at zero temperature [23]. Their introduction at zero temperature is, however, *ad hoc*, as they carry infinite action before smearing. It would be interesting to see whether they could be induced and localized (have a finite action) in QCD at high temperature. Lattice simulations in these directions could be helpful.

ACKNOWLEDGMENTS

The reported work was partially supported by the U.S. DOE Grant No. DE-FG-88ER40388.

APPENDIX A

We detail in this appendix the calculations that led to (2.9) in the text. The factors associated with the exact Dirac operator in (2.4), along with the line integral (2.3), act as effective sources in the action for the ϕ field:

$$\begin{aligned} S_{\text{eff}}[\phi] &= \int d^2y\{\tfrac{1}{2}\phi(\Box^2 - m^2\Box)\phi + \phi J\} \\ &= \frac{1}{2}\int d^2y\{\tilde{\phi}(\Box^2 - m^2\Box)\tilde{\phi} - J(\Box^2 - m^2\Box)^{-1}J\}, \end{aligned} \tag{A1}$$

with

$$\begin{aligned} \frac{1}{g}J(y) \equiv{}& e_x\delta^2(y-x) + e_0\delta^2(y) \\ &+ i\varepsilon_{\mu\nu}\int_0^x d\xi_\mu \frac{\partial}{\partial\xi^\nu}\delta^2(y-\xi). \end{aligned}$$

Here, the e_i's are either plus or minus depending on in which k sector the correlator is evaluated. They have the same sign in the $k = \pm 1$ sectors and opposite signs for the propagator in the $k = 0$ sector (2.16). The fermions contribute strongly at the end points (as δ functions) but the gauge field's contribution is smeared out along the transverse direction of the path.

The source-source term in Eq. (A1) given in terms of the bosonic Green's function, $K = (\Box^2 - m^2\Box)^{-1}$, and including the extra minus sign in the exponent is

$$g^2(K_{xx} + e_x e_0 K_{x0}) + ie_x I_1^x + ie_0 I_1^0 + I_2 \tag{A2}$$

with

$$I_1^{x_i} = -g^2\,\varepsilon_{\mu\nu}\int_0^x d\xi_\mu \partial_{\xi^\nu} K_{\xi x_i}\ ,$$

$$I_2 = \frac{g^2}{2}\varepsilon_{\mu\nu}\varepsilon_{\rho\sigma}\int_0^x\int_0^x d\xi_\mu d\eta_\rho \partial_{\xi^\nu}\partial_{\xi^\sigma} K_{\xi\eta}.$$

The first term in Eq. (A2) is just the self-energy contribution from the fermions along the path. The vanishing of I_1 follows from the mode expansion of the Green's function. Because of the identity $\varepsilon_{\mu\nu}\varepsilon_{\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}$, the self-energy of the gauge field, I_2, may be rewritten as

$$\begin{aligned} I_2 &= g^2\big(K_{xx} - K_{x0}\big) + \frac{g^2}{2}\int_0^x d\xi_\mu d\xi'_\mu \Box K_{\xi\xi'} \\ &\equiv g^2\big(K_{xx} - K_{x0}\big) + I_3(x,\beta), \end{aligned}$$

giving finally in our case for the bosonic contribution in the exponent

$$2g^2(K_{xx} + K_{x0}) + I_3(x,\beta)$$

in the $k = \pm 1$ sectors and

$$2g^2(K_{xx} - K_{x0}) + I_3(x,\beta)$$

in the $k = 0$ sector.

APPENDIX B

In this appendix, we detail the calculation that leads to the explicit form of the $I_3(x,\beta)$ integral both along the temporal and spatial directions. The double integral of the bosonic Green's function needed in Appendix A is

$$I_3(t,\beta) = \frac{g^2}{2}\int_0^t dt'\,dt''\Box K(t'-t'',x)|_{x=0}$$

in the temporal direction. Since $K = (\Box^2 - m^2\Box)^{-1}$, an additional $\Box$ in the numerator leaves only the massive part of the propagator. Using a complete set of states of ϕ this may be rewritten as

$$I_3(t,\beta) = -\frac{\pi m^2}{2V}\int_0^t\int_0^t dt'\,dt'' \sum_{(n_0,n_1)\neq(0,0)} \frac{e^{2\pi i n_0(t'-t'')/\beta}}{(2\pi n_0/\beta)^2+(2\pi n_1/L)^2+m^2}$$
$$= -\frac{\pi m^2 t^2}{2V}\sum_{n\neq 0}\frac{1}{(2\pi n/L)^2+m^2} - \frac{\pi}{2\tau}\left(\frac{m\beta}{2}\right)^2 \sum_{n_1=-\infty}^{\infty}\sum_{n_0\neq 0}\frac{\sin^2\pi n t/\beta}{(\pi n_0)^2\left[(\pi n_0)^2+(\pi n_1/\tau)^2+(m\beta/2)^2\right]}\,.$$

The n_0 sum in the second term may be done by integrating both sides of

$$\sum_{n\neq 0}\frac{\cos 2\pi n\phi}{(\pi n)^2+a^2} = \frac{1}{a}\frac{\cosh a(1-2|\phi|)}{\sinh a} - \frac{1}{a^2}$$

with respect to ϕ twice. The result in the temporal direction for finite L is

$$\frac{\pi t^2}{2V} - \frac{\pi m|t|}{4}\coth\frac{mL}{2} - \frac{\pi m^2}{2L}\sum_{n=-\infty}^{\infty}\frac{\coth\frac{\beta}{2}\xi(n)\left(\cosh t\xi(n)-1\right)-\sinh|t|\xi(n)}{\xi^3(n)} \tag{B1}$$

with $\quad \xi(n)=\sqrt{(2\pi n/L)^2+m^2}.$

Taking the limit $L\to\infty$, the sum turns into an integral and gives

$$I_3(t,\beta) = -\frac{\pi m|t|}{4} - \frac{m^2}{2}\int_0^\infty \frac{dk}{(k^2+m^2)^{\frac{3}{2}}}\left[\coth\frac{\beta}{2}\sqrt{k^2+m^2}(\cosh t\sqrt{k^2+m^2}-1) - \sinh|t|\sqrt{k^2+m^2}\right]. \tag{B2}$$

The part in the brackets may be rewritten as

$$e^{-|t|\sqrt{k^2+m^2}} - 1 + \frac{2}{e^{\beta\sqrt{k^2+m^2}}-1}(\cosh t\sqrt{k^2+m^2}-1)$$

showing the Bose-Einstein number occupation factor for finite temperature with a form factor.

For the spatial result, all that is needed is to exchange x with t and L with β in Eq. (B1). Taking $L\to\infty$ in that result produces

$$I_3(x^1,\beta) = -\frac{\pi m|x^1|}{4}\coth\frac{m\beta}{2} + \frac{\pi m^2}{2\beta}\sum_{n=-\infty}^{\infty}\frac{1-e^{-|x^1|\sqrt{(2\pi nT)^2+m^2}}}{\left((2\pi nT)^2+m^2\right)^{\frac{3}{2}}}$$
$$= -m^2\int_0^\infty \frac{dk}{\sqrt{k^2+m^2}}\frac{\sin^2\frac{kx^1}{2}}{k^2}\coth\frac{\beta}{2}\sqrt{k^2+m^2}.$$

In the first expression $T=1/\beta$ was used to show the Matsubara form. The second expression may be obtained by expanding the hyperbolic cotangent in the first expression as a series, combining the two series, and noting that

$$\int_0^\infty \frac{dk\,\sin^2\frac{ka}{2}}{k^2(k^2+b^2)} = \frac{\pi a}{b^2} - \frac{\pi}{4b^2}(1-e^{-ab})$$

to produce the integral.

[1] I. Zahed, Acta Phys. Pol. **B25**, 99 (1994), and references therein.
[2] B. Petersson, in *Lattice '92*, Proceedings of the International Symposium, Amsterdam, The Netherlands, 1992, edited by J. Smit and P. van Baal [Nucl. Phys. B (Proc. Suppl.) **30**, 66 (1993)].
[3] T.H. Hansson and I. Zahed, Nucl. Phys. **B374**, 177 (1992); G.E. Brown, A. Jackson, V. Koch, and E. Shuryak, Phys. Rev. D **45**, 3169 (1992).
[4] G. Boyd, S. Gupta, and F. Karsch, Nucl. Phys. **B385**, 481 (1992); G. Boyd, in *Lattice '92* [2], p. 335.
[5] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[6] M. Einhorn, Phys. Rev. D **14**, 3451 (1976).
[7] A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D **10**, 732 (1974).
[8] R. Pisarski, Nucl. Phys. **309**, 476 (1988).
[9] A. Fayyazuddin, T.H. Hansson, M.A. Nowak, J.J.M. Verbaarschot, and I. Zahed, Nucl. Phys. **B425**, 553 (1994).
[10] A. Abada and R. Schrock (unpublished); A. Abada, Ph.D. thesis, State University of New York at Stony Brook, 1992. We thank R. Schrock for bringing these works to our attention.
[11] A. Smilga, Phys. Rev. D **49**, 5480 (1994).
[12] H. Yamagishi, Phys. Lett. B **173**, 465 (1986).
[13] A. Sachs and I. Wipf, Helv. Phys. Acta **65**, 652 (1992).
[14] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).
[15] J.V. Steele, A. Subramanian, and I. Zahed, Nucl. Phys.

B (to be published).
[16] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985), p. 294.
[17] P. Becher, Ann. Phys. (N.Y.) **146**, 223 (1983).
[18] N.K. Nielsen and B. Schroer, Phys. Lett. **66B**, 373 (1977).
[19] E. Marinari, G. Parisi, and C. Rebbi, Nucl. Phys. **B190**, 734 (1981); C.J. Hamer, J. Kogut, D.P. Crewther, and M.M. Mazzolini, *ibid.* **B208**, 413 (1982); S.R. Carson and R.D. Kenway, Ann. Phys. (N.Y.) **166**, 364 (1986); R. Ben-Av *et al.*, Phys. Lett. B **253**, 185 (1991); A. Smilga, Phys. Rev. D **46**, 5598 (1992); V. Azcoiti *et al.*, in *Lattice '93*, Proceedings of the International Symposium, Dallas, Texas, edited by T. Draper *et al.* [Nucl. Phys. B (Proc. Suppl.) **34** (1994)]; H. Dilger and H. Joos, *ibid.*
[20] H. Leutwyler and A. Smilga, Phys. Rev. D **46**, 5607 (1992).
[21] F. Guerin and H.M. Fried, Phys. Rev. D **33**, 3039 (1985).
[22] J.V. Steele and I. Zahed (in preparation).
[23] C. Callan, R. Dashen, and D. Gross, Phys. Rev. D **17**, 2717 (1978).