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Equation of state for cool relativistic two-constituent superfluid dynamics

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The natural relativistic generalization of Landau's two-constituent superfluid theory can be formulated in terms of a Lagrangian $\mathcal{L}$ that is given as a function of the entropy current four-vector s^ρ and the gradient $\nabla_\rho \varphi$ of the superfluid phase scalar. It is shown that in the "cool" regime, for which the entropy is attributable just to phonons (not rotons), the Lagrangian function $\mathcal{L}\{\vec{s}, \nabla\varphi\}$ is given by an expression of the form $\mathcal{L} = P - 3\psi$ where P represents the pressure as a function just of $\nabla_\rho \varphi$ in the (isotropic) cold limit. The entropy current-dependent contribution ψ represents the generalized pressure of the (nonisotropic) phonon gas, which is obtained as the negative of the corresponding grand potential energy per unit volume, whose explicit form has a simple algebraic dependence on the sound or "phonon" speed $c_{\rm P}$ that is determined by the cold pressure function P.

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I. INTRODUCTION

The purpose of the present work is to derive the natural *cool limit form* of the equation of state that is needed to complete the formulation of the natural relativistic generalization of Landau's two constituent superfluid theory [1]. The qualification "cool" is to be understood here as referring to the low temperature limit in which account is taken only of lowest order deviations from a nearby *cold* configuration in which thermal effects are absent altogether.

Whereas the cold (strictly zero temperature) superfluid case is described by irrotational configurations of a simple perfect fluid model (i.e., one that is both isotropic and barytropic), on the other hand the allowance for thermal effects, even in the cool limit, necessitates the use of a two constituent fluid model, of the kind that was pioneered by workers such as London, and perfected by Landau. Accurate treatment of nonstationary applications would require allowance for viscosity of the "normal" constituent. However it is sufficient for many purposes, and entails no loss of accuracy at all in the case of equilibrium states such as simple cylindrical vortex configurations, to use a strictly conservative treatment as in Landau's original model [1].

In addition to the neglect of dissipation, the main simplification on which the analysis below is based is the neglect of the nonlinear excitations known as "rotons" which are of dominant importance in the "warm" regime nearer to the phase transition (beyond which lies the regime of "hot" states in which superfluidity is absent altogether). The presence of such effects makes it very difficult to derive the equation of state over the full range of the three relevant variables, which in addition to the temperature Θ say, could be taken to consist of the relevant conserved particle number density n say, and the relative velocity v of the two constituents. Although it is out of the question for neutron star matter, an experimental investigation of the equation of state should be feasible in practice for the laboratory case of helium-4. However even in this experimentally accessible case, the exploration of the relevant phase space (as parametrized by Θ, n, and v) does not yet appear to have been sufficiently systematic and thorough (see, e.g., [2]).

In contrast with the dubious quality of present day knowledge of "warm" superfluid dynamics, the state of knowledge of the "cool" limit is very satisfactory. In this limit, complications such as "rotons" can be ignored, the only important thermal effects being entirely attributable to simple "phonon" excitations, which can be adequately described by an essentially linear treatment whose original development is again largely attributable to the (by now experimentally well substantiated) work of Landau [3]. The detailed development of the necessary statistical mechanics has been summarized in a convenient form by Khalatnikov [4]. It will be shown below that although it was originally carried out in a nonrelativistic framework, the nature of this statistical analysis is such that it can be translated into a relativistic form without any change of form provided that sufficient care is used in defining the appropriate variables for the relativistic version. In consequence, as shown in the Appendix, Landau's expressions for the first and second sound speed can also be retained without change in the relativistic regime.

II. RELATIVISTIC GENERALIZATION OF TWO-CONSTITUENT SUPERFLUID DYNAMICS

It has recently been shown [5] that the natural relativistic generalization of the standard Landau theory of two constituent superfluid dynamics in its simplest strictly conservative version [1] can be formulated in a particularly convenient manner by taking as the starting point a Lagrangian scalar $\mathcal{L}$ that is given as an appropriate function (which was denoted by X in the original presentation [5]) of the entropy current vector s^μ and of the gradient

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$$\mu_\rho = \hbar \nabla_\rho \varphi \qquad (2.1)$$

of the superfluid phase scalar φ. Two of the required equations of motion are obtained in the form of the usual conservation laws as given by

$$\nabla_\rho n^\rho = 0 \ , \qquad (2.2)$$

for the particle current n^ρ constructed below, and

$$\nabla_\rho s^\rho = 0 \ . \qquad (2.3)$$

The system is completed by the equation

$$s^\rho \nabla_{[\rho} \Theta_{\sigma]} = 0 \qquad (2.4)$$

(using square brackets to denote index antisymmetrization) governing the evolution of a certain *thermal momentum* covector Θ_ρ. This last equation can alternatively be expressed (in a form that is useful as a starting point for the derivation of corresponding circulation and helicity conservation laws [6]) as the vanishing of the Lie derivative of the thermal momentum vector with respect to the associated *temperature* vector, β^ρ, i.e.,

$$\beta^\sigma \nabla_\sigma \Theta_\rho + \Theta_\sigma \nabla_\rho \beta^\sigma = 0 \ , \qquad (2.5)$$

where

$$\beta^\rho = -\left(s^\sigma \Theta_\sigma\right)^{-1} s^\rho \ . \qquad (2.6)$$

The thermal four-momentum covector Θ_ρ and the particle number current vector n^ρ are specified in this formulation as the algebraic functions of the entropy current vector s^ρ and of the superfluid four-momentum covector μ_ρ that are obtained by partial differentiation of the Lagrangian according to the infinitesimal variation formula

$$d\mathcal{L} = \Theta_\rho ds^\rho - n^\rho d\mu_\rho \ . \qquad (2.7)$$

III. ALTERNATIVE ACTION FUNCTIONS AND THE STRESS MOMENTUM-ENERGY TENSOR

The formulation that has just been summarized has the technical advantage of being particularly economical inasmuch as it involves only five independent component variables: namely, the phase scalar φ and the four independent components of the entropy current vector s^ρ. This feature of economy has recently been exploited for the purpose of setting up a correspondingly economical Hamiltonian formulation of the theory [7], and it has also been exploited as a guide to the formulation of an analogously economical theory for describing thermal effects in superconducting cosmic strings [8].

Although mathematically equivalent as far as its physical ("on shell") solutions are concerned, this more recent five-component formulation differs from the earlier convective variational formulation [9,10] (which has the alternative advantage of being less specialized) that treated the current vectors n^ρ and s^ρ on the same footing as independent variables, thus involving a total of eight independent space-time components. The presence of the three extra components made it necessary to include, as an extra dynamical equation, the irrotationality condition

$$\nabla_{[\rho} \mu_{\sigma]} = 0 \qquad (3.1)$$

that is the Poincaré integrability condition for the existence of a potential φ satisfying (2.1). In the convective variational formulation μ_ρ is on the same footing as Θ_ρ, both being obtained by partial differentiation according to the formula

$$d\Lambda = \Theta_\rho ds^\rho + \mu_\rho dn^\rho, \qquad (3.2)$$

where the convective variational master function Λ is obtainable from the Lagrangian $\mathcal{L}$ of the newer more specifically adapted approach by a Legendre type transformation given by the relation

$$\Lambda = \mathcal{L} + n^\rho \mu_\rho. \qquad (3.3)$$

The approach based on the Lagrangian $\mathcal{L}$ that will be used here is in fact a compromise, intermediate between the convective variational approach based on the master function Λ, and another independently developed (Clebsch-type) variational approach [11,12] in which both currents are treated as dependent variables, their specification being given by a generalized pressure function Ψ in terms of μ_ρ, as defined by (2.1), and of Θ_ρ, which in this version has independent status, by the partial differentiation formula

$$d\Psi = -s^\rho d\Theta_\rho - n^\rho d\mu_\rho, \qquad (3.4)$$

where the generalized pressure function Ψ itself is given by

$$\Psi = \mathcal{L} - \Theta_\rho s^\rho \ . \qquad (3.5)$$

Although the generalized pressure function Ψ will not act in a fundamental role in the formulation used here, it nevertheless plays an important part, notably in the expression for the stress momentum energy tensor. An important part is also played by a set of three scalar functions of state, $\mathcal{C}$, $\mathcal{B}$, and $\mathcal{A}$ (respectively describable as the *caloric* coefficient, the *bulk* coefficient, and the *anomaly* coefficient), that are defined as matrix elements in the relation expressing the momenta in terms of the velocities in the form

$$\Theta^\rho = \mathcal{C} s^\rho + \mathcal{A} n^\rho \ , \qquad \mu^\rho = \mathcal{A} s^\rho + \mathcal{B} n^\rho. \qquad (3.6)$$

These coefficients can be used to convert the canonical expression

$$T^\rho{}_\sigma = n^\rho \mu_\sigma + s^\rho \Theta_\sigma + \Psi g^\rho{}_\sigma \qquad (3.7)$$

for the stress momentum energy density tensor of the two-constituent superfluid into the manifestly symmetric though not so elegant form

$$T^{\rho\sigma} = \mathcal{B} n^\rho n^\sigma + \mathcal{A}\left(n^\rho s^\sigma + s^\rho n^\sigma\right) + \mathcal{C} s^\rho s^\rho + \Psi g^{\rho\sigma} \ , \qquad (3.8)$$

where $g^{\rho\sigma}$ is the contravariant inverse of the, flat or curved, space-time metric tensor $g_{\rho\sigma}$ that is to be used for index raising or lowering. For the purpose of the present work it will be more convenient to work with an alternative recombination expressible in terms of the *dilation* coefficient Φ^2 and the *determinant* coefficient $\mathcal{K}$ by

$$T^{\rho\sigma} = \Phi^2\left(\mu^\rho\mu^\sigma + \mathcal{K}s^\rho s^\sigma\right) + \Psi g^{\rho\sigma} , \tag{3.9}$$

where

$$\Phi^2 = \mathcal{B}^{-1} , \qquad \mathcal{K} = \mathcal{C}\mathcal{B} - \mathcal{A}^2 . \tag{3.10}$$

IV. THE COLD LIMIT AND THE SPEED OF SOUND

Before considering the treatment of the lowest order thermal corrections, it is to be recalled that in the cold (i.e., strictly zero temperature) limit, the superfluid behaves as a perfect fluid of simple barytropic type. This means that in terms of the unit flow vector u^ρ defined by

$$n^\rho = nu^\rho , \qquad u^\rho u_\rho = -c^2 , \tag{4.1}$$

where c is the speed of light, the stress tensor reduces to the isotropic form

$$T^{\rho\sigma} = \left(\rho + c^{-2}P\right)u^\rho u^\sigma + Pg^{\rho\sigma}, \tag{4.2}$$

in which the mass-density ρ and the (isotropic) pressure P are directly related by a single variable equation of state. The equation of state can be specified either by giving the rest frame energy density ρ as a function of the corresponding particle number density n in the form

$$\rho = \rho\{n\} , \tag{4.3}$$

or equivalently, in dual form, by giving the pressure P as a function of the corresponding effective mass (i.e., relativistic chemical potential) variable μ in the form

$$P = P\{\mu\} \tag{4.4}$$

(using small curly brackets to indicate functional dependence, as distinct from the multiplication that would be indicated by ordinary brackets). The connection between these two formulations is given by the familiar differential specifications

$$\mu = \frac{d\rho}{dn} , \qquad n = \frac{1}{c^2}\frac{dP}{d\mu} , \tag{4.5}$$

and the dually symmetric relation

$$\rho + c^{-2}P = n\mu . \tag{4.6}$$

For the simple ("barytropic") perfect fluid model that has just been formulated, the convective variational master function Λ referred to in the introduction is given directly by the first version of the equation of state as

$$\Lambda = -c^2\rho\{n\} \tag{4.7}$$

while the corresponding dual Lagrangian $\mathcal{L}$, which in this zero temperature limit has just the same form as the Clebsch-type potential function Ψ, will be given analogously by the dual version of the equation of state as

$$\mathcal{L} = \Psi = P\{\mu\} . \tag{4.8}$$

To complete the translation into the notation of the preceding section the particle momentum covector will be given simply by

$$\mu_\rho = \mu u_\rho , \tag{4.9}$$

and of course the entropy vector s^ρ and the thermal momentum covector Θ_ρ will both vanish, while the temperature vector β^ρ is singular in this zero temperature limit.

One of the most important quantities in this simple perfect fluid model is the characteristic sound or "phonon" speed, $c_{\rm P}$ say, i.e., the short wavelength limit of the propagation velocity relative to the preferred rest frame of small (necessarily longitudinal) perturbations. This characteristic speed is immediately derivable from either of the versions (4.3) or (4.4) of the equation of state via the familiar differential formula

$$c_{\rm P}^2 = \frac{dP}{d\rho} = c^2\frac{n}{\mu}\frac{d\mu}{dn} . \tag{4.10}$$

This can be used to construct the phonic (or sonic) metric tensor

$$\mathcal{G}^{\rho\sigma} = g^{\rho\sigma} + \left(\frac{1}{c^2} - \frac{1}{c_{\rm P}^2}\right)u^\rho u^\sigma , \tag{4.11}$$

whose null eigencovectors p_ρ, as defined by

$$\mathcal{G}^{\rho\sigma}p_\rho p_\sigma = 0 , \tag{4.12}$$

are tangential to the characteristic hypersurfaces of sound propagation in the medium, in the same way as the null covectors of the ordinary space-time metric $g^{\rho\sigma}$ are tangential to the characteristic hypersurfaces of light propagation in vacuum.

V. THE CONCEPT OF "NORMAL" AND "SUPERFLUID" DENSITY CONTRIBUTIONS

As soon as we want to allow for deviations from the cold limit case that has just been summarized, it is necessary to increase the number of independent variables in the fundamental equation of state from one to three. It might naively have been hoped that two would suffice for the cool limit with which we are concerned here but, as will be made apparent at once, this ceases to be possible unless one is willing to restrict attention to states for which there is no relative motion between the two constituents. In the convective variational formulation based on Λ as the fundamental state function, the three independent variables correspond to the three scalar invariants that can be constructed from the pair of independent current vectors, namely n^ρ whose direction determines the particle or "Eckart" rest frame, and s^ρ which similarly determines the thermal or "normal" rest frame, so that an obviously convenient way of choosing the three independent variables in a fundamental state function of the form

$$\Lambda = \Lambda\{n, x, s\} \tag{5.1}$$

will be to take them to consist of the particle rest frame number density n as given by

$$c^2 n^2 = -n_\rho n^\rho \ , \tag{5.2}$$

the cross product variable x as given by

$$c^2 x^2 = -n_\rho s^\rho \ , \tag{5.3}$$

and the thermal rest frame entropy density s as given by

$$c^2 s^2 = -s_\rho s^\rho \ . \tag{5.4}$$

In the dual formulation based on the Lagrangian $\mathcal{L}$ that will be used here, the relevant independent variables are the three scalar invariants that can be constructed from the entropy current vector s^ρ and the momentum covector μ_ρ whose orientation determines what is known in this context as the "superfluid" rest frame. It is obviously convenient to take one of the required scalars to be again the "normal" entropy density variable s as defined by (5.4), while taking the other two to consist of a new cross product variable y given by

$$c^2 y^2 = -\mu_\rho s^\rho \ , \tag{5.5}$$

together with the effective mass variable μ given by

$$c^2 \mu^2 = -\mu_\rho \mu^\rho \ , \tag{5.6}$$

so that the required generalization of (4.8), i.e., the relevant analogue of (5.1), will be given by an expression of the form

$$\mathcal{L} = \mathcal{L}\{\mu, y, s\}. \tag{5.7}$$

Starting from an expression of this form it can be seen from (3.6) and (3.10) that the secondary variables n^ρ and Θ_ρ will be given in terms of the primary variables μ_ρ and s^ρ of this formulation by

$$n^\rho = \Phi^2\big(\mu^\rho - \mathcal{A} s^\rho\big) \ , \qquad \Theta_\rho = \Phi^2\big(\mathcal{K} s_\rho + \mathcal{A}\mu_\rho\big) \ , \tag{5.8}$$

where by (2.7) the relevant dilation, determinant, and anomaly coefficients Φ^2, $\mathcal{K}$, and $\mathcal{A}$ are given by the partial differentiation formulas

$$c^2\Phi^2 = \frac{1}{\mu}\frac{\partial \mathcal{L}}{\partial \mu} \ , \quad c^2\Phi^2\mathcal{K} = -\frac{1}{s}\frac{\partial \mathcal{L}}{\partial s} \ , \quad c^2\Phi^2\mathcal{A} = -\frac{1}{2y}\frac{\partial \mathcal{L}}{\partial y} \ . \tag{5.9}$$

It was a generic Lagrangian function of this form (5.7) that was the starting point for the construction, in implicit form, of a Hamiltonian reformulation of the theory [7]. However in order to be able to make such a construction explicit it is necessary to sacrifice generality and deal separately with particular kinds of state function. From a purely mathematical point of view the simplest kind that can be envisaged is the "regular" category, as characterized [10] by the condition that the anomaly parameter $\mathcal{A}$ vanishes, which evidently, by the original definition (3.6), means that the momenta are aligned with the corresponding current vectors. It can be seen from (5.9) that in the formulation based on the Lagrangian $\mathcal{L}$ that we are using here this regularity condition is expressible as the condition that the state function should depend only on μ and s but not on the third cross product variable y. However although it can be hoped that such a mathematically attractive alignment ansatz may provide a useful approximation in contexts involving hot conducting fluids, it is certainly not at all an appropriate simplification in the context of superfluidity.

In order to see how to obtain a more suitable form of equation of state it will be useful to establish an appropriate translation relating the terminology of the traditional formalism that was developed in a nonrelativistic framework to that of the relativistic formalism used here. In particular it will be convenient to introduce appropriate relativistic generalizations of the concepts of the "normal" and "superfluid" mass densities ρ_N and ρ_S. One way of defining such a generalization is to base it on a corresponding vectorial decomposition [5] of the particle current n^ν, which in the nonrelativistic limit [13] is interpretable as being proportional to the Newtonian mass current $\rho^\nu = m n^\nu$ for some constant coefficient m representing the mass per particle. However such a definition has the disadvantage that in a relativistic context there will be a certain margin of ambiguity in the choice of the appropriate "rest mass" parameter m. It will be more convenient for most purposes, including that of the present work, to adopt a natural alternative possibility that gives the same result in the nonrelativistic limit but that is free of any such ambiguity of normalization.

The definition that will be adopted here, and that seems clearly most appropriate in the relativistic context, is based on the decomposition of the stress momentum energy density tensor in the form

$$T^{\rho\sigma} = \frac{\rho_S}{\mu_N^2}\mu^\rho\mu^\sigma + \frac{\rho_N}{s_S^2}s^\rho s^\sigma + \Psi g^{\rho\sigma} \ , \tag{5.10}$$

where s_S is the value of the entropy density in the "superfluid" rest frame defined by μ^ρ, and μ_N is the analogously defined superfluid chemical potential with respect to the "normal" rest frame defined by s^ρ, i.e.,

$$s_S = s\Big(1 - \frac{v^2}{c^2}\Big)^{-1/2} = \frac{y^2}{\mu} \ ,$$
$$\mu_N = \mu\Big(1 - \frac{v^2}{c^2}\Big)^{-1/2} = \frac{y^2}{s} \ , \tag{5.11}$$

in which the velocity v is the relative translation speed between the "normal" and "superfluid" frames as given by

$$\frac{v^2}{c^2} = 1 - \frac{s^2\mu^2}{y^4} \ . \tag{5.12}$$

The definitions implicit in (5.10) are such as to ensure that the momentum density (T^0_3, say) in the direction of the relative motion will have magnitude $\rho_N v$ in the "superfluid" frame, while it will have magnitude $\rho_S v$ in the "normal" frame. (One could define a corresponding generalization of the Newtonian concept of the mass current by setting $\rho^\nu = \rho_S\mu_N^{-1}\mu^\nu + \rho_N s_S^{-1} s^\nu$, but it would be of limited utility since it would not retain the exact conservation property of its Newtonian limit.) It can be seen by comparison with (3.9) that the values of the quantities so defined can be evaluated in terms of the partial derivatives appearing in (5.9) using the formulas

$$\rho_S = \Phi^2 \mu_N^2 \ , \qquad \rho_N = \Phi^2 \mathcal{K} s_S^2 \ . \tag{5.13}$$

VI. STATISTICAL MECHANICS OF THE PHONON GAS

As in the standard Landau analysis [3], it will be supposed that, in the cool limit with which we are concerned, the thermal excitations can be described as a bosonic gas of noninteracting phonons.

To see how this works out, and to fix the notation, we start by recalling that the essential step in such an analysis [3] is the evaluation of the expected value of the occupation numbers ν_r, say where r is an index running over the relevant quantum states. The standard trick for doing this is work in terms of groups, with collective index $R\{r\}$ say, consisting of a large number, G_R say, of nearby states characterized by approximately equal quantum numbers, the point of this being to ensure that although the mean occupation number $\bar{\nu}_R$ may be small, the total group occupation number

$$\mathcal{N}_R = G_R \bar{\nu}_R = \sum_{r:R\{r\}=R} \nu_r \tag{6.1}$$

will be large compared with unity. This makes it possible to replace the exact Bose statistical formula

$$\exp\{S\} = \prod_R \frac{(G_R + \mathcal{N}_R - 1)!}{(G_R - 1)!\,\mathcal{N}_R!} \ , \tag{6.2}$$

for the total entropy S of the whole system (in units such that Boltzmann's constant is set to unity) by the corresponding Stirling formula

$$S = \sum_R G_R [(\bar{\nu}_R + 1)\ln\{\bar{\nu}_R + 1\} - \bar{\nu}_R \ln \bar{\nu}_R] \ . \tag{6.3}$$

Introducing the dimensionless state functions $\sigma\{r\} = \sigma\{R\{r\}\}$ defined by setting

$$\bar{\nu}_R = \frac{1}{e^{\sigma\{R\}} - 1} \ , \qquad \sigma\{R\} = \ln\left\{\frac{\bar{\nu}_R + 1}{\bar{\nu}_R}\right\} \ , \tag{6.4}$$

formula (6.3) can be usefully rewritten as

$$S = \ln \mathcal{Z} + \sum_R \mathcal{N}_R \sigma\{R\} \ , \tag{6.5}$$

where the partition function $\mathcal{Z}$ is defined as a sum over all admissible combinations $\{\nu_r\}$ of individual occupation numbers by

$$\mathcal{Z} = \sum_{\{\nu_r\}} \exp\left\{\sum_r \nu_r \, \sigma\{r\}\right\} \ , \tag{6.6}$$

from which the logarithm required in (6.5) is obtained simply as

$$\ln \mathcal{Z} = \sum_R G_R \ln\{\bar{\nu}_R + 1\}. \tag{6.7}$$

The corresponding variation formulas are thus obtained even more simply as

$$\delta \ln \mathcal{Z} = -\sum_r \nu_r \delta\sigma\{r\} \ , \qquad \delta S = \sum_r \sigma\{r\}\, \delta\nu_r \ . \tag{6.8}$$

Having carried out these routine preliminary steps, one can immediately obtain the thermal equilibrium state specified by maximizing the entropy subject to possible constraints on the total occupation number $\mathcal{N}$ and the total energy momentum covector $\mathcal{P}_\rho$ as defined in terms of the corresponding microscopic energy momentum covectors p_ρ by

$$\mathcal{N} = \sum_R \mathcal{N}_R = \sum_r \nu_r \ , \qquad \mathcal{P}_\rho = \sum_r \nu_r \, p_\rho\{r\} \ . \tag{6.9}$$

In order to satisfy the ensuing requirement

$$\delta S + \alpha \delta \mathcal{N} + \beta^\rho \delta \mathcal{P}_\rho = 0 \ , \tag{6.10}$$

where the scalar α and the four vectorial components β^ρ are Lagrange multipliers, it evidently follows from (6.8) that the solution for the equilibrium distribution must be given simply by

$$\sigma = -\alpha - \beta^\rho p_\rho \ , \tag{6.11}$$

which by (6.5) implies the thermodynamic relation

$$S - \ln \mathcal{Z} = -\alpha \mathcal{N} - \beta^\rho \mathcal{P}_\rho \ . \tag{6.12}$$

In the case of phonons, as in that of the more familiar example of Planckian photons, there is no conservation law imposing any constraint on the total occupation number so the multiplier α will vanish, while the vector β^ρ will give the temperature Θ and relative flow velocity components v^i (i=1,2,3) with respect to the chosen reference frame according to the specifications

$$\alpha = 0 \ , \qquad \beta^0 = \frac{1}{\Theta} \ , \qquad \beta^i = \frac{1}{\Theta} v^i \ . \tag{6.13}$$

For the actual evaluation of the distribution thus obtained it is of course convenient to work in the continuum limit for which the summation goes over to a phase space integration in the simple form

$$\sum_R G_R = \sum r \rightarrow \mathcal{V} \int \frac{dp_1\, dp_2\, dp_3}{(2\pi\hbar)^3} \ , \quad \mathcal{V} = \int dx^1\, dx^2\, dx^3 \ . \tag{6.14}$$

This is the first point at which the phonon case with which we are concerned here deviates from the more familiar photon gas, in which an extra factor of 2 would be required to allow for the existence of two (transverse) polarization modes, whereas in the phonon case there is only a single (longitudinal) polarization mode.

The only integration for which (6.14) is actually needed in practice is that for the logarithm of the partition function $\mathcal{Z}$ as given by (6.7), since once this is known the other thermodynamic quantities involved will be obtainable from it by straightforward partial differentiation procedures. Defining the *grand potential* energy Ω, the *free* energy F, and the ordinary energy U of the system by formulas of the standard form [3]

$$\Omega = -\Theta \ln \mathcal{Z} \ , \qquad F = U - \Theta S \ , \qquad U = -\mathcal{P}_0 \tag{6.15}$$

(which differs, however, from the terminology of Khalatnikov [4] who uses the symbol F in place of Ω), the relation (6.12) can be rewritten as

$$F - \Omega = v^i \mathcal{P}_i \ , \tag{6.16}$$

while the variation laws (6.8) (which are of course to be understood as determined with respect to fixed boundary conditions, and hence in particular for a constant value of the volume $\mathcal{V}$) give rise to corresponding variation formulas of the form

$$\begin{aligned} \delta\Omega &= -S\delta\Theta - \mathcal{P}_i\,\delta v^i \ , \\ \delta F &= -S\delta\Theta + v^i\delta\mathcal{P}_i \ , \\ \delta U &= \Theta\delta S + v^i\delta\mathcal{P}_i \ . \end{aligned} \tag{6.17}$$

Although it is simpler so far as polarization is concerned, on the other hand the phonon case is more complicated than that of photons in that the presence of the background medium implies a breakdown of Lorentz invariance which is expressed by the need to use not the ordinary metric but the phonic metric (4.11) in the relevant nullity restriction (4.12). In order to be able to satisfy this condition by expressing the phonon energy ϵ by the simple formula

$$\epsilon = -p_0 = c_{\rm P} p \ , \qquad p^2 = p_1^2 + p_2^2 + p_3^2, \tag{6.18}$$

where $c_{\rm P}$ is the zero temperature sound speed as given by (4.10), we must now restrict the choice of reference frame to be one in which the underlying superfluid is at rest. This contrasts with the case of a photon gas, which is subject to an analogous formula, with c replacing $c_{\rm P}$, in an arbitrarily boosted frame. Further restricting the frame by choosing space axes aligned with the relative flow direction, one obtains the logarithm of the partition function $\mathcal{Z}$, in the explicit form

$$\ln \mathcal{Z} = -\mathcal{V} \int \frac{2\pi p^2 \sin\theta\, d\theta\, dp}{(2\pi\hbar)^3} \ln\left\{1 - e^{-\sigma}\right\} \ , \tag{6.19}$$

with

$$\sigma = -\beta^\rho p_\rho = \frac{c_{\rm P} p - v p_3}{\Theta} \ , \tag{6.20}$$

where

$$v^1 = v^2 = 0 \ , \qquad v^3 = v \ , \qquad p_3 = p\cos\theta \ . \tag{6.21}$$

VII. THERMODYNAMICS OF THE PHONON GAS

To relate the thermodynamic quantities obtained in the preceding section to the superfluid continuum variables described in the earlier sections, we start by introducing the generalized pressure function ψ determined by the grand potential Ω of the phonon gas, together with the entropy density s_S in the superfluid frame, which is of course to be identified with the component given by (5.11), according to the specifications

$$\Omega = -\mathcal{V}\psi \ , \qquad S = \mathcal{V} s_S, \tag{7.1}$$

while the corresponding preferred frame components $\mathcal{E}$ and Π of the phonon energy and momentum density are specified by setting

$$\mathcal{P}_0 = -\mathcal{V}\mathcal{E} \ , \qquad \mathcal{P}_3 = \mathcal{V}\Pi \ , \qquad \mathcal{P}_2 = \mathcal{P}_1 = 0 \ . \tag{7.2}$$

With these definitions the global thermodynamic equilibrium condition (6.12) can be rewritten in terms of the corresponding local field variables as

$$\mathcal{E} + \psi = \Theta\, s_S + \Pi\, v \ , \tag{7.3}$$

while by (6.17) the fundamental entropy maximization condition (6.10) gives the relevant local version of the first law of thermodynamics in the form

$$\delta\mathcal{E} = \Theta\, \delta s_S + v\, \delta\Pi \ , \tag{7.4}$$

whose conjugate, by (7.3), is evidently

$$\delta\psi = s_S\, \delta\Theta + \Pi\, \delta v \ . \tag{7.5}$$

Using this last formula one can obtain the values of s_S and Π as functions of Θ and v by partial differentiation of the generalized pressure function ψ, which is obtained from (6.19) in the form

$$\psi = \left(\frac{3}{\tilde\hbar c_{\rm P}}\right)^3 \left(\frac{\Theta}{4}\right)^4 \left(1 - \frac{v^2}{c_{\rm P}^2}\right)^{-2}, \tag{7.6}$$

where $\tilde\hbar$ is a constant that is given, within a numerical factor that is extremely close to unity, by the usual Dirac Planck constant $\hbar$, its exact expression being

$$\tilde\hbar = \left(\frac{1215}{128\pi^2}\right)^{1/3} \hbar = \frac{9}{4\pi}\left(\frac{5\pi}{6}\right)^{1/3} \hbar \simeq 0.99\hbar \ . \tag{7.7}$$

One thus obtains the expressions

$$s_S = \left(\frac{3\Theta}{4\tilde\hbar c_{\rm P}}\right)^3 \left(1 - \frac{v^2}{c_{\rm P}^2}\right)^{-2} \tag{7.8}$$

and

$$\Pi = \left(\frac{3}{4\tilde\hbar}\right)^3 \left(\frac{\Theta}{c_{\rm P}}\right)^4 \left(1 - \frac{v^2}{c_{\rm P}^2}\right)^{-3} \frac{v}{c_{\rm P}} \ , \tag{7.9}$$

which are formally identical with the analogous expressions as originally derived in a nonrelativistic framework [4].

It is to be remarked that the temperature vector (6.13) can be used to rewrite (7.6) in the neater form

$$\psi = \left(\frac{3}{4\tilde\hbar}\right)^3 c_{\rm P} \left(2\mathcal{G}^{-1}_{\ \rho\sigma}\beta^\rho\beta^\sigma\right)^{-2} \ , \tag{7.10}$$

where $\mathcal{G}^{-1}_{\rho\sigma}$ is the covariant inverse of the sonic metric given by (4.11).

VIII. CONCLUSION: THE EQUATION OF STATE FOR COOL SUPERFLUID DYNAMICS

We can use the expressions obtained in the previous section to evaluate the "normal" mass density contribution ρ_N as introduced in (5.10) in a manner that is formally consistent with the traditional definition [4] by identifying the momentum density $\varPi$ with the corresponding mixed component of the stress momentum energy density tensor in the preferred superfluid frame defined by μ_ρ according to the specification

$$T^{\circ}{}_{3} = \varPi = \rho_N v \ . \tag{8.1}$$

This leads to a result that is expressible in the form

$$\rho_N = \frac{4\tilde{\hbar}}{3c_{\rm P}} s_S^{4/3} \left(1 - \frac{v^2}{c_{\rm P}^2}\right)^{-1/3} . \tag{8.2}$$

We are now in a position to obtain the equation of state for the required cool superfluid Lagrangian $\mathcal{L}$ by integrating the partial differential equation

$$\frac{1}{s}\frac{\partial \mathcal{L}}{\partial s} = -\frac{c^2}{s_S^2}\rho_N \tag{8.3}$$

obtained from (5.9) and (5.13). This leads to the principal result of this work which is the formula

$$\mathcal{L} = P\{\mu\} - 3\psi\{\mu, y, s\} \ , \tag{8.4}$$

where ψ is the generalized pressure function of the phonon gas, which can be seen from (7.10) to be given by

$$3\psi = \tilde{\hbar} c_{\rm P}^{-1/3} |\mathcal{G}^{-1}_{\rho\sigma} s^\rho s^\sigma|^{2/3} \ . \tag{8.5}$$

Since the operation $\partial/\partial s$ is defined in terms of the scalar variables μ, y, and s, the verification that (8.4) satisfies (8.3) requires the conversion of (8.5) into terms of these three scalars. This can be done using the explicit expression

$$\mathcal{G}^{-1}_{\rho\sigma} = g_{\rho\sigma} + \frac{1}{c^2}\left(1 - \frac{c_{\rm P}^2}{c^2}\right) u_\rho u_\sigma \ , \tag{8.6}$$

for the covariant version of the phonic metric (4.11), which gives

$$\mathcal{G}^{-1}_{\rho\sigma} s^\rho s^\sigma = \left(c^2 - c_{\rm P}^2\right)\left(\frac{y^2}{\mu}\right)^2 - c^2 s^2 \ . \tag{8.7}$$

The solution of (8.3) is of course not unique (since one can always add in any function of μ and y) but (8.4) is the only solution compatible with the boundary requirements that it should go over correctly to the cold limit value (4.8) when both s and y vanish, and that the energy in the static limit $v = 0$ should differ from that of the cold solution with the same particle number density n by an amount given by the phonon gas contribution $\mathcal{E}$, which in the static limit is the same as 3ψ.

We conclude by remarking that the description of ψ as the generalized pressure function of the phonon gas is justified by the fact that the basic cool equation of state formula (8.4) translates into terms of the total generalized pressure function Ψ, as defined by (3.5), simply as

$$\Psi = P + \psi \ , \tag{8.8}$$

where the appropriate expression for the thermal correction term ψ is

$$\psi = \frac{c_{\rm P}}{4}\left(\frac{3}{4\tilde{\hbar}}\right)^3 \left(\mathcal{G}^{\rho\sigma}\Theta_\rho\Theta_\sigma\right)^2 \ , \tag{8.9}$$

in which it is to be recalled that (as described in Sec. IV) $c_{\rm P}$ and $\mathcal{G}^{\rho\sigma}$ are given as functions (only) of the superfluid momentum covector μ_ρ in a manner determined by the original equation of state for the pressure P in the zero temperature limit. The corresponding expression for the relationship between the entropy current vector s^ρ, the temperature vector β^ρ, and the thermal momentum covector Θ_ρ is

$$s^\rho = 4\psi\,\beta^\rho = 2\left(\frac{3}{4\tilde{\hbar}}\right)^{3/2} \left(c_{\rm P}\psi\right)^{1/2} \mathcal{G}^{\rho\sigma}\Theta_\sigma \ . \tag{8.10}$$

The first of these relations can be used to verify that the temperature vector β^ρ introduced as a Lagrange multiplier in Sec. VI agrees with the earlier definition (2.6).

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APPENDIX: EVALUATION OF THE FIRST AND SECOND SOUND SPEEDS

To obtain the characteristic speeds of propagation of small perturbations, it suffices to follow the lines already developed in the context of more general multiconstituent fluid theory [10] using a technique originally due to Hadamard. The speeds in question are those characterizing the set of possible directions for the normal covector, λ_ρ say, of a characteristic hypersurface of the system. The Hadamard analysis postulates that the relevant independent dynamical variables, in this case s^ρ and μ_ρ, should be continuous across the characteristic hypersurface, but that their derivatives have infinitesimal discontinuities that will be expressible in the form

$$\left[\nabla_\rho s^\sigma\right] = \lambda_\rho \hat{s}^\sigma \ , \qquad \left[\nabla_\rho \mu_\sigma\right] = \lambda_\rho \hat{\mu}_\sigma \ , \tag{A1}$$

in terms of a corresponding set of infinitesimal discontinuity amplitudes $\hat{s}^\rho$ and $\hat{\mu}^\rho$. For the dependent variables n^ρ and Θ_ρ, one then obtains the analogous relations

$$\left[\nabla_\rho n^\sigma\right] = \lambda_\rho \hat{n}^\sigma \ , \qquad \left[\nabla_\rho \Theta_\sigma\right] = \lambda_\rho \hat{\Theta}_\sigma \ . \tag{A2}$$

For a generic Lagrangian of the form (5.7) the independent dynamical variables μ_ρ and s^ρ will determine the dependent dynamical variables n^ρ and Θ_ρ by the relation (5.8), which can be rewritten succinctly as

$$n^\rho = B\mu^\rho - As^\rho\ , \qquad \Theta^\rho = Cs^\rho + A\mu^\rho, \tag{A3}$$

with

$$B = 2\frac{\partial\mathcal{L}}{c^2\partial\mu^2}\ , \qquad C = -2\frac{\partial\mathcal{L}}{c^2\partial s^2}\ , \qquad A = -\frac{\partial\mathcal{L}}{c^2\partial y^2}\ . \tag{A4}$$

It follows that the independent discontinuity amplitudes $\hat{\mu}_\rho$ and $\hat{s}^\rho$ will determine the corresponding dependent discontinuity amplitudes $\hat{n}^\rho$ and $\hat{\Theta}_\rho$ by an analogous linear relation of the form

$$\hat{n}_\rho = B_{\rho\sigma}\hat{\mu}^\sigma - A_{\sigma\rho}\hat{s}^\sigma, \qquad \hat{\Theta}_\rho = C_{\rho\sigma}\hat{s}^\sigma + A_{\rho\sigma}\hat{\mu}^\sigma\ , \tag{A5}$$

with

$$B_{\rho\sigma} = Bg_{\rho\sigma} - 2\frac{\partial B}{c^2\partial\mu^2}\mu_\rho\mu_\sigma + 4\frac{\partial A}{c^2\partial\mu^2}\mu_{(\rho}s_{\sigma)} + \frac{\partial A}{c^2\partial y^2}s_\rho s_\sigma\ ,$$

$$C_{\rho\sigma} = Cg_{\rho\sigma} - 2\frac{\partial C}{c^2\partial s^2}s_\rho s_\sigma - 4\frac{\partial A}{c^2\partial s^2}s_{(\rho}\mu_{\sigma)} - \frac{\partial A}{c^2\partial y^2}\mu_\rho\mu_\sigma\ , \tag{A6}$$

$$A_{\rho\sigma} = Ag_{\rho\sigma} - 2\frac{\partial A}{c^2\partial\mu^2}\mu_\rho\mu_\sigma - \frac{\partial A}{c^2\partial y^2}\mu_\rho s_\sigma - \frac{\partial C}{c^2\partial y^2}s_\rho s_\sigma - 2\frac{\partial C}{c^2\partial\mu^2}s_\rho\mu_\sigma\ .$$

The required characteristic equations are to be obtained by substituting the formulas (A1) and (A2) in the discontinuities of the relevant dynamical equations, which are the integrability condition (3.1) for (2.1), together with (2.2), (2.3), and (2.4). The last of these gives

$$s^\rho\lambda_{[\rho}\hat{\Theta}_{\sigma]} = 0\ , \tag{A7}$$

the flux conservation equations (2.2) and (2.3) give

$$\lambda_\rho\hat{n}^\rho = 0\ , \tag{A8}$$

and

$$\lambda_\rho\hat{s}^\rho = 0\ , \tag{A9}$$

while finally the irrotationality condition (3.1) gives

$$\lambda_{[\rho}\hat{\mu}_{\sigma]} = 0\ , \tag{A10}$$

which is interpretable as meaning that $\hat{\mu}_\rho$ must be proportional to λ_ρ. The weaker condition (A7) gives rise to two qualitatively different possibilities, the one of interest in relation to the first and second sound modes under consideration here being that $\hat{\Theta}_\rho$ should also be proportional to λ_ρ. This condition is expressible by replacing (A7) by the formal analogue of (A10), i.e.,

$$\lambda_{[\rho}\hat{\Theta}_{\sigma]} = 0\ . \tag{A11}$$

The alternative possibility is the trivial case of a thermal shearing mode as characterized by $s^\rho\lambda_\rho = 0$ and $s^\rho\hat{\Theta}_\rho = 0$. In the more general nonsuperfluid case [10] the ordinary particle constituent could also have a shear type mode, but this second kind of trivial mode is excluded in the superfluid case by the irrotationality condition (A10). Leaving aside this trivial case, we now restrict our attention to the sound type modes characterized by the restriction of (A7) to (A11).

To deal with the ensuing system of characteristic equations (A8), (A9), (A10), and (A11), we now put it in a more explicit form by the use of local "superfluid frame" Minkowski coordinates $\{x^\rho\} = \{x^0, x^1, x^2, x^3\}$ aligned with the relative flow direction at a particular point under consideration, so that the independent dynamical variables will be given there by

$$\{\mu^\rho\} = \mu\{1,0,0,0\}, \qquad \{s^\rho\} = s_S\{1,v,0,0\}. \tag{A12}$$

There will be no loss of generality in taking the characteristic covector to have the form

$$\{\lambda_\rho\} = \{-u,\ \cos\theta,\ \sin\theta,\ 0\}\ , \tag{A13}$$

where θ is the angle between the relative flow direction and the direction of propagation of the perturbation, and u is the characteristic velocity whose evaluation is the ultimate objective of the exercise. The ensuing set of characteristic equations can be organized in three subsets. First there is a "superfluid" subset obtained from (A8) and (A10) in the form

$$u\,\hat{n}^0 - \cos\theta\,\hat{n}^1 - \sin\theta\,\hat{n}^2 = 0\ , \tag{A14}$$

$$\cos\theta\,\hat{\mu}_0 + u\,\hat{\mu}_1 = 0\ , \qquad \sin\theta\,\hat{\mu}_1 - \cos\theta\,\hat{\mu}_2 = 0\ . \tag{A15}$$

Next there is a "thermal" (or "normal") subset obtained from (A9) and (A11) in the form

$$u\,\hat{s}^0 - \cos\theta\,\hat{s}^1 - \sin\theta\,\hat{s}^2 = 0\ , \tag{A16}$$

$$\cos\theta\,\hat{\Theta}_0 + u\,\hat{\Theta}_1 = 0\ , \qquad \sin\theta\,\hat{\Theta}_1 - \cos\theta\,\hat{\Theta}_2 = 0\ . \tag{A17}$$

Finally there is an orthogonal subset containing the remaining information from (A10) and (A11) in the form

$$\hat{\mu}_3 = 0\ , \qquad \hat{\Theta}_3 = 0\ . \tag{A18}$$

This last set (A18) will always decouple and can be solved separately so as to provide the conclusion that orthogonally to the plane defined by the relative flow and propagation directions, not just the momentum discontinuity amplitudes but also those of the currents must vanish: $\hat{n}^3 = \hat{s}^3 = 0$.

In the limit when the temperature Θ and the entropy magnitude $s = \mathcal{O}\{\Theta^3\}$ go to zero, one is left only with the first subset of characteristic equations (A14), (A15) corresponding to ordinary sound modes, but when entropy is present it will evidently be necessary to take account also of the second "normal" subset (A16), (A17). In general this second set will be coupled with the first by the cross terms in (A6), so that one will obtain a rather intractable quartic characteristic equation for the propagation speed u. However, it transpires that in the "cool" limit with which the present work is concerned, i.e., to lowest order in the temperature Θ, the first and second subsets will decouple, giving a pair of quadratic equations that can easily be solved to give the respective "first" and "second" sound speeds in explicit form.

To see how this comes about, we proceed by evaluating the coefficients involved using the cool equation of state as specified by substitution of (8.7) in (8.5). The lowest order coefficients can easily be worked out exactly as

$$C = 3(c_{\rm P}^2 - v^2)\mathcal{Q}\ , \qquad C_{00} = (3v^2 - c_{\rm P}^2)c_{\rm P}^2\mathcal{Q}\ ,$$

$$C_{11} = (3c_{\rm P}^2 - v^2)\mathcal{Q}\ , \qquad C_{01} = -2vc_{\rm P}^2\mathcal{Q}\ , \qquad \text{(A19)}$$

in terms of a factor

$$\mathcal{Q} = \frac{4\tilde{\hbar}}{9}(c_{\rm P}^2 - v^2)^{-4/3}c_{\rm P}^{-1/3}s_S^{-2/3}\ , \qquad \text{(A20)}$$

which is unbounded, growing proportionally to Θ^{-2}, in the limit as $\Theta \to 0$. The coefficients of next order are bounded in this limit, but are functionally more complicated. Nevertheless, to the order of accuracy that is needed they are simply expressible in terms of the dilation coefficient Φ^2 defined by (3.10) as

$$B = \Phi^2, \;\; B_{00} = -\frac{c^4}{c_{\rm P}^2}\Phi^2 + \mathcal{O}\{\Theta^4\}, \;\; B_{11} = \Phi^2 + \mathcal{O}\{\Theta^4\}\ . \qquad \text{(A21)}$$

To the required order of accuracy, the dilation coefficient itself (which effectively controls the dynamics of the zero temperature limit [14]) will be given simply by

$$\Phi^2 = \frac{n}{\mu} + \mathcal{O}\{\Theta^4\}\ . \qquad \text{(A22)}$$

Those of the remaining coefficients that are not exactly zero are not just bounded but are characterized more strictly by

$$A = \mathcal{O}\{\Theta\}\ , \qquad A_{00} = \mathcal{O}\{\Theta\}\ , \qquad A_{11} = \mathcal{O}\{\Theta\}\ ,$$
$$A_{01} = \mathcal{O}\{\Theta\}\ , \qquad A_{10} = \mathcal{O}\{\Theta\}\ , \qquad B_{01} = \mathcal{O}\{\Theta^4\}\ . \qquad \text{(A23)}$$

Thus they all tend to zero in the limit $\Theta \to 0$, and so will drop out as far as the present calculation is concerned.

Since the combination $\mathcal{Q}s_S^{2/3}$ is bounded, it can be seen from (A23) that the characteristic equation obtained as a condition of vanishing determinant for the combined system (A14), (A15), (A16), (A17) will be expressible to lowest order as a product of a pair of quadratic factors

$$\mathcal{F}_{\rm I} = B_{00}u^2 + c^2(B_{11}\cos^2\theta + B\sin^2\theta)\ , \qquad \text{(A24)}$$
$$\mathcal{F}_{\rm II} = \big[C(C_{11}u^2 + 2C_{01}\cos\theta\, u + C_{00}\cos^2\theta) + (C_{00}C_{11} - C_{01}{}^2)\sin^2\theta\big]s_S^{4/3}$$

in the form

$$\mathcal{F}_{\rm I}\mathcal{F}_{\rm II} = \mathcal{O}\{\Theta^4\}\ , \qquad \text{(A25)}$$

in which the left-hand side is finite while the right-hand side tends to zero as $\Theta \to 0$. At lowest order we thus obtain an effective decoupling into two factors one or the other of which must vanish separately. The first alternative, $\mathcal{F}_{\rm I} = \mathcal{O}\{\Theta^4\}$, will characterize ordinary "first" sound modes, and is expressible using the explicit expressions (A21) simply as

$$u^2 - c_{\rm P}^2 = \mathcal{O}\{\Theta^4\}\ . \qquad \text{(A26)}$$

It can thus be seen that the "first" sound speed will be given, independently of the propagation angle θ, by

$$u = \pm c_{\rm P} + \mathcal{O}\{\Theta^4\}\ , \qquad \text{(A27)}$$

consistently with the original interpretation of $c_{\rm P}$ as the ordinary sound speed in the zero temperature limit. The more interesting alternative, $\mathcal{F}_{\rm II} = \mathcal{O}\{\Theta^4\}$, will characterize "second" sound modes, and is expressible using the formulas (A19) as

$$(3c_{\rm P}^2 - v^2)u^2 - 4c_{\rm P}^2\cos\theta\, vu - c_{\rm P}^4 + c_{\rm P}^2(1 + 2\cos^2\theta)v^2 = \mathcal{O}\{\Theta^4\}\ . \qquad \text{(A28)}$$

The "second" sound speed is thereby found to be given by

$$u = \frac{2c_{\rm P}^2 v\cos\theta \pm c_{\rm P}\sqrt{(c_{\rm P}^2 - v^2)\left[3c_{\rm P}^2 - (1 + 2\cos^2\theta)v^2\right]}}{3c_{\rm P}^2 - v^2} + \mathcal{O}\{\Theta^4\}\ . \qquad \text{(A29)}$$

It is to be remarked that when the two constituents are relatively at rest, i.e., when $v = 0$, this expression for the second sound speed reduces just to

$$u = \pm\frac{c_{\rm P}}{\sqrt{3}} + \mathcal{O}\{\Theta^4\}\ , \qquad \text{(A30)}$$

whose form has been well known since the original work of Landau [3] in the Newtonian limit characterized by $c_{\rm P}^2 << c^2$. What is new here is the demonstration that this formula can be retained without any change for a relativistic superfluid in which the sound speed $c_{\rm P}$ may be comparable with the light speed c.

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