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Curvature corrections to dynamics of domain walls

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The most usual procedure for deriving curvature corrections to effective actions for topological defects is subjected to a critical reappraisal. A logically unjustified step (leading to overdetermination) is identified and rectified, taking the standard domain wall case as an illustrative example. Using the appropriately corrected procedure, we obtain a new exact (analytic) expression for the corresponding effective action contribution of quadratic order in the wall width, in terms of the intrinsic Ricci scalar R and the extrinsic curvature scalar K. The result is proportional to $cK^2 - R$ with the coefficient given by $c \simeq 2$. The resulting form of the ensuing dynamical equations is obtained in terms of the second fundamental form and the Dalembertian of its trace, K. It is argued that this does not invalidate the physical conclusions obtained from the "zero rigidity" ansatz $c = 0$ used in previous work.

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I. INTRODUCTION: THE IMPORTANCE OF DEFECT DYNAMICS

Topological and other vacuum defects are of interest and importance in many areas of physics today. In high-energy physics they generically occur during a symmetry-breaking process where different parts of a medium choose different minimal energy configurations, or vacua, and the noncompatibility of these different vacua forces a sheet, line, or point of energy in which the vacua meet at a *defect*, where the relevant vacuum order parameter becomes indeterminate. The phrase topological defect is used to embody the idea that it is the topology of the vacuum that simultaneously allows the formation, and prevents dissipation, of these objects; but a defect need not be topological. Many instances are known where a defect may be stable dynamically (i.e., classically, due to energy considerations) but not topologically; for example, semilocal defects [1] fall into this category. A defect can even be "topological" and unstable, as in the case of textures [2], but nonetheless of physical importance.

In cosmology, there are two main concerns when considering defects. One is their gravity and the other their dynamics. Any theory concerning large-scale transport of matter (such as in galaxy formation) must be able to allow for, constrain, or even rule out, the presence of strongly self-gravitating objects. But the primary concern still is dynamics. There may be defects (such as low-mass cosmic strings [3]) that have little impact gravitationally when in straight static configurations but that become gravitationally important when strongly curved, crinkled, or compact (as in the case of string loops). Questions of dynamics may also have an essential influence on decay rates. It is therefore worthwhile to study the purely dynamical aspects as a subject in its own right, leaving gravitational aspects to be included in subsequent investigations. It is this strategy that will be followed in the present analysis, whose scope will be limited to defects in a flat Minkowski background in the interest of conceptual clarity and mathematical simplicity.

Attempts to derive effective actions or equations of motion for topological defects have commonly focused on the strong-coupling limit, meaning that of large values of the coupling coefficient λ of the relevant Higgs field. In this limit, the defect becomes infinitesimally thin and effectively decouples from the other (infinitely massive) particles in the field theory. The study of the effective motion of topological defects has been extended [4–12] away from the limit $\lambda \to 0$ to cases for which the thickness is small but not exactly zero. The resultant effective action generically contains a "zero-thickness" term proportional to the area of the defect, and extrinsic curvature terms that appear at quadratic order in the thickness. It is the controversy about the way to evaluate these second-order terms that has prompted the present work.

While the earliest investigations [4–6] agreed in predicting that such extrinsic curvature terms should definitely exist, they failed to reach consensus, not only about their amplitudes and their completeness (meaning whether or not other "twist" terms of the same order were needed as well) but even about whether their signs cor-

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responded to "rigidity" or "antirigidity." The confusion became worse [7] after the publication of many subsequent studies [8–12] predicting or assuming zero rigidity, meaning the absence of any quadratic order corrections except for the term proportional to the *intrinsic* curvature term R (which in the case of a string is a pure divergence having no effect on the motion).

The present work makes a fresh start on the basis of a critical examination of the procedure used in the preceding work [4–10] in the simplest case, namely, that of a domain wall (for which the question of a twist contribution does not even arise). We adopt a "classical" approach that is appropriate to a strong-coupling limit, $\lambda \to \infty$. However, the validity of our analysis is by no means restricted to this limit, but extends to moderate and even small values of λ. Our results are applicable quite generally to any limit in which the wall curvature scale, L say, is large compared to the wall width, l say, even when the latter is not infinitesimal (though the method will not describe the interaction of the wall with the underlying scalar field).

We find that the approach used in the original investigations [4–6] was essentially sound, their discrepancies being mainly due to the difficulty of being sure that no terms were overlooked. However, while justified in having doubted the detailed conclusions of these pioneering investigations, the subsequent papers [8–10] strayed from strict logic in imposing an unduly restrictive simplification ansatz.

The present work corrects this step, providing a new evaluation of the second-order curvature contribution to the off-shell action, in the case of a simple domain wall. The advantage of considering the domain wall becomes apparent at this stage, for we are able to perform all operations analytically, obtaining exact values for all the parameters in the second-order effective action. It is found that the internal mechanics of the wall is characterized by a well-defined and *strictly negative* rigidity coefficient. This does not invalidate the use in the previous work [8–12] of the corresponding zero rigidity model as a permissible (though not obligatory) second-order approximation because the effect on the dynamical equations of the rigidity term in question is of higher order.

II. THE SCALAR FIELD MODEL

The simplest relativistic domain-wall model in common use is based on a bosonic field theory consisting of a real scalar Φ whose self-interaction is governed by the Lagrangian

$$\hat{\mathcal{L}} = -\tfrac{1}{2}(\nabla_\mu \Phi)\nabla^\mu \Phi - \lambda(\Phi^2 - \eta^2)^2 \qquad (1)$$

for positive constants η and λ, in a $(D+1)$-dimensional background, with coordinates x^μ $(\mu = 0, 1, \ldots, D)$, and Lorentz signature $(-,+,\ldots,+)$ spacetime metric $g_{\mu\nu}$. In the present work this metric is postulated to be *flat* (which means that gravitational effects are neglected). The Lagrangian (1) gives the well-known field equation

$$\nabla_\mu \nabla^\mu \Phi - 4\lambda\Phi(\Phi^2 - \eta^2) = 0, \qquad (2)$$

which has two distinct homogeneous "vacuum" solutions, $\Phi = \pm\eta$. Positive and negative domains, as characterized, respectively, by $\Phi > 0$ and $\Phi < 0$, are separated by domain walls that are identified as hypersurfaces, with internal coordinates $\sigma^i, (i = 0, \ldots, D-1)$, on which $\Phi = 0$.

The simplest domain-wall solution is given by the static plane wall ansatz expressible in terms of Minkowski background coordinates x^μ by

$$x^i = \sigma^i, \quad x^D = 0, \quad \nabla_i \phi = 0. \qquad (3)$$

Writing $z = x^D$ for the last coordinate (the only one that is not ignorable), the field equation reduces in this case to

$$\frac{d^2\Phi}{dz^2} - 4\lambda\Phi(\Phi^2 - \eta^2) = 0. \qquad (4)$$

Subject to the convention that the positive ϕ domain should be given by positive z, this equation has a unique asymptotically vacuum solution, which is given by

$$\Phi = \eta\phi_{(0)}, \quad \phi_{(0)} = \tanh\{(\eta\sqrt{2\lambda}z)\}. \qquad (5)$$

By substituting this in (1) and integrating over z one obtains the constant effective action per unit measure of the world sheet that is taken as the basis of the (Dirac-type) thin-membrane model that is generally expected to provide a good macroscopic description of the dynamical behavior of the wall under conditions such that the relevant dynamical length scales L are all very large compared with the dimension

$$l = \frac{1}{\eta\sqrt{2\lambda}} \qquad (6)$$

that characterizes the thickness of the wall.

The question motivating the present work is how to include the corrections to the thin-membrane model that one would expect to be needed when the relevant dimensionless curvature magnitude

$$\epsilon = \frac{l}{L} \qquad (7)$$

is still small, but not entirely negligible, as it must be for the simple membrane approximation to be valid.

Starting off in the same way as in an earlier analysis [8] (which was more general the present one in so much as it included allowance for weak self-gravitation) what we want do is to consider configurations obtained by perturbing the standard solution in such a way that the coordinates parallel to the wall are no longer exactly but only approximately ignorable. In other words

$$l\frac{\partial\Phi}{\partial x^D} \sim 1, \quad l\frac{\partial\Phi}{\partial x^i} = O(\epsilon), \qquad (8)$$

in the limit of large values of the lengthscale $L = l/\epsilon$ characterizing variation in directions parallel to the world sheet of the domain wall.

In order to proceed with the calculation, we split quantities into their components perpendicular and parallel to the defect world sheet, Σ say, where Φ vanishes in the middle of the wall. This is done formally by utilizing a Gauss-Codazzi formalism, the details of which were developed in the earlier analysis [8] and are paraphrased here.

We take n^μ to be a unit geodesic normal vector field to Σ, and we generalize the coordinate z by defining it to be the proper length along the integral curves of n^μ. Each constant z surface then has unit normal n_μ, fundamental tensor $h_{\mu\nu}$ (the background projection of the intrinsic metric), and extrinsic curvature $K_{\mu\nu}$ defined by

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \quad K_{\mu\nu} = h^\rho_\mu \nabla_\rho n_\nu. \tag{9}$$

Using the Gauss-Codazzi formalism, the equations of motion for the wall can be written in "$D+1$" fashion:

$$\mathcal{L}_n h_{\mu\nu} = 2K_{\mu\nu}, \tag{10a}$$

$$\mathcal{L}_n K_{\mu\nu} = K_{\mu\rho}K^\rho_\nu, \tag{10b}$$

$$\mathcal{L}_n\mathcal{L}_n\Phi + K\mathcal{L}_n\Phi + D_iD^i\Phi - 4\lambda\Phi(\Phi^2 - \eta^2) = 0, \tag{10c}$$

where σ^i are taken to be coordinates on the wall, D_i is the derivative operator for the wall hypersurface, and $\mathcal{L}_n$ is the Lie derivative along the vector field n^μ.

III. THE APPROXIMATION SCHEME

The foregoing system is a complete *exact* set of equations for the geometry and fields of the model, which we now intend to analyze along the lines described above. This means that after scaling out the dimensional dependence on wall width and curvature, we shall make a power series expansion of the physical quantities in terms of ϵ, the ratio of the wall width to its radius of curvature. We therefore start by setting

$$u = \frac{z}{l}, \quad \Phi = \eta\phi, \quad K_{\mu\nu} = \frac{1}{L}\kappa_{\mu\nu}. \tag{11}$$

In terms of these new variables we have $\mathcal{L}_n = l^{-1}\partial/\partial u$, and hence, using the abbreviation

$$' \equiv \frac{\partial}{\partial u}, \tag{12}$$

we obtain

$$h'_{\mu\nu} = 2\epsilon\kappa_{\mu\nu}, \tag{13a}$$

$$\kappa'_{\mu\nu} = \epsilon\kappa_{\mu\rho}\kappa^\rho_\nu, \tag{13b}$$

$$\phi'' - 2\phi(\phi^2 - 1) + \epsilon\kappa\phi' + \epsilon^2 D_iD^i\phi = 0, \tag{13c}$$

which is the starting point for a rigorous expansion in powers of ϵ.

It is worth digressing at this point to address a misconception that has arisen as to the interpretation of u in the zero thickness limit. Formally, "setting $\epsilon = 0$" is interpretable as either letting the wall thickness vanish, or letting the wall become flat. It has been suggested that it is incorrect to expand quantities in ϵ when $\epsilon \to 0$ corresponds to the former limit, since in this limit fields become discontinuous [13]. However, in the limit $l \to 0$, the coordinate u, while having an infinite range, corresponds to an infinitesimal physical range, that range being $(0^-, 0^+)$ in z space. Thus, the coordinate u takes the step function in z space and "blows it up" to give a continuous interpolation between the vacua on either side of the infinitesimally thin wall. Thus this limit is singular only in the literary, rather than the mathematical, sense.

We now proceed by expressing the rescaled quantities as power series in ϵ (with coefficients that are functions of the coordinates $\{\sigma^i, u\}$) in the form

$$\begin{aligned} \phi &= \phi_{(0)} + \epsilon\phi_{(1)} + \frac{\epsilon^2}{2}\phi_{(2)} + O\{\epsilon^3\}, \\ h_{\mu\nu} &= h_{(0)\mu\nu} + \epsilon h_{(1)\mu\nu} + \frac{\epsilon^2}{2}h_{(2)\mu\nu} + O\{\epsilon^3\}, \end{aligned} \tag{14}$$

and

$$\kappa_{\mu\nu} = \kappa_{(1)\mu\nu} + \frac{\epsilon}{2}\kappa_{(2)\mu\nu} + \frac{\epsilon^2}{6}\kappa_{(3)\mu\nu} + O\{\epsilon^4\}. \tag{15}$$

Substituting such a power expansion into (13) gives a sequence of equations obtained by setting the coefficients of successive powers of ϵ to zero.

To zeroth order, the geometry is independent of u, and the field equation reduces to (4), which is automatically satisfied by using expression (5) for $\phi_{(0)}$, which, in terms of the rescaled coordinate u, is simply

$$\phi_{(0)} = \tanh u. \tag{16}$$

After the lowest-order requirement (4), the next (and the last that will be needed here) in the sequence of requirements obtained from (13c) is the one governing the first-order field $\phi_{(0)}$, which satisfies the dynamical equation

$$\phi''_{(1)} - 2(3\phi^2_{(0)} - 1)\phi_{(1)} = -\kappa_{(1)}\phi'_{(0)}. \tag{17}$$

The driving term on the right of this linearized perturbation equation can be seen to be proportional to the lowest-order coefficient in the expansion for the extrinsic curvature scalar.

IV. THE QUESTION OF FIELD REGULARITY ON THE DEFECT LOCUS

Up to this point the present analysis agrees completely with that of the previous work [8–10], which went on from here to make the *crucial observation* that unless the scalar curvature coefficient $\kappa_{(1)}$ vanishes on the wall, Eq. (17) has no solution that is regular and bounded over the whole range from $u = -\infty$ to $u = +\infty$. It can be deduced from this that freely moving domain walls satisfying the field equations must obey the condition

$$\kappa_{(1)} = 0. \tag{18}$$

This is exactly what is required for consistency with the thin (Dirac-type) membrane treatment of the dynamics in the extreme limit when L/l is very large, for which the dynamic equations are well known to consist just of the "harmonicity" condition to the effect that the trace of the membrane curvature scalar K should vanish.

It is at the next stage of the work that discord arises. The ultimate motive for the present work, as indeed for previous work, is the derivation of higher-order corrections to the simple Dirac membrane approximation. The obviously natural and generally agreed strategy for doing this is to try to apply the same kind of procedure that was used in the zeroth-order membrane treatment whereby the spacetime action integral

$$\mathcal{I} = \int \hat{\mathcal{L}}\sqrt{-\det g}\, d^{D+1}x \tag{19}$$

is expressed in the form

$$\mathcal{I} = \int \mathcal{L}\sqrt{-\det h\{\sigma\}}\, d^{D}\sigma, \tag{20}$$

in which the off-world-sheet degrees of field freedom are eliminated from the world-sheet hypersurface Lagrangian density $\mathcal{L}$, which is to be obtained by integrating the ordinary spacetime Lagrangian density $\hat{\mathcal{L}}$ over the remaining dimension parametrized by z that is suppressed in (20) after fixing the off-wall values of the field variables by the requirement that the off-wall field equations should be satisfied to the required degree of accuracy.

Where this paper departs from previous work [8–10] is in the use made of the *crucial observation* cited above: on the basis of the supposition that the solution of (17) should be regular and bounded over $u \in \mathbb{R}$, it was argued previously that (18) should indeed be satisfied, i.e., that $\kappa_{(1)}$ *must* vanish. This is, in essence, a requirement that the field equations should be satisfied not just *off* the perturbed worldsheet but even *on* it. If we were already trying to solve for the motion of the wall, this would be an eminently reasonable, and indeed necessary, step to take, but we have not yet reached that stage. The aim of the game at this stage is to try to find an effective wall action that will be varied later on to get the equations of motion of the wall location. We must therefore be careful that we solve, or eliminate, only those degrees of freedom that are external to the wall, maintaining the fully unrestrained "off-shell" character of those virtual modes corresponding to the degrees of freedom of the wall itself. The more severe requirement postulated in the previous work [8–10] is interpretable as demanding that the worldsheet should satisfy the relevant dynamical equation, namely, (18) in the present instance, which is clearly not consistent with the requirement that the off-shell world-sheet configuration in the action (20) should be freely variable. The premature imposition of the dynamical condition (18) resulted in the unjustified suppression of a potentially important contribution to the action that needs to be evaluated. In order to avoid premature imposition of the dynamical equation (18) when evaluating the action one *must not* try to satisfy the first-order field equation (17) continuously over the whole range extending through the defect locus Σ itself, where Φ vanishes, but only in the separate domains outside this locus.

V. EVALUATION OF THE LINEARIZED SOLUTION AND THE CORRESPONDING ACTION

It follows from the preceding considerations that the appropriate procedure is just to require that the field equation be satisfied separately in the positive Φ domain $0 < u < \infty$ and in the negative Φ domain $-\infty < u < 0$, i.e., off the defect locus. The boundary conditions localizing the defect $\Phi = 0$ at the middle of the wall, where $u = 0$ and imposing a vacuum state at infinity are expressible formally as

$$\lim_{u\to 0^\pm} \Phi = 0 \quad\Rightarrow\quad \phi_{(0)} \to 0,\ \phi_{(1)} \to 0, \ldots \quad \text{as } u \to 0^\pm, \tag{21a}$$

$$\lim_{u\to \pm\infty} \Phi = \pm\eta \quad\Rightarrow\quad \phi_{(0)} \to 1,\ \phi_{(1)} \to 0, \ldots \quad \text{as } u \to \pm\infty. \tag{21b}$$

Subject to the foregoing requirements, the linearized field equation (17) is uniquely soluble. The required solution is given, without any ambiguity at all, by

$$\phi_{(1)} = \kappa_{(1)} f, \tag{22}$$

where the dimensionless function f of u has the explicit analytic form

$$f\{u\} = \pm\frac{1}{2}\tanh\{u\} - \frac{1}{2} + \left(\frac{2}{3} \pm \frac{u}{2}\right)\frac{1}{\cosh^2\{u\}} - \frac{1}{6}\exp\{\mp 2u\}, \tag{23}$$

in which the upper and lower sign choices apply, respectively, to the positive and negative domains, so that f is even under reflection (see Fig. 1), i.e., $f\{u\} = f\{-u\}$. At the origin $u = 0$ separating the two domains this function is constructed so as to vanish, $f\{0\} = 0$, but its gradient there has a nonvanishing limit, $(df/du)|_0 = \pm\frac{4}{3}$ so that it has a discontinuity across the wall given by $[df/du]^{+}_{-} = \frac{8}{3}$.

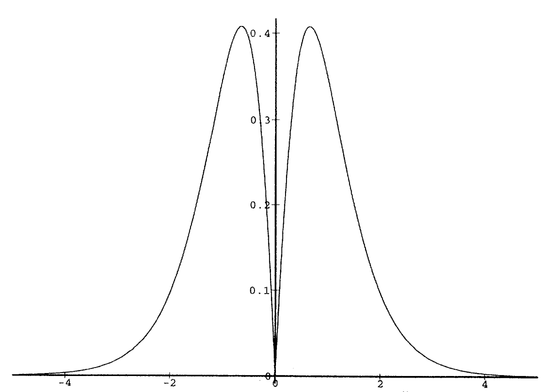

FIG. 1. Numerical evaluation of the function f.

In terms of the dimensionless function f, the solution for Φ itself (which is thus continuous but not continuously differentiable across the wall surface $z = 0$, see Fig. 2) will be given to the required order, with the dimensional parameters restored, by

$$\Phi = \eta \tanh\left\{\frac{z}{l}\right\} + \eta l K f\left\{\frac{z}{l}\right\} + O\{\epsilon^2\}. \quad (24)$$

In terms of the solution (24) it is now straightforward to evaluate the corresponding expression for the effective domain-wall surface Lagrangian $\mathcal{L}$ in (20), which will be given by

$$\mathcal{L} \equiv \mathcal{L}\{\sigma\} = \int \hat{\mathcal{L}} J\, dz, \quad (25)$$

where $\hat{\mathcal{L}}$ is the original Lagrangian density function (1) as evaluated for the solution (24), and J is the relevant Jacobean factor which is given by

$$J = \frac{\sqrt{-\det g}}{\sqrt{-\det h}|_{u=0}}. \quad (26)$$

Since the first-order contribution to $\mathcal{L}$ will vanish by the zeroth-order field equations, it is necessary to work out (25) to second order to get the lowest nontrivial corrections to the simple Dirac membrane treatment. To this degree of accuracy, the geometry is readily calculated from (13) as

$$h_{(0)\mu\nu} = h_{(0)\mu\nu}(\sigma), \quad h_{(1)\mu\nu} = 2u\kappa_{(1)\mu\nu}, \qquad h_{(2)\mu\nu} = 2u^2\kappa_{(1)\rho\nu}\kappa^{\rho}_{(1)\mu}, \quad (27a)$$

$$\kappa_{(1)\mu\nu} = \kappa_{(1)\mu\nu}\{\sigma\}, \quad \kappa_{(2)\mu\nu} = 2u\kappa_{(1)\mu\rho}\kappa^{\rho}_{(1)\nu}, \quad (27b)$$

and hence the Jacobean (26) is obtained via

$$\begin{aligned}\sqrt{-g} &= \sqrt{-g}|_{u=0} + \epsilon u(\sqrt{-g})'|_{u=0} + \epsilon^2(\sqrt{-g})''|_{u=0} + \cdots \\ &= \sqrt{-h}|_{u=0}\left[1 + \epsilon u\kappa_{(1)} + \frac{\epsilon^2 u^2}{2}(\kappa^2_{(1)} - \kappa_{(1)\mu\nu}\kappa^{\mu\nu}_{(1)}) + \cdots\right]\end{aligned} \quad (28)$$

as

$$\begin{aligned}J &= 1 + \epsilon u\kappa_{(1)} + \frac{\epsilon^2u^2}{2}(\kappa^2_{(1)} - \kappa_{(1)\mu\nu}\kappa^{\mu\nu}_{(1)}) + \cdots \\ &= 1 + \epsilon J_{(1)} + \frac{\epsilon^2}{2}J_{(2)} + \cdots.\end{aligned} \quad (29)$$

Since $\phi_{(0)}$ depends only on u and $\phi_{(1)}$ depends on the other coordinates σ^i only through $\kappa_{(1)}$ it follows that we have $\partial\phi_{(0)}/\partial x^i = 0$ and $\partial\phi_{(1)}/\partial x^i = O\{\epsilon\}$, and hence that $\partial\phi/\partial x^i = O\{\epsilon^2\}$. This implies that up to (and even beyond) the required degree of accuracy the Lagrangian density $\hat{\mathcal{L}}$ will be expressible simply as

$$\hat{\mathcal{L}} = -\lambda\eta^4[\phi'^2 + (\phi^2 - 1)^2] + O\{\epsilon^4\}. \quad (30)$$

We thus obtain

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_{(0)} + \epsilon\hat{\mathcal{L}}_{(1)} + \frac{\epsilon^2}{2}\hat{\mathcal{L}}_{(2)} + O\{\epsilon^3\}, \quad (31)$$

with

$$\hat{\mathcal{L}}_{(0)} = -\lambda\eta^4[\phi'^2_{(0)} + (\phi^2_{(0)} - 1)^2] = -2\lambda\eta^4\phi'^2_{(0)}, \quad (32a)$$

$$\begin{aligned}\hat{\mathcal{L}}_{(1)} &= -2\lambda\eta^4[\phi'_{(0)}\phi'_{(1)} + 2(\phi^2_{(0)} - 1)\phi_{(0)}\phi_{(1)}] \\ &= -2\lambda\eta^4(\phi'_{(0)}\phi_{(1)})',\end{aligned} \quad (32b)$$

and

$$\begin{aligned}\hat{\mathcal{L}}_{(2)} &= -2\lambda\eta^4[\phi'_{(0)}\phi'_{(2)} + \phi'^2_{(1)} + 2(\phi^2_{(0)} - 1)\phi_{(0)}\phi_{(2)} \\ &\quad + 2(3\phi^2_{(0)} - 1)\phi^2_{(1)}] \\ &= -2\lambda\eta^4[(\phi'_{(0)}\phi_{(2)})' + (\phi'_{(1)}\phi_{(1)})' + \kappa_{(1)}\phi_{(1)}\phi'_{(0)}],\end{aligned} \quad (32c)$$

using the field equations (4) and (17).

Using these expressions to simplify the corresponding expansion

$$\hat{\mathcal{L}}J = \hat{\mathcal{L}}_{(0)}\left(1 + \epsilon J_{(1)} + \frac{\epsilon^2}{2}J_{(2)}\right) + \hat{\mathcal{L}}_{(1)}(1 + \epsilon J_{(1)}) + \frac{\epsilon^2}{2}\hat{\mathcal{L}}_{(2)} + O\{\epsilon^3\}, \quad (33)$$

the required integrand $\hat{\mathcal{L}}J$ in (25) is found to be expressible to the required degree of accuracy by

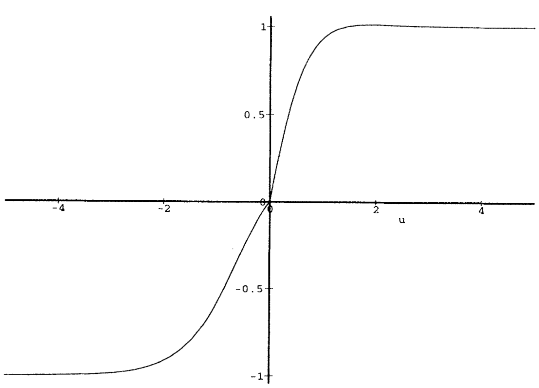

FIG. 2. Approximate evaluation of the dimensionless field ϕ for the (rather large) perturbation amplitude $lK = 0.5$.

$$\frac{\hat{\mathcal{L}}J}{\lambda\eta^4} = -2\phi_{(0)}'^2\left(1+\epsilon J_{(1)}+\frac{\epsilon^2}{2}J_{(2)}\right)+\epsilon^2\kappa_{(1)}\phi_{(0)}'\phi_{(1)} - \epsilon[\phi_{(0)}'(2\phi_{(1)}+\epsilon\phi_{(2)})+\epsilon(\phi_{(1)}'+2\kappa_{(1)}\phi_{(0)}u)\phi_{(1)}]'+O\{\epsilon^3\} \quad (34)$$

in each of the separate domains $-\infty < u < 0$ and $0 < u < \infty$.

The integral (25) will be expressible as the sum of contributions from each of the two separate domains in the form

$$\mathcal{L} = \int_{-\infty}^{0}\hat{\mathcal{L}}J\,du + \int_{0}^{\infty}\hat{\mathcal{L}}J\,du. \quad (35)$$

The condition that ϕ and hence also the separate expansion coefficients $\phi_{(0)}$, $\phi_{(1)}$, and $\phi_{(2)}$ should vanish at the domain boundary $u = 0$, together with the outer limit condition $\phi - \phi_{(0)} \to 0$, which implies $\phi_{(1)} \to 0$ and $\phi_{(2)} \to 0$, as $u \to \pm\infty$, implies that there is no contribution from the total derivative in (34). Thus the second-order correction $\phi_{(2)}$ does not contribute to the effective Lagrangian at this order. This is because at the only place where the equations of motion (and hence the first variation of the action to which this term is proportional) are not imposed, namely, the domain wall $\phi_{(2)}$ is C^1 and necessarily zero; hence, any potential contribution of this boundary term vanishes. It can also be seen that the first-order contribution of the integrand is an odd function of u, and thus that it cancels out between the two terms in (35), so that the final result is obtained in the expected form

$$\mathcal{L} = \mathcal{L}_{(0)} + \frac{\epsilon^2}{2}\mathcal{L}_{(2)} + O\{\epsilon^3\}, \quad (36)$$

with

$$\mathcal{L}_{(0)} = -\frac{2}{l}\eta^2 I_{\rm I}, \quad (37)$$

and

$$\mathcal{L}_{(2)} = \frac{-2}{l}\eta^2(\kappa_{(1)}^2 - \kappa_{(1)\mu\nu}\kappa_{(1)}^{\mu\nu})I_{\rm II} + \frac{2}{l}\eta^2\kappa_{(1)}^2 I_{\rm III}, \quad (38)$$

where the dimensionless constant coefficients are given as the integrals

$$I_{\rm I} = \int_{-\infty}^{\infty}\phi_{(0)}'^2 du = \tfrac{4}{3}, \quad (39a)$$

$$I_{\rm II} = \int_{-\infty}^{\infty}\phi_{(0)}'^2 u^2 du = \frac{\pi^2-6}{9}, \quad (39b)$$

$$I_{\rm III} = \int_{-\infty}^{\infty}\phi_{(0)}' f\,du = \tfrac{8}{9}. \quad (39c)$$

The difference between the present calculation and its predecessors [8–12] is the inclusion here of the extra term proportional to $I_{\rm III}$ in (38).

VI. THE CANONICALLY TRUNCATED MODEL

The outcome of the preceding calculation is that the second-order effective action obtained for the wall from (36) by truncating the uncalculated higher-order correction $O\{\epsilon^3\}$ will be expressible explicitly, with the dimensional factors restored, as

$$\mathcal{L} = -\frac{8}{3l}\eta^2[1 + C_{\rm I}R + C_{\rm II}K^2], \quad (40)$$

where R is the three-dimensional Ricci scalar of the internal metric h_{ij} of the wall, which is given by the well-known Gauss formula

$$R = K^2 - K^\mu_\nu K^\nu_\mu, \quad (41)$$

while the coefficients are constants, of the order of the square of the wall thickness l, which are given exactly by

$$C_{\rm I} = \frac{I_{\rm II}}{I_{\rm I}}\frac{l^2}{2} = \frac{\pi^2-6}{24}l^2, \quad C_{\rm II} = -\frac{I_{\rm III}}{I_{\rm I}}\frac{l^2}{2} = -\frac{1}{3}l^2. \quad (42)$$

Using the formula (A10) obtained in earlier work [9] (after rectification of a transcription error interchanging the parameters β and Δ that were then to be identified) or more rapidly by direct substitution of the expressions $K_{\mu\nu}^{\rho} = K_{\mu\nu}n^\rho$ and $K^\rho = Kn^\rho$ (for the second fundamental tensor and its contraction) in the general (dimensionally unrestricted) formulas that have been derived more recently [14], the equation of motion that ultimately results from the Lagrangian (40) is found to be given by

$$\begin{aligned} K = {} & C_{\rm I}(3KK^\mu_\nu K^\nu_\mu - K^3 - 2K^\mu_\nu K^\nu_\rho K^\rho_\mu) \\ & + C_{\rm II}(2KK^\mu_\nu K^\nu_\mu - K^3 + 2\,\Box\,K), \end{aligned} \quad (43)$$

(where $\Box$ denotes the world sheet Dalembertian) in which the final set of parentheses with coefficient $C_{\rm II}$ groups the contributions that were unjustifiably left out in the previous work [8–12]. It is to be remarked that one is free to work with units that adjust the numerical value of the length scale l in order to set either of the magnitudes (though not the signs) of the coefficients $C_{\rm I}$ and $C_{\rm II}$ to any chosen value such as unity: thus, apart from the signs (which, as discussed below, are of crucial importance), all that matters qualitatively is their magnitude ratio, c say, which is given by

$$c = \frac{C_{\rm II}}{C_{\rm I}} = \frac{I_{\rm III}}{I_{\rm II}} = \frac{8}{\pi^2-6} \simeq 2. \quad (44)$$

It is to be remarked that (unlike what can be seen to occur in the string case [14] because of the divergence property of its Ricci scalar) the exact satisfaction of the lowest-order dynamical equation, namely $K = 0$, is *not* by itself sufficient to ensure satisfaction of the corresponding higher-order system (43). The simple harmonicity condition $K = 0$ can, however, be seen to be sufficient in the restricted case of a *static* configuration in ordinary flat spacetime (with $D = 3$), since in these circumstances it automatically entails the cubic order condition $K^{\mu}_{\nu}K^{\nu}_{\rho}K^{\rho}_{\mu} = 0$ as well, which is evidently enough.

VII. IMPLICATIONS

The lowest-order contribution $-8\eta^2/3l$ to the Lagrangian (40) is the constant that by itself gives the simple Dirac membrane action. The next term, proportional to the world-sheet Ricci scalar R with coefficient $-(\pi^2 - 6)\eta^2 l/9$, is the purely "geometric" contribution whose derivation is described in the previous work [8–10] that was cited above. However, that work overlooked the final "deformation" term, proportional to K^2 with coefficient $8\eta^2 l/9$, which arises from the first-order correction term in (24) when this expression is substituted in (1) prior to the performance of the integration over the off-world-sheet dimension parametrized by z. In the more familiar example of buckling in a bent elastic rod, the deformation correction reduces the bending energy arising from the rigidity of the solid material involved and is therefore appropriately describable as an antirigidity effect. It is therefore reasonable also to describe the negativity of the coefficient $C_{\rm II}$ for the analogous deformation term in the present example as an antirigidity effect.

The idea implicit in the above terminology is that the contribution to the energy in a static configuration (that is not necessarily a solution) should be positive in the case of a rigidity term and negative in the case of an antirigidity term. However, one should be aware that the notion of rigidity (whose introduction in the present context is attributable to Polyakov [15] and also, independently, to Kleinert [16]) is potentially misleading, since one can conceive alternative defining conventions in terms of criteria for stable equilibrium, for which, however, the alternative term "stiffness" is perhaps more appropriate. A systematic study [14] of the effect of conceivable quadratic curvature corrections for closed maximally symmetric p-brane configurations—meaning *a* circle in the case of a string with $p = 1$, spheres in the case of a membrane with $p = 2$, and so on in hypothetical higher-dimensional cases—shows that in the case of strings the criterion of positive rigidity according to the defining convention postulated above agrees with the condition for the existence of static ring solutions, i.e., it is positive rigidity that provides "stiffness." On the other hand, in higher-dimensional cases with $p > 2$ it is antirigidity as defined above that is required for the existence of static equilibrium: the negativity of $C_{\rm II}$ in the theory considered here is thus interpretable as making a higher-dimensional wall "stiff" in the sense of allowing it to avoid collapse in a hyperspherical configuration. However, this notion of stiffness loses its meaning in the critical intermediate case, with $p = 2$, that applies to walls in ordinary four-dimensional spacetime, for which spherical equilibrium will *always be impossible*, regardless of whether the sign of the coefficient $C_{\rm II}$ is positive, which would correspond to rigidity, or negative as in the specific antirigid wall model considered here.

The question of existence of static solutions leads one to the question of their stability. Although the model characterized by (40) has no maximally symmetric static solution that is closed within a four-dimensional spacetime background, it is evident that there will always be one that is open, namely, the simple plane wall solution. It is also evident that at least locally there will be many other static, though less highly symmetric solutions, whose stability can be tested by linear perturbation theory. Although it may have some effect on their propagation speeds, the extra antirigidity term evidently cannot destabilize any of the large L (i.e., low-frequency, long wavelength) modes to which the validity of our derivation of the model (40) is restricted. However, the negativity of $C_{\rm II}$ will engender instability in modes whose characteristic curvature scale L is small enough to be comparable with the wall width l. This instability in the model characterized by (40) and (42) does not mean that the domain wall is actually unstable: it merely means that such rapidly varying modes cannot be treated adequately without allowance for the higher-order terms $O\{\epsilon^3\}$ that were thrown away in the truncation that was made in going from (36) to (40). This feature is a serious drawback from the point of view of the use of (40) in conjunction with (42) in practice: it implies the need, in numerical computations, to incorporate some artificial mechanism for damping out the unphysical short time scale instabilities that would otherwise occur.

This caveat, to the effect that the canonically truncated model of the preceding section should not be taken too literally but used with caution, provides the motivation for seeking a more practically convenient alternative. A reasonable way of getting round the difficulty in the practical calculation of curvature corrections to domain-wall dynamics in the long-wavelength limit is to take advantage of the reassuring observation that, whereas it only has to satisfy $lK = O\{\epsilon\}$ "off shell," this dimensionless combination must satisfy the more severe requirement

$$lK = O\{\epsilon^3\} \tag{45}$$

for any configuration that is actually a solution of the dynamical equation (43). The corresponding reduced curvature scalar must therefore satisfy $\kappa = o\{\epsilon^2\}$, the latter being expressible, by (15), as the vanishing not only of $\kappa_{(1)}$ but even of $\kappa_{(2)}$, which is more than enough to ensure that the litigious regularity condition (18) (that was imposed prematurely, before the variation, in the previous work [8–10]) will after all be satisfied "on shell" as one would expect. It follows that whether it be obtained from the truncated Lagrangian (40) or from the original expansion (36), a solution of the dynamical equations will be characterized up to second-order corrections by

$$lK + \frac{\pi^2 - 6}{12} l^3 K^{\mu}_{\nu}K^{\nu}_{\rho}K^{\rho}_{\mu} = O\{\epsilon^4\}. \tag{46}$$

This is evidently the same as would be obtained by taking the deformation coefficient $C_{\rm II}$ to vanish, i.e., setting $c = 0$ as was done in previous work [8–12] instead of using the value $c \simeq 2$ derived by the logically consistent procedure used above.

The conclusion is that although, strictly speaking, the internal mechanics of the wall is really characterized by the "antirigidity" property represented by the well-defined negative value of $C_{\rm II}$ as given by (43), nevertheless this effect does not influence the dynamics to the order of accuracy under consideration. It is therefore quite permissible to use the simpler and better-behaved zero rigidity model specified by setting $C_{\rm II} = 0$ in (40) as advocated in previous work [8–12]. There is, however, nothing obligatory about this option: it would also be permissible (for example, if it were thought helpful for numerical computations) to use an overstabilized model characterized by a positive value of $C_{\rm II}$, provided it did not exceed the order of magnitude limitation $|C_{\rm II}| \lesssim l^2$.

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