QCD strings as a constrained Grassmannian σ model

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We present calculations for the effective action of a string world sheet in R^3 and R^4 utilizing its correspondence with the constrained Grassmannian σ model. Quantum fluctuations of both minimal and harmonic surfaces with punctures are computed. The calculation of their contributions in both R^3 and R^4 is reduced to the study of the grand partition function of a two-dimensional modified Coulomb gas system.

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I. INTRODUCTION

Two-dimensional nonlinear σ models share many features with four-dimensional Yang-Mills theories. They both are scale invariant, asymptotically free, and possess multi-instanton solutions [1–3]. In spite of these similarities, not much has been done to explore if there is a deeper relation between these theories. Over the years string models [4] have been proposed to describe QCD flux tubes, deemed responsible for quark confinement. It has been widely recognized that QCD strings should take into account the extrinsic geometry of the string world sheet [5, 6]. General properties of strings with extrinsic curvature action $S = \frac{2}{\alpha_0}\int \sqrt{g}|\ H\ |^2 d^2\xi$ have been analyzed [5]. For example, it was shown that this term is asymptotically free. However, so far it has not been established if the rigid string theories are appropriate to model QCD flux tubes. See, for example [7].

The present authors have in a series of publications [8–11] studied the extrinsic geometry of string world sheet immersed in background n-dimensional space from the point of view of Grassmannian σ models. The set of all tangent planes to the world sheet of strings immersed in R^n and regarded as a two-dimensional (2D) Riemann surface endowed with the induced metric, is equivalent to the Grassmannian manifold $G_{2,n} \simeq \frac{\mathrm{SO}(n)}{\mathrm{SO}(2)\times \mathrm{SO}(n-2)} \simeq Q_{n-2} \subset \mathrm{CP}^{n-1}$. Note that $G_{2,n}$ can be realized as a quadratic Q_{n-2} in CP^{n-1}. It is this representation that we use throughout our work. However, it is not an ordinary σ model, since not every field in $G_{2,n}$ forms a tangent plane to the world sheet. This forces the $G_{2,n}$ fields to satisfy $(n-2)$ integrability conditions which have been derived explicitly in [8, 10, 12, 13] by the use of Gauss mapping [12, 13]. The Gauss map is a mapping of the tangent planes to a conformally immersed world sheet $X^\mu(z,\bar{z})$ into the Grassmannian $G_{2,n}$, realized as a quadratic in CP^{n-1}. There is a third order differential constraint on the $G_{2,n}$ fields and $(n-3)$ algebraic constraints on the derivative of the Gauss map. Note that the integrability conditions on the $G_{2,n}$ fields allow us to study the string model in terms of a constrained $G_{2,n}$ σ model. These constraints ensure that the σ model does in fact represent a string world sheet. We are stressing this point, because many authors have incorporated the constraint by requiring that the normals N^μ (σ-model fields) to the surface satisfy the condition $\partial_z X^\mu \cdot N^\mu = 0$, thereby making it difficult to implement this constraint without dealing with X^μ coordinates. Both the Nambu-Goto (NG) action and the extrinsic curvature action can be written in terms of their images in $G_{2,n}$ through Gauss maps and the integrability conditions can be implemented by Lagrange multipliers. Thus the problem of string dynamics visualized as the dynamics of the world sheet immersed in a background R^n, can be transformed into, at least at the classical level, that of a constrained Grassmannian σ model. To complete the picture, the immersion coordinates $X^\mu(z,\bar{z})$ can be reconstructed from the constrained $G_{2,n}$ σ model fields [8].

From the above discussion, the advantages of studying QCD strings as a constrained σ model should be clear. The extrinsic curvature action, which usually leads to fourth derivative theory in X^μ, becomes a σ-model action in terms of $G_{2,n}$ fields; the higher derivatives arising only through the differential integrability condition. In quantizing this theory we need to know the proper measure to use for the functional integral over $G_{2,n}$ fields. Polchinski and Strominger [14] point out that the standard string quantization should not be used in describing QCD strings. Thus the measure for X^μ integration is not completely known. Because of this uncertainty we take the viewpoint that we can describe QCD flux tubes by constrained $G_{2,4}$ σ model (underlying field theory) with the usual σ-model measure. We shall show in this paper that while the resulting quantum theory shares features with the unconstrained σ model, the constraint has nontrivial effect on the nonperturbative aspects of the theory. It will further be seen in this article that a major advantage of formulating the QCD flux tubes through Gauss map is that it allows one to do path integrals over a select class of surfaces having prescribed extrinsic geo-

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metric properties.

We consider below strings in background R^3 and R^4 only. The corresponding Grassmannian manifolds are $G_{2,3} \simeq \mathrm{CP}^1$ and $G_{2,4} \simeq \mathrm{CP}^1 \times \mathrm{CP}^1$, respectively. Two classes of surfaces, minimal and harmonic, are considered. Minimal surfaces are noncompact and have zero scalar mean curvature h [12, 13]. They correspond to minimum action solution to the NG action (area term) i.e., $\Box X^\mu = 0 \Leftrightarrow h = 0$. Minimal surfaces describe the dynamics of open strings. In the σ-model language, minimal surfaces are described by instantons. In the instanton configuration, the σ-model fields are holomorphic. An N-instanton solution that is meromorphic arises as the Gauss map of a world sheet with $2N$ punctures. This describes N-open string interactions at the tree level (genus zero). Harmonic surfaces correspond to solutions to the equations of motion of the image of the extrinsic curvature action in the Grassmannian $G_{2,n}$ [12, 13]. However they do not generally minimize the extrinsic curvature action when expressed in terms of X^μ as $\int \sqrt{g}(\Box X^\mu)^2 d^2\xi$ with $g_{\alpha\beta}$ as the induced metric. The equation of motion following from this action is

$$\sqrt{g}\Box^2 X^\mu + \sqrt{g}\Box X^\mu(\Box X^\nu)^2 + 2(\partial_{\bar{z}} X^\mu)(\Box X^\nu)(\partial_z \Box X^\nu) + 2(\partial_z X^\mu)(\Box X^\nu)(\partial_{\bar{z}} \Box X^\nu) = 0. \quad (1)$$

For immersion in R^3 there is only one normal to the surface defined through $\Box X^\mu = hN^\mu$. The above equation can be easily generalized to immersion in R^4, by using $\Box X^\mu = h_1 N_1^\mu + h_2 N_2^\mu$ where h_1, h_2 are the projections of H^μ on to the two normals [8]. Returning to immersion in R^3, for constant h surfaces (1) reads as $\Box N^\mu + h^2 N^\mu = 0$. Expressing N^μ in terms of $G_{2,3}$ fields [8], we find that this is satisfied only when the $G_{2,3}$ fields are antiholomorphic. Similar conclusion is reached for immersion in R^4. So, for harmonic surfaces, the choice of Grassmannian fields as antiholomorphic minimizes the extrinsic curvature action whether it is written in terms of X^μ or the Grassmannian fields. The Gaussian curvature for these surfaces is a constant and the principal curvatures are same, thus the world sheet topologically corresponds to a two-sphere. Harmonic surfaces describe the dynamics of closed strings. In the language of the σ model these surfaces correspond to anti-instantons, i.e., σ-model fields that are antiholomorphic. An N-anti-instanton solution arises as the Gauss map of a world sheet with $2N$ punctures and describes N-closed string interactions at the tree level.

We compute the one-loop quantum fluctuations following Fateev, Frolov, and Schwarz [15] (see also [16]) around instantons (minimal surfaces) and anti-instantons (harmonic surfaces) in R^3 in Sec. II and in R^4 in Sec. III. The resulting effective action is found to be the grand partition function of a modified two-dimensional classical Coulomb gas system (MCGS) for immersion of both minimal and harmonic surfaces in both R^3 and in R^4. The term modified is used here to describe the system in which the instanton quarks and antiquarks have logarithmic interaction with a strength thrice as large as the strength of the quark-quark and antiquark-antiquark interactions. The $2N$ punctures correspond to the location of N instanton-quarks and N instanton-antiquarks. The implications of these results to QCD strings are also discussed.

II. QUANTUM FLUCTUATIONS—IMMERSION IN R^3

The Gauss map of a 2D Riemann surface conformally immersed in R^3 has been considered in detail in [8, 12, 13]. By conformal immersion it is meant that the induced metric is in the conformal gauge ($g_{zz} = g_{\bar{z}\bar{z}} = 0; g_{z\bar{z}} \neq 0$), where $z = \xi_1 + i\xi_2$ with (ξ_1, ξ_2) as isothermal coordinates on the world sheet. The Gauss map is described by

$$\partial_z X^\mu = \psi\{1 - f^2,\ i(1 + f^2),\ 2f\}, \quad (2)$$

where f is the CP^1 field and the complex function ψ is determined by the extrinsic geometry and f [8]. The NG and extrinsic curvature actions in terms of $G_{2,3} \simeq \mathrm{CP}^1$ σ-model field f [8] are

$$S = \sigma \int \frac{1}{h^2(z,\bar{z})} \frac{|f_{\bar{z}}|^2}{(1 + |f|^2)^2} \frac{i}{2} dz \wedge d\bar{z} + \frac{2}{\alpha_0} \int \frac{|f_{\bar{z}}|^2}{(1 + |f|^2)^2} \frac{i}{2} dz \wedge d\bar{z}, \quad (3)$$

where σ is the string tension and α_0 is a dimensionless coupling whose renormalized expression is asymptotically free [8]. The integrability condition on f is

$$\mathrm{Im}\left(\frac{f_{z\bar{z}}}{f_{\bar{z}}} - \frac{2\bar{f} f_z}{1 + |f|^2} \right)_{\bar{z}} = 0, \quad (4)$$

whenever $f_{\bar{z}} \neq 0$. The notation used throughout in this paper is $f_z \equiv \partial_z f$, etc. For minimal surfaces, there is no integrability condition on f and for harmonic surfaces, f satisfies a stronger integrability condition given in (7).

A. Minimal surfaces in R^3

Our aim is to calculate instanton contributions to the generating functional for strings conformally immersed in R^3 (and in R^4). A convenient background configuration is provided by minimal surfaces for which $h = 0$. The classical action is taken to be the extrinsic curvature action written in terms of the σ-model field represented by the second term in (3). In other words we set the bare string tension to zero. This simplifies considerably the analysis. As pointed out earlier, for a minimal conformal immersion of a surface, the Gauss map f is holomorphic. Let us consider as classical background, the instanton background [15]:

$$f(z) = c\, \frac{\prod_{i=1}^N (z - a_i)}{\prod_{i=1}^N (z - b_i)}, \quad (5)$$

where $\{c, a_i, b_i\}$ are the (complex) instanton parameters. In general, an arbitrary holomorphic function need not represent tangent planes to some conformal minimal surface. However, it is possible to produce a ψ defined in (2) for the above choice of f. This pair $\{\psi, f\}$ does represent the Gauss map of a conformal minimal surface with $2N$ punctures at $\{a_i\}$ and $\{b_i\}$. In the case of the instantons of an abstract σ model, the scale factor c is arbitrary. In the case of the world sheet it is possible to check that the induced metric is singular at $c = 0$. For this reason we restrict $c \neq 0$.We can rescale $c = 1$. Only the punctures have physical significance of denoting the interactions of the string on the surface. The quantum fluctuations $\nu(z, \bar{z})$ around (5) are defined by

$$f(z) \to f(z) + \nu(z, \bar{z}). \tag{6}$$

The fluctuated field should arise as the Gauss map of a surface obtained as fluctuation of the given minimal surface. Consequently we need to implement an integrability condition for $\nu(z, \bar{z})$. At the classical level there is no integrability condition for the Gauss map of a minimal surface. Thus the Lagrange multiplier field needed must be quantum. We restrict the fluctuations to correspond to harmonic surfaces described by small but nonvanishing constant h. A harmonic Gauss map satisfies the equation

$$f_{z\bar{z}} - \frac{2\bar{f} f_z f_{\bar{z}}}{1+|f|^2} = 0, \tag{7}$$

which is also the equation of motion of the extrinsic curvature action in (3). It can be shown [8] that if (7) is satisfied then h is constant. The integrability condition (7) for harmonic Gauss maps is stronger than the one given in (4). Since the Lagrange multiplier field needed to implement (7) is quantum, it is sufficient to linearize it in the form

$$\nu_{z\bar{z}} - \frac{2\bar{f} f_z \nu_{\bar{z}}}{1+|f|^2} = 0. \tag{8}$$

The physical picture that emerges here is this. We are studying the quantum effects of small fluctuations of a conformally immersed minimal surface with $2N$ punctures, where the fluctuations are restricted to those that result in harmonic surfaces with constant h. Expanding the action up to quadratic terms in fluctuations, we find

$$S = \frac{2\pi}{\alpha_0} N + \frac{4}{\alpha_0} \int \bar{\tilde{\nu}} \triangle_f \tilde{\nu} \sqrt{\mathcal{G}} d^2 z + \int \lambda \left(\nu_{z\bar{z}} - \frac{2\bar{f} f_z \nu_{\bar{z}}}{1+|f|^2} \right) d^2 z + \text{H.c.}, \tag{9}$$

where

$$\triangle_f = -\frac{1}{\sqrt{\mathcal{G}}} \rho \partial_z \rho^{-2} \partial_{\bar{z}} \rho,$$
$$\tilde{\nu} = \frac{2\nu}{\rho} \prod_{i=1}^{N} (z - b_i)^2,$$
$$\rho = \rho_0 \prod_{i=1}^{N} |z - b_i|^2,$$
$$\rho_0 = 1 + |f|^2, \tag{10}$$

where $\mathcal{G}_{\alpha\beta}$ is a metric introduced to avoid infrared divergences [15], which will eventually be taken as $\delta_{\alpha\beta}$. The multiplier term in (9) can be rewritten as

$$\int \bar{\tilde{\lambda}} \triangle_f \tilde{\nu} d^2 z + \text{H.c.}, \tag{11}$$

where

$$\tilde{\lambda} = \frac{\rho\lambda}{2} \prod_{i=1}^{N} (z - b_i)^2. \tag{12}$$

The quantum generating functional can be obtained upon functional integration over $\tilde{\nu}, \bar{\tilde{\nu}}, \lambda, \bar{\lambda}$ the exponential of S. This procedure is standard [15] and we quote the result. We get the following expression for the partition function:

$$Z = \sum_{N=0}^{\infty} (N!)^{-2} \exp\left(-\frac{2\pi N}{\alpha_0}\right) \int d\mu_0 \det\left(\frac{4}{\alpha_0} \triangle_f\right)^{-1}, \tag{13}$$

where $d\mu_0$ is the instanton measure and a sum over instantons of all winding numbers is introduced. It should be noted here that the effect of the integrability condition is to produce an additional factor $(\det\triangle_f)^{-\frac{1}{2}}$ arising from (11). We can follow the procedure of Fateev, Frolov, and Schwarz [15] and evaluate the determinant in (13). The result is

$$Z = \sum_{N=0}^{\infty} (N!)^{-2} \left[\left(\frac{4}{\alpha_0}\right) \exp\left(-\frac{\pi}{\alpha_0}\right) \exp(2\ln\Lambda) \right]^{2N} \int \prod_{j=1}^{N} d^2 a_j d^2 b_j (\det M)^{-1} \times \exp\left(\sum_{i<j} \ln|a_i - a_j|^2 + \sum_{i<j} \ln|b_i - b_j|^2 - 3\sum_{i,j} \ln|a_i - b_j|^2 \right). \tag{14}$$

In (14), Λ is a cutoff parameter introduced to regularize the determinant. This can be removed by renormalizing the coupling constant according as

$$\alpha_R(\mu) = \frac{\alpha_0}{1 - 2(\frac{\alpha_0}{\pi}) \ln\frac{\Lambda}{\mu}}, \tag{15}$$

where μ is the renormalization point. Note that the integrability condition on the Gauss map leads to the factor $2(= d-1)$ in front of $\alpha_0/4\pi$ in (15) rather than to $1(= d-2)$ as for the unconstrained σ model. This is in agreement with our earlier calculations [8]. The effect of the constraint on the nonperturbative sector can be read off from (14) on comparing it with the corresponding result in the usual σ model [15]. The strength of the logarithmic interaction between the instanton quarks located at $\{a_i\}$ and the instanton antiquarks at $\{b_i\}$ is three times stronger than in the usual CP^1 model. In the case of CP^1, the partition function is that of a classical 2D Coulomb gas system. The effect of the constraint is to enhance the attractive interaction between the instanton quark-antiquark pair. Furthermore, there is an extra determinant in [14], i.e., det$M$ where, $M_{ij} = \int \bar{z}^i z^j \rho^{-2} d^2 z$. We have checked that the (finite) contribution of det$M$ is independent of the instanton parameters. The final result is

$$Z = \sum_N \int \frac{\kappa^{2N}}{(N!)^2} \exp[-E_N(a,b)] \prod_{j=1}^{N} d^2 a_j d^2 b_j, \quad (16)$$

where

$$\begin{aligned} E_N(a,b) = &- \sum_{i<j} \ln | a_i - a_j |^2 \\ &- \sum_{i<j} \ln | b_i - b_j |^2 \\ &+3 \sum_{i,j} \ln | a_i - b_j |^2 \end{aligned} \quad (17)$$

and

$$\kappa = \frac{4k_0\mu^2}{\alpha_R(\mu)} \exp\left(-\frac{\pi}{\alpha_R(\mu)}\right). \quad (18)$$

μ in (18) is the renormalization point and k_0 is a constant which depends on the cutoff method. The coupling constant in the denominator in (18) is set as the renormalized coupling constant [17–20]. We thus find that the generating functional is the grand partition function of a classical two-dimensional gas of instanton quarks and antiquarks with Coulombic interactions at a temperature $\beta = 1$. However, it is not the usual Coulomb gas picture that appears in CP^1 instantons. (16) together with (17) can be termed the partition function of a modified Coulomb gas system (MCGS).

Let us introduce the renormalization-group invariant mass m by

$$m = \mu \exp\left(-\frac{\pi}{2\alpha_R(\mu)}\right). \quad (19)$$

To understand the meaning of m note that

$$m = \int \frac{d^2p}{2\pi} \exp\left(-\frac{p^2}{2m}\right). \quad (20)$$

We can thus write

$$m^{2N} = \prod_{j=1}^{2N} \int \frac{d^2p_j}{2\pi} \exp\left(-\frac{p_j^2}{2m}\right), \quad (21)$$

which is just the kinetic energy term for the $2N$ particles of momentum p_i and mass m [21]. Thus the renormalization-group invariant mass may be interpreted as the mass of the instanton quarks and antiquarks. It is also clear from (18) that the parameter $\kappa(\mu)$ defined by

$$\kappa(\mu) = \frac{4k_0 m^2}{\alpha_R(\mu)} \quad (22)$$

plays the role of the fugacity of the MCGS and thus the "thermodynamic" properties of this system, will depend on the renormalization scale.

In the case of the Coulomb gas arising in the ordinary CP^1 model, a number of properties are known. There is a phase transition for $\beta_c = 2$. For $\beta < \beta_c$, the system is in the plasma phase with a mass gap, while for $\beta > \beta_c$ it is in the molecular phase with long-range order and no mass gap. Since the Coulomb gas of instanton quarks and antiquarks in the unconstrained CP^1 model is at $\beta = 1$, this system is in the plasma phase with a mass gap. It has been argued [7] that such a phase is not suitable for describing QCD strings. Let us try to understand the picture emerging in the case of the constrained σ-model description of the string world sheet. Unfortunately, the properties of the MCGS have not been analyzed in statistical mechanics exactly. Let us assume that the thermodynamic limit exists for the MCGS. We can use the iterated mean-field result of Kosterlitz and Thouless [22] to estimate the critical temperature as a function of the fugacity $\kappa \equiv \exp(-\beta\mu')$ (μ' is the chemical potential). It is determined by

$$q^2\beta_c \simeq 2 + 2.6\pi \exp[-(\beta_c\mu')], \quad (23)$$

where $q^2 = 3$ for the MCGS, while $q^2 = 1$ for the Coulomb gas.

It can be seen from (23) that for the Coulomb gas system for which $q^2 = 1$, the critical temperature $\beta_c \geq 2$ for any value of the fugacity. Let us apply this formula for the MCGS that we have obtained. We find in this case that $\beta_c > 0.667$. In fact $\beta_c < 1$ when $\beta_c\mu' \geq 2$ (approximately). From the formula (18) for the fugacity, we see that $\kappa \propto \frac{1}{\alpha_R(\mu)}$ for a fixed m. In the infrared region the coupling constant becomes large [see Eq. (15)], thus making it possible for $\beta\mu'$ to become larger than 2. When this happens, then the modified Coulomb gas system that describes the quantum corrections to conformally immersed minimal surfaces, in the large distances limit, can find itself in a phase with long-range correlations and no mass gap since $\beta = 1$ is larger than β_c. On the other hand, in the UV region the renormalized coupling constant $\alpha_R(\mu)$ is small and in this case β_c exceeds 1 and the system will be in the plasma phase with a mass gap in the short-distance limit.

We conclude from these arguments that the string world sheet is stable against small fluctuations in the infrared region and avoids crumpling. The fluctuations of the Grassmannian field represent fluctuations of the surface along the normal direction. The long-range correlations would thus correspond to long-range normal-normal correlations.

B. Harmonic surfaces in R^3

The Gauss map of surfaces immersed in R^3 is said to be harmonic if f satisfies (7). For harmonic Gauss maps [12, 13], the mean curvature scalar h is constant ($\neq 0$) and the given surface is compact. Harmonic maps thus describe closed string dynamics [4]. The integrability condition for harmonic Gauss maps is also the equation of motion

(7). In order to minimize the extrinsic curvature action in terms of X^μ, f in (7) should be antiholomorphic. This result is also the content of Chern's theorem [23]. In this case, as mentioned in the Introduction, the surface is actually a Riemann sphere and thus we are considering closed string dynamics at the tree level. As a solution to harmonic Gauss map (7), we consider the anti-instanton configuration

$$f(\bar{z}) = \bar{c}\,\frac{\prod_{i=1}^{N}(\bar{z}-\bar{a}_i)}{\prod_{i=1}^{N}(\bar{z}-\bar{b}_i)}, \tag{24}$$

where $\{\bar{c}, \bar{a}_i, \bar{b}_i\}$ are the anti-instanton parameters. It is readily checked that the induced metric or more precisely, the inverse metric, has a singularity at $\bar{c}=0$ and thus we should have $\bar{c}\neq 0$. We scale $\bar{c}=1$. (24) represents an N-fold covering of the world sheet in the Z plane into a unit sphere in $\mathbb{R}^3$ with $\{a_i\}$and $\{b_i\}$ mapped into the north and south poles, respectively. The classical action (3) for a given constant h can be written as

$$S = \frac{2}{\alpha_0}\int \frac{|f_{\bar{z}}|^2}{(1+|f|^2)^2}\frac{i}{2}dz\wedge d\bar{z} + \int \lambda\left(f_{z\bar{z}} - \frac{2\bar{f}f_z f_{\bar{z}}}{1+|f|^2}\right)\frac{i}{2}dz\wedge d\bar{z} + \text{H.c.}, \tag{25}$$

where we have redefined $\frac{2}{\alpha_0}+\frac{\sigma}{h^2}$ as $\frac{2}{\alpha_0}$. The integrability condition (7) for harmonic surfaces has been implemented in (25) through the multiplier field λ. The equations of motion for the total action (25) contain in addition to (7) an homogeneous equation for λ. We choose the trivial solution $\lambda_{cl}=0$ in the calculations below. This choice is reasonable for, at the classical level, the equation of motion for f ensures implementation of the constraint without the need for the multiplier field. Quantum fluctuations around this background configuration can be handled exactly as in the previous sub-section. We assume that the quantum fluctuations generate surfaces which are also harmonic. As before, since $\lambda_{cl}=0$, the constraint field is quantum and we need only consider the linearized form of the integrability condition (8).

The effective action in the case of harmonic immersion can now be calculated and it reduces to the partition function of the modified Coulomb gas system at $\beta=1$ found earlier for the minimal surfaces.

III. QUANTUM FLUCTUATIONS, IMMERSION IN R^4, AND QCD STRINGS

The Grassmannian σ-model approach to the string dynamics presented in Sec. II, is extended to string world sheet immersed in R^4. We have two normals N_i^μ ($\mu=1,2,3,4; i=1,2$) at each point on the surface and so there are two extrinsic curvature tensors $H^\mu_{\alpha\beta i}$ for the surface. The scalar mean curvature $h=\sqrt{(}h_1^2+h_2^2)$ and the detailed expressions for h_1, h_2 and the extrinsic curvature action in terms of Gauss map are derived in [8]. In the case of R^4, Q_2, which is equivalent to $\mathrm{CP}^1\times\mathrm{CP}^1$, is parameterized by two complex functions f_1 and f_2 and the Gauss map is given by

$$\partial_z X^\mu = \psi[1+f_1f_2, i(1-f_1f_2), f_1 - f_2, -i(f_1+f_2)], \tag{26}$$

where ψ is determined by the extrinsic geometry and f_1 and f_2. The two integrability conditions are

$$\mathrm{Im}\left(\sum_{i=1}^{2}\frac{f_{iz\bar{z}}}{f_{iz}} - \frac{2\bar{f}_i f_{iz}}{1+|f_i|^2}\right)_{\bar{z}} = 0 \tag{27}$$

and

$$|F_1| = |F_2|, \tag{28}$$

where $F_i = f_{i\bar{z}}/(1+|f|^2)$ and whenever $f_{i\bar{z}}\neq 0$. In our considerations [8] the world sheet is described locally by $X^\mu(z,\bar{z})$ and the Gauss map allows us to express NG and extrinsic curvature actions as a $G_{2,4}$ σ-model action:

$$S = \int\left(\frac{\sigma}{h^2(z,\bar{z})}+\frac{2}{\alpha_0}\right)\sum_{i=1}^{2}\frac{|f_{i\bar{z}}|^2}{(1+|f_i|^2)^2}d^2z. \tag{29}$$

This together with (27) and (28) describes the dynamics of the string world sheet in background R^4. The scalar mean curvature h is given by [8]

$$(\ln h)_z = \sum_{i=1}^{2}\left(\frac{f_{iz\bar{z}}}{f_{i\bar{z}}} - \frac{2\bar{f}_i f_{iz}}{1+|f_i|^2}\right), \tag{30}$$

and the Gauss map (21) is said to be harmonic if

$$f_{iz\bar{z}} - \frac{2\bar{f}_i f_{iz} f_{i\bar{z}}}{1+|f_i|^2} = 0\ , \quad i=1,2. \tag{31}$$

From (30) and (31) it follows that when the Gauss map is harmonic, h is constant.

A. Minimal immersion in R^4

The Gauss map of minimal surfaces ($h=0$) in R^4 has been studied in detail [12, 13, 24, 25] and accordingly, if $F_1=F_2\equiv 0$, then the Gauss map represents a minimal surface in R^4, provided the surface is noncompact. A solution to $F_1=F_2\equiv 0$ is given by holomorphic functions $f_1(z)$ and $f_2(z)$. There are no integrability conditions classically. The holomorphic functions $f_1(z)$ and $f_2(z)$ are chosen as

$$f_i(z) = \frac{\prod_{j=1}^{N_i}(z-a_{ij})}{\prod_{j=1}^{N_i}(z-b_{ij})}, \quad i=1,2, \tag{32}$$

representing background instantons, with parameters $\{a_{ij}, b_{ij}\}$ for i = 1,2.

Quantum fluctuations $\nu_i(z,\bar{z})$ around the instanton background are defined through

$$f_i(z)\rightarrow f_i(z)+\nu_i(z,\bar{z}), \quad i=1,2\,. \tag{33}$$

The fluctuated surface also arises as the Gauss map of a surface obtained from the given classical minimal surface. So we need integrability conditions on ν_i. Since classically there are no integrability conditions for a minimal surface, the Lagrange multiplier fields to implement the conditions on ν_i must be quantum. As in Sec. II A we restrict the fluctuations (33) to represent harmonic surfaces (i.e., the fluctuated surface has constant scalar mean curvature). Thus the fields ν_i are required to satisfy (31) in its linearized version:

$$\nu_{iz\bar{z}} - \frac{2\bar{f}_i f_{iz}\nu_{i\bar{z}}}{1+\mid f_i\mid^2} = 0, \quad i = 1,2, \tag{34}$$

which are implemented by two multipliers. For surfaces immersed in R^4 we need to examine the algebraic integrability condition (28) as well, for the fields (33). It can be readily checked that there are no linear terms in ν_i arising from the constraint (28) in the instanton background.

As a classical action we consider the action for the underlying $G_{2,4}$ σ model which can be rewritten as

$$S = \frac{2\pi}{\alpha_0}(N_1+N_2) + \frac{4}{\alpha_0}\int\sum_{i=1}^{2}\frac{\mid f_{iz}\mid^2}{(1+\mid f_i\mid^2)^2}d^2z, \tag{35}$$

where N_1 and N_2 are the winding numbers of the two CP^1 instantons. Expanding (35) using (32) and (33), and implementing (34), the effective action is found to be

$$S = \frac{2\pi}{\alpha_0}(N_1+N_2) - \frac{4}{\alpha_0}\int\left(\bar{\tilde{\nu}}_1\triangle_{f_1}\tilde{\nu} + \bar{\tilde{\nu}}_2\triangle_{f_2}\tilde{\nu}\right)d^2z + \int\bar{\lambda}_1\left(\nu_{1z\bar{z}} - \frac{2\bar{f}_1 f_{1z}\nu_{1\bar{z}}}{1+\mid f_1\mid^2}\right)d^2z + \text{H.c.} + \int\bar{\lambda}_2\left(\nu_{2z\bar{z}} - \frac{2\bar{f}_2 f_{2z}\nu_{2\bar{z}}}{1+\mid f_2\mid^2}\right)d^2z + \text{H.c.}, \tag{36}$$

where

$$\triangle_{f_i} = -\frac{1}{\sqrt{g}}\rho_i\partial_z(\rho_i^{-2}\partial_{\bar{z}}\rho_i),$$
$$\tilde{\nu}_i = \frac{2\nu_i}{\rho_i}\prod_{j=1}^{N_i}(z-b_{ij})^2,$$
$$\rho_i = \rho_{0i}\prod_{j=1}^{N_i}\mid z-b_{ij}\mid^2,$$
$$\rho_{0i} = 1+\mid f_i\mid^2. \tag{37}$$

By comparing (36) with (9), it is seen that a doubling corresponding to the two CP^1 instantons occur. The evaluation of the partition function then is similar to that in Sec. II A, now with measures for the two CP^1 instantons and the result is,

$$Z = \sum_{N_1,N_2}\int\frac{\kappa^{2(N_1+N_2)}}{(N_1!)^2(N_2!)^2}\exp[-E_{N_1}(a_1,b_1) - E_{N_2}(a_2,b_2)]\prod_{j=1}^{N_1}d^2a_{1j}d^2b_{1j}\prod_{k=1}^{N_2}d^2a_{2k}d^2b_{2k}, \tag{38}$$

where

$$E_{N_i}(a_i,b_i) = -\sum_{k<j}^{N_1}\ln\mid a_{ik}-a_{ij}\mid^2 - \sum_{k<j}^{N_1}\ln\mid b_{ik}-b_{ij}\mid^2 + 3\sum_{k,j}^{N_1}\ln\mid a_{ik}-b_{ij}\mid^2, \quad i=1,2. \tag{39}$$

Equation (38) together with (39) thus represents the partition function of two uncoupled modified Coulomb gas system, arising from the two CP^1 instantons.

B. Surfaces of constant mean curvature

We consider 2D surfaces in R^4 described by harmonic Gauss maps as representing closed QCD strings. For the harmonic Gauss map, as can be seen from (30), the scalar mean curvature h is constant. The two CP^1 fields satisfy the harmonic map equation (31). When h is constant, the NG and extrinsic curvature actions can be written as

$$S = \left(\frac{\sigma}{h^2} + \frac{2}{\alpha_0}\right)\int\sum_{i=1}^{2}\frac{\mid f_{i\bar{z}}\mid^2}{(1+\mid f_i\mid^2)^2}d^2z, \tag{40}$$

whose equations of motion are the same as (31). However, as pointed out in the Introduction, the extrinsic curvature action expressed in terms of $X^\mu(z,\bar{z})$ acquires a minimum only when f_1 and f_2 are antiholomorphic. In this case, the surface representing the world sheet is a compact two-sphere. The two antiholomorphic functions are chosen as the two CP^1 anti-instanton configurations

$$f_i(\bar{z}) = \frac{\prod_{j=1}^{N_i}(\bar{z}-\bar{a}_{ij})}{\prod_{j=1}^{N_i}(\bar{z}-\bar{b}_{ij})}, \quad i=1,2, \tag{41}$$

where $\{\bar{a}_{ij},\bar{b}_{ij}\}$ for i=1,2 are the anti-instanton parameters. The above background satisfies trivially the harmonic map equation (31). The algebraic integrability condition (28) has to be satisfied at the classical level, in order for (41) to represent a Gauss map. This puts conditions on the positions of the two CP^1 instantons.

Consider now the quantum fluctuations. Let us examine the integrability conditions. We restrict fluctuations to represent surfaces of constant scalar mean curvature. In this case the constraint reads as

$$\sum_{i=1}^{2}\left(\frac{f_{iz\bar{z}}}{f_{i\bar{z}}} - \frac{2\bar{f}_i f_{iz}}{1+\mid f_i\mid^2}\right) = 0, \tag{42}$$

whose linearized version is

$$\sum_{i=1}^{2}\left(\frac{\nu_{iz\bar{z}}}{f_{i\bar{z}}} - \frac{2\bar{f}_i\nu_{iz}}{1+\mid f_i\mid^2}\right) = 0. \tag{43}$$

The second integrability condition $\mid F_1\mid^2 = \mid F_2\mid^2$ is likewise expanded to give its linearized version. Expanding (40) and implementing (43) along with the linearized version of $\mid F_1\mid^2 = \mid F_2\mid^2$ we obtain,

$$S_{\rm eff}=\frac{2\pi}{\beta_0}(N_1+N_2)-\frac{4}{\beta_0}\int\sum_{i=1}^{2}\left(\bar{\tilde\nu}_i\triangle_{f_i}\tilde\nu_i\right)d^2z+\int\lambda\sum_{i=1}^{2}\left(\frac{\nu_{iz\bar z}}{f_{i\bar z}}-\frac{2\bar f_i\nu_{iz}}{1+|f_i|^2}\right)d^2z+\text{H.c.},$$
$$-\int\bar\nu_1(\partial_z\chi)\frac{f_{1\bar z}}{(1+|f_1|^2)^2}d^2z-\text{H.c.},+\int\bar\nu_2(\partial_z\chi)\frac{f_{2\bar z}}{(1+|f_2|^2)^2}d^2z+\text{H.c.},\tag{44}$$

where λ and χ are the quantum multipliers and $\frac{2}{\beta_0}=\frac{\sigma}{h^2}+\frac{2}{\alpha_0}$. The terms involving the multiplier fields are rewritten as

$$S_{\rm eff}=\frac{2\pi}{\beta_0}(N_1+N_2)-\frac{4}{\beta_0}\int\left(\bar{\tilde\nu}_1\triangle_{f_1}\tilde\nu_1+\bar{\tilde\nu}_2\triangle_{f_2}\tilde\nu_2\right)d^2z+\int\bar{\tilde\lambda}_1\triangle_{f_1}\tilde\nu_1+\text{H.c.}$$
$$+\int\bar{\tilde\lambda}_2\triangle_{f_2}\tilde\nu_2+\text{H.c.}-\int\bar{\tilde\nu}_1\chi_1-\text{H.c.}+\int\bar{\tilde\nu}_2\chi_2+\text{H.c.},\tag{45}$$

where

$$\tilde\lambda_i=\frac{\rho_i\lambda}{2\prod_{j=1}^{N_i}(z-b_{ij})^2f_{i\bar z}}\tag{46}$$

and

$$\chi_i=\frac{\rho_i f_{i\bar z}\chi'}{2\rho_{0i}^2\prod_{j=1}^{N_i}(\bar z-\bar b_{ij})^2},\tag{47}$$

with $\chi'=\partial_z\chi$. The expressions for $\triangle,\tilde\nu$ in (45) are the same as in (37) with $z\to\bar z$, $b\to\bar b$. The partition function is obtained by the functional integral of the exponential of $S_{\rm eff}$ over all the quantum fields. In order to perform this, the quantum action is rewritten by shifting the fields, as

$$S_q=-\frac{4}{\beta_0}\left(\int(\bar{\tilde\nu}_1-\beta_0\bar{\tilde\lambda}_1+\beta_0\bar\xi_1)\triangle_{f_1}(\tilde\nu_1-\beta_0\tilde\lambda_1+\beta_0\xi_1)d^2z+\int(\bar{\tilde\nu}_2-\beta_0\bar{\tilde\lambda}_2+\beta_0\bar\xi_2)\triangle_{f_2}(\tilde\nu_2-\beta_0\tilde\lambda_2+\beta_0\xi_2)d^2z\right.$$
$$\left.-\int[\beta_0(\bar{\tilde\lambda}_1-\bar\xi_1)]\triangle_{f_1}[\beta_0(\tilde\lambda_1-\xi_1)]d^2z-\int[\beta_0(\bar{\tilde\lambda}_2+\bar\xi_2)]\triangle_{f_2}[\beta_0(\tilde\lambda_2+\xi_2)]d^2z\right),\tag{48}$$

where $\xi_i=\triangle'^{-1}_{f_i}\chi_i$, with the prime denoting the determinant of nonzero eigenvalues of $\triangle_{f_i}$. Note that $\tilde\lambda_1$ and $\tilde\lambda_2$ as well as ξ_1 and ξ_2 are not independent. The integration over $\{\tilde\nu_1,\tilde\nu_2,\lambda,\chi'\}$ can be transformed into their linear combinations in (48). This introduces the Jacobian of the transformation, which is easily calculated as $\det^{-1}(\triangle'^{-1}_{f_1}+\triangle'^{-1}_{f_2})$. Thus the partition function for (48) becomes

$$Z=\beta_0^{-2(N_1+N_2)}\exp\left(-\frac{2\pi}{\beta_0}\right)\int\det\triangle_{f_1}^{-1}\det\triangle_{f_2}^{-1}$$
$$\times\det(\triangle'^{-1}_{f_1}+\triangle'^{-1}_{f_2})^{-1}\,d\mu_{01}\,d\mu_{02}.\tag{49}$$

The role of the integrability conditions is first to produce additional $(\det\triangle_{f_i})^{-\frac{1}{2}}$ as in Sec. III A and secondly to produce $\det(\triangle'^{-1}_{f_1}+\triangle'^{-1}_{f_2})^{-1}$ from the Jacobian , with the classical instanton parameters related by (28).

We now discuss the implications of (49). If the Jacobian is ignored, then we would have obtained MCGS for the two CP^1 instantons with no interaction between them. In this way the integrability condition (28) couples the two CP^1 instantons at the quantum level as well. $\det\triangle_{f_1}^{-1}$ and $\det\triangle_{f_2}^{-1}$ are evaluated using the methods of Fateev, Frolov, and Schwarz [15]. We infer the form of $\det^{-1}(\triangle_{f_1}^{-1}+\triangle_{f_2}^{-1})$ by the following procedure. Let us note that the interaction term is symmetric in the indices 1 and 2. When $\triangle_{f_1}=\triangle_{f_2}$, (49) reduces to the results for immersion in R^3 (as it should) with one instanton measure removed. The regularized expression for ln $\det\triangle_{f_1}^{-1}$ is

$$-4\sum_{j,k}\ln|a_{1j}-b_{1k}|^2,$$

and similar expression for ln $\det\triangle_{f_2}^{-1}$, apart from $\det M_i$. These observations suggest that ln $\det(\triangle_{f_1}+\triangle_{f_2}^{-1})$ should be of the form

$$2\sum_{j,k}\ln|a_{1j}-b_{2k}|^2+2\sum_{j,k}\ln|a_{2j}-b_{1k}|^2$$
$$-A\sum_{j,k}\ln|a_{1j}-a_{2k}|^2-B\sum_{j,k}\ln|b_{1j}-b_{2k}|^2.\tag{50}$$

Then when $a_{1j}=a_{2k}$; $b_{1j}=b_{2k}$ and with one instanton measure removed, we recover the results for immersion in R^3, modulo an infinite constant. In (50) we have included the last two terms as possible additional interactions among the two CP^1–anti-instantons with arbitrary coefficients. Then, the partition function Z can be written as

$$Z = \sum_{N_1,N_2} \int \frac{\kappa^{N_1+N_2}}{(N_1!)^2(N_2!)^2} \exp[-E_{N_1}(a_1,b_1) - E_{N_2}(a_2,b_2) - V_{N_1,N_2}] \prod_{j=1}^{N_1} d^2a_{1j}d^2b_{1j} \prod_{k=1}^{N_2} d^2a_{2k}d^2b_{2k}, \tag{51}$$

where

$$E_{N_1}(a_1,b_1) = -\sum_{i<j}^{N_1} \ln|a_{1i}-a_{1j}|^2 - \sum_{i<j}^{N_1} \ln|b_{1i}-b_{1j}|^2 + 3\sum_{i,j}^{N_1} \ln|a_{1i}-b_{1j}|^2,$$
$$E_{N_2}(a_2,b_2) = -\sum_{i<j}^{N_2} \ln|a_{2i}-a_{2j}|^2 - \sum_{i<j}^{N_2} \ln|b_{2i}-b_{2j}|^2 + 3\sum_{i,j}^{N_2} \ln|a_{2i}-b_{2j}|^2,$$
$$V_{N_1,N_2} = -2\sum_{i,j}^{N_1,N_2} \ln|a_{1i}-b_{2j}|^2 - 2\sum_{i,j}^{N_1,N_2} |a_{2i}-b_{1j}|^2 + A\sum_{i,j}^{N_1,N_2} \ln|a_{1i}-a_{2j}|^2 + B\sum_{i,j}^{N_1,N_2} \ln|b_{1i}-b_{2j}|^2. \tag{52}$$

The above partition function thus represents two coupled MCGS. In the absence of interactions, the system is identical to the system in Sec. III A. A detailed study of the interaction in (53) will shed further light on the behavior of this system.

We now comment on the renormalization of the coupling constant β_0 for immersion in R^4. Regularization of the determinants in (49) leads to the following result for $\beta_R(\mu)$:

$$\beta_R(\mu) = \frac{\beta_0}{1 - 3\left(\frac{\beta_0}{\pi}\right)\ln\frac{\Lambda}{\mu}}. \tag{53}$$

The factor 3 in the denominator in (54) is consistent with our observation following (15). For surfaces immersed in R^4, the self-intersection number, which is a topological invariant, is given in terms of the Gauss map by

$$\mathcal{I} = \frac{1}{2}\int \{(|F_1|^2 - |\hat{F}_1|^2) - (|F_2|^2 - |\hat{F}_2|^2)\} d^2z\ , \tag{54}$$

where $\hat{F}_i = f_{iz}/(1+|f_i|^2)$; i=1,2 and this vanishes identically for the background configuration studied here. The self-intersection number plays the role of the θ term in the QCD Lagrangian [5, 26]. In order to study the θ term, we need to consider immersed surfaces of non-zero self-intersection number.

IV. SUMMARY

One-loop multi-instanton and anti-instanton effects in the theory of conformally immersed 2D string world sheet in background R^3 and R^4 have been evaluated in this paper. N-instanton solutions arise as the Gauss map of minimally immersed surfaces, i.e., surfaces with vanishing mean curvature scalar h, with $2N$ punctures. Anti-instanton solutions arise as Gauss map of harmonic surfaces (h=const). In both cases the semiclassical solutions are chosen to represent Gauss map of genus zero surfaces. The minimal surface with $2N$ punctures describes open string interactions at the tree level, while the harmonic surfaces are appropriate for describing closed string interactions.

We have used the language of Gauss map in this paper. The Gauss map is the mapping of a conformally immersed surface in R^n into the Grassmannian manifold $G_{2,n}$. The Grassmannian σ-model fields satisfy certain integrability conditions. The problem of the string dynamics or of random surfaces, especially the ones influenced by the extrinsic geometry, is reduced to the problem of a constrained σ model. The instantons referred to above are instantons of the σ model. Semiclassical treatment about these instantons is equivalent to doing functional integral over the above classes of 2D surfaces.

The generating functional for both minimal and harmonic immersions in R^3 and in R^4 has been evaluated. It is found that it is the grand partition function of a classical system of instanton quarks and antiquarks interacting through logarithmic (Coulombic) potentials at an inverse temperature $\beta = 1$. We call this system the modified Coulomb gas system (MCGS). It differs from the corresponding result in the ordinary CP^1 model where one finds a classical 2D Coulomb gas system. MCGS reflects the effect of the integrability condition on the partition function. It increases the strength of the interactions between instanton quark and antiquark pairs. The fugacity of the MCGS is proportional to $1/\alpha_R(\mu)$, where $\alpha_R(\mu)$ is the renormalized running coupling constant of the extrinsic curvature term and μ is the renormalization energy scale.

The physics implied in these results appear to be interesting in the context of QCD. In particular, it seems to suggest that in the infrared region when $\alpha_R(\mu)$ becomes large, corresponding to a large chemical potential, the critical β_c will be less than one. In this case, the system which is at $\beta = 1$ will be found in a molecular phase characterized by long-range order and zero mass gap. This implies that the string world sheet is stable against small fluctuations and avoids crumpling in the large distance limit. On the opposite end, in the UV region, the system is likely to be found in the plasma phase which has a mass gap. These properties seem desirable for a description of QCD strings.

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