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$O(1/N_f)$ corrections to the Thirring model in $2 < d < 4$

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The Thirring model, that is, a relativistic field theory of fermions with a contact interaction between vector currents, is studied for dimensionalities $2 < d < 4$ using the $1/N_f$ expansion, where N_f is the number of fermion species. The model is found to have no ultraviolet divergences at leading order provided a regularization respecting current conservation is used. Explicit $O(1/N_f)$ corrections are computed, and the model is shown to be renormalizable at this order in the massless limit; renormalizability appears to hold to all orders due to a special case of Weinberg's theorem. This implies that there is a universal amplitude for four particle scattering in the asymptotic regime. Comparisons are made with both the Gross-Neveu model and QED.

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I. FOUR-FERMI THEORIES IN $d \in (2,4)$

It has been believed for some time that a renormalizable expansion for four-Fermi models exists for dimensions larger than two, which is naively the upper critical dimension [1–5]. Rather than using the coupling constant g^2, which has inverse dimensions of mass for $d > 2$, to organize the expansion, the dimensionless parameter $1/N_f$, where N_f is the number of fermion species, is used. The standard example is the Gross-Neveu model:

$$\mathcal{L} = \bar{\psi}_i \partial\!\!\!/ \psi_i - \frac{g^2}{2N_f}(\bar{\psi}_i\psi_i)^2 \ . \tag{1.1}$$

In this case, which has been widely studied [1–8], spontaneous fermion mass generation occurs for values of $g^2 > g_c^2 = O(\Lambda^{2-d})$, where Λ is some ultraviolet cutoff in the model. It is preferable to discuss the problem with d a continuous parameter, $d \in (2,4)$; the scaling properties of the model are then more transparent. Of course, only $d = 3$ can correspond to a physically realizable system. If the coupling is now fine-tuned to the neighborhood of g_c, then light fermions propagate and interact via exchange of a composite scalar state of mass $2m$, where m is the fermion mass. (Actually a perfectly acceptable model also arises by approaching g_c from the massless phase.) Because the model is strongly interacting at $g = g_c$, the ultraviolet asymptotic behavior of the scalar propagator, obtained by resumming a sequence of fermion-antifermion bubble diagrams which are dominant at leading order in $1/N_f$, is nonstandard:

$$\lim_{k^2\to\infty} D_\sigma(k) \propto \frac{1}{k^{d-2}} \ . \tag{1.2}$$

The behavior (1.2) of D_σ when input to a standard power-counting analysis [4,5] implies that the superficial degree of divergence of Feynman diagrams describing corrections of higher order in $1/N_f$ does not depend on the number of interaction vertices, which in turn suggests that the expansion is exactly renormalizable. This property has been explicitly verified at $O(1/N_f)$ [5,7–9]. Physically, the renormalizability of the model may be understood as a consequence of its being the infrared fixed point under renormalization group flow of a model of fermions interacting with elementary (i.e., not auxiliary) scalar fields via a Yukawa interaction [3,7,10]. This model is superrenormalizable. The IR fixed point of the Yukawa model is identical to a UV fixed point of the Gross-Neveu model as $g^2 \to g_c^2$. The relation between renormalizability and hyperscaling relations between the model's critical exponents (which are polynomials in $1/N_f$) was stressed in [7,8]. The exponents are currently all known to $O(1/N_f^2)$ [11], some to $O(1/N_f^3)$ [12], and have been verified for $d = 3$ by numerical simulation first for $N_f = 12$ [8] and most recently for $N_f = 2$ [13]. The situation is analogous to that in interacting scalar field theories; there the IR fixed point of the superrenormalizable ϕ^4 theory in $d \in (2,4)$ is identical to the UV fixed point of a corresponding nonlinear σ model [1,3], which once again has an unexpected renormalizability in $1/N_f$ [14,15].

In this paper I wish to concentrate on another, distinct interacting fermion theory, a generalization of the massive Thirring model. The Lagrangian is

$$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + m)\psi_i + \frac{g^2}{2N_f}(\bar{\psi}_i\gamma_\mu\psi_i)^2 \ . \tag{1.3}$$

This model has also been studied in the $1/N_f$ expansion [16–19]. In this case there is no phase transition corresponding to spontaneous mass generation, but instead the "vacuum polarization" fermion bubble diagrams correcting the intermediate boson propagator prove to be UV finite, despite a superficial Λ^{d-2} divergence, due to fermion current conservation. The situation is analogous to QED, where due to current conservation, the transverse projection operator $(\delta_{\mu\nu}k^2 - k_\mu k_\nu)$ can always be factored from the vacuum polarization, reducing the effective degree of divergence by two. In QED the result is that the photon remains massless at each order of perturbation theory.

It is interesting to contrast the Thirring model with the Gross-Neveu model. In the latter case the superficial Λ^{d-2} divergences do not cancel, and result in an additive renormalization of the coupling g^2, and hence the need to fine-tune to recover the continuum limit. In the Thirring case, the Λ^{d-2} divergences vanish for essentially kinematic reasons, the coupling g^2 is not renormalized (as shown below), and the continuum limit appears to exist, to leading order in $1/N_f$, for all values of g^2. As in the Gross-Neveu model, the expression for the intermediate boson propagator in the Thirring model, which is now a vector, can be resummed, to give the same asymptotic form as Eq. (1.2). Therefore the power-counting arguments also suggest that the $1/N_f$ expansion is renormalizable. However, the underlying physics is different; there is no phase transition corresponding to a fixed point condition. It is plausible, as we shall briefly discuss, that there is a superrenormalizable model which yields equivalent physics in its IR limit, namely, large-N_f QED. In this paper, however, we shall discuss renormalizability entirely within the context of the $1/N_f$ expansion of the four-Fermi theory.

The potential problem which might arise for renormalization at higher orders in four-Fermi models was highlighted in [8]. At next-to-leading order, the corrections to the boson propagator are given by two-loop diagrams, exemplified in Fig. 4. These are superficially Λ^{d-2} divergent, as discussed above, but there are also subleading divergences of the form (on dimensional grounds) $k^{d-2}\ln\Lambda$, divergent contributions which are nonpolynomial in external momentum, and which hence cannot be compensated by the addition of a local counterterm. These terms, if not removed by explicit cancellation with other divergent graphs at the same order (which was demonstrated for the Gross-Neveu model in [8]), would spoil the renormalizability of the model, and result in a nonlocal interaction being generated as the cutoff is removed. As we shall see, this constraint on subdivergences is a feature of a graphical expansion in which diagrams of one and two loops appear at the same order. It will turn out that the cancellation of nonpolynomial divergences is a natural consequence of Weinberg's theorem [20] applied to theories with nonstandard propagators. There appears to be no barrier to extending a renormalizability proof to all orders using standard arguments.

The rest of the paper is organized as follows. In Sec. II we review the work of [18,19] in setting up the $1/N_f$ expansion for the Thirring model at leading order, extending their work, which was for the special (but physical) case $d=3$, to the interval $d\in(2,4)$. We shall give a closed form expression for the auxiliary vector propagator, and examine it in various limits, including the important deep Euclidean limit $k^2\to\infty$. The mass of the resulting vector boson is discussed as a function of g and m. In Sec. III the divergence structure of the model is discussed, $O(1/N_f)$ corrections computed, and the renormalization of the model at this order given. The condition that the fermion current is conserved translates into a requirement that the vacuum polarization is two-loop finite: this is verified explicitly. In Sec. IV we compare the Thirring model with the Gross-Neveu model and show why the cancellation of nonpolynomial divergences is to be expected as a result of Weinberg's theorem: the result is that both models have a very similar asymptotic structure corresponding to an interacting UV fixed point, despite the contrast at low energies. Finally, comparisons are drawn between the Thirring model and QED, and possible implications for a nontrivial fixed point for the latter are discussed.

II. LEADING ORDER RESULTS

Consider the Lagrangian for the Thirring model in the bosonized form

$$\mathcal{L}=\bar\psi_i\partial\!\!\!/\,\psi_i+m\bar\psi_i\psi_i+\frac{ig}{\sqrt{N_f}}A_\mu\bar\psi_i\gamma_\mu\psi_i+\tfrac{1}{2}A_\mu A_\mu\ , \tag{2.1}$$

where sums on repeated spacetime indices μ and flavor indices i are understood. The field A_μ is a vector auxiliary: it may be integrated over to recover the original Lagrangian (1.3). In d-dimensional Euclidean space, $d\in(2,4)$, we define $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}\mathbb{1}$, $\delta_{\mu\mu}=d$, $\mathrm{tr}\mathbb{1}=4$; that is, we assume four component spinors and hence avoid complications due to parity violation and generation of a Chern-Simons term in $d=3$ [18,19]. The Feynman rules are thus

$$\begin{aligned}&\text{fermion propagator}:\ S_F=(i\,p\!\!\!/+m)^{-1}\ ,\\ &\text{interaction vertex}:\ \Gamma_\mu=-\frac{ig}{\sqrt{N_f}}\gamma_\mu\ .\end{aligned} \tag{2.2}$$

To leading order in $1/N_f$ the auxiliary propagator receives a contribution from "vacuum polarization," that is, a fermion-antifermion bubble (Fig. 1), so we write

$$\text{vector propagator}:\ D^{-1}_{\mu\nu}(k)=\delta_{\mu\nu}-\Pi_{\mu\nu}(k)\ , \tag{2.3}$$

where the leading order vacuum polarization is given by

$$\Pi_{\mu\nu}(k)=\frac{g^2}{N_f}N_f\int_p\mathrm{tr}\gamma_\mu\frac{1}{i(p\!\!\!/+a\,k\!\!\!/)+m}\times\gamma_\nu\frac{1}{i[p\!\!\!/+(a-1)\,k\!\!\!/]+m}\ . \tag{2.4}$$

The constant a defining the momentum routing is kept arbitrary at present. For $d\in(2,4)$ momentum integration is defined as

$$\int_p f(p^2)=S_d\int_0^\infty p^{d-1}f(p^2)dp\ , \tag{2.5}$$

with

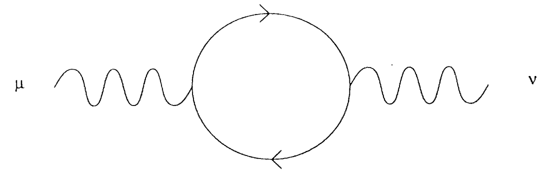

FIG. 1. Leading order contribution to the vector auxiliary two-point function. Solid lines represent fermions, wavy lines the vector auxiliary.

$$\int_p p_\mu p_\nu f(p^2) = \frac{\delta_{\mu\nu}}{d}\int_p p^2 f(p^2) \ ,$$
$$\int_p p_\mu p_\nu p_\lambda p_\kappa f(p^2) = \frac{\delta_{\mu\nu}\delta_{\lambda\kappa}+\delta_{\mu\lambda}\delta_{\nu\kappa}+\delta_{\mu\kappa}\delta_{\nu\lambda}}{d(d+2)} \times \int_p p^4 f(p^2) \ , \tag{2.6}$$

etc., and

$$S_d \equiv \frac{2}{(4\pi)^{d/2}\Gamma(\frac{d}{2})} \ . \tag{2.7}$$

At this stage no regularization is specified. We now perform the trace and then apply Schwinger parametrization:

$$\Pi_{\mu\nu}(k) = 4g^2\int_0^\infty d\alpha\, d\beta \int_p \exp\{-\alpha((p+ak)^2+m^2) - \beta([p+(a-1)k]^2+m^2)\}$$
$$\times[-2p_\mu p_\nu + 2a(1-a)k_\mu k_\nu + (1-2a)(p_\mu k_\nu + p_\nu k_\mu) + \delta_{\mu\nu}(p^2 + a(a-1)k^2 + k\cdot p + m^2)] \ . \tag{2.8}$$

Since $\int_p$ is now finite, the momentum p may be shifted and the integration performed; the result is

$$\Pi_{\mu\nu}(k) = \frac{4g^2}{(4\pi)^{d/2}}\int_0^\infty \frac{d\alpha\, d\beta}{(\alpha+\beta)^{d/2}}\exp\left[-(\alpha+\beta)m^2 - \frac{\alpha\beta}{\alpha+\beta}k^2\right]$$
$$\times\left\{-\frac{2\alpha\beta}{(\alpha+\beta)^2}(k^2\delta_{\mu\nu} - k_\mu k_\nu) + \delta_{\mu\nu}\left[m^2 + \frac{\alpha\beta}{(\alpha+\beta)^2}k^2 + \frac{d-2}{2(\alpha+\beta)}\right]\right\} \ . \tag{2.9}$$

Note that all dependence on the momentum routing parameter a has disappeared. Now the integral of the second term in curly brackets, proportional to $\delta_{\mu\nu}$, may be reexpressed as

$$\frac{\partial}{\partial x}\int_0^\infty \frac{d\alpha\, d\beta}{(\alpha+\beta)^{d/2+1}}\frac{1}{x^{d/2-1}}\exp\left[-x\frac{\alpha\beta}{\alpha+\beta}k^2 - x(\alpha+\beta)m^2\right]\Bigg|_{x=1} \ . \tag{2.10}$$

However, it can be seen that the integral in (2.10) is formally independent of x, by rescaling α and β. Strictly, the integral diverges and must be made finite by use of a Pauli-Villars regulator field (e.g., see [21], Chap. 7). Its contribution to $\Pi_{\mu\nu}$ thus vanishes and we can write

$$\Pi_{\mu\nu} = \mathcal{P}_{\mu\nu}(k)\Pi(k^2) \ , \tag{2.11}$$

with the transverse projection operator $\mathcal{P}_{\mu\nu}(k)$ defined as

$$\mathcal{P}_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \ . \tag{2.12}$$

The remaining integrations over α and β are straightforward, and the result for $\Pi(k^2)$ is finite:

$$\Pi(k^2) = -\frac{8g^2k^2}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\frac{x(1-x)}{[x(1-x)k^2+m^2]^{2-d/2}} = -g^2\frac{4\Gamma\left(2-\frac{d}{2}\right)}{3(4\pi)^{d/2}m^{4-d}}k^2F\left(2;2-\frac{d}{2};\frac{5}{2};-\frac{k^2}{4m^2}\right) \ , \tag{2.13}$$

where F is the hypergeometric function.

For $d<4$ we see that the vacuum polarization tensor $\Pi_{\mu\nu}(k)$ can be evaluated exactly at leading order, with the assumption of a regularization which respects current conservation. It is interesting to compare (2.13) with known results in $d=4$ and $d=2$. In the limit $d\to 4_-$,

$$\Pi_{\mu\nu}(k) = -\frac{g^2}{6\pi^2}\left(\frac{1}{(4-d)} - \frac{\gamma_E}{2}\right)k^2\mathcal{P}_{\mu\nu}(k) \ , \tag{2.14}$$

where γ_E is the Euler constant. This is almost the textbook result for one-loop vacuum polarization in dimensionally regularized four-dimensional QED (QED_4), except that since we have had no need to introduce a renormalization scale to make the coupling g^2 dimensionless, then there is no term in $\ln k^2$. If we use the linear transformation properties of F to examine the limit $m^2\to 0$ (e.g., [22], Chap. 15), it is also possible subsequently to take the limit $d\to 2_+$ with the result

$$\Pi_{\mu\nu}(k) = -2\frac{g^2}{\pi}\mathcal{P}_{\mu\nu}(k) \ . \tag{2.15}$$

This is exactly twice the result for the one-flavor massless Schwinger model (in that case Fig. 1 generates a dynamical photon mass $g/\sqrt{\pi}$); the extra factor of 2 arises from our insistence on four component spinors.

Now we set $d=3$ to obtain

$$\Pi_{\mu\nu}(k) = -\frac{g^2}{2\pi}\mathcal{P}_{\mu\nu}(k)\left[m + \frac{1}{2(k^2)^{1/2}}(k^2-4m^2)\times\arctan\left(\frac{(k^2)^{1/2}}{2m}\right)\right] . \qquad (2.15')$$

This is identical to the result of Yang [18] and Gomes *et al.* [19], with the momentum k analytically continued to Euclidean space. It is important to note that for the whole range $d\in(2,4)$ the asymptotic form of $\Pi_{\mu\nu}(k)$ is not polynomial in k^2: viz.,

$$\lim_{k^2\to\infty}\Pi_{\mu\nu}(k) \sim -g^2\mathcal{P}_{\mu\nu}(k)\frac{(k^2)^{d/2-1}}{A_d} , \qquad (2.16)$$

with the numerical constant A_d given by

$$A_d = \frac{d-1}{d-2}\frac{(4\pi)^{d/2}}{4\Gamma\left(2-\frac{d}{2}\right)B\left(\frac{d}{2},\frac{d}{2}-1\right)} , \qquad (2.17)$$

where B is the beta function. The form (2.16) was first found by Hikami and Muta [17], modulo a difference of definition of $\mathrm{tr}\mathbb{1}$, and a factor of d.

Let us now return to expression (2.3) for the inverse vector propagator. Using (2.11) we can invert to yield the propagator

$$D_{\mu\nu}(k) = \mathcal{P}_{\mu\nu}(k)\frac{1}{1-\Pi(k^2)} + \frac{k_\mu k_\nu}{k^2} . \qquad (2.18)$$

As argued in [16,18,19], the second term in $D_{\mu\nu}(k)$, which is longitudinal, behaves as a constant in the limit $k\to\infty$, and might naively be expected to lead to poor ultraviolet behavior. However, since the vector auxiliary interacts with a physical current which is conserved, S-matrix elements and observables constructed as gauge-invariant combinations of the fields ψ and A_μ (in the sense used in QED) should not be affected by this problem, although Green functions in general might be. We shall see this when we calculate $O(1/N_f)$ corrections in the next section. To aid calculation (but not to *define* the vector propagator, as would be the case in QED), following [19], we introduce a gauge-fixing term $(1/2\mu^2)(\partial_\mu A_\mu)^2$ to the Lagrangian (2.1), which has the effect of moderating the UV behavior of the second term for finite μ, which has dimensions of mass. The vector propagator becomes

$$D_{\mu\nu}(k;\mu) = \mathcal{P}_{\mu\nu}(k)\frac{1}{1-\Pi(k^2)} + \frac{\mu^2}{\mu^2+k^2}\frac{k_\mu k_\nu}{k^2} . \qquad (2.19)$$

The scale μ is in effect a regulator which should not appear in final expressions. The limit $\mu\to\infty$ recovers the original Thirring model, whereas the limit $\mu\to 0$ specifies a "Landau gauge" which will render the two-loop calculation in the next section much easier.[1] Note also that in the limit $k^2\to 0$, $\Pi(k^2)\to 0$ and $D_{\mu\nu}(k)\to\delta_{\mu\nu}$: hence the infrared problems associated with QED are not present here.

On the assumption that the longitudinal piece of $D_{\mu\nu}$ has no physical consequence, we focus on the transverse piece and identify a pole condition for the mass of the vector M_V:

$$1 - g^2M_V^2\frac{4\Gamma\left(2-\frac{d}{2}\right)}{3(4\pi)^{d/2}m^{4-d}}F\left(2;2-\frac{d}{2};\frac{5}{2};\frac{M_V^2}{4m^2}\right) = 0 . \qquad (2.20)$$

In general this is a transcendental equation. It can be solved in two limits. For strong coupling $g^2\gg m^{2-d}$ the vector channel will be dominated by a tightly bound fermion-antifermion state, so $M_V^2\ll m^2$. Therefore we can expand F to obtain

$$\frac{M_V^2}{m^2} \simeq \frac{m^{2-d}}{g^2}\left(\frac{3(4\pi)^{d/2}}{4\Gamma\left(2-\frac{d}{2}\right)}\right) . \qquad (2.21)$$

For arbitrarily weak coupling, real solutions of (2.20) can only exist if the hypergeometric function is able to grow arbitrarily large. We expect a weakly bound state to have mass given by

$$M_V = 2m - \varepsilon . \qquad (2.22)$$

As $M_V\to(2m)_-$, the hypergeometric function diverges only for $d<3$. In this case we can once again perform a linear transformation on F to get

$$1 - g^2M_V^2\frac{\sqrt{\pi}\Gamma\left(\frac{3-d}{2}\right)}{m^{4-d}(4\pi)^{d/2}}\left(\frac{\varepsilon}{m}\right)^{(d-3)/2}\left[1+O\left(\frac{\varepsilon}{m}\right)\right] = 0 , \qquad (2.23)$$

i.e., the binding energy is

$$\varepsilon = m\left(g^2m^{d-2}\frac{\Gamma\left(\frac{3-d}{2}\right)}{2^{d-2}\pi^{(d-1)/2}}\right)^{2/(3-d)} . \qquad (2.24)$$

The case $d=3$ must be handled separately; we use expression (2.15′) to find the binding energy is essentially singular in g^2:

$$\varepsilon = 2m\exp\left(-\frac{2\pi}{mg^2}\right) . \qquad (2.25)$$

Finally, for $d>3$ the hypergeometric function remains finite as $k^2\to -4m^2$. In this case the bound state van-

[1]The invariance of the Thirring model under local gauge transformations has recently been put on a firmer footing [23].

ishes (i.e., the would-be pole coalesces with the branch cut in F) for values of g below a critical g_c given by

$$g_c^2 = m^{2-d}\frac{(4\pi)^{d/2}(d-1)(d-3)}{16\Gamma\left(2-\frac{d}{2}\right)} . \tag{2.26}$$

In the subcritical region $D_{\mu\nu}$ has no poles on the physical sheet; the vector can only be regarded as a resonant intermediate state in four-Fermi scattering.

In the deep Euclidean region $k^2 \to \infty$ things simplify considerably: the vector propagator has the form

$$\lim_{k^2\to\infty} D_{\mu\nu}(k) = \mathcal{P}_{\mu\nu}(k)\frac{A_d}{g^2}\frac{1}{(k^2)^{(d/2)-1}} . \tag{2.27}$$

In this limit the four-Fermi scattering amplitude has the form $A_d J_\mu(q) J_\mu(q+k)/N_f k^{d-2}$. As we shall argue in the final section, this interaction receives no corrections in the $1/N_f$ expansion, and is thus a universal form characterizing the short-distance structure of the model; in other words it defines a UV fixed point. In this respect it resembles the Gross-Neveu model as discussed in [8] (though note the definition of A_d is distinct). In the next section when the renormalization of the model at $O(1/N_f)$ is discussed, the form (2.27) will be used throughout.

III. RENORMALIZATION AT $O(1/N_f)$

In this section I will discuss the renormalization of the model to next-to-leading order in the $1/N_f$ expansion. First let me review why we might expect such a program to be feasible. The short-distance fluctuations of the model are encoded in the asymptotic form for the vector propagator (2.27). Suppose we analyze the superficial degree of divergence of a higher order diagram with N_ψ external fermion lines, N_A external auxiliary lines, and V vertices. If we use the form (2.27), then standard power counting analysis gives the superficial degree of divergence ω:

$$\omega = d - \frac{d-1}{2}N_\psi - N_A . \tag{3.1}$$

It is interesting to compare this result with that for canonical boson asymptotics, $D(k) \sim 1/k^2$, which applies, say, for QED:

$$\omega_{\rm can} = d - \frac{d-1}{2}N_\psi - \frac{d-2}{2}N_A - \frac{4-d}{2}V . \tag{3.2}$$

For $d < 4$ the degree of divergence falls as the number of vertices increases: this is characteristic of a superrenormalizable theory. Only when ω is independent of V can a perturbative expansion be exactly renormalizable (i.e., divergent graphs appear at every order of the expansion, but can always be made finite by retuning a finite set of counterterms).

Using (3.1) we can compile a list of potentially dangerous graphs for $d \in (2,4)$. The $O(1/N_f)$ contributions are shown in Figs. 2–4. The fermion self-energy $\psi\psi$ (Fig. 2) has $\omega = 1$, as in QED_4, but since that divergence is odd in loop momentum, the true divergence is logarithmic ($\omega = 0$). The vertex correction $\psi\psi A$ (Fig. 3) also has $\omega = 0$, but the four-vector $AAAA$ scattering, which is superficially divergent in QED_4, here has $\omega = d-4$ and so is safe. One-point and three-point vector scatterings vanish by Furry's theorem, leaving the vector two-point function AA (Fig. 4), with $\omega = d-2$ as the only other superficially divergent case. One can then argue [19] that in a regularization which respects current conservation, one can always extract a factor $k^2\mathcal{P}_{\mu\nu}(k)$ from these diagrams to give $\omega = d-4$, and hence no new divergence. The model (2.1) can then be renormalized simply by rescaling the ψ and A fields, and retuning the fermion mass. I shall show in this section that this conclusion is correct, though the argument is not quite so straightforward. Because the vacuum polarization is nonpolynomial in k^2, it is in fact only permissible to extract $\mathcal{P}_{\mu\nu}(k)$ from inside the graph, which does not improve the power counting. As we shall see, there are divergent contributions both of degree $\omega = d-2$ and $\omega = 0$. The applicability of power counting to four-Fermi models is discussed further in the final section.

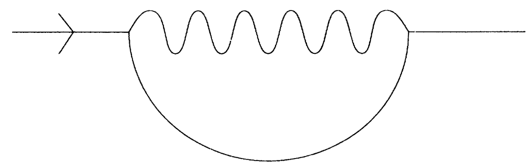

FIG. 2. $O(1/N_f)$ contribution to the fermion self-energy.

It is worth contrasting the four-Fermi case with the situation in pure scalar theories. In the renormalization of the nonlinear σ model in $d \in (2,4)$ [15], the power counting gives

$$\omega = d - \frac{d-2}{2}N_\psi - 2N_A , \tag{3.3}$$

where now ψ denotes the elementary scalar field and A the auxiliary scalar boson. The only superficial divergences are $\psi\psi(\omega = 2)$ and $\psi\psi A(\omega = 0)$; the auxiliary propagator AA has $\omega = d-4$ and hence is superficially convergent. On dimensional grounds there is no reason to expect any nonpolynomial divergences.

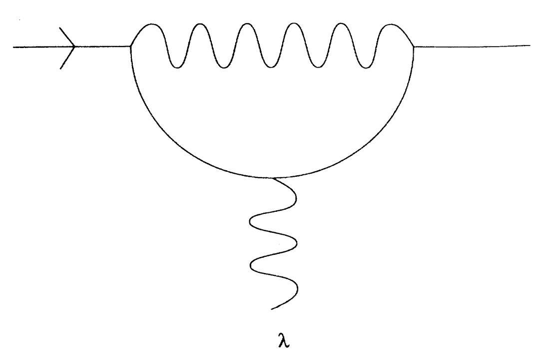

FIG. 3. $O(1/N_f)$ contribution to the vertex.

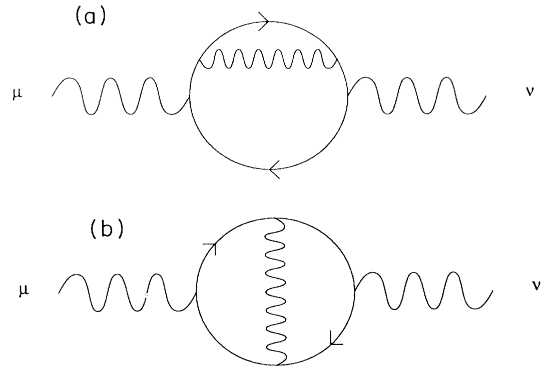

FIG. 4. (a)$O(1/N_f)$ contribution to the vector two-point function. (b) $O(1/N_f)$ contribution to the vector two-point function.

The procedure for renormalizing the model follows the treatment in [8]: first we redefine the Lagrangian

$$\mathcal{L} = Z_\psi \bar{\psi}_i(\partial\!\!\!/ + M)\psi_i + \frac{ig}{\sqrt{N_f}} Z_\psi Z_A^{1/2} \bar{\psi}_i A\!\!\!/ \psi_i + \frac{1}{2} Z_A \left(A_\mu^2 + \frac{1}{\mu^2}(\partial_\mu A_\mu)^2 \right) . \tag{3.4}$$

The constants Z_ψ, Z_A, M and in principle g and μ are all cutoff dependent, and must be adjusted at each order of the $1/N_f$ expansion to keep physical matrix elements finite. As we have seen, at leading order an adequate choice is $Z_\psi = 1$; $M = m$, the physical fermion mass; $\mu \to \infty$; and g, Z_A unconstrained. Z_A simply defines the scale of an auxiliary field at leading order and hence has no physical relevance. The first divergent Green function we must examine is the fermion self-energy (Fig. 2):

$$\Sigma(k) = -\frac{g^2}{N_f} \frac{Z_\psi^2 Z_A}{Z_\psi Z_A} \int_p \gamma_\mu \frac{1}{i(p\!\!\!/ + k\!\!\!/) + M} \gamma_\nu D_{\mu\nu}(p;\mu) . \tag{3.5}$$

Note that with the definition (3.4), the vector propagator at leading order is $Z_A^{-1} D_{\mu\nu}(k;\mu)$, with $D_{\mu\nu}$ defined by (2.19). On rearranging we find

$$\Sigma(k) = -\frac{g^2}{N_f} Z_\psi \int_p \gamma_\mu \frac{-i(p\!\!\!/ + k\!\!\!/) + M}{(p+k)^2 + M^2} \gamma_\nu \left[A(p^2)\mathcal{P}_{\mu\nu}(p) + B(p^2;\mu)\frac{p_\mu p_\nu}{p^2} \right] , \tag{3.6}$$

with

$$\lim_{k^2 \to \infty} A(k^2) \sim \frac{A_d}{g^2 (k^2)^{d/2-1}}, \quad B(k^2) = \frac{\mu^2}{\mu^2 + k^2} . \tag{3.7}$$

We will treat the parts depending on $A(p^2)$ and $B(p^2;\mu)$ separately. For the first piece, apart from a term which is odd in p and hence vanishes on $\int_p$, the leading contribution is $O(p^{-1}dp)$, and hence logarithmically divergent. With the choice of a simple momentum cutoff $|p| < \Lambda$, and the definitions (2.5) and (2.6), we find

$$\Sigma^A(k) = -Z_\psi \frac{C_d}{N_f} \frac{(d-1)^2}{2(d-2)} \left[i k\!\!\!/ \frac{d-4}{d} + M \right] \ln \frac{\Lambda}{M} + \text{finite} , \tag{3.8}$$

with the constant C_d defined as in [8]:

$$C_d = \frac{1}{B\left(\frac{d}{2}, 2 - \frac{d}{2}\right) B\left(\frac{d}{2}, \frac{d}{2} - 1\right)} . \tag{3.9}$$

Note C_d is positive definite for $d \in (2,4)$. For $d = 3$, C_d has the value $4/\pi^2$. The piece depending on $B(p^2;\mu)$ can be recast in the form

$$\Sigma^B(k) = -\frac{g^2 Z_\psi \mu^2}{N_f} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{d/2}} \int_0^1 dx [x(1-x)k^2 + xM^2 + (1-x)\mu^2]^{d/2-2} \left[i(1+x) k\!\!\!/ + M - \frac{2i}{d} k\!\!\!/ \right] + O(\mu^{d-6}) . \tag{3.10}$$

In the limit $\mu^2 \to \infty$,

$$\Sigma^B(k) = -\frac{g^2 Z_\psi \mu^{d-2}}{N_f} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{(4\pi)^{d/2}} \frac{2}{(d-2)} (i k\!\!\!/ + M) + O(\mu^{d-4}) . \tag{3.11}$$

Note that Σ^A and Σ^B have different characteristics: Σ^A is cutoff dependent but independent of the coupling constant g, whereas Σ^B is finite for finite μ, but depends on both μ and g. The Thirring model limit $\mu^2 \to \infty$ cannot be taken for Σ in isolation.

We can now write for the full inverse fermion propagator

$$S_F^{-1}(k) = Z_\psi [i k\!\!\!/ + M - \Sigma(k)] \equiv i k\!\!\!/ + m , \tag{3.12}$$

thus defining the wave function renormalization constant Z_ψ and the renormalized (i.e., physical) mass m in terms of the bare mass M:

$$Z_\psi = 1 - \frac{C_d}{N_f}\frac{(d-1)^2(d-4)}{2d(d-2)}\ln\frac{\Lambda}{m} - \frac{g^2\mu^{d-2}}{N_f}\frac{2\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}(d-2)} + \text{finite} , \tag{3.13}$$

$$m = M\left(1 + \frac{C_d}{N_f}\frac{2(d-1)^2}{d(d-2)}\ln\frac{\Lambda}{m} + \text{finite}\right) > M . \tag{3.14}$$

Thus we find an expression for the physical mass m independent of the regulator μ, and a wave function constant Z_ψ which depends on μ. This is, of course, very similar to what is found for QED: the physical parameter m is renormalized in a gauge-invariant fashion, whereas the unphysical Z_ψ is not. The term proportional to C_d in (3.13) was derived in [17], but the μ-dependent piece was neglected.

The results (3.13) and (3.14) for the wave function and mass renormalizations are identical to those obtained in large-N_f QED in $d \in (2,4)$ evaluated at its critical point [24]. This is strong evidence for the equivalence of the Thirring model to large-N_f QED considered at its infrared fixed point. Both mass and wave function renormalizations (in arbitrary covariant gauge) are now known in QED to $O(1/N_f^2)$ [25]; presumably these results will also be valid for the Thirring model.

Next we calculate the $O(1/N_f)$ contribution $\Gamma_\lambda^{[1]}$ to the vertex (Fig. 3). For zero external momentum we have

$$\begin{aligned}\Gamma_\lambda^{[1]} &= \frac{ig^3}{N_f^{3/2}}\frac{Z_\psi^3 Z_A^{3/2}}{Z_\psi^2 Z_A}\int_p \gamma_\mu\frac{1}{i\not p + M}\gamma_\lambda\frac{1}{i\not p + M}\gamma_\nu D_{\mu\nu}(p;\mu) \\ &\simeq -ig\frac{Z_\psi Z_A^{1/2}}{\sqrt{N_f}}\frac{g^2}{N_f}\int_p \frac{1}{(p^2+M^2)^2}D_{\mu\nu}(p;\mu)\gamma_\mu\not p\gamma_\lambda\not p\gamma_\nu ,\end{aligned} \tag{3.15}$$

where the second line follows because we are only interested in the divergent part. Using the same procedure as before, we find, for the full vertex,

$$\begin{aligned}\Gamma_\lambda &= \Gamma_\lambda^{[0]} + \Gamma_\lambda^{[1]} \\ &= -ig\frac{Z_\psi Z_A^{1/2}}{\sqrt{N_f}}\gamma_\lambda\left[1 + \frac{1}{N_f}\left(C_d\frac{(d-1)^2(d-4)}{2d(d-2)}\ln\frac{\Lambda}{m} + g^2\mu^{d-2}\frac{2\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}(d-2)}\right) + \text{finite}\right] .\end{aligned} \tag{3.16}$$

However, the constant Z_ψ has already been determined in (3.13), and is found to exactly cancel both Λ- and μ-dependent terms in (3.16). Hence

$$\Gamma_\lambda = -\frac{ig}{\sqrt{N_f}}Z_A^{1/2}\gamma_\lambda[1 + O(1/N_f)\times\text{finite}] . \tag{3.17}$$

Once again, this is familiar situation from QED: current conservation plus gauge invariance ensures that the divergent and gauge-dependent parts of the self-energy and vertex corrections cancel, i.e., $Z_1 = Z_2$ (as noted in Sec. II, there are no problems with infrared divergences in the Thirring case). We expect this cancellation to persist at higher orders. So, to maintain the finiteness of fermion self-energy and vertex corrections to $O(1/N_f)$, our only requirement of Z_A and g so far is that the combination $Z_A^{1/2}g$ be finite. However, we have not yet exhausted the list of superficially divergent graphs. We next consider the $O(1/N_f)$ corrections to the vector propagator, which consists of two two-loop diagrams [Figs. 4(a) and 4(b)]:

$$\begin{aligned}\Pi_{\mu\nu}^{[1]}(k) &= -\frac{g^4}{N_f^2}\times\frac{Z_\psi^4 Z_A^2}{Z_\psi^2 Z_A}\times N_f\times[2I_{\mu\nu}^a(k) + I_{\mu\nu}^b(k)] , \\ I_{\mu\nu}^a(k) &= \int_{pq}\text{tr}\left(\gamma_\mu\frac{1}{i\not p}\gamma_\alpha\frac{1}{i(\not p+\not q)} \right. \\ &\quad \left.\times\gamma_\beta\frac{1}{i\not p}\gamma_\nu\frac{1}{i(\not p+\not k)}\right)D_{\alpha\beta}(q) , \\ I_{\mu\nu}^b(k) &= \int_{pq}\text{tr}\left(\gamma_\mu\frac{1}{i\not p}\gamma_\alpha\frac{1}{(\not p+\not q)}\gamma_\nu\frac{1}{i(\not p+\not q+\not k)}\right. \\ &\quad \left.\times\gamma_\beta\frac{1}{(i\not p+\not k)}\right)D_{\alpha\beta}(q) .\end{aligned} \tag{3.18}$$

The fermion masses, which have little impact on the ultraviolet behavior of the integrands, have been neglected (see below). Now, given the asymptotic form (2.27) for $D_{\alpha\beta}(q) \sim A_d/g^2q^{d-2}$, it is easy to see that the combination of renormalization constants multiplying $I_{\mu\nu}^{a,b}$ reduces to $Z_A g^2$, which from previous considerations must be cutoff independent. The conclusion is that $\Pi^{[1]}_{\mu\nu}$ must be UV finite if the model is to be consistently renormalized at this order. The constant Z_A, being just the scale of an auxiliary field, is undetermined in the model, and the value of g also appears to be irrelevant as regards the UV behavior of the model (Yang [18] points out that g cannot be renormalized, since it appears in the Lagrangian (3.4) in a gauge-variant manner after rescaling $A_\mu \mapsto A_\mu/g$).

Now, as shown in [8], diagrams such as those of Fig. 4 generically have divergences of two forms, one independent of k, proportional to Λ^{d-2} + const $\times M^{d-2}\ln\Lambda$, and the other proportional to $k^{d-2}\ln\Lambda$. We must show that for the Thirring model both occur with zero coefficient. First we deal with the momentum-independent piece, following the technique used in the Appendix of [5].

Consider the expression (2.4) for the one-loop vacuum polarization $\Pi^{[0]}_{\mu\nu}(k)$, and in particular the result of differentiating it with respect to external momentum k. By using the identity

$$\frac{\partial}{\partial k_\mu}\frac{1}{i(\not p+\not k)+M} = -\frac{1}{i(\not p+\not k)+M}\gamma_\mu\frac{1}{i(\not p+\not k)+M}\ , \quad (3.19)$$

we see that each differentiation is equivalent to a zero momentum insertion of $-iA_\mu$ (modulo a factor of $g/\sqrt{N_f}$). Thus

$$\frac{\partial^2}{\partial k_\mu \partial k_\nu}\Pi^{[0]}_{\alpha\beta}(k) = -[2(1-a)^2+2a^2]J^a_{\mu\nu\alpha\beta}(k) +2a(1-a)J^b_{\mu\nu\alpha\beta}(k)\ , \quad (3.20)$$

where $J^{a,b}_{\mu\nu\alpha\beta}(k)$ are represented in Fig. 5. However, as we showed in Sec. II, $\Pi^{[0]}_{\alpha\beta}$, and hence $J^{a,b}_{\mu\nu\alpha\beta}$, are finite analytic functions of k which are independent of the momentum routing a. Hence

$$0\equiv \frac{\partial}{\partial a}\frac{\partial^2}{\partial k_\mu\partial k_\nu}\Pi^{[0]}_{\alpha\beta}(k) = 2(1-2a)[2J^a_{\mu\nu\alpha\beta}(k)+J^b_{\mu\nu\alpha\beta}(k)]\ . \quad (3.21)$$

Now we contract the right-hand side (RHS) of (3.21) with $D_{\alpha\beta}(k)$ and perform $\int_k$ to obtain the two-loop integrals $I^{a,b}_{\mu\nu}$ at zero momentum. Since the RHS must vanish for arbitrary a, we conclude

$$2I^a_{\mu\nu}(0)+I^b_{\mu\nu}(0)=0\ , \quad (3.22)$$

This argument shows that the momentum-independent part of $\Pi^{[1]}_{\mu\nu}(k)$ is identically zero, and hence that we do not need to worry about Λ^{d-2} divergences. However, each diagram is individually divergent, as can be seen by setting $a=0$ in (3.20), then evaluating $\int_k J^a_{\mu\nu\alpha\beta}(k)D_{\alpha\beta}(k)$; we find, for the leading divergence,

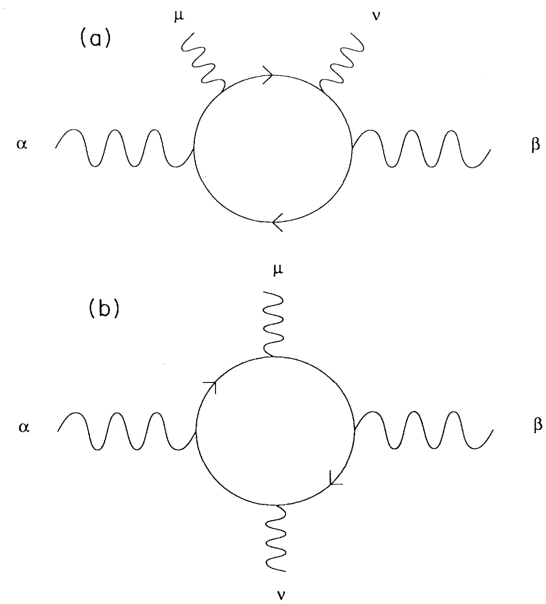

FIG. 5. (a)Diagram representing $J^a_{\mu\nu\alpha\beta}(k)$. (b) Diagram representing $J^b_{\mu\nu\alpha\beta}(k)$.

$$2\Pi^{[1a]}_{\mu\nu}(0) = \delta_{\mu\nu}\frac{4\Lambda^{d-2}}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\frac{(d-1)^2(d-3)}{d(d-2)}\ , \quad (3.23)$$

where Λ is a momentum cutoff for $\int_k$.

Of course, we might have anticipated that the Λ^{d-2} superficial divergence vanishes due to current conservation, as it did at leading order, and indeed as it does in QED to all orders, where $\Pi_{\mu\nu}(0)=0$ ensures that the photon propagator retains a zero mass pole, as required by gauge invariance. However, no such argument constrains the momentum-dependent divergence $k^{d-2}\ln\Lambda$, for instance, in QED_4, $k^2\ln\Lambda$ divergences are physical and responsible for charge renormalization, and it is not *a priori* clear what will happen in a model with nonstandard asymptotics in $d<4$. To examine this case it is necessary to perform an explicit two-loop calculation, using the techniques developed in [8], which are now outlined.

After performing the trace over Dirac indices, each integral may be reduced to several components of the form

$$\text{const}\times\int_q\int_p\int_{\Sigma_i x_i}\frac{P_{\mu\nu\alpha\beta}(q,k;x_i)D_{\alpha\beta}(q;\mu)}{[p^2+Q(q,k;x_i)]^n}\ , \quad (3.24)$$

using Feynman parametrization and momentum shift in p. Here P and Q are polynomial functions of momenta, x_i are Feynman parameters which are to be integrated over a domain $\Sigma_i x_i \le 1$, and n is integer. The algebra is considerably simplified by the choice of "Landau gauge" $\mu^2\to 0$, which means that all terms proportional to q_α, q_β can be discarded. The integral over the fermion loop momentum p can always be performed for $n\ge 2$:

$$\int_p \frac{1}{[p^2+Q]^n} = \frac{Q^{d/2-n}\Gamma(n-\frac{d}{2})}{\Gamma(n)(4\pi)^{d/2}} . \quad (3.25)$$

Momentum dependent divergences arise when there are two or more Feynman parameters, in which case there is an intermediate integral of the form

$$\int_0^{1-\Sigma_{j\neq i}x_j} dx_i \frac{A+Bx_i+Cx_i^2+\cdots}{(a+bx_i+cx_i^2)^s} , \quad (3.26)$$

where s is noninteger, and the coefficients b,c are $O(q^2)$, where q is the remaining loop momentum, but the coefficient a is $O(q^0k^2)$. The contribution to (3.26) from the $x_i \to 0$ limit of the integral is then

$$-\frac{Ab}{(s-1)\Delta a^{s-1}}\left(1+\frac{3-2s}{2-s}\frac{2ac}{\Delta}\right) - \frac{B}{(s-1)(s-2)\Delta a^{s-2}} + O\left(\frac{1}{q^6}\right) , \quad (3.27)$$

with $\Delta \equiv 4ac-b^2$. The exact expression on which this approximation is based is given in the Appendix of [8]. Note that all factors, including Δ, must be expanded consistently to $O(1/q^4)$ due to the presence of $O(q^4)$ terms in the numerator polynomial P. Once this limit has been isolated, the remaining integral over q is of the form

$$\int_q \frac{R_{\alpha\beta\mu\nu}(k,q)k^{d-4}}{q^2} D_{\alpha\beta}(q) , \quad (3.28)$$

where R is $O(k^2q^0)$. The form (2.27) for $D_{\alpha\beta}$ is now sufficient to evaluate $\int_q$ with a momentum cutoff: it yields a logarithmic divergence. Any remaining integrals over Feynman parameters give combinations of beta functions in d. The final result is

$$\begin{aligned} 2\Pi^{[1a]}_{\mu\nu}(k) &= -\Pi^{[1b]}_{\mu\nu}(k) \\ &= \frac{g^2 Z_A}{N_f}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)A_d}{(4\pi)^d\Gamma(d)}\frac{32(d-1)(d-4)}{d}\mathcal{P}_{\mu\nu}(k)(k^2)^{(d/2)-1}\ln\frac{\Lambda}{k} \\ &= g^2 Z_A \mathcal{P}_{\mu\nu}(k)\frac{(k^2)^{(d/2)-1}}{A_d}\left(\frac{C_d}{N_f}\frac{(d-1)^2(d-4)}{d(d-2)}\ln\frac{\Lambda}{k}\right) \end{aligned} \quad (3.29)$$

on rearranging. We note that each diagram is individually transverse and equal and opposite, respectively, to twice the μ-independent parts of the self-energy correction (3.13) (a), or the vertex correction (3.16) (b). The main result, of course, is that the two-loop vacuum polarization $\Pi^{[1]}_{\mu\nu}(k)$ is UV finite in the massless theory. Strictly, we have not demonstrated the independence of this result on the "gauge fixing" parameter μ, and must rely on reasoning that the vacuum polarization, which yields charge renormalization in QED, is gauge invariant. Perhaps more importantly, we have not considered divergences of the form $M^{d-4}k^2\ln\Lambda$: these probably exist and correct the mass of the vector M_V following the discussion (2.20)–(2.26).

We have now exhausted the list of divergent Green functions at $O(1/N_f)$, and proven that the model is renormalizable at this order (at least in the massless limit); indeed the only physical (i.e., gauge invariant) renormalization that must be made is that of the fermion mass (3.14).

IV. DISCUSSION

We begin the final section by thinking about why the cancellation of $k^{d-2}\ln\Lambda$ divergences in the two-loop vacuum polarization takes place. The way the calculation has been presented here suggests a cancellation between the two diagrams $\Pi^{[1a]}_{\mu\nu}$ and $\Pi^{[1b]}_{\mu\nu}$ (Fig. 4). However, a similar analysis of the Gross-Neveu model (1.1) [8] suggests it is more natural to think of a cancellation between $\Pi^{[1a]}_{\mu\nu}$ and twice the self-energy (3.13), and between $\Pi^{[1b]}_{\mu\nu}$ and twice the vertex correction (3.16). For the analogous diagrams in the Gross-Neveu case (note the constant A_d is different),

$$\begin{aligned} \Pi^{[1a]}(k) &= -2\frac{(k^2)^{(d/2)-1}}{A_d}\frac{\partial\Sigma^{[1]}(k)}{\partial(i\not k)} \\ &= \frac{(k^2)^{(d/2)-1}}{A_d}\frac{(d-2)}{d}\frac{C_d}{N_f}\ln\frac{\Lambda}{k} , \\ \Pi^{[1b]}(k) &= -2\frac{(k^2)^{(d/2)-1}}{A_d}\frac{\Gamma^{[1]}(k)}{(-g/\sqrt{N_f})} \\ &= \frac{(k^2)^{(d/2)-1}}{A_d}\frac{C_d}{N_f}\ln\frac{\Lambda}{k} . \end{aligned} \quad (4.1)$$

In fact, what has occurred is a cancellation between the subdivergences of Fig. 4 and the diagrams which would result from inserting local counterterms arising from the divergences of Eqs. (3.13) and (3.16) in the leading order vacuum polarization Fig. 1; factors of 2 come because there are two fermion lines and two vertices to correct.

In a treatment of renormalization which proceeds by subtraction of divergent parts in an ordered fashion, the Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) scheme, this cancellation is transparent. In the presentation here I have chosen a physically more intuitive approach, rescaling the fields and coupling parameters in the bare Lagrangian to keep Green functions finite at each order, but of course the two schemes are completely equivalent. However, it is important to note that the argument depends on a novel application of Weinberg's theorem [20]. The theorem states, in effect, that a one particle irreducible (1PI) graph such as $\Pi^{[1]}_{\mu\nu}$ with positive degree of divergence ω will be balanced by an overall counterterm which is polynomial in external momentum [in this case $O(\Lambda^{d-2}k^0)$], provided that its subdivergences are first subtracted, in this case by applying the appropriate vertex and self-energy corrections. These subdivergences may be nonpolynomial in momentum, but the theorem guarantees that the cancellation will be exact. The crucial point is that Weinberg proved the theorem for a wide class of integrands; the usual integrals built from the standard Feynman propagators proportional to $(p^2+m^2)^{-1}$, $(i\not{p}+m)^{-1}$, etc., form just one class. There is no general requirement that the propagators be algebraic in the internal momenta, and integrals including the resummed propagator $\propto k^{2-d}$ are also included.[2]

To the reader familiar with the calculation of the two-loop vacuum polarization in QED$_4$ (e.g., [21], Chap. 8), it is worth making a further comment. The analogue of the $k^{d-2}\ln\Lambda$ divergences are *not* (as one might first think) the $k^2\ln\Lambda$ terms which lead to a physical charge renormalization (these are polynomial in k and "belong" to the diagram as a whole), but rather the nonpolynomial $\ln k\ln\Lambda$ divergences, which in general are canceled by counterterm subtractions, and in the particular case of QED cancel between the two diagrams (4a) and (4b). The discontinuous behavior as $d\to 4$ illustrates why this is a critical dimension for the model.

So, we see that in a sense the main achievement of this paper is simply the verification of a peculiar case of Weinberg's theorem. There appears to be no fundamental obstacle to formulating a proof of renormalizability of the $1/N_f$ expansion for four-Fermi theories to all orders. The only complication arises, as we have seen, because graphs with different numbers of loops arise at a given order, which means that counterterm subtractions *of the same order* must be applied to yield a finite result, or in the language of [8], nontrivial cancellations between divergent graphs at the same order must occur. As we argued in the last section, this does not occur in the $1/N_f$ expansion of the nonlinear σ model.

We deduce that if the expansion is renormalizable, the logarithmic corrections to fermion and vector propagators and the vertex always cancel at each order in $1/N_f$. An important physical consequence, which also follows from Weinberg's theorem, is that in both Gross-Neveu and Thirring models, the amplitude for four-Fermi scattering receives no $1/N_f$ corrections in the deep Euclidean limit [8]. In both cases it assumes a universal from $A_d/(N_f k^{d-2})$, which thus characterizes an interacting ultraviolet fixed point of the renormalization group. Both models resemble each other at high energies. At low energies they differ, and now we return our focus to the Thirring model. In Sec. II we saw that at leading order the coupling g is completely unconstrained, and that the model can be formulated either as weakly or strongly coupled. After radiative corrections, this may no longer be true. Due to the mass renormalization (3.14), the fermion mass operator acquires an anomalous dimension of $O(1/N_f)$. The result is that

$$\frac{m}{M}\propto\left(\frac{\Lambda}{m}\right)^{(C_d/N_f)[2(d-1)^2/d(d-2)]}, \tag{4.2}$$

that is, for fixed physical mass m, the bare mass M must be tuned to zero as the cutoff is removed. Accordingly the ratio g^2/M^{2-d}, which governs whether the model is weakly or strongly coupled at leading order, must grow small. However, since the low energy nature of the model is characterized by the ratio M_V/m, it will be necessary to compute $O(1/N_f)$ corrections to M_V to determine whether the model is driven to strong or weak coupling at higher orders. It would also be interesting to examine the stability of (4.2) under corrections of $O(1/N_f^2)$ [25].

Finally we speculate on the relevance of this model to strongly coupled QED, both in 3 and 4 dimensions. This paper has been concerned exclusively with $1/N_f$ perturbation theory. Indeed, we saw in Sec. III that the coincidence of $O(1/N_f)$ corrections for the fermion mass and wave function with those obtained in large-N_f QED suggests a correspondence between the models exactly analogous to that between the Gross-Neveu and Yukawa models discussed in the Introduction. However, in [19,23,26], the leading order vector propagator (2.15$'$) was used in the Schwinger-Dyson equation to solve for dynamical fermion mass generation self-consistently. The result, $\propto\exp(-N_f)$, is nonperturbative in $1/N_f$. It is suggested that the solution may shed light on spontaneous chiral symmetry breaking in QED$_3$, which is suspected to be critically dependent on N_f [27]. Since QED$_3$ is superrenormalizable, the continuum limit is thought to exist in the limit of weak coupling; so far we can reach no conclusion for the Thirring model. It is also worth noting that since all $O(1/N_f)$ corrections to the vector propagator are UV finite, then in the asymptotic regime the quenched approximation (which must be made to solve the Schwinger-Dyson equation) is exact. This is therefore a model with many similarities to QED in which charge screening is naturally switched off at short distances, an effect which must be postulated in studies of a nontrivial fixed point of QED$_4$ due to fermion mass generation at strong coupling. This deserves further study. Another possibility is to introduce both scalar and vector current interactions with independent couplings, to generate a fermionic analogue of the gauged Nambu–Jona-Lasinio model, which would be renormalizable in $d\in(2,4)$. It would then be interesting to examine the limit $d\to 4_-$ and compare the results with other approaches [28].

[2]In fact, one requirement of Weinberg's original proof is that the momenta are defined in an integer-dimensioned vector space; the extension to noninteger d has not been established to my knowledge. There are two responses: either set $d=3$ at this stage to yield a "physical" theory, or note that it is always possible to route a simple loop momentum through any internal auxiliary line, in which case it may be possible to analytically continue the integrand to $d=3$ where the theorem holds. We shall not pursue these rather abstract issues further.

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