Gravitational radiation from a particle in circular orbit around a black hole. V. Black-hole absorption and tail corrections

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A particle of mass μ moves on a circular orbit of a nonrotating black hole of mass M. Under the restrictions $\mu/M \ll 1$ and $v \ll 1$, where v is the orbital velocity (in units in which $c = 1$), we consider the gravitational waves emitted by such a binary system. The framework is that of black-hole perturbation theory. We calculate $\dot{E}$, the rate at which the gravitational waves remove energy from the system. The total energy loss is given by $\dot{E} = \dot{E}^{\infty} + \dot{E}^{H}$, where $\dot{E}^{\infty}$ denotes that part of the gravitational-wave energy which is carried off to infinity, while $\dot{E}^{H}$ denotes the part which is absorbed by the black hole. We show that the black-hole absorption is a small effect: $\dot{E}^{H}/\dot{E} \simeq v^8$. This is explained by the presence of a potential barrier in the vicinity of the black hole: Most of the waves propagating initially toward the black hole are reflected off the barrier; the black hole is therefore unable to absorb much. The black-hole absorption, and indeed any other effect resulting from imposing ingoing-wave boundary conditions at the event horizon, are sufficiently small to be irrelevant to the construction of matched filters for gravitational-wave measurements. To derive this result we extend the techniques previously developed by Poisson and Sasaki for integrating the Regge-Wheeler equation. The extension consists of an explicit consideration of the horizon boundary conditions, which were largely ignored in the previous work. Finally, we compare the wave generation formalism which derives from perturbation theory to the post-Newtonian formalism of Blanchet and Damour. Among other things we consider the corrections to the asymptotic gravitational-wave field which are due to wave-propagation (tail) effects. The results obtained using perturbation theory are identical to that of post-Newtonian theory.

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I. INTRODUCTION AND SUMMARY

A. Gravitational waves from coalescing compact binaries

Coalescing compact binary systems, composed of neutron stars and/or black holes, have been identified as the most promising source of gravitational waves for kilometer-size interferometric detectors [1,2].

The construction of three such detectors should be completed by the turn of the century. The American LIGO (Laser Interferometer Gravitational-Wave Observatory) project [3] involves two 4 km detectors, one situated in Hanford, Washington, the other in Livingston, Louisiana; construction has begun at the Hanford cite. The French-Italian VIRGO project [4] involves a single 3 km detector, to be built near Pisa, Italy.

The emission of gravitational waves by a compact binary system causes the orbits to shrink and the orbital frequency to increase [1]. The gravitational-wave frequency is given by twice the orbital frequency; it also increases as the system evolves. The LIGO and VIRGO detectors are designed to operate in the frequency band between approximately 10 Hz and 1000 Hz [3]. This corresponds to the last several minutes of inspiral, and possibly the final coalescence (depending on the size of the masses involved), of a compact binary system [5].

During this late stage of orbital evolution the orbital velocity is large (v/c ranging from approximately 0.07 to 0.35 if the binary system is that of two neutron stars [6]), and so is the gravitational field [the dimensionless gravitational potential is of order $(v/c)^2$]. Accurate modeling of the inspiral must therefore take general-relativistic effects carefully into account. On the other hand, other complications, such as tidal interactions (important only during the last few orbital cycles [7,8]) and orbital eccentricity (reduced to very small values by gravitational radiation reaction [9,10]) can be safely ignored.

The gravitational waves emitted by a coalescing compact binary carry information about the source — the wave forms depend on the parameters describing the source. These include the orientation and position of the source in the sky, its distance, and the masses and spins of the companions. A major goal in detecting these waves is to extract this information [1,2,11].

Because the wave forms can be expressed, at least in principle, as known functions of the source parameters, the technique of matched filtering can be used to esti-

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mate the values of these parameters [1,5,11–13]. For this technique to work, the gravitational-wave signal must be accurately predicted in the full range of frequencies described above. It can be estimated that the waves oscillate a number of times of order 10^4 as the frequency sweeps from 10 Hz to 1000 Hz [6]. To yield a large signal-to-noise ratio the template must stay in phase with the measured signal. The accurate prediction of the *phasing* of the wave is therefore a central goal for theorists [5,13]; an accuracy better than one part in 10^4 is required.

Gravitational radiation reaction causes f, the gravitational-wave frequency, to increase as the system evolves. The phasing of the wave is determined by $\dot{f}(f)$, the rate of change of frequency as a function of frequency. If the orbital energy E is known as a function of f, then $\dot{E}(f)$ determines the phasing. Below we will assume that the relation $E(f)$ is known (or can be calculated), and focus on the energy loss.

B. Energy loss: post-Newtonian theory

The energy radiated by a compact binary system can be calculated, approximately, using post-Newtonian theory [6]. The post-Newtonian approximation is based upon an assumption about the smallness of the orbital velocity. In units in which $c = 1$ (we also put $G = 1$), we demand $v \ll 1$. On the other hand, nothing is assumed about the mass ratio: If M denotes total mass and μ reduced mass, then μ/M is allowed to cover the whole range $(0, 1/4]$. For the problem under consideration v is small only during the early portion of the inspiral, when f is near 10 Hz. It is therefore expected that calculations will have to be pushed to extremely high order in v. An intensive effort to do just this is now underway [6].

The phrase "post-Newtonian theory" is used here loosely. A wave generation formalism suitable for high-order calculations was developed by Blanchet and Damour [14–16]. This combines a post-Newtonian expansion (based, in effect, on the assumption $c \to \infty$), for the region of spacetime corresponding to the near zone, with a post-Minkowskian expansion (based, in effect, on the assumption $G \to 0$), for the region of spacetime outside the source. The external metric is matched to the near-zone metric, and the radiative part of the gravitational field is extracted. The radiative field is characterized by an infinite set of multipole moments; the relation between these radiative moments and the source moments is revealed by matching.

The rate at which energy is radiated can be calculated from the radiative moments, and expressed as an expansion in powers of v. (In fact, the expansion also involves $\ln v$ at high order [14,17,18].) Thus far, the energy loss has been calculated accurately to order v^4, or post^2-Newtonian order, beyond the leading-order (Newtonian) quadrupole formula expression. Schematically,

$$\dot{E} = \dot{E}_{\rm QF}\Big[1 + O(v^2) + O(v^3) + O(v^4) + \cdots\Big], \qquad (1.1)$$

where $\dot{E}_{\rm QF} = (32/5)(\mu/M)^2 v^{10}$ [1]. Here and below, we define $v \equiv (M\Omega)^{1/3} = (\pi M f)^{1/3}$, where Ω is the angular velocity.

In Eq. (1.1) the $O(v^2)$ term was first calculated by Wagoner and Will [19] and is due to post-Newtonian corrections to the equations of motion and wave generation. The $O(v^3)$ term comes from two different effects: (i) tail terms in the waves (the gravitational waves scatter off the spacetime curvature as they propagate from the near zone to the far zone), first calculated by Poisson [20] using perturbation theory (see below), and subsequently by Wiseman [21] and Blanchet and Shäfer [22] using post-Newtonian theory; and (ii) spin-orbit interactions (present if at least one of the companions is rotating), first calculated by Kidder, Will, and Wiseman [23]. The $O(v^4)$ term originates from spin-spin interactions (if both stars are rotating), as was calculated by Kidder, Will, and Wiseman [23], and from post^2-Newtonian corrections to the equations of motion and wave generation, as was recently calculated by Blanchet, Damour, Iyer, Will, and Wiseman [24]. An additional contribution to the $O(v^4)$ term will be present if at least one of the stars has a nonvanishing quadrupole moment; this effect has not yet been calculated, except for the special case considered by Shibata *et al.* [25] (see below).

C. Energy loss: perturbation theory

Post-Newtonian theory is based upon a slow-motion, weak-field approximation. Because the orbital velocity is large during the late stages of the inspiral, and because the post-Newtonian expansion is not expected to converge rapidly {if at all; the coefficients in Eq. (1.1) are typically large and grow with the order [17]}, it is not clear *a priori* to which order in v the calculations must be taken in order to obtain results of sufficient accuracy. To answer this question requires a different method of analysis.

In perturbation theory it is assumed that the binary's mass ratio is very small, $\mu/M \ll 1$, and that the massive companion is a black hole (rotating or nonrotating). On the other hand, the orbital velocity is not restricted: the motion is arbitrarily fast, and the gravitational field arbitrarily strong. To calculate, within perturbation theory, the gravitational waves emitted by such binary systems is the purpose of this series of papers [20,26–29], as well as that of the work reviewed below.

The perturbation-theory approach can be summarized as follows.

A particle of mass μ moves in the gravitational field of a black hole of mass M. (By "particle" we mean an object whose internal structure is of no relevance to the problem.) The particle possesses a stress-energy tensor $T^{\alpha\beta}$ which creates a perturbation in the gravitational field. If $\mu \ll M$ the Einstein field equations can be solved perturbatively about the black-hole solution. The perturbations propagate as gravitational waves. By solving the perturbation equations one can calculate the wave forms, as well as the energy and angular momentum transported. The energy and angular momentum are partly carried off to infinity, and partly absorbed by the black hole.

The motion of the particle must be specified before the perturbation equations are integrated. (The particle's stress-energy tensor, which acts as a source term in the equations, is a functional of the world line.) In general, the motion is affected by the loss of energy and angular momentum to gravitational waves — there is a radiation reaction force [29]. However, the radiation reaction can be neglected to lowest order in μ/M, and the motion can be taken to be a geodesic of the black-hole spacetime. The stress-energy tensor is then completely specified.

Because the source term in the perturbation equations is known exactly, to integrate these equations amounts to solving an equation for wave propagation in curved spacetime. (In post-Newtonian theory, wave propagation is only part of the problem. One must also solve for the near-zone physics.) For large velocities the wave equation must be integrated numerically. When combined with a slow-motion approximation, the equations of perturbation theory can be integrated analytically.

D. Perturbation theory: a survey

The theory of black-hole perturbations (gravitational and otherwise) has been the topic of a vast literature. A self-contained summary can be found in Chandrasekhar's book [30]. In this series of papers we have adopted Teukolsky's formulation of perturbation theory [31], in which the perturbation field is Ψ_4, a complex-valued component of the Weyl tensor. The equation satisfied by Ψ_4 is known as the Teukolsky equation.

The perturbation formalism was summarized in paper I [20], and applied to the specific case of circular motion around a nonrotating black hole. Methods were developed for integrating the wave equation analytically in the low-frequency limit, and $\dot{E}/\dot{E}_{\rm QF}$ was calculated to order v^3 in the post-Newtonian expansion. These analytical methods were extended by Sasaki [32], and Tagoshi and Sasaki [17] have obtained $\dot{E}/\dot{E}_{\rm QF}$ to order v^8; see Sec. V D. Much was learnt from this work about the structure of the post-Newtonian expansion.

In paper II [26] the perturbation equations were integrated numerically, and the energy loss was calculated exactly (apart from numerical error) for a wide range of orbital frequencies. By fitting a post-Newtonian expansion to the curve $\dot{E}(f)$ the quality of the approximation could be determined. It was concluded that for the purpose of matched filtering, the expansion of $\dot{E}/\dot{E}_{\rm QF}$, Eq. (1.1), must be extended so as to include terms of order v^6 or higher. These numerical calculations were repeated with much higher accuracy by Tagoshi and Nakamura [18], who confirmed this conclusion. The analytical work of Tagoshi and Sasaki [17] is in complete agreement with the numerical work.

The gravitational waves emitted by a particle in eccentric motion around a Schwarzschild black hole were calculated, and the orbital evolution under radiation reaction considered, by Tanaka *et al.* [33]. They discovered that while radiation reaction always reduces the eccentricity in slow-motion situations, strong-field radiation reaction can cause the eccentricity to increase. This was studied in detail in paper III [27] for the special case of small eccentricities, and by Cutler, Kennefick, and Poisson [29] (we shall treat this paper as a honorary member of this series) for the general case. This increase of eccentricity can be understood as a precursor effect to the eventual plunging of the orbit, at the end of the inspiral.

Generalization of the work described above to the case of a rotating black hole was the topic of several papers. In the limit of slow rotation, analytical methods were used in paper IV [28] to calculate the corrections to the wave forms and energy loss due to the black hole's rotation. These results were generalized by Shibata *et al.* [25], who considered a rapidly rotating black hole, as well as orbits slightly inclined with respect to the hole's equatorial plane. Both papers assumed circular orbits. These results were extended to the case of slightly eccentric orbits by Tagoshi [34]. Numerical results were obtained by Shibata [35], who also considered the case of eccentric orbits [36].

E. Black-hole absorption

We now turn to a description of the work contained in this paper.

As mentioned previously, solving the perturbation equations amounts to solving a wave equation in the background of a Schwarzschild black hole. Boundary conditions must be imposed when integrating this equation. The correct choice is to impose a no-incoming-radiation condition, which forces the system to lose energy to, and not gain energy from, gravitational waves. In practice, this condition is implemented by imposing outgoing-wave boundary conditions at infinity, and ingoing-wave boundary conditions at the black-hole horizon.

In the analytic work of Poisson [20], Sasaki [32], and Tagoshi and Sasaki [17], the horizon boundary conditions were not found to play a significant role. The black hole could have been replaced by some regular distribution of matter, and the horizon boundary conditions replaced by a regularity condition at the origin (this is the type of condition that is imposed in post-Newtonian calculations [14]), and the results would have been identical. Because of this situation, the question of correctly imposing the horizon boundary conditions was not fully examined.

The apparent irrelevance of the horizon boundary conditions turns out to be a consequence of two facts. The first is that only $\dot{E}^\infty$, the energy radiated to infinity, was calculated in the previous analytical work. The total energy loss $\dot{E}$ must also include $\dot{E}^H$, the energy flowing through the black-hole horizon. The second is that the post-Newtonian expansion of $\dot{E}^\infty/\dot{E}_{\rm QF}$ was calculated only up to order v^8. Sasaki [32] has shown [this is essentially a consequence of Eq. (1.4) below] that the horizon boundary conditions affect $\dot{E}^\infty/\dot{E}_{\rm QF}$ only at order v^{18} and beyond.

One of the main objectives of this paper is to calculate $\dot{E}^H$, the rate at which the black hole absorbs energy, to leading order in a slow-motion approximation. This calculation is carried out in Sec. V, using material derived in preceding sections. We find

$$\dot{E}^H = \dot{E}_{\rm QF}\left[v^8 + O(v^{10})\right], \tag{1.2}$$

so that $\dot{E}^H$ contributes terms of order v^8 and higher to the post-Newtonian expansion of $\dot{E}/\dot{E}_{\rm QF}$. It can therefore be said that the black-hole absorption is an effect occurring at post^4-Newtonian order in the energy loss. This effect utterly dominates that of the horizon boundary conditions on the energy radiated to infinity.

Nevertheless, the black-hole absorption is a small effect. In view of the requirement for matched filtering — energy loss accurate to post^3-Newtonian order [18,26] (see subsection ID) — it can be safely ignored. The smallness of $\dot{E}^H/\dot{E}^\infty$ can be attributed to the presence of a potential barrier in the vicinity of the black hole. This barrier is a manifestation of the spacetime curvature; it influences the propagation of waves near the black hole. Since gravitational waves generated by binary motion have a frequency f such that $\pi M f = v^3 \ll 1$, most of the waves propagating initially toward the black hole are reflected off the potential barrier (see subsection IF). As a result, the black hole is unable to absorb much.

That the black-hole absorption is a small effect is ultimately due to the extreme degree of compactness of black holes. How large the absorption would be for a (substantially less compact) neutron star is unknown. This question warrants further examination. [*Noted added in proof.* This question was recently examined by A. Reisenegger and P. Goldreich, Astrophys. J. **426**, 688 (1994).]

F. Integration of the Regge-Wheeler equation

While integrating the perturbation equations amounts to solving a wave equation, solving that equation amounts to integrating the Regge-Wheeler equation [37] for the radial function $X_{\omega\ell}(r)$. Here, r is the Schwarzschild radial coordinate, ω the wave's angular frequency, and ℓ the spherical-harmonic index. The Regge-Wheeler equation takes the form

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 + V(r)\right]X_{\omega\ell}(r) = 0, \tag{1.3}$$

where $d/dr^* = (1-2M/r)d/dr$ and $V(r) = (1-2M/r)[\ell(\ell+1)/r^2 - 6M/r^3]$. A significant portion of this paper (Sec. III) is devoted to developing techniques for integrating Eq. (1.3). These techniques rely on an assumption of low frequency, $M\omega \ll 1$, which is appropriate for slow-motion situations.

Our techniques allow us to calculate the reflection and transmission coefficients associated with the potential barrier $V(r)$. Suppose an incident wave $e^{-i\omega(t+r^*)}$ is sent from $r=\infty$ toward the black hole. The wave partially reflects off the potential barrier. The reflected wave $\mathcal{R}_{\omega\ell}e^{-i\omega(t-r^*)}$ comes back to $r=\infty$, while the transmitted wave $\mathcal{T}_{\omega\ell}e^{-i\omega(t+r^*)}$ goes through the black-hole horizon. In Sec. III E we find that the reflection and transmission coefficients are given by

$$\mathcal{R}_{\omega\ell} = (\text{phase})\left\{1 + O\left[|M\omega|^{2(\ell+1)}\right]\right\} \tag{1.4}$$

and

$$\mathcal{T}_{\omega\ell} = \frac{2(\ell-2)!(\ell+2)!}{(2\ell)!(2\ell+1)!!}|2M\omega|^{\ell+1}(\text{phase}) \times\left\{1 + \pi M|\omega| + O\left[(M\omega)^2\right]\right\}. \tag{1.5}$$

Expressions for the phase factors, accurate to order $M\omega$, can be found in Sec. III E. (In the language of that section, $\mathcal{R}_{\omega\ell} = A^{\rm out}/A^{\rm in}$, and $\mathcal{T}_{\omega\ell} = 1/A^{\rm in}$.)

Leading-order expressions for the reflection and transmission coefficients were previously derived by Fackerell [38]. (See also Price [39] and Thorne [40].) His results correspond to the $M\omega \to 0$ limit of Eqs. (1.4) and (1.5), and therefore omit the $O(M\omega)$ part of the phase factors and the $\pi M|\omega|$ correction. These corrections, we should point out, are not needed for the derivation of Eq. (1.2).

G. Perturbation theory as a wave generation formalism

It is interesting to compare the wave generation formalism which derives from perturbation theory to the post-Newtonian formalism of Blanchet and Damour [14–16].

The Teukolsky equation can be separated by decomposing the perturbation field Ψ_4 into spin-weighted spherical harmonics [41]. From the solution at large distances it is possible to reconstruct the (traceless-transverse) gravitational-wave field $h_{ab}^{\rm TT}$, which is then naturally expanded in tensor spherical harmonics (Sec. II B). Apart from an overall factor of $1/r$, the coefficients are functions of retarded time only, and play the role of radiative multipole moments. These moments can be expressed in terms of an integral over the source. In general the radiative moments are not simply related to the source moments. However, to leading order in v in a slow-motion approximation, the relation *is* simple. This is established in Sec. IV A.

The situation is analogous in the Blanchet-Damour formalism. Here, the radiative field is also expanded in multipole moments, but for convenience a symmetric-trace-free (STF) representation is favored. (An expansion in STF moments is entirely equivalent to an expansion in tensor spherical harmonics [42].) In general the radiative moments are not simply related to the source moments, except for the limiting case of slowly moving sources.

The Blanchet-Damour formalism uses two coordinate systems. The first, $\{t, \boldsymbol{x}\}$, is rooted to the source, and chosen so as to make near-zone calculations simple. However, the true light cones of the spacetime do not everywhere coincide with the near-zone light cones (described by $t = |\boldsymbol{x}|$), and this generates artificial divergences in the radiation field. To remedy this, Blanchet and Damour [16] introduce radiative coordinates $\{T, \boldsymbol{X}\}$ which are adapted to the true light-cone structure of the spacetime. In terms of these coordinates, the radiation field is finite. The coordinate systems are approximately related by $T - |\boldsymbol{X}| = t - |\boldsymbol{x}| - 2M\ln(|\boldsymbol{x}|/b)$, where effects nonlinear in the gravitational-wave field have been neglected. The log term is naturally interpreted as a Shapiro time delay, and b is an arbitrary parameter which serves to fix the origin of T.

No such coordinate transformation is needed in perturbation theory. One can work with the usual Schwarzschild coordinates, and the radiative multipole moments are naturally expressed as functions of retarded time $u = t - r^* = t - r - 2M\ln(r/2M - 1)$; see Sec. II B. The Shapiro time delay is therefore automatically incorporated into the retarded time, and the radiative field is automatically finite.

The Blanchet-Damour formalism has been used to calculate the tail part of the radiation field [14]. Such tails arise because the curvature of spacetime outside the source scatters the gravitational waves as they propagate toward the far zone. As a result, the radiation field at time T does not depend only on the state of the source at retarded time $T - |\boldsymbol{X}|$, but also on its entire past history. This effect modifies both the amplitude and the phase of the radiative multipole moments.

The tail effect can also be investigated using perturbation theory, which we do in Sec. IV B. The results are identical to that of post-Newtonian theory.

We therefore see that perturbation theory can be used to recognize and clarify many of the issues that must be addressed when dealing with post-Newtonian wave generation. This, we feel, is a useful aspect of our work.

H. Organization of the paper

We begin in Sec. II with a brief summary of the perturbation formalism. This formalism has been manipulated, in the course of this series of papers, so as to make it as easy to use as possible. We believe that the present formulation is as convenient as can possibly be. We will not present derivations in this section, but refer the reader to earlier references. Section II A is devoted to a description of the inhomogeneous Teukolsky equation and its integration by means of a Green's function. A procedure to obtain this Green's function in terms of solutions to the Regge-Wheeler equation is described in schematic terms. The procedure is detailed in the Appendix. Section II B describes how to obtain the gravitational-wave field from the large-distance behavior of the Teukolsky function. The radiative multipole moments are introduced. Finally, Sec. II C gives expressions for the rates at which the gravitational waves carry energy and angular momentum, both to infinity and down the black hole.

In Sec. III we present our techniques for integrating the Regge-Wheeler equation in the limit of low frequencies. Two sets of boundary conditions are imposed. The function $X^H_{\omega\ell}(r)$ satisfies ingoing-wave boundary conditions at the black-hole horizon, while $X^\infty_{\omega\ell}(r)$ satisfies outgoing-wave boundary conditions at infinity [43]. Most of the section is devoted to $X^H_{\omega\ell}(r)$. Some notation is introduced in Sec. III A. The Regge-Wheeler equation is integrated near $r = 2M$ in Sec. III B, and is integrated in the limit $M\omega \ll 1$ in Sec. III C. Matching is carried out in Sec. III D. The reflection and transmission coefficients are then calculated in Sec. III E. Finally, Sec. III F contains a brief discussion of the function $X^\infty_{\omega\ell}(r)$.

In Sec. IV we illustrate how black-hole perturbation theory can be used as a wave generation formalism. We consider slowly moving sources, and first (Sec. IV A) obtain expressions, valid to leading order in v, for the radiative multipole moments. Then, in Sec. IV B, we consider the corrections to the radiative moments which are due to wave-propagation (tail) effects.

In Sec. V we present calculations pertaining to gravitational waves produced by a particle in circular motion around a black hole. These calculations are carried out within the slow-motion approximation, to leading order in v. In Secs. V A and V B we derive expressions for the radiative multipole moments. In Sec. V C we calculate the contribution from each moment to the energy radiated. We consider both the flux at infinity, and that at the black-hole horizon. Finally, in Sec. V D, we consider the role of the black-hole absorption in the post-Newtonian expansion of the energy loss.

II. THE PERTURBATION FORMALISM

The stress-energy tensor associated with a moving particle perturbs the gravitational field of a nonrotating black hole. We describe this perturbation using the Teukolsky formalism [31], which we review below. The following presentation will be brief; missing details can be found in Refs. [20,27,29], and in the Appendix.

A. The Teukolsky equation and its solution

In the Teukolsky formalism, gravitational perturbations of the Schwarzschild black hole are represented by the complex-valued function $\Psi_4 = -C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$. Here, $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, and $n^\alpha = \frac{1}{2}(1, -f, 0, 0)$, $\bar{m}^\alpha = (0, 0, 1, -i\csc\theta)/\sqrt{2}r$ (in the $\{t, r, \theta, \phi\}$ Schwarzschild coordinates) are members of an orthonormal null tetrad. We have introduced $f = 1 - 2M/r$, and a bar denotes complex conjugation.

The Weyl scalar can be decomposed into Fourier-harmonic components according to

$$\Psi_4 = \frac{1}{r^4}\int_{-\infty}^{\infty} d\omega \sum_{\ell m} R_{\omega\ell m}(r)\, {}_{-2}Y_{\ell m}(\theta,\phi) e^{-i\omega t}, \quad (2.1)$$

where ${}_sY_{\ell m}(\theta,\phi)$ are spin-weighted spherical harmonics [41]. The sums over ℓ and m are restricted to $-\ell \le m \le \ell$ and $\ell \ge 2$. The radial function $R_{\omega\ell m}(r)$ satisfies the inhomogeneous Teukolsky equation [31]

$$\left[r^2 f \frac{d^2}{dr^2} - 2(r - M)\frac{d}{dr} + U(r)\right] R_{\omega\ell m}(r) = T_{\omega\ell m}(r), \quad (2.2)$$

with

$$U(r) = f^{-1}\left[(\omega r)^2 - 4i\omega(r-3M)\right] - (\ell-1)(\ell+2). \tag{2.3}$$

The source term to the right-hand side of Eq. (2.2) is constructed from the particle's stress-energy tensor,

$$T^{\alpha\beta}(x) = \mu \int d\tau\, u^\alpha u^\beta \delta^{(4)}\left[x - x'(\tau)\right], \tag{2.4}$$

where x represents the spacetime event and $x'(\tau)$ the particle's world line with tangent vector $u^\alpha = dx'^\alpha/d\tau$ (τ denotes proper time). The first step is to obtain the projections ${}_0T = T_{\alpha\beta}n^\alpha n^\beta$, ${}_{-1}T = T_{\alpha\beta}n^\alpha \bar{m}^\beta$, and ${}_{-2}T = T_{\alpha\beta}\bar{m}^\alpha\bar{m}^\beta$. Then one calculates the Fourier-harmonic components ${}_sT_{\omega\ell m}(r)$ according to

$${}_sT_{\omega\ell m}(r) = \frac{1}{2\pi}\int dt\, d\Omega\, {}_sT\, {}_s\bar{Y}_{\ell m}(\theta,\phi)e^{i\omega t}, \tag{2.5}$$

where $d\Omega$ is the element of solid angle. Finally, $T_{\omega\ell m}(r)$ is obtained by applying a certain differential operator to each ${}_sT_{\omega\ell m}(r)$, and then summing over s [20,44]. Schematically, $T = \sum_s {}_sD_sT$, where ${}_sD$ (other indices suppressed for notational simplicity) are the differential operators. (See the Appendix for details.)

The inhomogeneous Teukolsky equation (2.2) can be integrated by means of a Green's function [45]. The Green's function is constructed from two linearly independent solutions to the *homogeneous* equation, so that Ψ_4 satisfies a no-incoming-radiation condition. Schematically (indices suppressed), we have $G(r,r') = R^H(r_<)R^\infty(r_>)$, where $G(r,r')$ is the Green's function, $R^H(r)$ the solution to the homogeneous equation satisfying ingoing-wave boundary conditions at the black-hole horizon, $R^\infty(r)$ the solution satisfying outgoing-wave boundary conditions at infinity, and $r_<$ ($r_>$) denotes the lesser (greater) of r and r'. Schematically also, the solution takes the form $R(r) = \int dr'\, G(r,r')T(r') = \sum_s \int dr'\, G(r,r')_sD_sT(r')$.

It is then useful to define the adjoint operators ${}_sD^\dagger$ so that $R(r)$ can be more conveniently expressed as $R(r) = \sum_s \int dr'\, {}_sT(r')_sD^\dagger G(r,r')$. The last step consists of invoking the Chandrasekhar transformation [46], which relates solutions to the homogeneous Teukolsky equation to that of the Regge-Wheeler equation [37]. Thus, $G(r,r')$ can be conveniently written in terms of linearly independent solutions of the Regge-Wheeler equation. (Relevant details can be found in the Appendix.)

The Regge-Wheeler equation [37] takes the form

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V(r)\right]X_{\omega\ell}(r) = 0, \tag{2.6}$$

where $d/dr^* = f d/dr$ and $V(r) = f[\ell(\ell-1)/r^2 - 6M/r^3]$. The solutions of interest are $X^H_{\omega\ell}(r)$, which satisfies ingoing-wave boundary conditions at the black-hole horizon, and $X^\infty_{\omega\ell}(r)$, which satisfies outgoing-wave boundary conditions at infinity [43]. More precisely, these functions are defined so as to have the asymptotic behavior

$$\begin{aligned} &X^H_{\omega\ell}(r\to 2M) \sim e^{-i\omega r^*}, \\ &X^H_{\omega\ell}(r\to\infty) \sim A^{\rm in}_{\omega\ell}e^{-i\omega r^*} + A^{\rm out}_{\omega\ell}e^{i\omega r^*}, \\ &X^\infty_{\omega\ell}(r\to\infty) \sim e^{i\omega r^*}. \end{aligned} \tag{2.7}$$

Here, $r^* = r + 2M\ln(r/2M - 1)$, and $A^{\rm in}_{\omega\ell}$ and $A^{\rm out}_{\omega\ell}$ are constants. It follows from the conservation of the Wronskian that $X^\infty_{\omega\ell}(r\to 2M) \sim -\bar{A}^{\rm out}_{\omega\ell}e^{-i\omega r^*} + A^{\rm in}_{\omega\ell}e^{i\omega r^*}$.

Schematically, the Chandrasekhar transformation [46] takes the form $R^{H,\infty} = CX^{H,\infty}$, where C is a second-order differential operator. Thus, the effective Green's functions ${}_sD^\dagger G(r,r')$ can be expressed in terms of $X^{H,\infty}(r)$ and their derivatives. Because the Regge-Wheeler functions satisfy a second-order differential equation, the final expressions involve the functions and their first derivatives only. (See the Appendix for details.)

At large radii, the solution to the inhomogeneous Teukolsky equation is given by [27]

$$R_{\omega\ell m}(r\to\infty) \sim \mu\omega^2 Z^H_{\omega\ell m} r^3 e^{i\omega r^*}. \tag{2.8}$$

Near the black-hole horizon [27],

$$R_{\omega\ell m}(r\to 2M) \sim \mu\omega^3 Z^\infty_{\omega\ell m} r^4 f^2 e^{-i\omega r^*}. \tag{2.9}$$

After carrying out the manipulations described above, the "amplitudes" $Z^{H,\infty}_{\omega\ell m}$ are found to be given by

$$Z^{H,\infty}_{\omega\ell m} = i\pi\left[\mu\omega\kappa^{H,\infty}_{\omega\ell}A^{\rm in}_{\omega\ell}\right]^{-1}\sum_s {}_sp_\ell \times \int_{2M}^\infty dr\, r f^{-2}{}_sT_{\omega\ell m}(r)\, {}_s\Gamma_{\omega\ell}X^{H,\infty}_{\omega\ell}(r). \tag{2.10}$$

We have introduced the symbols

$$\begin{aligned} \kappa^H_{\omega\ell} &= \tfrac{1}{4}\left[(\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega\right], \\ \kappa^\infty_{\omega\ell} &= -16(1-2iM\omega)(1-4iM\omega)(M\omega)^3, \end{aligned} \tag{2.11}$$

and

$${}_sp_\ell = \begin{cases} 2\left[(\ell-1)\ell(\ell+1)(\ell+2)\right]^{1/2}, & s=0, \\ 2\left[2(\ell-1)(\ell+2)\right]^{1/2}, & s=-1, \\ 1, & s=-2. \end{cases} \tag{2.12}$$

We also have

$$\begin{aligned} {}_0\Gamma_{\omega\ell} &= 2(1-3M/r+i\omega r)rf\frac{d}{dr} + f\left[\ell(\ell+1) - 6M/r\right] + 2i\omega r(1-3M/r+i\omega r), \\ {}_{-1}\Gamma_{\omega\ell} &= -f\left\{\left[\ell(\ell+1)+2i\omega r\right]rf\frac{d}{dr} + \ell(\ell+1)(f+i\omega r) - 2(\omega r)^2\right\}, \\ {}_{-2}\Gamma_{\omega\ell} &= f^2\left\{2\left[(\ell-1)(\ell+2)+6M/r\right]rf\frac{d}{dr} + (\ell-1)(\ell+2)\left[\ell(\ell+1)+2i\omega r\right] + 12fM/r\right\}. \end{aligned} \tag{2.13}$$

As noted previously, ${}_s\Gamma_{\omega\ell}$ are *first-order* differential operators. That this is so is convenient both for analytical and numerical evaluation of $Z^{H,\infty}_{\omega\ell m}$.

B. Waveform and multipole moments

The gravitational-wave field can be obtained from the asymptotic behavior of Ψ_4 at large distances. Choosing the θ and ϕ directions as polarization axes, the two fundamental polarizations of the gravitational waves can be expressed as [27]

$$h \equiv h_+ - ih_\times = \frac{2\mu}{r}\sum_{\ell m}\tilde{Z}^H_{\ell m}(u)\,_{-2}Y_{\ell m}(\theta,\phi), \qquad (2.14)$$

where $u = t - r^*$, and

$$\tilde{Z}^H_{\ell m}(u) = \int_{-\infty}^{\infty} d\omega\, Z^H_{\omega\ell m}e^{-i\omega u}. \qquad (2.15)$$

The traceless-transverse gravitational-wave tensor is then

$$h^{\rm TT}_{ab} = hm_a m_b + \bar{h}\bar{m}_a\bar{m}_b, \qquad (2.16)$$

where latin indices denote spatial components.

It is clear that the $\tilde{Z}^H_{\ell m}(u)$ represent the radiative multipole moments of the gravitational field. To relate these quantities to the commonly used definitions, we substitute Eq. (2.14) into (2.16) and rewrite in terms of the Mathews tensor spherical harmonics [42]. The result is Eq. (4.3) of Ref. [42],

$$h^{\rm TT}_{ab} = \frac{1}{r}\sum_{\ell m}\Big[\mathcal{I}_{\ell m}(u)T^{E2,\ell m}_{ab} + \mathcal{S}_{\ell m}(u)T^{B2,\ell m}_{ab}\Big], \qquad (2.17)$$

where the T tensors are the spherical harmonics. We thus find that the mass multiple moments are given by

$$\mathcal{I}_{\ell m}(u) = \sqrt{2}\mu\Big[\tilde{Z}^H_{\ell m}(u) + (-1)^m\overline{\tilde{Z}^H}_{\ell,-m}(u)\Big], \qquad (2.18)$$

while the current multipole moments are given by

$$\mathcal{S}_{\ell m}(u) = \sqrt{2}i\mu\Big[\tilde{Z}^H_{\ell m}(u) - (-1)^m\overline{\tilde{Z}^H}_{\ell,-m}(u)\Big]. \qquad (2.19)$$

When the source is confined to the equatorial plane, the identity $\bar{Z}^H_{-\omega,\ell,-m} = (-1)^\ell Z^H_{\omega\ell m}$ holds (see paper III [27] for a derivation). This can be used to simplify the expressions for the mass and current multipole moments.

C. Energy and angular momentum fluxes

Equations (2.8) and (2.9) can be used to calculate the rates at which energy and angular momentum are radiated to infinity and absorbed by the black hole. Here we consider only the special case of circular orbits; more general results can be found in Ref. [27].

It will be shown in Sec. V that for circular orbits, the frequency spectrum of the gravitational waves contains all the harmonics of the orbital frequency $\Omega \equiv d\phi/dt$, so that

$$Z^{H,\infty}_{\omega\ell m} = Z^{H,\infty}_{\ell m}\delta(\omega - m\Omega). \qquad (2.20)$$

For this special case, the energy flux at infinity is given by [47]

$$\dot{E}^\infty = \frac{1}{4\pi}\left(\frac{\mu}{M}\right)^2\sum_{\ell m}(M\omega)^2|Z^H_{\ell m}|^2, \qquad (2.21)$$

while the energy flux at the event horizon is [47]

$$\dot{E}^H = \frac{1}{4\pi}\left(\frac{\mu}{M}\right)^2\sum_{\ell m}\alpha_{\ell m}|Z^\infty_{\ell m}|^2, \qquad (2.22)$$

where

$$\alpha_{\ell m} = \frac{2^{12}\left[1+4(M\omega)^2\right]\left[1+16(M\omega)^2\right]}{\left|(\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega\right|^2}(M\omega)^8. \qquad (2.23)$$

In Eqs. (2.21) and (2.22), a dot denotes differentiation with respect to coordinate time t, and $\omega \equiv m\Omega$. The total amount of energy carried by the gravitational waves is $\dot{E} = \dot{E}^\infty + \dot{E}^H$.

The angular momentum fluxes can be obtained from the relations $\dot{E}^\infty = \Omega\dot{L}^\infty$ and $\dot{E}^H = \Omega\dot{L}^H$, valid for any radiating system in rigid rotation [48].

III. INTEGRATION OF THE REGGE-WHEELER EQUATION

Our objective in this section is to integrate Eq. (2.6) in the limit $M\omega \ll 1$. Techniques for solving this problem were developed by Poisson in the first paper in this series [20], and then extended by Sasaki [32]. Here we extend these techniques further, by explicitly considering the issue of the boundary conditions at the black-hole horizon.

Most of this section will be devoted to the "ingoing-wave" Regge-Wheeler function $X^H_{\omega\ell}(r)$. The "outgoing-wave" function $X^\infty_{\omega\ell}(r)$ will be briefly considered in subsection III F. For notational simplicity we will, throughout this section, suppress the use of the $\omega\ell$ suffix.

A. Preliminaries

For concreteness we assume that the wave frequency ω is positive. Results for negative frequencies must be derived separately [49]. For convenience we define

$$z = \omega r, \quad \varepsilon = 2M\omega, \quad z^* = z + \varepsilon\ln(z-\varepsilon). \qquad (3.1)$$

Both z and ε are dimensionless, and we note that $z^* = \omega r^* + \varepsilon\ln\varepsilon$; r^* was introduced in Eq. (2.7).

With these definitions the Regge-Wheeler equation becomes

$$\left\{\frac{d^2}{dz^{*2}} + 1 - f\left[\frac{\ell(\ell+1)}{z^2} - \frac{3\varepsilon}{z^3}\right]\right\}X(z) = 0, \qquad (3.2)$$

where $f = 1 - \varepsilon/z$. The ingoing-wave Regge-Wheeler function satisfies the boundary conditions

$$X^H(z\to\varepsilon) \sim e^{i\varepsilon\ln\varepsilon}e^{-iz^*},$$
$$X^H(z\to\infty) \sim A^{\rm in}e^{i\varepsilon\ln\varepsilon}e^{-iz^*} + A^{\rm out}e^{-i\varepsilon\ln\varepsilon}e^{iz^*}. \qquad (3.3)$$

To carry out the integration of the Regge-Wheeler equation it is useful to follow Sasaki [32] and introduce an auxiliary function $Y^H(z)$ defined by

$$Y^H(z) = z^{-1} e^{i\varepsilon \ln(z-\varepsilon)} X^H(z). \tag{3.4}$$

The great advantage of dealing with $Y^H(z)$ instead of $X^H(z)$ directly will become apparent below. The auxiliary function satisfies

$$\left\{ z(z-\varepsilon)\frac{d^2}{dz^2} + \left[2(1-i\varepsilon)z - \varepsilon\right]\frac{d}{dz} + \left[z^2 - \ell(\ell+1) + \frac{\varepsilon}{z}(4 - iz + z^2)\right]\right\} Y^H(z) = 0. \tag{3.5}$$

Equations (3.3) and (3.4) further imply

$$Y^H(z \to \varepsilon) \sim \varepsilon^{-1} e^{i\varepsilon(\ln\varepsilon - 1)} \tag{3.6}$$

and

$$Y^H(z \to \infty) \sim A^{\rm in} e^{i\varepsilon\ln\varepsilon} z^{-1} e^{-iz} + A^{\rm out} e^{-i\varepsilon\ln\varepsilon} z^{-1} e^{i(z + 2\varepsilon\ln z)}. \tag{3.7}$$

For future reference we shall now rewrite Eq. (3.7) in a different form. We first expand the right-hand side in powers of ε, which below will treated as a small number. (This assumption has not yet been used.) We also invoke the spherical Bessel functions $j_\ell(z)$ and $n_\ell(z)$, which will play a prominent role below, via the asymptotic relations $z^{-1}e^{\pm iz} \sim (\pm i)^{\ell+1}[j_\ell(z\to\infty) \pm i n_\ell(z \to \infty)]$. We thus obtain

$$Y^H(z\to\infty) \sim \left[a_+ + 2i\tilde{a}\varepsilon\ln z + O(\varepsilon^2)\right] j_\ell(z\to\infty) - \varepsilon\left[i a_- + 2\tilde{a}\ln z + O(\varepsilon)\right] n_\ell(z\to\infty). \tag{3.8}$$

In terms of $a_\pm$ we have

$$\begin{aligned} A^{\rm in} &= \tfrac{1}{2}(i)^{\ell+1}(a_+ + \varepsilon a_-)e^{-i\varepsilon\ln\varepsilon}, \\ A^{\rm out} &= \tfrac{1}{2}(-i)^{\ell+1}(a_+ - \varepsilon a_-)e^{i\varepsilon\ln\varepsilon}, \\ \tilde{a} &= \tfrac{1}{2}(a_+ - \varepsilon a_-). \end{aligned} \tag{3.9}$$

B. Auxiliary function: solution for $z \ll 1$

We first integrate Eq. (3.5) in the limit $z \ll 1$. We shall also treat ε as a small number, but leave arbitrary the ratio z/ε. Accordingly, we neglect the z^2 terms within the large square brackets of Eq. (3.5): the first one can be neglected in front of $\ell(\ell+1)$, while the second is negligible in front of 4. All other terms are kept in the equation.

The resulting equation can easily be solved if we change the independent variable to $x = 1 - z/\varepsilon$ and the dependent variable to $Z = (\varepsilon/z)^2 Y$. We then obtain

$$x(x-1)Z'' + \left[2(3-i\varepsilon)x - (1-2i\varepsilon)\right]Z' + \left[6 - \ell(\ell+1) - 5i\varepsilon\right]Z = 0, \tag{3.10}$$

where a prime denotes differentiation with respect to x. This is the hypergeometric equation, with parameters

$$\begin{aligned} a &= -(\ell-2) - i\varepsilon + O(\varepsilon^2), \\ b &= \ell + 3 - i\varepsilon + O(\varepsilon^2), \\ c &= 1 - 2i\varepsilon. \end{aligned} \tag{3.11}$$

The two linearly independent solutions are $F(a,b;c;x)$ and $x^{1-c}F(a+1-c, b+1-c; 2-c; x)$, where F is the hypergeometric function. However, only the first solution is regular at $x = 0$ ($z = \varepsilon$). Compatibility with Eq. (3.6) therefore implies that the second solution must be rejected. We therefore obtain

$$Y^H(z \ll 1) = \varepsilon^{-1} e^{i\varepsilon(\ln\varepsilon - 1)}(z/\varepsilon)^2 F(a,b;c;1-z/\varepsilon). \tag{3.12}$$

Equation (3.12) enforces the correct boundary condition at the black-hole horizon.

For later use we now evaluate $Y^H(z)$ in the limit $\varepsilon \ll z \ll 1$. First, a straightforward calculation, using Eq. (15.3.7) of Abramowitz and Stegun [50], reduces Eq. (3.12) to

$$Y^H(\varepsilon \ll z \ll 1) = \varepsilon^{-1} e^{i\varepsilon(\ln\varepsilon-1)} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\left(\frac{z}{\varepsilon}\right)^{\ell+i\varepsilon} \times \left\{1 - \left[\frac{(\ell-2)(\ell+2)}{2\ell} + i\varepsilon\right]\frac{\varepsilon}{z} + O\left[(\varepsilon/z)^2\right]\right\}. \tag{3.13}$$

Next, we expand the complex exponential, and the Γ functions according to $\Gamma(n+\delta) = (n-1)![1 + \psi(n)\delta + O(\delta^2)]$, in powers of ε. [Here, n is an integer and $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$, where $\gamma \simeq 0.57721$ is the Euler constant, is the digamma function.] Finally, we obtain

$$Y^H(\varepsilon \ll z \ll 1) = \frac{(2\ell)!}{(\ell-2)!(\ell+2)!}\frac{z^\ell}{\varepsilon^{\ell+1}}\left[1 + i\varepsilon(\alpha_\ell + \ln z) - \frac{(\ell-2)(\ell+2)}{2\ell}\frac{\varepsilon}{z} + O(\varepsilon^2)\right]. \tag{3.14}$$

We have defined

$$\alpha_\ell = 2\gamma + \psi(\ell-1) + \psi(\ell+3) - 1. \tag{3.15}$$

C. Auxiliary function: solution for $\varepsilon \ll 1$

We now integrate Eq. (3.5) in the limit $\varepsilon \ll 1$. It will also be assumed that $z \gg \varepsilon$, but z is no longer restricted to be much smaller than unity.

We first rewrite Eq. (3.5) by sending to the right-hand side all terms which are linear in ε; the left-hand side is then independent of ε. To solve this equation we proceed iteratively [32], by setting

$$Y^H(z) = \sum_{n=0}^{\infty} \varepsilon^n Y^{H(n)}(z). \tag{3.16}$$

A short calculation then shows that each $Y^{H(n)}(z)$ satisfies

$$\left[z^2\frac{d^2}{dz^2}+2z\frac{d}{dz}+z^2-\ell(\ell+1)\right]Y^{H(n)}(z)=S^{H(n)}(z), \tag{3.17}$$

where

$$S^{H(n)}(z)=\frac{1}{z}\left[z^2\frac{d^2}{dz^2}+z(1+2iz)\frac{d}{dz}-(4-iz+z^2)\right]Y^{H(n-1)}(z). \tag{3.18}$$

Equation (3.17) is an inhomogeneous spherical Bessel equation. It is the simplicity of this equation which motivated the introduction of the auxiliary function [32].

The zeroth-order solution $Y^{H(0)}(z)$ satisfies the homogeneous spherical Bessel equation, and must therefore be a linear combination of $j_\ell(z)$ and $n_\ell(z)$. Compatibility with Eq. (3.14) then demands

$$Y^{H(0)}(z)=Bj_\ell(z), \tag{3.19}$$

where B is constant to be determined.

The procedure to obtain $Y^{H(1)}(z)$ was described in detail in Ref. [32]. In short, Eq. (3.19) is substituted into Eq. (3.18) to obtain $S^{H(1)}$, and Eq. (3.17) is integrated using the Green's function $G(z,z')=j_\ell(z_<)n_\ell(z_>)$. The resulting integrals can be evaluated explicitly, and the final result is [32]

$$\begin{aligned}Y^{H(1)}(z)/B=&\left(a^{(1)}+i\ln z-\mathrm{Si}2z\right)j_\ell+\left(b^{(1)}-\ln 2z+\mathrm{Ci}2z\right)n_\ell+z^2(n_\ell j_0-j_\ell n_0)j_0\\&+\sum_{p=1}^{\ell-2}\left(\frac{1}{p}+\frac{1}{p+1}\right)z^2(n_\ell j_p-j_\ell n_p)j_p-\left[\frac{(\ell-2)(\ell+2)}{2\ell(2\ell+1)}+\frac{2\ell-1}{\ell(\ell-1)}\right]j_{\ell-1}+\frac{(\ell-1)(\ell+3)}{2(\ell+1)(2\ell+1)}j_{\ell+1}.\end{aligned} \tag{3.20}$$

Here, $a^{(1)}$ and $b^{(1)}$ are constants of integration to be determined, $\mathrm{Si}x=\int_0^x dt\,t^{-1}\sin t$ is the sine integral, and $\mathrm{Ci}x=\gamma+\ln x+\int_0^x dt\,t^{-1}(\cos t-1)$ is the cosine integral.

D. Matching

The constants B, $a^{(1)}$, and $b^{(1)}$ can be determined by matching $Y^H(z)=Y^{H(0)}(z)+\varepsilon Y^{H(1)}(z)+O(\varepsilon^2)$ to Eq. (3.14). This involves the evaluation of Eqs. (3.19) and (3.20) in the limit $z\ll 1$. This calculation is straightforward, since the limiting expressions for the spherical Bessel functions, as well as that for the sine and cosine integrals, are well known. We find that $b^{(1)}=-\gamma$ and that $a^{(1)}$ is arbitrary — it can be absorbed into the $O(\varepsilon)$ part of B. Without loss of generality we may set $a^{(1)}=0$.

We note in passing that consistency between Eqs. (3.14), (3.19), and (3.20) demands that the following equation holds:

$$\sum_{n=2}^{\infty}(-1)^n\frac{n-1}{2n(2n)!}(2z)^{2n}-\sum_{p=1}^{\ell-2}\left(\frac{1}{p}+\frac{1}{p+1}\right)z^2{j_p}^2=\frac{z^{2\ell}[1+O(z^2)]}{(2\ell-1)!!(2\ell-3)!!\ell(\ell-1)}. \tag{3.21}$$

We have not been able to directly establish the validity of Eq. (3.21), but have checked that it does indeed hold for several special cases.

Finally, matching yields the value of B, which is

$$B=\frac{(2\ell)!(2\ell+1)!!}{(\ell-2)!(\ell+2)!}\frac{1}{\varepsilon^{\ell+1}}\left[1+i\varepsilon\alpha_\ell+O(\varepsilon^2)\right], \tag{3.22}$$

where α_ℓ was introduced in Eq. (3.15). The integration of Eq. (3.5), accurately to first order in ε and with boundary condition (3.6), is now completed. In Ref. [32], Sasaki has indicated how to proceed to higher order in ε.

E. Calculation of $A^{\rm in}$ and $A^{\rm out}$

We are now in a position to evaluate $Y^H(z)$ in the limit $z\to\infty$ and to compare with Eq. (3.8) so as to obtain expressions for the amplitudes $A^{\rm in}$ and $A^{\rm out}$. Again the calculation is straightforward, since the asymptotic expressions for the spherical Bessel functions (as well as that for the sine and cosine integrals) are well known.

We use such expressions as $j_{\ell-1}\sim -j_{\ell+1}\sim -n_\ell$ and $z^2(n_\ell j_p-j_\ell n_p)j_p\sim\frac{1}{2}[1-(-1)^{\ell-p}]n_\ell$ to calculate

$$\begin{aligned}Y^H(z\to\infty)\sim B\Big\{&[1+\varepsilon(i\ln z-\pi/2)]j_\ell(z\to\infty)\\&-\varepsilon(\ln 2z-\beta_\ell)n_\ell(z\to\infty)+O(\varepsilon^2)\Big\},\end{aligned} \tag{3.23}$$

where

$$\beta_\ell=\frac{1}{2}\left[\psi(\ell)+\psi(\ell+1)+\frac{(\ell-1)(\ell+3)}{\ell(\ell+1)}\right]. \tag{3.24}$$

Comparison with Eqs. (3.8) and (3.9) finally yields

$$\begin{aligned}A^{\rm in}&=\frac{(2\ell)!(2\ell+1)!!}{2(\ell-2)!(\ell+2)!}\left(\frac{i}{\varepsilon}\right)^{\ell+1}e^{-i\varepsilon(\ln 2\varepsilon-\alpha_\ell-\beta_\ell)}\times\left[1-\frac{\pi}{2}\varepsilon+O(\varepsilon^2)\right],\\A^{\rm out}&=\frac{(2\ell)!(2\ell+1)!!}{2(\ell-2)!(\ell+2)!}\left(\frac{-i}{\varepsilon}\right)^{\ell+1}e^{i\varepsilon(\ln 2\varepsilon+\alpha_\ell-\beta_\ell)}\times\left[1-\frac{\pi}{2}\varepsilon+O(\varepsilon^2)\right].\end{aligned} \tag{3.25}$$

Analytical expressions for $A^{\rm in}$ and $A^{\rm out}$ in the low-frequency regime were first obtained by Fackerell [38]; his results correspond to the $\varepsilon \to 0$ limit of Eq. (3.25). That these quantities scale like $\varepsilon^{-(\ell+1)}$ in that regime was first discovered by Price [39] and Thorne [40]. The $O(\varepsilon)$ corrections to Fackerell's results, we believe, are presented here for the first time.

It is important to notice that $A^{\rm out}$ is not quite the complex conjugate of $A^{\rm in}$. (The physical relevance of the phase factor will be discussed in Sec. IV B.) In magnitude, Eq. (3.25) implies that they are equal, $|A^{\rm out}| = |A^{\rm in}|[1 + O(\varepsilon^2)]$, up to a fractional accuracy of order ε^2. In fact, a much stronger result follows from the Wronskian relation $|A^{\rm in}|^2 - |A^{\rm out}|^2 = 1$. Simple manipulations and use of Eq. (3.25) indeed reveal that $|A^{\rm out}|$ and $|A^{\rm out}|$ are equal up to a fractional accuracy of order $\varepsilon^{2(\ell+1)}$.

F. The outgoing-wave Regge-Wheeler function

We now integrate the Regge-Wheeler equation for $X^{\infty}_{\omega\ell}(r)$, the outgoing-wave function. The method is entirely analogous to that described in the preceding subsections. We briefly sketch the most important steps.

We first introduce an auxiliary function $Y^{\infty}(z)$ defined by

$$Y^{\infty}(z) = z^{-1} e^{-i\varepsilon \ln(z-\varepsilon)} X^{\infty}(z). \tag{3.26}$$

From Eq. (2.7) we see that the auxiliary function must satisfy the boundary condition

$$Y^{\infty}(z \to \infty) \sim e^{-i\varepsilon \ln \varepsilon} z^{-1} e^{iz}. \tag{3.27}$$

It is immediate that $\bar{Y}^{\infty}(z)$ satisfies the same differential equation, Eq. (3.5), as $Y^H(z)$. Since the boundary conditions are imposed at $z = \infty$, we shall not need to solve Eq. (3.5) in the limit $z \ll 1$ (as described in subsection III B). Instead we can proceed with an analysis similar to that contained in subsection III C.

The zeroth-order solution $Y^{\infty(0)}$ must be identified, up to a normalization constant, with the spherical Hankel function $h^{(1)}_\ell(z)$, whose asymptotic behavior is identical to Eq. (3.27). We therefore have

$$Y^{\infty(0)}(z) = C h^{(1)}_\ell(z), \tag{3.28}$$

where C is a constant to be determined. Taking the complex conjugate, we get $\bar{Y}^{\infty(0)}(z) = \bar{C} h^{(2)}_\ell(z) = \bar{C}[j_\ell(z) - i n_\ell(z)]$. The integration of Eq. (3.17) for $\bar{Y}^{\infty(1)}(z)$ proceeds as previously described. We obtain

$$\begin{aligned} Y^{\infty(1)}(z)/C = {} & c^{(1)} h^{(1)}_\ell + \Big(d^{(1)} - \mathrm{Si}2z + i\mathrm{Ci}2z\Big) h^{(2)}_\ell + z^2(n_\ell j_0 - j_\ell n_0) h^{(1)}_0 + \sum_{p=1}^{\ell-2} \left(\frac{1}{p} + \frac{1}{p+1}\right) z^2 (n_\ell j_p - j_\ell n_p) h^{(1)}_p \\ & - \left[\frac{(\ell-2)(\ell+2)}{2\ell(2\ell+1)} + \frac{2\ell-1}{\ell(\ell-1)}\right] h^{(1)}_{\ell-1} + \frac{(\ell-1)(\ell+3)}{2(\ell+1)(2\ell+1)} h^{(1)}_{\ell+1}, \end{aligned} \tag{3.29}$$

where $c^{(1)}$ and $d^{(1)}$ are constants of integration.

Matching to Eq. (3.27) involves the evaluation of Eq. (3.29) in the limit $z \to \infty$, which requires manipulations similar to the ones leading to Eq. (3.23). We eventually obtain that $c^{(1)}$ is arbitrary and can be set to zero without loss of generality, that $d^{(1)} = \pi/2$, and that

$$C = (i)^{\ell+1} \left[1 + i\varepsilon(\beta_\ell + \gamma - \ln \varepsilon) + O(\varepsilon^2)\right], \tag{3.30}$$

where β_ℓ is given in Eq. (3.24). This completes the determination of $Y^{\infty}(z)$ to first order in ε.

IV. RADIATIVE MULTIPOLE MOMENTS IN THE SLOW-MOTION APPROXIMATION

In this section we apply the results of Secs. II and III to the limiting case of a slowly moving source. We first derive leading-order (Newtonian) expressions for the radiative multipole moments — $Z^H_{\omega\ell m}$ in the frequency domain, $\tilde{Z}^H_{\ell m}(u)$ in the time domain; see Sec. II B. We then improve on these expressions by considering the corrections due to wave-propagation (tail) effects, which are of post$^{3/2}$-Newtonian order. These corrections do not depend on the specific details of the source, but are a manifestation of the curvature of spacetime outside the source. The larger, source-specific, post-Newtonian corrections will not be considered in this paper.

For the purpose of the following calculation we do not assume a specific form for the stress-energy tensor $T^{\alpha\beta}$ that perturbs the black hole's gravitational field. In particular, we do not assume that the perturbations are produced by a moving particle and that $T^{\alpha\beta}$ is of the form (2.4). [The following results will nevertheless contain factors of μ^{-1}, the inverse of the particle's mass. This is because such a factor has been inserted for convenience into the definition of $Z^H_{\omega\ell m}$; see Eq. (2.10). Such meaningless occurrences of μ^{-1} can be avoided by setting $\mu \equiv 1$ throughout this section.] We do assume, however, that the source is slowly moving, in a sense made precise below.

A. Leading-order calculation

Let the source be characterized by some internal velocity v. We demand $v \ll 1$. We assume that the source is gravitationally bound to the black hole, so that $M/r \approx v^2$, where r is a typical value of the radial coordinate inside the source. Finally, if ω is a typical frequency of the gravitational waves, then slow motion implies $\omega r \approx v$, and $M\omega \approx v^3$. In terms of the variables introduced in Sec. III, we have $z \approx v$, $\varepsilon/z \approx v^2$, and

$\varepsilon \approx v^3$. This implies that such results as Eqs. (3.14) and (3.25) can be used.

In the slow-motion approximation, the stress-energy tensor is dominated by the component ${}_0T = T_{\alpha\beta}n^\alpha n^\beta = \frac{1}{4}\rho + O(\rho v)$, where ρ is the source's mass density. By comparison, ${}_{-1}T = O(\rho v)$ and ${}_{-2}T = O(\rho v^2)$; these components will be neglected. Equation (2.5) then implies ${}_0T_{\omega\ell m}(r) = \frac{1}{4}\int d\Omega\tilde{\rho}(\omega, \boldsymbol{x})\bar{Y}_{\ell m}(\theta,\phi) + O(\rho v)$, where

$$\tilde{\rho}(\omega, \boldsymbol{x}) = \frac{1}{2\pi}\int dt \rho(t, \boldsymbol{x})e^{i\omega t} \tag{4.1}$$

is the mass density in the frequency domain.

The slow-motion approximation also implies that Eqs. (2.11) and (2.13) reduce to $\kappa^H_{\omega\ell} = \frac{1}{4}(\ell-1)\ell(\ell+1)(\ell+2) + O(v^3)$ and ${}_0\Gamma_{\omega\ell} = 2rd/dr + \ell(\ell+1) + O(v)$. Equations (3.4), (3.14), and (3.25) can then be combined to give

$$\frac{{}_0\Gamma_{\omega\ell}X^H_{\omega\ell}(r)}{A^{\rm in}_{\omega\ell}} = \frac{2(\ell+1)(\ell+2)}{(2\ell+1)!!}(-i\omega r)^{\ell+1}, \tag{4.2}$$

to leading order in v. Substituting these results into Eq. (2.10) yields

$$Z^H_{\omega\ell m} = \frac{4\pi}{(2\ell+1)!!\mu}\left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell}\right]^{1/2}(-i\omega)^\ell \times \int d^3x\,\tilde{\rho}(\omega,\boldsymbol{x})r^\ell\bar{Y}_{\ell m}(\theta,\phi), \tag{4.3}$$

the radiative moments in the frequency domain.

Finally, taking the Fourier transform of Eq. (4.3) we obtain, to leading order in v,

$$\tilde{Z}^H_{\ell m}(u) = \frac{4\pi}{(2\ell+1)!!\mu}\left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell}\right]^{1/2}\left(\frac{d}{du}\right)^\ell \times \int d^3x\,\rho(u,\boldsymbol{x})r^\ell\bar{Y}_{\ell m}(\theta,\phi), \tag{4.4}$$

the radiative moments in the (retarded) time domain. As expected, the radiative moments of order ℓ are given, up to a numerical factor, by the ℓth time derivative of the source moments, $\int d^3x\,\rho r^\ell \bar{Y}_{\ell m}$. This leading-order calculation agrees with that of Sec. V C of Ref. [42]. (It should be noted that we only have obtained the mass multipole moments. To calculate the current moments would involve keeping terms of order ρv in our approximations.)

B. Tail correction

We now improve upon the leading-order calculation by considering the corrections to Eq. (4.3) which are due to wave-propagation effects (tails; for the purpose of this discussion it is simpler to work in the frequency domain). More precisely, we consider the corrections to Eq. (4.3) which are of order $M\omega \approx v^3$, the smallness parameter appearing in the Regge-Wheeler equation. We shall ignore, for reasons given above, the corrections of order v and v^2.

The steps are simple. We repeat the calculation described in the previous subsection, but with the following changes. (i) We rewrite Eq. (2.11) as $\kappa^H_{\omega\ell} = \frac{1}{4}(\ell-1)\ell(\ell+1)(\ell+2)e^{-12iM\omega/(\ell-1)\ell(\ell+1)(\ell+2)}$, to a fractional accuracy of order $(M\omega)^2$. (ii) We similarly modify $X^H_{\omega\ell}(r)$ by multiplying the leading-order expression by $e^{2i\alpha_\ell M\omega}$, with α_ℓ given in Eq. (3.15). This step is dictated by Eq. (3.14); according to our rules we may ignore the other correction terms. (iii) We modify $A^{\rm in}_{\omega\ell}$ by multiplying the leading-order expression by $(1-\pi M|\omega|)e^{-2iM\omega(\ln 4M|\omega|-\alpha_\ell-\beta_\ell)}$, as dictated by Eq. (3.25); β_ℓ is defined in Eq. (3.24).

After simple manipulations, the final result is that the tail corrections to the radiative multipole moments take the form

$$Z^H_{\omega\ell m} \to Z^H_{\omega\ell m}(1+\pi M|\omega|)e^{2iM\omega(\ln 4M|\omega|-\mu_\ell)}, \tag{4.5}$$

where $Z^H_{\omega\ell m}$ stands for the leading-order expression for the radiative moments. We have introduced $\mu_\ell = \beta_\ell - 6[(\ell-1)\ell(\ell+1)(\ell+2)]^{-1}$, or

$$\mu_\ell = \psi(\ell-1) + \frac{\ell^3+7\ell^2+12\ell+8}{2\ell(\ell+1)(\ell+2)}, \tag{4.6}$$

after using Eq. (3.24).

A result identical to Eq. (4.5) was first derived for the frequency-domain quadrupole ($\ell = 2$) moments by Blanchet and Shäfer [22], and then generalized to moments of arbitrary order by Blanchet [51]. Those results were derived within the context of post-Newtonian theory (see Secs. I B and I G). Our results, which confirm that of Blanchet and Shäfer, were obtained using a completely different method of analysis, that of black-hole perturbation theory. It is, of course, pleasing that such different methods yield identical results.

Blanchet's tail correction [51] is identical to Eq. (4.5) except for a different representation of the multipole moments (he uses a symmetric-trace-free representation), and a different phase. Blanchet's expression substitutes a constant $\kappa_\ell - \gamma$ in place of μ_ℓ, where [16]

$$\kappa_\ell = \sum_{k=1}^{\ell-2}\frac{1}{k} + \frac{2\ell^2+5\ell+4}{\ell(\ell+1)(\ell+2)}. \tag{4.7}$$

It is easy to see that the two constants are related by $\mu_\ell = \kappa_\ell - \gamma + 1/2$. Our result therefore differs from Blanchet's by a phase factor $e^{-iM\omega}$ independent of ℓ. This overall phase has no physical significance: it can be absorbed into a shift in the origin of the retarded time u.

V. CIRCULAR ORBITS IN THE SLOW-MOTION APPROXIMATION

In this section we use the perturbation formalism to calculate the energy radiated by a particle in circular motion around a Schwarzschild black hole. We perform this calculation in the slow-motion approximation, to leading order in the velocity v. We consider both the energy radiated to infinity, and that absorbed by the black hole. Part of the calculations presented in this section were also carried out in paper I [20]. We repeat these calculations here for completeness. The remaining calculations are presented here for the first time. Equations (5.17) and (5.18) below were quoted without derivation in a footnote of paper III [27].

A. $Z^{H,\infty}_{\omega\ell m}$ for $\ell + m$ even

We begin with the calculation of the amplitudes $Z^{H}_{\omega\ell m}$ and $Z^{\infty}_{\omega\ell m}$, to leading order in v (to be defined precisely below), for the case where $\ell + m$ is an even integer. The case $\ell + m$ odd will be treated in subsection V B. That these cases must be considered separately will become apparent below.

As was discussed in Sec. IV A, the dominant component of the stress-energy tensor in a slow-motion approximation is ${}_0T = T_{\alpha\beta}n^\alpha n^\beta$. Equation (2.4) then yields

$$ {}_0T = \frac{\mu}{{r_0}^2}\frac{(u^\alpha n_\alpha)^2}{u^t}\delta(r-r_0)\delta(\cos\theta)\delta(\phi-\Omega t). \quad (5.1)$$

Here, r_0 is the radius of the circular orbit which, without loss of generality, has been put in the equatorial plane $\theta = \pi/2$; u^α is the four-velocity, and $\Omega \equiv d\phi/dt = (M/{r_0}^3)^{1/2}$ is the angular velocity.

We define v as

$$ v = \Omega r_0 = (M/r_0)^{1/2} = (M\Omega)^{1/3}, \quad (5.2)$$

and demand $v \ll 1$. Up to corrections of order $M/r_0 = v^2$, Eq. (5.1) reduces to ${}_0T = (\mu/4{r_0}^2)\delta(r - r_0)\delta(\cos\theta)\delta(\phi - \Omega t)$. Substitution into Eq. (2.5) and integration yields

$$ {}_0T_{\omega\ell m}(r) = \frac{\mu}{4{r_0}^2}\,{}_0Y_{\ell m}(\tfrac{\pi}{2},0)\delta(r-r_0)\delta(\omega-m\Omega). \quad (5.3)$$

That ${}_sT_{\omega\ell m}$, and hence $Z^{H,\infty}_{\omega\ell m}$, is proportional to $\delta(\omega - m\Omega)$ has been anticipated in Eq. (2.20). (This result is exact, and not a consequence of the slow-motion approximation. See Ref. [20] for details.) We shall now factor out this δ function, and work with the quantities $Z^{H,\infty}_{\ell m}$ defined in Eq. (2.20).

Substituting Eqs. (4.2) and (5.3) into (2.10) and integrating, we obtain

$$ Z^{H}_{\ell m} = 4\pi(-i)^\ell \frac{m^\ell}{(2\ell+1)!!}\left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell}\right]^{1/2} \times {}_0Y_{\ell m}(\tfrac{\pi}{2},0)v^\ell. \quad (5.4)$$

Equation (5.4) is valid up to fractional corrections of order v^2 (see Refs. [17,20] for details.)

A similar calculation yields $Z^{\infty}_{\ell m}$. We first use the results of Sec. III E to obtain $X^{\infty}_{\omega\ell}(r) = (i)^{\ell+1}\omega r h^{(1)}_\ell(\omega r)$, where $h^{(1)}_\ell$ is the spherical Hankel function, to leading order in $M\omega = O(v^3)$. Then, up to fractional corrections of order $(\omega r)^2 = O(v^2)$, we have $X^{\infty}_{\omega\ell}(r) = (i)^\ell(2\ell-1)!!(\omega r)^{-\ell}$. Next, we use the leading-order expressions for ${}_0\Gamma_{\omega\ell}$ and $A^{\rm in}_{\omega\ell}$, Eqs. (2.13) and (3.25), to obtain

$$ \frac{{}_0\Gamma_{\omega\ell}X^{\infty}_{\omega\ell(r)}}{A^{\rm in}_{\omega\ell}} = -2^{\ell+2}i\frac{(\ell-1)\ell(\ell-2)!(\ell+2)!}{(2\ell)!(2\ell+1)}\frac{(M\omega)^{\ell+1}}{(\omega r)^\ell}, \quad (5.5)$$

to leading order in v. Finally, substitution of Eqs. (5.3) and (5.5) into (2.10) yields

$$ Z^{\infty}_{\ell m} = -2^{\ell-3}\pi\frac{[(\ell-1)\ell]^{3/2}[(\ell+1)(\ell+2)]^{1/2}}{m^3(2\ell)!(2\ell+1)} \times (\ell-2)!(\ell+2)!\,{}_0Y_{\ell m}(\tfrac{\pi}{2},0)v^{2\ell-7}, \quad (5.6)$$

to leading order in v.

An explicit expression for ${}_0Y_{\ell m}(\frac{\pi}{2},0)$ can be obtained by expressing the spherical harmonics in terms of the associated Legendre polynomials, and using the fact that $P^m_\ell(0) = (-1)^{\frac{1}{2}(\ell-m)}(\ell+m-1)!!/(\ell-m)!!$ if $\ell+m$ is even, and $P^m_\ell(0) = 0$ if $\ell+m$ is odd [52]. Thus,

$$ {}_0Y_{\ell m}(\tfrac{\pi}{2},0) = (-1)^{\frac{1}{2}(\ell+m)}\left[\frac{2\ell+1}{4\pi}\right]^{1/2} \times \frac{[(\ell-m)!(\ell+m)!]^{1/2}}{(\ell-m)!!(\ell+m)!!} \quad (5.7)$$

if $\ell+m$ is even, and ${}_0Y_{\ell m}(\frac{\pi}{2},0) = 0$ otherwise. We therefore see that the dominant, $s = 0$, contribution to $Z^{H,\infty}_{\ell m}$ vanishes identically if $\ell + m$ is odd.

B. $Z^{H,\infty}_{\omega\ell m}$ for $\ell + m$ odd

If $\ell + m$ is odd the dominant contribution to $Z^{H,\infty}_{\ell m}$ comes from the component ${}_{-1}T = T_{\alpha\beta}n^\alpha \bar{m}^\beta$ of the stress-energy tensor. A calculation analogous to that of the previous subsection yields ${}_{-1}T = (i\mu v/2\sqrt{2}{r_0}^2)\delta(r - r_0)\delta(\cos\theta)\delta(\phi - \Omega t)$, so that

$$ {}_{-1}T_{\omega\ell m}(r) = \frac{i\mu v}{2\sqrt{2}{r_0}^2}\,{}_{-1}Y_{\ell m}(\tfrac{\pi}{2},0)\delta(r-r_0)\delta(\omega-m\Omega). \quad (5.8)$$

From Eq. (2.13) we have ${}_{-1}\Gamma_{\omega\ell} = -\ell(\ell+1)(rd/dr+1)$. It follows that

$$ \frac{{}_{-1}\Gamma_{\omega\ell}X^{H}_{\omega\ell}(r)}{A^{\rm in}_{\omega\ell}} = -2(-i)^{\ell+1}\frac{\ell(\ell+1)(\ell+2)}{(2\ell+1)!!}(\omega r)^{\ell+1} \quad (5.9)$$

and

$$ \frac{{}_{-1}\Gamma_{\omega\ell}X^{\infty}_{\omega\ell}(r)}{A^{\rm in}_{\omega\ell}} = -2^{\ell+2}i\frac{(\ell-1)\ell(\ell+1)}{(2\ell)!(2\ell+1)}(\ell-2)!(\ell+2)! \times (M\omega)^{\ell+1}(\omega r)^{-\ell}. \quad (5.10)$$

Here, $X^{H}_{\omega\ell}(r)$, $X^{\infty}_{\omega\ell}(r)$, and $A^{\rm in}_{\omega\ell}$ were approximated as in the previous subsection.

We arrive at

$$ Z^{H}_{\ell m} = 8\pi(-i)^{\ell+1}\left[\frac{\ell+2}{\ell-1}\right]^{1/2}\frac{m^\ell}{(2\ell+1)!!} \times {}_{-1}Y_{\ell m}(\tfrac{\pi}{2},0)v^{\ell+1} \quad (5.11)$$

and

$$Z^{\infty}_{\ell m} = -2^{\ell-2} i\pi \frac{[(\ell-1)(\ell+2)]^{1/2}}{m^3(2\ell)!(2\ell+1)}(\ell+1)\ell!(\ell+2)! \times {}_{-1}Y_{\ell m}(\tfrac{\pi}{2},0)v^{2\ell-6}, \tag{5.12}$$

to leading order in v.

An explicit expression for ${}_{-1}Y_{\ell m}(\frac{\pi}{2},0)$ can be obtained as follows. First, we calculate the $s=-1$ spherical harmonics [41] using ${}_{-1}Y_{\ell m}(\theta,\phi) = -[\ell(\ell+1)]^{-1/2}(\partial_\theta - i\csc\theta\partial_\phi){}_0Y_{\ell m}(\theta,\phi)$. We then express the $s=0$ harmonics in terms of the associated Legendre polynomials. Since $P^m_\ell(0)=0$ when $\ell+m$ is odd, we obtain ${}_{-1}Y_{\ell m}(\frac{\pi}{2},0) = (-1)^m[(2\ell+1)/4\pi\ell(\ell+1)]^{1/2}[(\ell-m)!/(\ell+m)!]^{1/2}P^{m\prime}_\ell(0)$ if $\ell+m$ is odd. (Here, a prime denotes differentiation with respect to the argument.) Finally, use of the relation [52] $P^{m\prime}_\ell(0) = \frac{1}{2}P^{m+1}_\ell(0) - \frac{1}{2}(\ell+m)(\ell-m+1)P^{m-1}_\ell(0)$ yields

$${}_{-1}Y_{\ell m}(\tfrac{\pi}{2},0) = (-1)^{\frac{1}{2}(\ell+m-1)}\left[\frac{2\ell+1}{4\pi\ell(\ell+1)}\right]^{1/2} \times \frac{(\ell-m)!!(\ell+m)!!}{\left[(\ell-m)!(\ell+m)!\right]^{1/2}} \tag{5.13}$$

if $\ell+m$ is odd. For $\ell+m$ even ${}_{-1}Y_{\ell m}(\frac{\pi}{2},0)$ is nonzero, but its expression will not be needed.

C. Energy radiated

It is now a straightforward task to calculate the rate at which the gravitational waves remove energy from the system. We use the results of the preceding subsections, together with Eqs. (2.21)–(2.23). For convenience we shall normalize our expressions for $\dot{E}^\infty$ and $\dot{E}^H$ to the quadrupole-formula expression, $\dot{E}_{\rm QF} = (32/5)(\mu/M)^2v^{10}$ [1]. We therefore define the numbers $\eta^{H,\infty}_{\ell m}$ such that

$$\dot{E}^{H,\infty} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}\sum_{\ell m}\tfrac{1}{2}\eta^{H,\infty}_{\ell m}. \tag{5.14}$$

The factor of 1/2 is inserted for convenience because of the symmetry $\eta^{H,\infty}_{\ell,-m} = \eta^{H,\infty}_{\ell m}$ which can be derived from $\bar{Z}^{H,\infty}_{-\omega,\ell,-m} = (-1)^\ell Z^{H,\infty}_{\omega\ell m}$ (see Ref. [27] for a derivation).

Calculation yields, to leading order in v,

$$\eta^\infty_{\ell m} = \frac{5\pi}{4}\frac{m^{2(\ell+1)}}{(2\ell+1)!!^2}\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell}\left[{}_0Y_{\ell m}(\tfrac{\pi}{2},0)\right]^2 v^{2(\ell-2)} \tag{5.15}$$

if $\ell+m$ is even, and

$$\eta^\infty_{\ell m} = 5\pi\frac{m^{2(\ell+1)}}{(2\ell+1)!!^2}\frac{(\ell+2)}{(\ell-1)}\left[{}_{-1}Y_{\ell m}(\tfrac{\pi}{2},0)\right]^2 v^{2(\ell-1)} \tag{5.16}$$

if $\ell+m$ is odd.

Calculation also yields, to leading order in v,

$$\eta^H_{\ell m} = \left[\frac{2\ell!}{m^\ell}\right]^2 v^{2(\ell+2)}\,\eta^\infty_{\ell m} \tag{5.17}$$

if $\ell+m$ is even, and

$$\eta^H_{\ell m} = \left[\frac{2(\ell+1)!}{\ell m^\ell}\right]^2 v^{2(\ell+2)}\,\eta^\infty_{\ell m} \tag{5.18}$$

if $\ell+m$ is odd.

D. Black-hole absorption

In a slow-motion approximation, the dominant contribution to the energy radiated comes from the $\ell=2$, $|m|=2$ (mass quadrupole) terms in Eq. (5.14). The dominant contribution to the black-hole absorption is given by $\eta^H_{22} = v^8\eta^\infty_{22} = v^8$. Therefore

$$\dot{E}^H/\dot{E}^\infty = v^8\left[1+O(v^2)\right]. \tag{5.19}$$

(That the correction must be of order v^2 can be established by direct calculation.)

In a slow-motion approximation, the total energy radiated, $\dot{E} = \dot{E}^\infty + \dot{E}^H$, can be expressed as a post-Newtonian expansion of the form

$$\dot{E} = \dot{E}_{\rm QF}\Big[1 + O(v^2) + O(v^3) + O(v^4) + O(v^5) + O(v^6) + O(v^6\ln v) + O(v^7) + O(v^8) + O(v^8\ln v) + \cdots\Big]. \tag{5.20}$$

The first three terms of this expansion, through $O(v^3)$, were calculated in paper I [20]; all other terms were calculated by Tagoshi and Sasaki [17]. (See Sec. I B and I D for a more detailed discussion.)

We therefore see that the contribution $\dot{E}^H$ to the total energy radiated occurs at quite a high order in the post-Newtonian expansion: $O(v^8)$, or post^4-Newtonian order. What is more, the black-hole absorption is also a small contribution to the term of order v^8. The coefficient of the $O(v^8)$ term in the post-Newtonian expansion of $\dot{E}^\infty$ was calculated, in the limit of small mass ratios, by Tagoshi and Sasaki [17]. They find that it is approximately equal to -117.5044. The coefficient of the $O(v^8)$ term in the expansion of $\dot{E}^H$ is given in Eq. (5.19): it is equal to unity. The black-hole absorption therefore contributes less than one percent of the $O(v^8)$ term in Eq. (5.20).

The black-hole absorption is a small effect indeed.

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APPENDIX: DERIVATION OF EQ. (2.10)

The source term to the right-hand side of the inhomogeneous Teukolsky equation, Eq. (2.2), is given explicitly by [20,44]

$$T_{\omega\ell m}(r) = 2\pi \sum_s {}_sp_\ell \, {}_sD_\omega \, {}_sT_{\omega\ell m}(r), \tag{A1}$$

where the constants ${}_sp_\ell$ are listed in Eq. (2.12), the functions ${}_sT_{\omega\ell m}(r)$ given in Eq. (2.5), and where

$${}_sD_\omega = \begin{cases} r^4 \,, & s=0 \,, \\ r^2 f \mathcal{L} r^3 f^{-1} \,, & s=-1 \,, \\ r f \mathcal{L} r^4 f^{-1} \mathcal{L} r \,, & s=-2 \,. \end{cases} \tag{A2}$$

are differential operators. Here, $\mathcal{L} = f d/dr + i\omega = d/dr^* + i\omega$.

Equation (2.2) is integrated by constructing a Green's function from two linearly independent solutions to the homogeneous equation [45]. These are denoted $R^H_{\omega\ell}(r)$ and $R^\infty_{\omega\ell}(r)$, and have the following asymptotic behaviors: $R^H_{\omega\ell}(r \to 2M) \sim (\omega r)^4 f^2 e^{-i\omega r^*}$; $R^H_{\omega\ell}(r \to \infty) \sim Q^{\rm in}_{\omega\ell}(\omega r)^{-1} e^{-i\omega r^*} + Q^{\rm out}_{\omega\ell}(\omega r)^3 e^{i\omega r^*}$ ($Q^{\rm in}_{\omega\ell}$ and $Q^{\rm out}_{\omega\ell}$ are constants); $R^\infty_{\omega\ell}(r \to \infty) \sim (\omega r)^3 e^{i\omega r^*}$. A straightforward application of the general theory of Green's functions then shows that the solution to the inhomogeneous Teukolsky equation reduces to Eqs. (2.8) and (2.9), with

$$Z^{H,\infty}_{\omega\ell m} = \left[2i\mu\omega^2 Q^{\rm in}_{\omega\ell}\right]^{-1} \times \int_{2M}^\infty dr\, r^{-4} f^{-2} R^{H,\infty}_{\omega\ell}(r) T_{\omega\ell m}(r). \tag{A3}$$

Our goal in this Appendix is to rewrite Eq. (A3) into the form (2.10). The first step is to substitute Eq. (A1) into (A3) and to introduce the adjoint operators ${}_sD^\dagger_\omega$ such that this can be written in the equivalent form

$$Z^{H,\infty}_{\omega\ell m} = \pi\left[i\mu\omega^2 Q^{\rm in}_{\omega\ell}\right]^{-1} \sum_s {}_sp_\ell \times \int_{2M}^\infty dr\, r^{-4} f^{-2} \, {}_sT_{\omega\ell m}(r) \, {}_sD^\dagger_\omega R^{H,\infty}_{\omega\ell}(r). \tag{A4}$$

Equation (A4) can be obtained from (A3) by performing a number of integration by parts, and the adjoint operators as thus determined. They are found to be given by

$${}_sD^\dagger_\omega = \begin{cases} r^4 \,, & s=0 \,, \\ -r^7 \bar{\mathcal{L}} r^{-2} \,, & s=-1 \,, \\ r^5 f \bar{\mathcal{L}} r^4 f^{-1} \bar{\mathcal{L}} r^{-3} \,, & s=-2 \,, \end{cases} \tag{A5}$$

where $\bar{\mathcal{L}} = d/dr^* - i\omega$. It is useful to note that the operator adjoint to $\mathcal{L}$ is $\mathcal{L}^\dagger = -r^4 f \bar{\mathcal{L}}(r^4 f)^{-1}$. With this known, Eq. (A5) can be obtained directly from (A2).

The next step is to relate the functions $R^{H,\infty}_{\omega\ell}(r)$ to the Regge-Wheeler functions $X^{H,\infty}_{\omega\ell}(r)$, introduced in Eq. (2.7). A straightforward calculation (see Ref. [29] for details) shows that these are related by the Chandrasekhar transformation [46]

$$R^{H,\infty}_{\omega\ell}(r) = \chi^{H,\infty}_{\omega\ell} C_\omega X^{H,\infty}_{\omega\ell}(r), \tag{A6}$$

where

$$\chi^H_{\omega\ell} = \frac{16(1-2iM\omega)(1-4iM\omega)}{(\ell-1)\ell(\ell+1)(\ell+2) - 12iM\omega}(M\omega)^3, \qquad \chi^\infty_{\omega\ell} = -\frac{1}{4}, \tag{A7}$$

and where

$$C_\omega = \omega r^2 f \mathcal{L} f^{-1} \mathcal{L} r \tag{A8}$$

is a second-order differential operator. Equation (A6) implies

$$Q^{\rm in}_{\omega\ell} = -4(1-2iM\omega)(1-4iM\omega)(M\omega)^3 A^{\rm in}_{\omega\ell}; \tag{A9}$$

the constant $A^{\rm in}_{\omega\ell}$ was introduced in Eq. (2.7).

In order to re-express $Z^{H,\infty}_{\omega\ell m}$ as in Eq. (2.10) we must operate with ${}_sD^\dagger_\omega C_\omega$ on the Regge-Wheeler functions. These operations involve successive applications of $\mathcal{L}$, and the Regge-Wheeler equation (2.6) is substituted as often as necessary to simplify the expressions. To carry out these manipulations efficiently, it is useful to first develop the algebra of the $\mathcal{L}$ and $\bar{\mathcal{L}}$ operators. We now list the most useful rules.

When acting on any solution $X_{\omega\ell}(r)$ of the Regge-Wheeler equation, two successive applications of $\mathcal{L}$ or $\bar{\mathcal{L}}$ reduce to

$$\begin{aligned} \mathcal{L}\mathcal{L} &= 2i\omega\mathcal{L} + V(r), \\ \bar{\mathcal{L}}\mathcal{L} &= \mathcal{L}\bar{\mathcal{L}} = V(r), \\ \bar{\mathcal{L}}\bar{\mathcal{L}} &= -2i\omega\bar{\mathcal{L}} + V(r). \end{aligned} \tag{A10}$$

Here, $V(r)$ is the Regge-Wheeler potential, introduced in Eq. (2.6). When carrying out the calculations it is also useful to invoke the commutation relations $[\mathcal{L}, g(r)] = [\bar{\mathcal{L}}, g(r)] = f dg/dr$, where g is any function of r.

Using these rules the calculations are straightforward. The final answer takes the form (2.10) if we define ${}_s\Gamma_{\omega\ell} = \omega^{-1} r^{-5} {}_sD^\dagger_\omega C_\omega$, and $\kappa^{H,\infty}_{\omega\ell} = -Q^{\rm in}_{\omega\ell}/\chi^{H,\infty}_{\omega\ell} A^{\rm in}_{\omega\ell}$.

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