Asymptotic behavior of complex scalar fields in a Friedmann-Lemaitre universe

David Scialom and Philippe Jetzer
Institute of Theoretical Physics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland
(Received 14 November 1994)

We study the coupled Einstein-Klein-Gordon equations for a complex scalar field with and without a quartic self-interaction in a zero curvature Friedmann-Lemaitre universe. The equations can be written as a set of four coupled first-order nonlinear differential equations, for which we establish the phase portrait for the time evolution of the scalar field. For that purpose we find the singular points, including those lying at infinity, of the differential equations of the phase space and study the corresponding asymptotic behavior of the solutions. This knowledge is of relevance, since it provides the initial conditions needed to solve numerically the differential equations. For some singular points lying at infinity we recover the expected emergence of an inflationary stage.

PACS number(s): 04.62.+v, 98.80.Cq, 98.80.Hw

I. INTRODUCTION

The recent developments in particle physics and cosmology suggest that scalar fields may have played an important role in the evolution of the early Universe, for instance, in primordial phase transitions, and that they may constitute part of the dark matter. Moreover, scalar fields are predicted by most of the particle physics models based on the unification of the fundamental forces, as, for instance, in superstring theories. Scalar particles are needed in cosmological models based on inflation, whose relevance is supported by the results of the Cosmic Background Explorer (COBE) Differential Microwave Radiometer (DMR) measurements that are consistent with an Harrison-Zel'dovich (scale invariant $n = 1$) spectrum [1]. These facts, in particular inflation, motivated the study of the coupled Einstein-scalar field equations to determine the time evolution and also the gravitational equilibrium configurations of the scalar fields. The latter one in particular for massive complex scalar fields, which may form so-called boson stars [2,3].

A detailed study of the solutions of the Einstein equations for a homogeneous isotropic Friedmann-Lemaitre universe with a real scalar field has been done in particular by Belinsky *et al.* [4–6]; see also Ref. [7]. In this paper we extend, following Refs. [4–6], these investigations to a complex scalar field.

This analysis is important in order to see the degree of generality of solutions possessing an inflationary stage and also due to the fact that these solutions constitute the background, starting from which one can study in the early Universe the time evolution of perturbations for the scalar field [8,9]. This is a fact that is of relevance if scalar fields make up part of the dark matter. If this is the case, they may also form compact objects, such as Bose stars, or trigger the formation of observed large scale structures in the Universe.

Since we consider a complex scalar field with or without a quartic self-interaction in a zero curvature Friedmann-Lemaitre universe, we get for the Einstein-Klein-Gordon equations a set of four first-order nonlinear differential equations, for which we study the phase portrait for the time evolution of the scalar field. We first determine the singular points of the differential equations and then find analytically the asymptotic behavior for the solutions near these points. For the singular point not lying at infinity of the phase space, we can use for $m \neq 0$, in an adapted coordinate system, the averaging method in order to get the asymptotic behavior of the solution, whereas for the points lying at infinity, we first compactify the phase space on the lower hemisphere of the three-dimensional sphere. We can then apply the Poincaré-Dulac theorem to get the corresponding asymptotic behavior. From the solution we then see if there is inflation and how long it lasts. For some singular points lying at infinity, we recover the expected emergence of an inflationary stage.

The paper is organized as follows. In Sec. II we present the basic equations, which we will use. In Sec. III first we study the singular point not lying at infinity of the phase space for a massive scalar field and then the singular points lying at infinity both with and without a quartic self-interaction. Section IV is devoted to the massless scalar field with a quartic self-interaction, and a short summary concludes the paper.

II. BASIC EQUATIONS

We consider a massive complex scalar field with quartic self-interaction in a Friedmann-Lemaitre universe with the action

$$S = \int \left(-\frac{R}{16\pi G} + e_\mu \varphi e^\mu \varphi^* + m^2 \varphi\varphi^* + \lambda(\varphi\varphi^*)^2 \right) \times \sqrt{-g} d^4x, \quad (1)$$

where g is the determinant of the metric

$$ds^2 = g_{\mu\nu}\theta^\mu\theta^\nu = -\theta^0\theta^0 + \delta_{ij}\theta^i\theta^j, \quad (2)$$

with

$$\theta^0 = dt, \quad \theta^i = \frac{a(t)dx^i}{1 + (k/4)r^2}, \quad r^2 = \sum_{i=1}^{3} x_i^2$$

and e_μ is the dual basis of θ^μ. By varying the action with respect to $g^{\mu\nu}$ we get the Einstein field equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3}$$

with

$$T_{\mu\nu} = e_\mu\varphi e_\nu\varphi^* + e_\nu\varphi e_\mu\varphi^* - g_{\mu\nu}[g^{\alpha\beta}e_\alpha\varphi e_\beta\varphi^* + m^2\varphi\varphi^* + \lambda(\varphi\varphi^*)^2]. \tag{4}$$

The (00) component of Eq. (3) leads to the constraint equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}[\dot\varphi\dot\varphi^* + m^2\varphi\varphi^* + \lambda(\varphi\varphi^*)^2], \tag{5}$$

with $H = \dot a/a$, and an overdot means a derivative with respect to time. For the (ij) component we have

$$\left(-2\dot H - 3H^2 - \frac{k}{a^2}\right)\delta_{ij} = 8\pi G[\dot\varphi\dot\varphi^* - m^2\varphi\varphi^* - \lambda(\varphi\varphi^*)^2]\delta_{ij}. \tag{6}$$

By varying the action with respect to φ^* and φ we get the Klein-Gordon equation

$$\ddot\varphi + 3H\dot\varphi + m^2\varphi + 2\lambda(\varphi\varphi^*)\varphi = 0, \tag{7}$$

and its complex conjugate. The system is fully determined by the independent equations (5) and (7). In fact, one can easily show that Eq. (6) follows from Eqs. (5) and (7). For the massive scalar field case we use the dimensionless variables

$$\begin{aligned} t &\to \eta = mt, \\ \lambda &\to \Lambda = \frac{\lambda}{8\pi G m^2}, \\ \varphi &\to x_1 + ix_2 = \sqrt{8\pi G/3}\varphi, \\ \dot\varphi &\to y_1 + iy_2 = \sqrt{8\pi G/3}\,\frac{\dot\varphi}{m}, \\ H &\to z = \frac{H}{m}, \\ k &\to \tilde k = \frac{k}{m^2}. \end{aligned} \tag{8}$$

This way, we get for Eqs. (5) and (7) the set

$$\frac{\tilde k}{a^2} + z^2 = y_1^2 + y_2^2 + x_1^2 + x_2^2 + 3\Lambda(x_1^2 + x_2^2)^2, \tag{9}$$

$$y_1' = -3zy_1 - x_1 - 6\Lambda x_1(x_1^2 + x_2^2), \tag{10}$$

$$x_1' = y_1, \tag{11}$$

$$y_2' = -3zy_2 - x_2 - 6\Lambda x_2(x_1^2 + x_2^2), \tag{12}$$

$$x_2' = y_2, \tag{13}$$

where a prime means a derivative with respect to η. The only singular point [defined as the point (x_1, x_2, y_1, y_2) for which the right-hand side of Eqs. (9)–(13) vanishes], which we denote by A, not lying at infinity of the phase space (defined by $\varphi, \dot\varphi$ or equivalently x_1, x_2, y_1, y_2) is the coordinate origin. Equations (9)–(13) are invariant under the transformations

$$x_1 \to -x_1 \text{ and } y_1 \to -y_1, \tag{14a}$$

$$x_2 \to -x_2 \text{ and } y_2 \to -y_2, \tag{14b}$$

$$x_2 \leftrightarrow x_1 \text{ and } y_2 \leftrightarrow y_1. \tag{14c}$$

For every solution that describes an expanding Universe (i.e., $z > 0$) there is a corresponding solution describing a collapsing Universe. This can be seen by performing one of the following transformations on the set of Eqs. (9)–(13):

$$\eta \to -\eta,\quad z \to -z,\quad x_1 \to -x_1,\quad x_2 \to -x_2, \tag{15a}$$

$$\eta \to -\eta,\quad z \to -z,\quad x_1 \to -x_1,\quad y_2 \to -y_2, \tag{15b}$$

$$\eta \to -\eta,\quad z \to -z,\quad y_1 \to -y_1,\quad x_2 \to -x_2, \tag{15c}$$

$$\eta \to -\eta,\quad z \to -z,\quad y_1 \to -y_1,\quad y_2 \to -y_2. \tag{15d}$$

For the massless scalar field case we use the dimensionless variables

$$\begin{aligned} t &\to \eta_0 = \frac{t}{\sqrt{8\pi G}}, \\ \varphi &\to x_{10} + ix_{20} = \sqrt{8\pi G/3}\varphi, \\ \dot\varphi &\to y_{10} + iy_{20} = \frac{8\pi G}{\sqrt3}\varphi, \\ H &\to z_0 = \sqrt{8\pi G}H, \\ k &\to \tilde k_0 = 8\pi G k, \end{aligned} \tag{16}$$

and λ remains unchanged. This way, we get for Eqs. (5) and (7) the set

$$\frac{\tilde k_0}{a^2} + z_0^2 = y_{10}^2 + y_{20}^2 + 3\lambda(x_{10}^2 + x_{20}^2)^2, \tag{17}$$

$$y_{10}' = -3z_0y_{10} - 6\lambda x_{10}(x_{10}^2 + x_{20}^2), \tag{18}$$

$$x_{10}' = y_{10}, \tag{19}$$

$$y'_{20} = -3z_0 y_{20} - 6\lambda x_{20}(x_{10}^2 + x_{20}^2), \tag{20}$$

$$x'_{20} = y_{20}, \tag{21}$$

where a prime here means a derivative with respect to η_0. These equations are also invariant under the transformations given by Eqs. (14) and (15), and the coordinate origin is the only singular point not lying at infinity of the phase space.

III. MASSIVE SCALAR FIELD IN A ZERO CURVATURE FRIEDMANN-LEMAITRE UNIVERSE

Next we study the asymptotic behavior of the solutions of Eqs. (9)–(13) near the singular points. Because of the increasing complexity of the analysis involved for a complex scalar field with respect to a real one, we restrict ourselves to the case $k = 0$. In Sec. V we briefly comment on the extension to $k = \pm 1$. With $\tilde{k} = 0$ in Eq. (9) we can then rewrite Eqs. (10)–(13) in spherical coordinates, defined as

$$\begin{aligned} x_1 &= r\cos\vartheta_3\cos\vartheta_1, \\ y_1 &= r\cos\vartheta_3\sin\vartheta_1, \\ x_2 &= r\sin\vartheta_3\cos\vartheta_2, \\ y_2 &= r\sin\vartheta_3\sin\vartheta_2, \end{aligned} \tag{22}$$

with $\vartheta_1, \vartheta_2 \in [0, 2\pi)$, and $\vartheta_3 \in [0,\pi)$. This way we obtain

$$z^2 = r^2 + 3\Lambda r^4(\cos^2\vartheta_3\cos^2\vartheta_1 + \sin^2\vartheta_3\cos^2\vartheta_2)^2, \tag{23}$$

$$\vartheta'_1 = -3z\sin\vartheta_1\cos\vartheta_1 - 1 - 6\Lambda r^2(\cos^2\vartheta_3\cos^2\vartheta_1 + \sin^2\vartheta_3\cos^2\vartheta_2)\cos^2\vartheta_1, \tag{24}$$

$$\vartheta'_2 = -3z\sin\vartheta_2\cos\vartheta_2 - 1 - 6\Lambda r^2(\cos^2\vartheta_3\cos^2\vartheta_1 + \sin^2\vartheta_3\cos^2\vartheta_2)\cos^2\vartheta_2, \tag{25}$$

$$\vartheta'_3 = \sin\vartheta_3\cos\vartheta_3[-3z(\sin^2\vartheta_2 - \sin^2\vartheta_1) - 6\Lambda r^2(\cos^2\vartheta_3\cos^2\vartheta_1 + \sin^2\vartheta_3\cos^2\vartheta_2)(\sin\vartheta_2\cos\vartheta_2 - \sin\vartheta_1\cos\vartheta_1)], \tag{26}$$

$$\begin{aligned} r' = &-3rz(\cos^2\vartheta_3\sin^2\vartheta_1 + \sin^2\vartheta_3\sin^2\vartheta_2) - 6\Lambda r^3(\cos^2\vartheta_3\cos^2\vartheta_1 + \sin^2\vartheta_3\cos^2\vartheta_2) \\ &\times(\cos\vartheta_1\sin\vartheta_1\cos^2\vartheta_3 + \cos\vartheta_2\sin\vartheta_2\sin^2\vartheta_3). \end{aligned} \tag{27}$$

Because of the transformations given in Eq. (14), which leave the equations invariant, we can restrict the domain of definition for ϑ_1, ϑ_2, and ϑ_3, respectively, to $[0,\pi)$, $[0,\pi)$, and $[\pi/2,\pi)$. At the singular point A lying at $r = 0$, the above equations reduce to $\vartheta'_1 = -1$, $\vartheta'_2 = -1$, $\vartheta'_3 = 0$, and $r' = 0$. To get the asymptotic behavior of the solution near A, we apply the averaging method (for details see Ref. [10]). Thus, we have to solve the differential equation

$$r' = -\tfrac{3}{2}r^2, \tag{28}$$

which is obtained by averaging Eq. (27) over the angular variables. We get the asymptotic behavior for the solution near A:

$$\begin{aligned} x_1 &= \frac{2}{3\eta}\cos\vartheta_{30}\cos(\eta - \eta_1), \\ y_1 &= -\frac{2}{3\eta}\cos\vartheta_{30}\sin(\eta - \eta_1), \\ x_2 &= \frac{2}{3\eta}\sin\vartheta_{30}\cos(\eta - \eta_2), \\ y_2 &= -\frac{2}{3\eta}\sin\vartheta_{30}\sin(\eta - \eta_2). \end{aligned} \tag{29}$$

where ϑ_{30}, η_1, and η_2 are integration constants. We see that A is an asymptotically stable winding point. Setting $\vartheta_{30} = 0$ corresponds to consider only a real scalar field, and we recover the solution discussed by Belinsky *et al.* [4–6].

We study now all singular points lying at infinity in phase space. First, we consider the case with no quartic self-interaction term.

A. Properties of the singular points lying at infinity for $\Lambda = 0$

In order to find the singular points lying at infinity, we perform a transformation, which maps them on the boundary of a unit three-sphere, defined as

$$r \to \rho = \frac{r}{1+r}, \quad d\eta \to d\tau = \frac{d\eta}{1-\rho}, \tag{30}$$

with $\rho \in [0,1)$. With this transformation Eqs. (24)–(27) become

$$\frac{d\rho}{d\tau} = -3\rho^2(1-\rho)(\cos^2\vartheta_3\sin^2\vartheta_1 + \sin^2\vartheta_3\sin^2\vartheta_2), \tag{31}$$

$$\frac{d\vartheta_1}{d\tau} = -3\rho\sin\vartheta_1\cos\vartheta_1 - (1-\rho), \tag{32}$$

$$\frac{d\vartheta_2}{d\tau} = -3\rho \sin\vartheta_2 \cos\vartheta_2 - (1-\rho), \tag{33}$$

$$\frac{d\vartheta_3}{d\tau} = 3\rho \sin\vartheta_3 \cos\vartheta_3(\sin^2\vartheta_1 - \sin^2\vartheta_2). \tag{34}$$

Since the system of differential equations is well defined for $\rho = 1$, we can extend the domain of definition for ρ to the boundary. This way, we have now a compactified phase space. At infinity [i.e., setting $\rho = 1$ in the above Eqs. (31)–(34)] we find two singular curves and two singular points, which are given by

$$l_1: \quad \vartheta_1 = 0,\ \vartheta_2 = 0,\ \frac{\pi}{2} \le \vartheta_3 < \pi,$$

$$l_2: \quad \vartheta_1 = \frac{\pi}{2},\ \vartheta_2 = \frac{\pi}{2},\ \frac{\pi}{2} \le \vartheta_3 < \pi,$$

$$p_1: \quad \vartheta_1 = \frac{\pi}{2},\ \vartheta_2 = 0,\ \vartheta_3 = \frac{\pi}{2},$$

$$p_2: \quad \vartheta_1 = 0,\ \vartheta_2 = \frac{\pi}{2},\ \vartheta_3 = \frac{\pi}{2}.$$

To study the asymptotic behavior of the solutions near l_1 is rather involved, due to the fact that every point $l_0 = (1,0,0,\vartheta_{30}) \in l_1$ is nonhyperbolic (see Ref. [11] for details). We first perform a variable shift in Eqs. (31)–(34) defined as follows: $\delta\rho = \rho - 1$, $\delta\vartheta_1 = \vartheta_1$, $\delta\vartheta_2 = \vartheta_2$, and $\delta\vartheta_3 = \vartheta_3 - \vartheta_{30}$. We then expand the differential equations in the new coordinates up to third order in a sufficiently small neighborhood around the singular point l_0. Next, we define the linear coordinate transformation $x = \delta\rho$, $y = -\frac{1}{3}\delta\rho + \delta\vartheta_1$, $v = -\frac{1}{3}\delta\rho + \delta\vartheta_2$, and $w = \delta\vartheta_3$, such that we get a set of differential equations of the form

$$(\vec{y})' = D\vec{y} + \vec{p}, \tag{35}$$

where $\vec{y} = (x, y, v, w)$. D is a diagonal matrix, and $\vec{p}$ is a vector whose components are polynomes in x, y, v and w, containing monomes of degrees 2 and 3. We can now use the Poincaré-Dulac theorem [12] to classify the monomes in $\vec{p}$ into resonant and nonresonant ones. For the nonresonant monomes of degree n, there is a polynomial change of coordinates of degree n, so that they are transformed into polynomes of at least degree $n+1$. This is not the case for the resonant monomes. The polynomial change of coordinates is found by solving the so-called homological equation (see Ref. [12] for more details). Performing the polynomial change of coordinates on the differential equations, the nonresonant terms of degrees 2 and 3 become of higher order and are thus neglected. Retaining only terms up to third order, we obtain the set of equations

$$\frac{d\tilde{x}}{d\tau} = \frac{1}{3}\tilde{x}^3, \tag{36}$$

$$\frac{d\tilde{y}}{d\tau} = -3\tilde{y} - 3\tilde{x}\tilde{y} + \frac{2}{3}\sin^2\vartheta_{30}(\tilde{x}^2\tilde{y} - \tilde{x}^2\tilde{v}), \tag{37}$$

$$\frac{d\tilde{v}}{d\tau} = -3\tilde{v} - 3\tilde{x}\tilde{v} - \frac{2}{3}\cos^2\vartheta_{30}(\tilde{x}^2\tilde{y} - \tilde{x}^2\tilde{v}), \tag{38}$$

$$\frac{d\tilde{w}}{d\tau} = 0, \tag{39}$$

where $(x,y,v,w) = (\tilde{x},\tilde{y},\tilde{v},\tilde{w}) +$ polynomes of degree 2 or higher in $(\tilde{x},\tilde{y},\tilde{v},\tilde{w})$. Since the solution of Eq. (36) does monotonically increase as a function of τ, l_0 is a saddle point. The above equations can now be solved analytically, and, furthermore, by transforming back to the original variables one gets, for the asymptotic behavior near any point l_0 of l_1,

$$\varphi = \frac{-M_p m t}{\sqrt{3}} e^{i\vartheta_{30}}, \tag{40}$$

$$\dot{\varphi} = \frac{-M_p m}{\sqrt{3}} e^{i\vartheta_{30}}, \tag{41}$$

$$H = \frac{-m^2 t}{3}, \tag{42}$$

for $t \to -\infty$, corresponding to the initial singularity and with $M_p = 1/\sqrt{8\pi G}$. This solution corresponds to an outgoing line, the so-called separatrix. Using Eq. (42) and the definition of H, we have

$$\frac{a(t_f)}{a(t_i)} = \exp[\tfrac{1}{6}m^2(t_i^2 - t_f^2)] \tag{43}$$

for $t_f > t_i$.

The effective equation of state near l_0 tends to $\epsilon = -p$, where $\epsilon = T_{00}$ and $p = \frac{1}{3}(T_{11} + T_{22} + T_{33})$. The solutions near the separatrix are characterized by the fact that $\dot{\varphi}\dot{\varphi}^* \ll m^2\varphi\varphi^*$ and by the phase ϑ_{30} being constant. If for a time $t_f - t_i$ a trajectory T satisfies these two conditions and, at time t_f, T is near to the separatrix, then it follows with Eq. (5) that $\varphi \sim \sqrt{3}M_p/mHe^{i\vartheta_{30}}$. Using Eq. (7), it is easy to show that

$$\frac{a(t_f)}{a(t_i)} = \left|\frac{\dot{\varphi}(t_i) + C_1}{\dot{\varphi}(t_f) + C_1}\right|^{1/3}, \tag{44}$$

for $t_f > t_i$, with $C_1 = (M_p m/\sqrt{3})e^{i\vartheta_{30}}$. Every trajectory that lies close enough to the separatrix will thus meet the criteria for inflation.

To establish the asymptotic behavior of the solution near the singular saddle point p_1, the same strategy as for the line l_1 has to be applied. We therefore give here only the result, which is

$$\varphi = \frac{-iM_p m t}{\sqrt{3}}, \tag{45}$$

$$\dot{\varphi} = \frac{-iM_p m}{\sqrt{3}}, \tag{46}$$

$$H = \frac{-m^2 t}{3}, \tag{47}$$

for $t \to -\infty$. This solution also corresponds to an outgoing separatrix. A more detailed analysis shows that the singular point can only be reached if $\varphi_3 \equiv \pi/2$. Otherwise, starting from points near p_1 with $\vartheta_3 \neq \pi/2$, the phase of the scalar field varies strongly. In this region of the phase-space inflation is driven by the imaginary part of the scalar field. Also, here the phase of φ remains constant along the separatrix, and we get the same equation of state as for the preceding case. Applying the transformations defined in Eq. (14) to Eq. (45), instead of p_1 we get the singular point $\tilde{p}_1 = (1, 0, \pi/2, 0)$, for which the corresponding solution is now real. Hence, the analysis made in Refs. [4–6] applies here as well.

For all points $b = (1, \pi/2, \pi/2, \vartheta_3)$ in l_2, we can directly solve the linearized differential equations. It turns out that ϑ_3 remains constant and that on the plane $\vartheta_3 = \vartheta_{30}$=const the solution expands away from b. Using the inverse transformations of Eqs. (30) and (8), we obtain the asymptotic behavior of the scalar field and of the Hubble parameter:

$$\varphi = \frac{M_p}{\sqrt{3}} \ln\left(\frac{t}{t_0}\right) e^{i\vartheta_{30}}, \tag{48}$$

$$\dot{\varphi} = \frac{M_p}{\sqrt{3}\, t} e^{i\vartheta_{30}}, \tag{49}$$

$$H = \frac{1}{3t}, \tag{50}$$

for $t \to 0^+$ corresponding to the initial cosmological singularity and where t_0 is an integration constant. The equation of state near these points tends to $\epsilon = p$ (i.e., stiff matter). From Eqs. (5) and (6), as long as $\dot{\varphi}\dot{\varphi}^* \gg m^2 \varphi\varphi^*$, we get that $\dot{H} = -3H^2$. Solving this differential equation and using the definition of H, we obtain, as expected,

$$\frac{a(t_f)}{a(t_i)} = \left(\frac{t_f}{t_i}\right)^{1/3}. \tag{51}$$

Following the same method used for the points in l_2, we get, for the saddle point p_2,

$$\varphi = M_p\sqrt{3}\left[C_2 + i\frac{1}{3}\ln\left(\frac{t}{t_0}\right)\right], \tag{52}$$

$$\dot{\varphi} = M_p\sqrt{3}\left[\frac{-C_2 m^2 t}{2} + \frac{i}{3t}\right], \tag{53}$$

$$H = \frac{1}{3t}, \tag{54}$$

for $t \to 0^+$, with C_2 and t_0 being integration constants. The results of the analysis for l_2 also apply here.

This completes the study of the phase portraits for $\Lambda = 0$, for which we found two singular curves l_1, l_2 and two singular points p_1, p_2. All other singular curves and points, due to the transformations given in Eq. (14), can be reduced to one of this cases. For the curve l_1 and the point p_1 the solutions correspond to an outgoing separatrix, where inflation occurs. Along these separatrices the phase of φ remains constant. For the curve l_2, for which the effective equation of state corresponds to stiff matter, the phase also remains constant. Setting $\varphi_{30} = 0$ in the solutions around l_1 and l_2, we recover the results found by Belinsky *et al.* [4–6] for a real scalar field. Contrary to the previous ones, the solution around p_2, for which we also get an effective equation of state for stiff matter, cannot be obtained by simply adding a constant phase to the corresponding asymptotic solution for the real scalar field. Next we turn to the case where there is a quartic self-interaction term.

B. Properties of the singular points lying at infinity for $\Lambda \neq 0$

We perform on Eqs. (24)–(27) the transformation defined by Eq. (30). We then obtain the set of equations

$$\frac{d\rho}{d\tau} = -3\rho^2 f(\cos^2\vartheta_3 \sin^2\vartheta_1 + \sin^2\vartheta_3 \sin^2\vartheta_2) - 6\Lambda\rho^3 g(\sin\vartheta_1 \cos\vartheta_1 \cos^2\vartheta_3 + \sin\vartheta_2 \cos\vartheta_2 \sin^2\vartheta_3), \tag{55}$$

$$\frac{d\vartheta_1}{d\tau} = -\frac{1}{1-\rho}[(1-\rho)^2 + 3\rho f \sin\vartheta_1 \cos\vartheta_1 + 6\Lambda\rho^2 g \cos^2\vartheta_1], \tag{56}$$

$$\frac{d\vartheta_2}{d\tau} = -\frac{1}{1-\rho}[(1-\rho)^2 + 3\rho f \sin\vartheta_2 \cos\vartheta_2 + 6\Lambda\rho^2 g \cos^2\vartheta_2], \tag{57}$$

$$\frac{d\vartheta_3}{d\tau} = -\frac{1}{1-\rho}[3\rho f \sin\vartheta_3 \cos\vartheta_3(\sin^2\vartheta_2 - \sin^2\vartheta_1) + 6\Lambda\rho^2 g \cos\vartheta_3 \sin\vartheta_3(\cos\vartheta_2 \sin\vartheta_2 - \cos\vartheta_1 \sin\vartheta_1)], \tag{58}$$

where

$$f = \sqrt{(1-\rho)^2 + 3\Lambda\rho^2 g^2} \tag{59}$$

and

$$g = \cos^2\vartheta_3 \cos^2\vartheta_1 + \sin^2\vartheta_3 \cos^2\vartheta_2. \tag{60}$$

Since the right-hand side of Eq. (55) is well defined and continuous for $\rho = 1$, we can find all sets of angles $(\vartheta_{10}, \vartheta_{20}, \vartheta_{30})$ for which the condition $d\rho/d\tau = 1$ at $\rho = 1$ is satisfied. For each solution $(\vartheta_{10}, \vartheta_{20}, \vartheta_{30})$ we have to check if $\lim_{\rho\to 1^-} f_i(\rho, \vartheta_{10}, \vartheta_{20}, \vartheta_{30}) = 0$, where f_i stands for the right-hand side of Eqs. (56)–(58). If this is the case, the point $(1, \vartheta_{10}, \vartheta_{20}, \vartheta_{30})$ is a singular point at infinity. Again, we find two singular curves and two singular points:

$$L_1: \ \ \vartheta_1 = \arctan\left(\frac{-2\sqrt{3\Lambda}}{3}\right) + \pi, \ \ \vartheta_2 = \arctan\left(\frac{-2\sqrt{3\Lambda}}{3}\right) + \pi, \ \ \frac{\pi}{2} \le \vartheta_3 < \pi, \ \ \rho \to 1, \tag{61}$$

$$L_2: \ \ \vartheta_1 = \frac{\pi}{2}, \ \ \vartheta_2 = \frac{\pi}{2}, \ \ \frac{\pi}{2} \le \vartheta_3 < \pi, \ \ \rho \to 1, \tag{62}$$

$$P_1: \ \ \vartheta_1 = \frac{\pi}{2}, \ \ \vartheta_2 = \arctan\left(\frac{-2\sqrt{3\Lambda}}{3}\right) + \pi, \ \ \vartheta_3 = \frac{\pi}{2}, \ \ \rho \to 1, \tag{63}$$

$$P_2: \ \ \vartheta_1 = 0, \ \ \vartheta_2 = \frac{\pi}{2}, \ \ \vartheta_3 = \frac{\pi}{2}, \ \ \rho \to 1. \tag{64}$$

Notice that the limit $\Lambda \to 0$ is rather subtle, since in the differential equations there are terms involving $\Lambda/(1-\rho)$, which are not defined when $\rho \to 1$ and $\Lambda \to 0$ simultaneously. We expand the differential equations around an arbitrary point P_0 of L_1 defined by the coordinates

$$\rho, \vartheta_1 = \arctan\left(\frac{-2\sqrt{3\Lambda}}{3}\right) + \pi, \ \ \vartheta_2 = \arctan\left(\frac{-2\sqrt{3\Lambda}}{3}\right) + \pi, \vartheta_3 = \vartheta_{30},$$

where ϑ_{30} is in $[\pi/2, \pi)$. In order to have finite partial derivatives, when ρ tends to 1, the angular variables ϑ_1 and ϑ_2 are kept fixed. When linearizing Eq. (58) around P_0, we see that also the angular variable ϑ_3 has to remain constant. Inserting the values of the angular variables defined in Eq. (22), we get for the scalar field

$$\varphi = \frac{-3M_p m e^{i\vartheta_{30}}}{\sqrt{4\lambda M_p^2 + 3m^2}} r, \tag{65}$$

$$\dot\varphi = -2\sqrt{\lambda/3} M_p \varphi, \tag{66}$$

for $r \to \infty$, where r is a dimensionless parameter defined by Eq. (22). Since the phase of φ is constant, we get can integrate Eq. (66) and thus get φ as a function of t rather than r. This way we get

$$\varphi = \varphi_0 e^{i\vartheta_{30}} \exp(-2M_p\sqrt{\lambda/3}t), \tag{67}$$

where $t \to -\infty$ and φ_0 is a negative integration constant. Using Eq. (5), one gets the asymptotic behavior for the Hubble parameter. This solution corresponds to an outgoing separatrix, and the equation of state tends to $\epsilon = -p$. A trajectory, which lies sufficiently close to the separatrix, can be parametrized by $\dot\varphi \sim \alpha\varphi$, where $\alpha \sim -2M_p\sqrt{\lambda/3}$. As long as $H^2 \sim (1/3M_p^2)\lambda(\varphi\varphi^*)^2$, with Eq. (7) we find, along the trajectory,

$$\frac{a(t_f)}{a(t_i)} = \exp\left(\frac{-\alpha^2}{3\alpha + 2\sqrt{3\lambda}M_p}(t_f - t_i)\right), \tag{68}$$

for $t_f > t_i$. The factor in the exponential multiplying the time difference is just the Hubble expansion rate, which is positive and tends to infinity as α reduces to $-2M_p\sqrt{\lambda/3}$. Thus, any solution that lies sufficiently close to the separatrix will go through an inflationary stage.

Using the same strategy as before for the point P_1, we obtain

$$\varphi = i\varphi_0 \exp(-2M_p\sqrt{\lambda/3}t), \tag{69}$$

where $t \to -\infty$ and φ_0 is a negative integration constant. This solution also corresponds to an outgoing separatrix. The singular point can only be reached if $\vartheta_3 \equiv \pi/2$; thus the real part of the scalar field vanishes. The analysis made for L_1 applies as well; again there is an inflationary stage if the solution gets close to the separatrix.

For the lines L_2 we expand the set of differential equations to first order around a point B defined by the coordinates $(\rho, \pi/2, \pi/2, \vartheta_{30})$. For $\rho \to 1$, the linearized equations reduce to those obtained for the line l_2. Therefore, the solutions are given by Eqs. (48)–(50). The inclusion of a quartic term in the Lagrangian does not affect the singular line. This is expected, since near L_2 the potential term is negligible compared to the kinetic term $\dot\varphi\dot\varphi^*$. The same remark holds for the singular point P_2, so that the asymptotic behavior is given by Eqs. (52)–(54).

We see that the presence of a quartic self-interaction term does not substantially change the main features of the phase portrait. We again find two singular curves and two singular points. All other solutions can be reduced to these by the transformations given in Eq. (14). The solu-

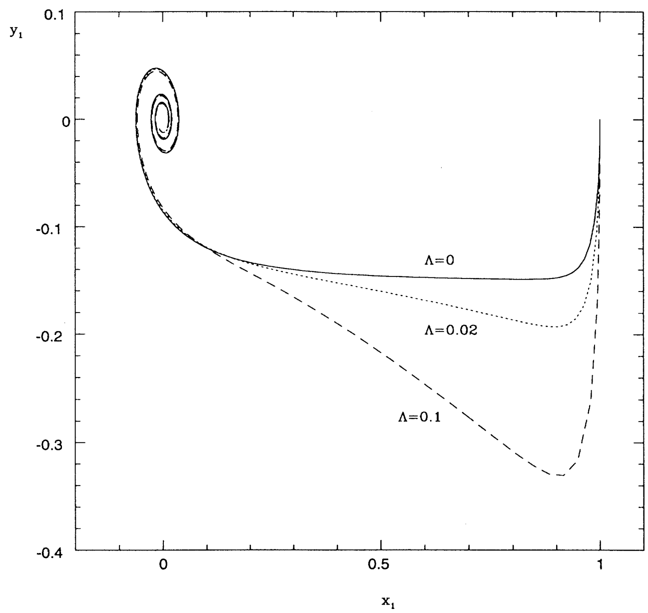

FIG. 1. Plot of the dimensionless variable y_1 as function of x_1 for different values of Λ. It corresponds to the phase portrait of the real part of the scalar field. All curves start with the same initial condition given by the asymptotic behavior near p_2 [see Eq. (52)]. The real part of the scalar fields depends on the value of C_2, for which we take the value $C_2 = 1$.

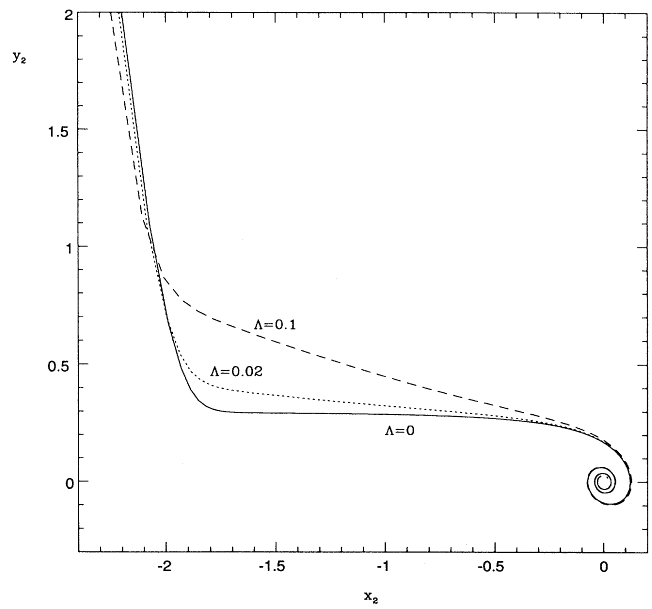

FIG. 2. Plot of the phase portrait of the imaginary part of the scalar field in dimensionless variables for the same cases as in Fig. 1. The asymptotic behavior of the imaginary part near p_2 depends on the value of t_0, for which we choose $t_0 = 100$.

tions which correspond to outgoing separatrices show an inflationary stage. The behavior of the solutions around these separatrices is not much affected by the quartic term. Its main influence is to shift the position of the singular points, where inflation occurs. For the solutions for which no inflation occurs, that is, around L_2 and P_2, the results are the same as for l_2 and p_2, respectively. We obtain the solutions of Belinsky *et al.* in Ref. [5] for the real scalar field by setting $\vartheta_3 = 0$ as mentioned in Sec. III A.

In Figs. 1 and 2 we plot the numerical solutions of the differential equations (10)–(13) for different values of Λ. For all solutions we take the same initial conditions as given by the asymptotic behavior near the singular point p_2. The plots are valid for every value of $m \neq 0$, since Eqs. (10)–(13) do not depend on it. The numerical solutions have only a physical meaning in the region where $T_{00} < M_p^4$. Hence, depending on the value of m, it might be that only a part of the figure is of physical relevance. Next, we analyze the asymptotic behavior of the solutions of Eqs. (18)–(21) for a massless scalar field with a quartic self-interaction nearby the singular points.

IV. MASSLESS SCALAR FIELD IN A ZERO CURVATURE FRIEDMANN-LEMAITRE UNIVERSE

Again the coordinate origin is the only singular point not lying at infinity of the phase space. This point is asymptotically stable, since we have the Liapunov function (see Ref. [13] for details)

$$l(x_{10}, x_{20}, y_{10}, y_{20}) = y_{10}^2 + y_{20}^2 + 3\lambda(x_{10}^2 + x_{20}^2)^2,$$

for which the derivative with respect to η_0 is strictly negative, except at the origin. Applying the transformation defined in Eq. (22) to Eqs. (17)–(21), we get

$$z_0^2 = r^2\{1 + [3\lambda r^2(\cos^2\vartheta_3 \cos^2\vartheta_1 + \sin^2\vartheta_3 \cos^2\vartheta_2) - 1](\cos^2\vartheta_3 \cos^2\vartheta_1 + \sin^2\vartheta_3 \cos^2\vartheta_2)\}, \tag{70}$$

$$\vartheta_1' = -3z_0 \sin\vartheta_1 \cos\vartheta_1 - 6\lambda r^2(\cos^2\vartheta_3 \cos^2\vartheta_1 + \sin^2\vartheta_3 \cos^2\vartheta_2)\cos^2\vartheta_1 - \sin^2\vartheta_1, \tag{71}$$

$$\vartheta_2' = -3z_0 \sin\vartheta_2 \cos\vartheta_2 - 6\lambda r^2(\cos^2\vartheta_3 \cos^2\vartheta_1 + \sin^2\vartheta_3 \cos^2\vartheta_2)\cos^2\vartheta_2 - \sin^2\vartheta_2, \tag{72}$$

$$\begin{aligned}\vartheta_3' = \sin\vartheta_3 \cos\vartheta_3[&-3z_0(\sin^2\vartheta_2 - \sin^2\vartheta_1) - 6\lambda r^2(\cos^2\vartheta_3 \cos^2\vartheta_1 + \sin^2\vartheta_3 \cos^2\vartheta_2)\\ &\times(\sin\vartheta_2 \cos\vartheta_2 - \sin\vartheta_1 \cos\vartheta_1) + (\sin\vartheta_2 \cos\vartheta_2 - \sin\vartheta_1 \cos\vartheta_1)],\end{aligned} \tag{73}$$

$$r' = -3rz_0(\cos^2\vartheta_3\sin^2\vartheta_1 + \sin^2\vartheta_3\sin^2\vartheta_2) + r(\cos\vartheta_1\sin\vartheta_1\cos^2\vartheta_3 + \cos\vartheta_2\sin\vartheta_2\sin^2\vartheta_3)$$
$$-6\lambda r^3(\cos^2\vartheta_3\cos^2\vartheta_1 + \sin^2\vartheta_3\cos^2\vartheta_2)(\cos\vartheta_1\sin\vartheta_1\cos^2\vartheta_3 + \cos\vartheta_2\sin\vartheta_2\sin^2\vartheta_3). \quad (74)$$

In a sufficiently small neighborhood around the origin, which we denote by W, we consider the projection of the solutions of Eqs. (71)–(74) on the (x_{10}, y_{10}) plane. The angular variable of this plane is ϑ_1 and its behavior is given by the solutions of the equation $\vartheta_1' = -\sin^2\vartheta_1$, which is just Eq. (71) with $r = 0$. Therefore, knowing that the coordinate origin is asymptotically stable and that ϑ_1' is almost everywhere strictly negative, it follows that the solutions are winding towards the point $A_0 = (x_{10} = 0,\ y_{10} = 0)$. One gets the same behavior when the solutions of Eqs. (71)–(74) in W are projected on the (x_{20}, y_{20}) plane.

We now turn to the singular points lying at infinity. One has to apply to Eqs. (71)–(74) the transformations given in Eq. (30), where η has to be replaced by η_0. We obtain the equations

$$\frac{d\rho}{d\tau} = -3\rho^2 f(\cos^2\vartheta_3\sin^2\vartheta_1 + \sin^2\vartheta_3\sin^2\vartheta_2) + [-6\lambda\rho^3 g + \rho(1-\rho)^2]$$
$$\times[\sin\vartheta_1\cos\vartheta_1\cos^2\vartheta_3 + \sin\vartheta_2\cos\vartheta_2\sin^2\vartheta_3], \quad (75)$$

$$\frac{d\vartheta_1}{d\tau} = -\frac{1}{1-\rho}[3\rho f\sin\vartheta_1\cos\vartheta_1 + 6\lambda\rho^2 g\cos^2\vartheta_1 + (1-\rho)^2\sin^2\vartheta_1], \quad (76)$$

$$\frac{d\vartheta_2}{d\tau} = -\frac{1}{1-\rho}[3\rho f\sin\vartheta_2\cos\vartheta_2 + 6\lambda\rho^2 g\cos^2\vartheta_2 + (1-\rho)^2\sin^2\vartheta_2], \quad (77)$$

$$\frac{d\vartheta_3}{d\tau} = -\frac{\cos\vartheta_3\sin\vartheta_3}{1-\rho}\{3\rho f(\sin^2\vartheta_2 - \sin^2\vartheta_1) + [6\lambda\rho^2 g - (1-\rho)^2](\cos\vartheta_2\sin\vartheta_2 - \cos\vartheta_1\sin\vartheta_1)\}, \quad (78)$$

where from now on

$$f = \sqrt{(1-\rho)^2(1-g) + 3\lambda\rho^2 g^2}, \quad (79)$$

and g is still given by Eq. (60). The singular points lying at infinity of the phase space are found using the same method as for $\Lambda \neq 0$. We again get two singular curves and two singular points, which we denote by l_{10}, l_{20}, p_{10}, and p_{20}, and their coordinates are given by Eqs. (61)–(64), but now with Λ replaced by λ. To obtain the asymptotic behavior around these singular points, we apply the same method used in Sec. III B.

The asymptotic behavior around a singular point lying in l_{10} is given by Eqs. (65) and (66), but where m is now replaced by M_p. The angular variables must be kept fixed in order to have finite partial derivatives, when $\rho \to 1$. The analysis made for L_1 is also valid here and l_{10} has inflationary stages. Setting $\vartheta_3 = 0$ we obtain automatically the asymptotic behavior for a massless real scalar field with a quartic self-interaction for which we recover the inflationary stage. This is a fact that was established heuristically by Linde in Ref. [14]. One gets the behavior of the solutions around the singular point p_{10} from the one near P_1 in the same way as discussed above for l_{10} from L_1.

The asymptotic behavior of the solutions around all points $c = (1, \pi/2, \pi/2, \vartheta_{30})$ of l_{20} or around the point p_{20} is found directly by solving the corresponding linearized differential equations. For the singular points on l_{20} it turns out that the solutions are given by Eqs. (48)–(50), whereas for p_{20} we get the solutions from Eqs. (52)–(54) inserting $m = 0$. The form of the solutions around the line l_{20} and p_{20} was expected to be similar to that obtained for l_2 and p_2, because the potential term is negligible with respect to the kinetic energy term $\dot\varphi\dot\varphi^*$.

V. CONCLUDING REMARKS

The extension of the above analysis to the singular points for $k = \pm 1$, although, in principle, straightforward, is much more involved. One could, for instance, using Eq. (5), eliminate the curvature term in Eq. (6) and consider this modified equation together with Eq. (7). This gives then a set of five nonlinear first-order differential equations. As a consequence, when performing the transformation to spherical coordinates needed in order to compactify the phase space, one gets one additional angular variable. It turns out that around some singular points the expansion of the differential equations must be done at least up to fourth order. The only singular point not lying at infinity of the phase space is at the coordinate origin. We conjecture that the asymptotic behavior near the singular points lying at infinity with $k \neq 0$ will not satisfy the criteria for inflation. This does not imply that inflation cannot occur in an open or a closed universe, but that every trajectory must come close enough to one of the separatrix found for $k = 0$ in order to go through an inflationary stage. This fact has been shown for the real scalar field case (see Refs. [4–6]).

In this paper we have extended to complex scalar fields the analysis of the initial conditions in an homogeneous and isotropic Friedmann-Lemaitre universe. The main features found for real scalar fields hold also for complex scalar fields, in particular, the existence of inflationary stages. The fact that along the separatrices the phase of φ remains constant is important and shows that inflation is essentially driven by one component of the complex scalar field. Therefore, the results on inflation valid for a real scalar field (see for instance Ref. [15] and references therein) apply also on the component of the complex fields, which drives inflation. The behavior around the singular points p_2, p_{20}, and P_2 is more involved and cannot be obtained by just adding a phase to the corresponding solutions for the real scalar field. We also notice that for a massive scalar field the presence of a quartic self-interaction term does not change substantially the main features of the phase portrait.

ACKNOWLEDGMENTS

We thank N. Straumann for very useful and clarifying discussions. This work was supported by the Swiss National Science Foundation.

[1] G. F. Smoot *et al.*, Astrophys. J. **396**, L1 (1992).
[2] P. Jetzer, Phys. Rep. **220**, 163 (1992).
[3] A. Liddle and M. Madsen, Int. J. Mod. Phys. D **1**, 101 (1992).
[4] V. A. Belinsky, L. P. Grishchuk, I. M. Khalatnikov, and Ya. B. Zeldovich, Phys. Lett. **155B**, 232 (1985).
[5] V. A. Belinsky, L. P. Grishchuk, Ya. B. Zeldovich, and I. M. Khalatnikov, Sov. Phys. JETP **62**, 195 (1985).
[6] V. A. Belinsky and I. M. Khalatnikov, Sov. Phys. JETP **66**, 441 (1987).
[7] T. Piran and R. M. Williams, Phys. Lett. **163B**, 331 (1985).
[8] N. Deruelle, C. Gundlach, and D. Langlois, Phys. Rev. D **45**, 3301 (1992); N. Deruelle, C. Gundlach, and D. Polarski, Class. Quantum Grav. **9**, 1511 (1992).
[9] V. Mukhanov, H. Feldman, and R. Brandenberger, Phys. Rep. **215**, 203 (1992).
[10] V. I. Arnold, *Geometrical Methods in Theory of Differential Equations* (Springer-Verlag, New York, 1983), p. 142.
[11] D. K. Arrowsmith and C. M. Place, *An Introduction to Dynamical System* (Cambridge University, Cambridge, England, 1990), p. 79.
[12] V. I. Arnold, *Geometrical Methods in Theory of Differential Equations* (Springer-Verlag, New York, 1983), p. 191.
[13] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983), p. 4.
[14] A. D. Linde, Phys. Lett. **129B**, 177 (1983).
[15] K. A. Olive, Phys. Rep. **190**, 307 (1990).