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# One-loop quantum gravity in Schwarzschild space-time

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The quantum theory of linearized perturbations of the gravitational field of a Schwarzschild black hole is presented. The fundamental operators are seen to be the perturbed Weyl scalars $\dot{\Psi}_0$ and $\dot{\Psi}_4$ associated with the Newman-Penrose description of the classical theory. Formulas are obtained for the expectation values of the modulus squared of these operators in the Boulware, Unruh, and Hartle-Hawking quantum states. The differences between the renormalized expectation values of both $|\dot{\Psi}_0|^2$ and $|\dot{\Psi}_4|^2$ in the three quantum states are evaluated numerically.

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## I. INTRODUCTION

In this paper we shall study quantized, linear perturbations of the gravitational field of a Schwarzschild black hole. Since this represents a study of the one-loop approximation to a theory of quantum gravity, it should provide useful insight into what the full theory should look like. In addition, there is a growing body of evidence which suggests that the influence of nonconformally invariant quantum fields, foremost of which is linearized gravity, strongly dominates that of conformally invariant quantum fields in the neighborhood of a space-time singularity. For example, it has been shown recently [1] that the value of the renormalized energy density for the graviton field in the vicinity of a conical singularity in an otherwise flat space-time (the idealized cosmic string) is roughly 10 times that for the electromagnetic field and 100 times that for the conformally coupled scalar field. A knowledge of how gravitons behave near the singularity at the center of a black hole is therefore also likely to be crucial to our understanding of quantum gravity.

The classical theory of linearized perturbations of a Schwarzschild black hole was developed by Regge and Wheeler [2], Zerilli [3], and others. Here we follow the approach of Teukolsky [4], who gave a complete set of solutions to the perturbation equations for a Kerr black hole within the Newman-Penrose formalism. In this formalism the system is described by two quantities, the perturbed Weyl scalars $\dot{\Psi}_0$ and $\dot{\Psi}_4$ (here and in the following, an overdot indicates a linearly perturbed quantity). These are the field variables of interest describing the semiclassical theory, since (a) they carry in their real and imaginary parts information on all the dynamical degrees of freedom of the perturbed field, (b) they are invariant under both gauge transformations and infinitesimal tetrad transformations which leave the metric perturbation intact [5] as befits a physical quantity, (c) they are simply expressed (albeit in a particular gauge) in terms of the metric perturbation which provides the most direct route to quantization [6], and (d) they directly measure the energy flux of classical perturbations of the black hole across the horizon and at infinity [5,7].

In this paper our main concern will be to extend the work of Candelas *et al.* [8] by first presenting formulas for the expectation values of $|\dot{\Psi}_0|^2$ and $|\dot{\Psi}_4|^2$ with respect to the three physically relevant quantum states for the Schwarzschild black hole, namely, the Boulware, Unruh, and Hartle-Hawking states, and then evaluating these expressions numerically. Since $|\dot{\Psi}_0|^2$ and $|\dot{\Psi}_4|^2$ are quadratic in the field variables their expectation values will be infinite; however differences between the expectation values of either $|\dot{\Psi}_0|^2$ or $|\dot{\Psi}_4|^2$ with respect to the various quantum states mentioned above are finite and these differences are the objects which we will calculate. (Indeed, some would argue that such differences are more likely to be of physical significance than the results obtained from renormalization procedures.) In the process of achieving these main goals we also fill several gaps in the literature concerning quantized perturbations of black holes; in particular we present both low-frequency

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analytic formulas and a set of Wronskian relations for the reflection and transmission coefficients associated with the theory.

The format of the paper is as follows. The two sections immediately following this Introduction summarize the essentials of what is already known about the classical and semiclassical theories of linear perturbations of black holes. Thus in Sec. II we present Teukolsky's complete set of classical solutions for the perturbed Weyl scalars. Since Schwarzschild space-time is static and spherically symmetric, the temporal and angular dependences of each of these solutions are given by $\exp(-i\omega t)$ and spherical harmonic functions, respectively; the radial component however is a solution of a complex, second order, ordinary differential equation for which no closed-form general solution exists. We discuss the behavior of the radial component both as one approaches the event horizon and spatial infinity. We end Sec. II by writing down formulas for the classical energy flux due to the perturbations across the event horizon and at infinity.

Section III is a review of the semiclassical theory. We present the complete set of solutions for the metric perturbations due to Chrzanowski [6] and verify that these yield Teukolsky's complete set of perturbed Weyl scalars. The question of orthonormality of the mode set is rigorously addressed, the perturbations are quantized and expressions for differences between the expectation values of $|\dot{\Psi}_0|^2$ and $|\dot{\Psi}_4|^2$ in the Boulware, Unruh, and Hartle-Hawking states are presented in terms of the radial functions.

In Sec. IV we prepare the expressions obtained for the various expectation values for numerical evaluation. The first step is to derive a power series representation for the general solution to the radial equation; the particular radial solutions occurring in the expectation values may then be substituted for these power series, weighted by appropriate constants called reflection and transmission amplitudes by virtue of the analogy with classical scattering.

Before the differences between the expectation values can be evaluated numerically, the reflection and transmission amplitudes must be determined explicitly. This is achieved in Sec. IV by comparing the power series representations for the particular radial solutions at large values of the radial coordinate with their known asymptotic forms. Graphs are presented for these amplitudes against frequency and angular number, and the results are checked for both internal and external consistency; the former by means of Wronskian relations derived between the coefficients in Appendix D and analytic formulas for the amplitudes valid at low frequencies derived in Appendix E; the latter by reproducing Page's [9] results for the luminosity due to graviton emission from the black hole.

In Sec. VI we give an analysis of the asymptotics of the expressions for the differences between the expectation values of the perturbed Weyl scalars.

In Sec. VII we present graphs for these differences, and discuss the main difficulties encountered in the numerical computation.

Throughout we use geometrized Planck units ($G = c = \hbar = 1$) and follow the sign conventions of Misner, Thorne, and Wheeler [10].

## II. REVIEW OF THE CLASSICAL THEORY

For space-times possessing a high degree of symmetry, it is often possible to reduce considerably the number and complexity of the equations of general relativity by projecting them onto a null, complex tetrad which encodes that symmetry. This is the motivation which lies behind the Newman-Penrose (NP) formulation of general relativity [11].

In the case of a Schwarzschild black hole of mass $M$, whose space-time is described by the metric

$$g_{\mu\nu} = \mathrm{diag}\left[-\frac{\triangle}{r^2}, \frac{r^2}{\triangle}, r^2, r^2\sin^2\theta\right]_{\mu\nu}, \qquad \triangle = r(r-2M),$$

in terms of the usual spherical polar coordinates $(t, r, \theta, \phi)$, an appropriate choice of tetrad which mirrors the temporal and spherical symmetry of the space-time is the *Kinnersley tetrad*

$$e_{(1)}{}^{\mu} = \left(\frac{r^2}{\triangle}, 1, 0, 0\right),$$
$$e_{(2)}{}^{\mu} = \frac{1}{2}\left(1, -\frac{\triangle}{r^2}, 0, 0\right), \qquad (2.1)$$
$$e_{(3)}{}^{\mu} = e_{(4)}{}^{\mu *} = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right).$$

The simplification afforded by using this tetrad is exemplified by the fact that all spin coefficients vanish apart from

$$\rho = -\frac{1}{r}, \quad \mu = -\frac{\triangle}{2r^3}, \quad \gamma = \frac{M}{2r^2},$$
$$-\alpha = \beta = \frac{\cot\theta}{2\sqrt{2}r}, \qquad (2.2)$$

and all tetrad components of the Weyl tensor are zero apart from

$$\Psi_2 = \frac{M}{r^3}. \qquad (2.3)$$

(More than one set of conventions exists for the NP formalism; those employed in this paper are spelled out in Appendix A to avoid any confusion.)

Teukolsky [5] took full advantage of the simplifications afforded by the NP formalism when investigating perturbations of the gravitational field of a Kerr black hole. He obtained linearly perturbed versions of the NP analogues of the basic equations of general relativity for the black hole metric and, by working in the Kinnersley tetrad, was able to decouple the equations for the ingoing and outgoing radiative parts of the perturbed Weyl tensor, $\dot{\Psi}_0$ and $\dot{\Psi}_4$. These were found to satisfy the "master perturbation equation," which, in the Schwarzschild limit, reads

$$\left[\frac{r^4}{\triangle}\frac{\partial^2}{\partial t^2} + \frac{2rs(r-3M)}{\triangle}\frac{\partial}{\partial t} - \triangle\frac{\partial^2}{\partial r^2} - 2(s+1)(r-M)\frac{\partial}{\partial r} - \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{2is\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} + s^2\cot^2\theta - s\right]\Phi_s = 0, \qquad (2.4)$$

where $s = \pm 2, \Phi_2 \equiv \dot{\Psi}_0$, and $\Phi_{-2} \equiv r^4\dot{\Psi}_4$. It is remarkable that this equation can also be shown to describe the behavior of a test scalar field ($s = 0, \Phi_0 \equiv \varphi$) or electromagnetic field ($s = \pm 1, \Phi_1 \equiv \phi_0, \Phi_{-1} \equiv r^2\phi_2$, where $\phi_0$ and $\phi_2$ are the tetrad components $F_{\mu\nu}e_{(1)}{}^{\mu}e_{(3)}{}^{\nu}$ and $F_{\mu\nu}e_{(4)}{}^{\mu}e_{(2)}{}^{\nu}$, respectively, of the Maxwell tensor $F_{\mu\nu}$) on the Schwarzschild background. For this reason we shall keep $s$ arbitrary whenever possible in the ensuing discussion.

By separating variables in (2.4) the following complete set of solutions is obtained:

$$\Phi_s = e^{-i\omega t}\,{}_sR_{l\omega}(r)\,{}_sY_l^m(\theta,\phi), \qquad (2.5)$$

where $\omega \in [0,\infty)$ and $l$, $m$ are integers satisfying the inequalities $l \geq |s|$, $-l \leq m \leq l$. ${}_sY_l^m(\theta,\phi)$ is a *spin-weighted spherical harmonic*, whose relevant properties are reviewed in Appendix B. ${}_sR_{l\omega}(r)$ satisfies the ordinary differential equation

$$\left[\triangle^{-s}\frac{d}{dr}\left(\triangle^{s+1}\frac{d}{dr}\right) + \frac{\omega^2r^4 + 2is\omega r^2(r-3M)}{\triangle} - (l-s)(l+s+1)\right]{}_sR_{l\omega}(r) = 0. \qquad (2.6)$$

The solutions of (2.6) cannot be obtained analytically in closed form (we shall solve it numerically later). However much can be learned about their properties by exploiting the analogy of (2.6) to a classical scattering problem, which becomes apparent when we write (2.6) in the form [5]

$$\left[\frac{d^2}{dr_*{}^2} + {}_sV_{l\omega}\right]{}_sQ_{l\omega} = 0. \qquad (2.7)$$

Here $r_*$ is the *tortoise coordinate*

$$r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right),$$

${}_sQ_{l\omega}(r) = \triangle^{\frac{s}{2}}r\,{}_sR_{l\omega}(r)$ and ${}_sV_{l\omega}(r)$ is the (complex) potential

$${}_sV_{l\omega}(r) = \omega^2 + \frac{2is\omega(r-3M)}{r^2} - \frac{s^2M^2}{r^4} - \frac{r-2M}{r^3}\left[l(l+1) + \frac{2M}{r}\right]. \qquad (2.8)$$

As $r \to \infty$ the potential (2.8) can be approximated by

$${}_sV_{l\omega}(r) \sim \omega^2 + \frac{2i\omega s}{r}$$

so that (2.7) possesses solutions which behave like

$${}_sQ_{l\omega}(r) \sim r^{\pm s}e^{\mp i\omega r_*}$$

which gives

$${}_sR_{l\omega}(r) \sim r^{-1}e^{-i\omega r_*} \quad \text{or} \quad r^{-2s-1}e^{+i\omega r_*}. \qquad (2.9)$$

Similarly, in the limit as $r \to 2M$,

$${}_sV_{l\omega} \sim \left(\omega - \frac{is}{4M}\right)^2 \qquad (2.10)$$

so that

$${}_sQ_{l\omega}(r) \sim \triangle^{\pm\frac{s}{2}}e^{\pm i\omega r_*} \qquad (2.11)$$

and

$${}_sR_{l\omega}(r) \sim \triangle^{-s}e^{-i\omega r_*} \quad \text{or} \quad e^{+i\omega r_*}. \qquad (2.12)$$

To end this section, we justify the assertion made in the Introduction that the perturbed Weyl scalars $\dot{\Psi}_0$ and $\dot{\Psi}_4$ measure the energy flux of classical perturbations of the Schwarzschild black hole across the horizon and at infinity.

**Hartle and Hawking [12] have shown that the energy flux transmitted across the horizon of a fluctuating classical black hole is given by the formula**

$$\frac{d^2E_\omega^{\text{hor}}}{dt\,d\Omega} = \frac{M^2}{\pi}\left|\dot{\sigma}_H(2M)\right|^2, \qquad (2.13)$$

where $\dot{\sigma}_H(r)$ is the perturbed shear of the null congruence $\left(e_{(1)}{}^{\mu}\right)_H$ which generates the future horizon, the calculation having been performed in the *Hartle-Hawking tetrad* which is defined as follows in terms of the Kinnersley tetrad (2.1):

$$\left(e_{(1)}{}^{\mu}\right)_H = \frac{\triangle}{2r^2}e_{(1)}{}^{\mu}, \qquad \left(e_{(2)}{}^{\mu}\right)_H = \frac{2r^2}{\triangle}e_{(2)}{}^{\mu},$$
$$\left(e_{(3)}{}^{\mu}\right)_H = \left(e_{(4)}{}^{\mu *}\right)_H = e_{(3)}{}^{\mu}.$$

It can also be shown from the perturbed Newman-Penrose equations that, for perturbations of frequency $\omega$ for which $\left(\dot{e}_{(1)}{}^{\mu}\right)_H \propto \left(e_{(1)}{}^{\mu}\right)_H$,

$$\dot{\sigma}_H(2M) = -\lim_{r\to 2M}\left[\left(\frac{\triangle}{r^2}\right)^2\frac{M\dot{\Psi}_{0\,\omega}}{(1+4iM\omega)}\right] \qquad (2.14)$$

(where $\dot{\Psi}_0$ is with respect to the Kinnersley tetrad). We thus obtain the formula

$$\frac{d^2E_\omega^{\text{hor}}}{dt\,d\Omega} = \frac{1}{2^8\pi M^4\,(16M^2\omega^2+1)}\lim_{r\to 2M}\left[\triangle^4\left|\dot{\Psi}_{0\,\omega}\right|^2\right]. \qquad (2.15)$$

At spatial infinity the space-time becomes flat and one can therefore carry over standard results from the theory of gravity linearized about Minkowski space, enabling one to derive the formulas [5]

$$\frac{d^2E^{\rm in}_\omega}{dt\,d\Omega} = \frac{1}{64\pi\omega^2}\lim_{r\to\infty}\left[r^2\left|\dot\Psi_{0\,\omega}\right|^2\right], \tag{2.16a}$$

$$\frac{d^2E^{\rm out}_\omega}{dt\,d\Omega} = \frac{1}{4\pi\omega^2}\lim_{r\to\infty}\left[r^2\left|\dot\Psi_{4\,\omega}\right|^2\right], \tag{2.16b}$$

for the incoming and outgoing energy fluxes infinitely far from the black hole. Formulas (2.15) and (2.16) demonstrate that $|\dot\Psi_0|^2$ and $|\dot\Psi_4|^2$ directly measure the energy flux of gravitational perturbations of a Schwarzschild black hole across the horizon and at infinity. This physical interpretation, together with the other features already discussed, makes $\Psi_0$ and $\Psi_4$ the natural choice as field variables for the semiclassical theory.

## III. REVIEW OF THE SEMICLASSICAL THEORY

Teukolsky has provided us with a complete set of perturbed Weyl scalars $\dot\Psi_0$ and $\dot\Psi_4$ (henceforth collectively labeled $\dot\Psi_A$). One could quantize the theory directly in terms of this set [9], however we choose to follow Candelas *et al.* [8] by working in terms of the metric perturbation. Thus we require a complete set of solutions for the metric perturbations $h_{\mu\nu}$. The relationship between $\dot\Psi_A$ and $h_{\mu\nu}$ is obtained by perturbing the defining equations for the Weyl scalars. Consider, for example,

$$\Psi_0 = -C_{(1)(3)(1)(3)} = -C_{\mu\nu\rho\lambda}e_{(1)}{}^{\mu}e_{(3)}{}^{\nu}e_{(1)}{}^{\rho}e_{(3)}{}^{\lambda} \tag{3.1}$$

which when perturbed linearly becomes

$$\dot\Psi_0 = -\dot C_{\mu\nu\rho\lambda}e_{(1)}{}^{\mu}e_{(3)}{}^{\nu}e_{(1)}{}^{\rho}e_{(3)}{}^{\lambda} - C_{\mu\nu\rho\lambda}\big(\dot e_{(1)}{}^{\mu}e_{(3)}{}^{\nu}e_{(1)}{}^{\rho}e_{(3)}{}^{\lambda} + e_{(1)}{}^{\mu}\dot e_{(3)}{}^{\nu}e_{(1)}{}^{\rho}e_{(3)}{}^{\lambda} + e_{(1)}{}^{\mu}e_{(3)}{}^{\nu}\dot e_{(1)}{}^{\rho}e_{(3)}{}^{\lambda} + e_{(1)}{}^{\mu}e_{(3)}{}^{\nu}e_{(1)}{}^{\rho}\dot e_{(3)}{}^{\lambda}\big). \tag{3.2}$$

Since the tetrad spans the tangent space we can write $\dot e_{(a)}{}^{\mu} = A_{(a)}{}^{(b)}e_{(b)}{}^{\mu}$ for some functions $A_{(a)}{}^{(b)}$; it follows that the last four terms on the right-hand side of (3.2) vanish, as the only nonzero, unperturbed Weyl scalar is $\Psi_2 = -C_{(1)(3)(4)(2)}$. In the first term one can use expressions provided by Barth and Christensen [13] for the perturbed Riemann and Ricci tensors to express $\dot C_{\mu\nu\rho\lambda}$ in terms of $h_{\mu\nu}$ and (after some tedious algebraic manipulation) arrive at the following formula for $\dot\Psi_0$ in terms of $h_{(a)(b)}$:

$$\dot\Psi_0 = \tfrac{1}{2}\left\{(\delta+2\alpha)\delta h_{(1)(1)} - (2D-3\rho)(\delta+2\alpha)h_{(1)(3)} + (D-\rho)(D-\rho)h_{(3)(3)}\right\}. \tag{3.3a}$$

A similar argument for $\dot\Psi_4$ yields

$$\dot\Psi_4 = \tfrac{1}{2}\left\{(\delta^*+2\alpha)\delta^* h_{(2)(2)} - (2\Delta+3\mu+4\gamma)(\delta^*+2\alpha)h_{(2)(4)} + (\Delta+\mu+2\gamma)(\Delta+\mu)h_{(4)(4)}\right\}. \tag{3.3b}$$

Despite appearances Eqs. (3.3) can be inverted to yield $h_{\mu\nu}$ in terms of $\dot\Psi_A$, following a procedure due to Wald [14] which is facilitated by a choice of gauge reflecting the symmetry of the background space-time. It is then straightforward to obtain a complete set of solutions to the equation of motion for $h_{\mu\nu}$ from Teukolsky's complete set of Weyl scalars $\dot\Psi_A$. Rather than repeat this lengthy derivation here however, we simply state the result (originally obtained by Chrzanowski [6] using less direct methods) and verify that it does indeed constitute a complete set of solutions to the perturbed Einstein equations, by substituting into (3.3) and reproducing Teukolsky's solution set (2.5).

We write Chrzanowski's complete, complex mode set as

$$\left\{h^{\Lambda}_{\mu\nu}(l,m,\omega,P;x), h^{\Lambda *}_{\mu\nu}(l,m,\omega,P;x)\right\}_{\Lambda,l,m,\omega,P}, \tag{3.4}$$

where $\Lambda\in\{\text{in},\text{up}\}$ and $P=\pm1$. The explicit form of the modes for which $\Lambda=$ in is

$$h^{\rm in}_{\mu\nu}(l,m,\omega,P;x) = N^{\rm in}\Big\{\Theta_{\mu\nu}\,{}_{+2}Y^m_l(\theta,\phi) + P\Theta^*_{\mu\nu}\,{}_{-2}Y^m_l(\theta,\phi)\Big\}\,{}_{-2}R^{\rm in}_{l\omega}(r)\,e^{-i\omega t} \tag{3.5a}$$

in the *ingoing radiation gauge* $h_{\mu\nu}e_{(1)}{}^{\nu}=0$, $h_\nu{}^\nu=0$; when $\Lambda=$ up we have

$$h^{\rm up}_{\mu\nu}(l,m,\omega,P;x) = N^{\rm up}\Big\{\Upsilon_{\mu\nu}\,{}_{-2}Y^m_l(\theta,\phi) + P\Upsilon^*_{\mu\nu}\,{}_{+2}Y^m_l(\theta,\phi)\Big\}\,{}_{+2}R^{\rm up}_{l\omega}(r)\,e^{-i\omega t} \tag{3.5b}$$

in the *outgoing radiation gauge* $h_{\mu\nu}e_{(2)}{}^{\nu}=0$, $h_\nu{}^\nu=0$. Here $N^\Lambda$ are constants which will be fixed by the quantization prescription, and $\Theta_{\mu\nu}$, $\Upsilon_{\mu\nu}$ are the second-order differential operators:

$$\Theta_{\mu\nu} = -e_{(1)\mu}e_{(1)\nu}(\delta^* - 2\alpha)(\delta^* - 4\alpha) - e_{(4)\mu}e_{(4)\nu}(D - \rho)(D + 3\rho)$$
$$+\tfrac{1}{2}\left(e_{(1)\mu}e_{(4)\nu} + e_{(4)\mu}e_{(1)\nu}\right)\left[D(\delta^* - 4\alpha) + (\delta^* - 4\alpha)(D + 3\rho)\right],$$
$$\Upsilon_{\mu\nu} = \rho^{-4}\left\{-e_{(2)\mu}e_{(2)\nu}(\delta - 2\alpha)(\delta - 4\alpha) - e_{(3)\mu}e_{(3)\nu}(\Delta + 5\mu - 2\gamma)(\Delta + \mu - 4\gamma)\right.$$
$$\left.+\tfrac{1}{2}\left(e_{(2)\mu}e_{(3)\nu} + e_{(3)\mu}e_{(2)\nu}\right)\left[(\delta - 4\alpha)(\Delta + \mu - 4\gamma) + (\Delta + 4\mu - 4\gamma)(\delta - 4\alpha)\right]\right\}. \quad (3.6)$$

The quantities ${}_{-2}R^{\text{in}}_{l\omega}(r)$ and ${}_{+2}R^{\text{up}}_{l\omega}(r)$ appearing in (3.5) are those particular solutions of the radial equation (2.6) with $s = -2$ and $s = +2$, respectively, which are specified by the boundary conditions

$$ {}_{-2}R^{\text{in}}_{l\omega}(r) \sim \begin{cases} B^{\text{in}}_{l\omega}\Delta^2 e^{-i\omega r_*} & \text{as } r \to 2M, \\ r^{-1}e^{-i\omega r_*} + A^{\text{in}}_{l\omega} r^3 e^{+i\omega r_*} & \text{as } r \to \infty, \end{cases} \quad (3.7a)$$

and

$$ {}_{+2}R^{\text{up}}_{l\omega}(r) \sim \begin{cases} A^{\text{up}}_{l\omega}\Delta^{-2} e^{-i\omega r_*} + e^{+i\omega r_*} & \text{as } r \to 2M, \\ B^{\text{up}}_{l\omega} r^{-5} e^{+i\omega r_*} & \text{as } r \to \infty. \end{cases} \quad (3.7b)$$

In the light of (2.9) and (2.12) it is clear that ${}_{-2}R^{\text{in}}_{l\omega}(r)$ and ${}_{+2}R^{\text{up}}_{l\omega}(r)$ are uniquely specified by the above conditions; explicit formulas for $A^{\Lambda}_{l\omega}$ and $B^{\Lambda}_{l\omega}$ will be produced in a later section.

Use of the subscripts "in" and "up" derives from the analogy with classical scattering demonstrated in the previous section; from (3.7a) we can interpret ${}_{-2}R^{\text{in}}_{l\omega}(r)e^{-i\omega t}$ as a unit-amplitude spherical wave propagating *inward* from infinity and being partially reflected back out to infinity and partially transmitted across the horizon, whereas from (3.7b) we see that ${}_{+2}R^{\text{up}}_{l\omega}(r)e^{-i\omega t}$ represents a unit-amplitude spherical wave propagating *upward* from the past horizon, and being partially reflected back and partially transmitted out to infinity. $|A^{\Lambda}_{l\omega}|^2$ is the reflection coefficient and and $|B^{\Lambda}_{l\omega}|^2$ is the transmission coefficient for the scattering process. The situation is depicted in Fig. 1.

The fact that the "in" and "up" modes (3.5) are expressed in different gauges will not cause any difficulty later as we shall only use the modes to construct objects which are gauge independent (namely, $|\dot{\Psi}_A|^2$).

We now verify explicitly that Chrzanowski's mode set comprises a complete set of solutions to the perturbed field equations for Schwarzschild space-time, and in the process derive a set of equations which will prove valuable for calculating expectation values of the perturbed Weyl scalars later.

First substitute (3.5a) for $h_{(a)(b)}$ in Eq. (3.3a). Since we are working in the ingoing radiation gauge for which $h_{(a)(1)} = 0$, we have simply

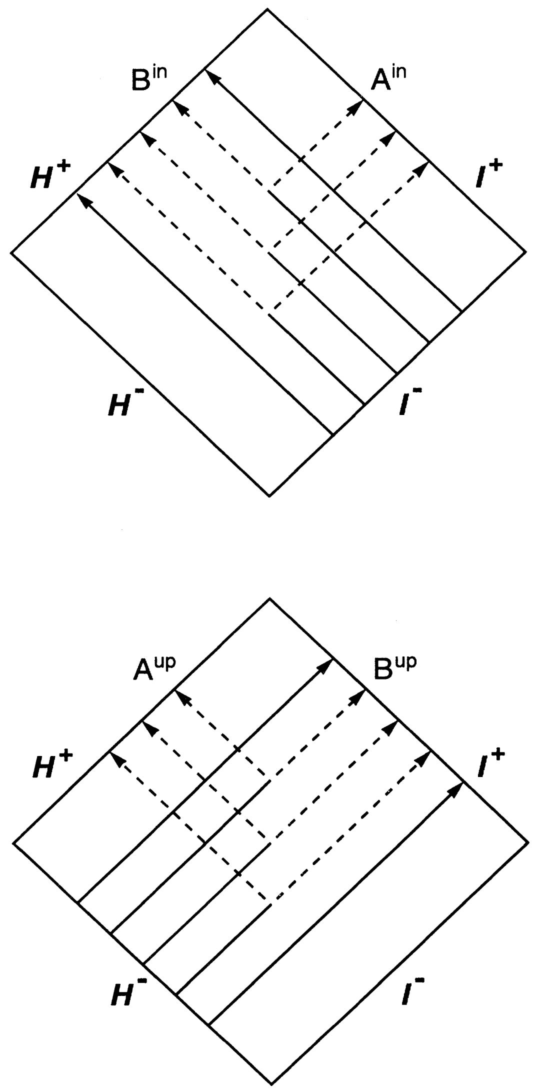


FIG. 1. Representation of the "in" and "up" modes on a Penrose diagram of the exterior Schwarzschild space-time. The "in" modes emanate from $\mathcal{I}^-$ and are partially transmitted across $\mathcal{H}^+$ and partly reflected out to $\mathcal{I}^+$. The "up" modes emanate from $\mathcal{H}^-$ and are partially transmitted out to $\mathcal{I}^+$ and partly reflected across $\mathcal{H}^+$.

$$\dot{\Psi}_0\left[h^{\rm in}(l,m,\omega,P;x)\right] = \tfrac{1}{2}(D-\rho)(D-\rho)h^{\rm in}_{(3)(3)}(l,m,\omega,P;x)$$
$$= -\tfrac{1}{2}N^{\rm in}(D-\rho)(D-\rho)(D-\rho)(D+3\rho)\,{}_{+2}Y_l^m(\theta,\phi)\,{}_{-2}R^{\rm in}_{l\omega}(r)e^{-i\omega t},$$

where (3.5a) and (3.6) have been used in obtaining the second line.

Recalling that

$$D = e_{(1)}{}^{\mu}\partial_\mu = \frac{r^2}{\triangle}\frac{\partial}{\partial t} + \frac{\partial}{\partial r}$$

and $\rho = -r^{-1}$, we see that $D\rho = \rho^2$ and consequently

$$\dot{\Psi}_0\left[h^{\rm in}(l,m,\omega,P;x)\right]$$
$$= -\tfrac{1}{2}N^{\rm in}DDDD\,{}_{+2}Y_l^m\,{}_{-2}R^{\rm in}_{l\omega}(r)e^{-i\omega t}$$
$$= -\tfrac{1}{2}N^{\rm in}\,{}_{+2}Y_l^m(\theta,\phi)e^{-i\omega t}\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}\,{}_{-2}R^{\rm in}_{l\omega}(r), \quad (3.8)$$

where

$$\mathcal{D} = \frac{d}{dr} - \frac{i\omega r^2}{\triangle}. \quad (3.9)$$

We now use the following result of Press and Teukolsky [15]; if ${}_{-2}R_{l\omega}(r)$ is any solution of the radial equation (2.6) with $s=-2$, then $\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}\,{}_{-2}R_{l\omega}(r)$ will be a solution of (2.6) with $s=+2$. So let ${}_{+2}R^{\rm in}_{l\omega}(r)$ be that particular solution of the $s=+2$ radial equation which is given by

$${}_{+2}R^{\rm in}_{l\omega}(r) = 4\mathcal{D}\mathcal{D}\mathcal{D}\mathcal{D}\,{}_{-2}R^{\rm in}_{l\omega}(r) \quad (3.10)$$

(the 4 is present to achieve consistency with the normalizations used in Refs. [15,6]). Then (3.8) becomes simply

$$\dot{\Psi}_0\left[h^{\rm in}(l,m,\omega,P;x)\right]$$
$$= -\tfrac{1}{8}N^{\rm in}\,{}_{+2}R^{\rm in}_{l\omega}(r)\,{}_{+2}Y_l^m(\theta,\phi)e^{-i\omega t}. \quad (3.11)$$

In a similar manner one can substitute (3.5a) into Eq. (3.3b) for $\dot{\Psi}_4$, and also (working in the outgoing gauge now) insert Eq. (3.5b) for the "up" mode into both (3.3a) and (3.3b). Altogether one obtains [8]

$$\dot{\Psi}_0^{\rm in}(l,m,\omega,P;x) = -\frac{1}{8}N^{\rm in}\,{}_{+2}R^{\rm in}_{l\omega}(r)\,{}_{+2}Y_l^m(\theta,\phi)e^{-i\omega t}, \quad (3.12a)$$

$$\dot{\Psi}_4^{\rm in}(l,m,\omega,P;x) = -\frac{1}{8r^4}N^{\rm in}\left(\mathrm{Re}\,C_{l\omega} + 12iM\omega P\right)\,{}_{-2}R^{\rm in}_{l\omega}(r)\,{}_{-2}Y_l^m(\theta,\phi)e^{-i\omega t}, \quad (3.12b)$$

$$\dot{\Psi}_0^{\rm up}(l,m,\omega,P;x) = -\frac{1}{8}N^{\rm up}\left(\mathrm{Re}\,C_{l\omega} - 12iM\omega P\right)\,{}_{+2}R^{\rm up}_{l\omega}(r)\,{}_{+2}Y_l^m(\theta,\phi)e^{-i\omega t}, \quad (3.12c)$$

$$\dot{\Psi}_4^{\rm up}(l,m,\omega,P;x) = -\frac{1}{8r^4}N^{\rm up}\left|C_{l\omega}\right|^2\,{}_{-2}R^{\rm up}_{l\omega}(r)\,{}_{-2}Y_l^m(\theta,\phi)e^{-i\omega t}, \quad (3.12d)$$

where we are using the obvious notation $\dot{\Psi}_A^{\Lambda}(l,m,\omega,P;x)$ for $\dot{\Psi}_A\left[h^{\Lambda}(l,m,\omega,P;x)\right]$ and

$$C_{l\omega} = (l-1)l(l+1)(l+2) + 12iM\omega. \quad (3.13)$$

${}_{-2}R^{\rm up}_{l\omega}(r)$ is defined by the equation

$${}_{-2}R^{\rm up}_{l\omega}(r) = \frac{\triangle^2}{4\left|C_{l\omega}\right|^2}\mathcal{D}^\dagger\mathcal{D}^\dagger\mathcal{D}^\dagger\mathcal{D}^\dagger\left[\triangle^2\,{}_{+2}R^{\rm up}_{l\omega}(r)\right] \quad (3.14)$$

and is known [15] to be a solution of the $s=-2$ radial equation. We stress that since $\dot{\Psi}_A$ is gauge independent, the final expressions (3.12) do not depend on the particular gauges chosen to facilitate their derivation.

Equation (3.12) is clearly in perfect agreement with Teukolsky's solution (2.5), and we conclude that the mode set (3.5) does indeed constitute a complete set of solutions to the equation of motion for the metric perturbation.

The constants $N^{\Lambda}$ appearing in (3.5) are fixed in the quantum theory by demanding that the mode set satisfy the orthonormality conditions

$$\left\langle \frac{1}{\sqrt{16\pi}}h^{\Lambda}(l,m,\omega,P;x), \frac{1}{\sqrt{16\pi}}h^{\Lambda'}(l',m',\omega',P';x)\right\rangle$$
$$= \delta_{\Lambda\Lambda'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega')\delta_{PP'}, \quad (3.15)$$

where the inner product $\langle\ ,\ \rangle$ is defined as follows for arbitrary complex symmetric tensor fields $\psi_{\alpha\beta}$ and $\phi_{\alpha\beta}$:

$$\langle\psi,\phi\rangle = \frac{i}{2}\int_S d\Sigma^\mu\Big(\psi^{\alpha\beta *}\nabla_\mu\tilde{\phi}_{\alpha\beta} - \phi^{\alpha\beta}\nabla_\mu\tilde{\psi}^*_{\alpha\beta}$$
$$+2\tilde{\phi}_{\alpha\mu}\nabla_\beta\tilde{\psi}^{\alpha\beta *} - 2\tilde{\psi}^*_{\alpha\mu}\nabla_\beta\tilde{\phi}^{\alpha\beta}\Big) \quad (3.16)$$

($\tilde{\psi}_{\alpha\beta}$, $\tilde{\phi}_{\alpha\beta}$ are the trace-free parts of $\psi_{\alpha\beta}$, $\phi_{\alpha\beta}$, respectively). $\mathcal{S}$ is an arbitrary spacelike hypersurface in the exterior Schwarzschild space-time, so one can divide the left-hand side of (3.15) into three integrals:

$$\int_{\mathcal{S}} = \int_{\tilde{\mathcal{S}}} - \int_{\mathcal{H}^-} - \int_{\mathcal{I}^-},$$

where $\tilde{\mathcal{S}}$ is the surface enclosing the volume $\mathcal{V}$ shown in Fig. 2.

The integral on the closed hypersurface $\tilde{\mathcal{S}}$ converts to an integral over the space-time region $\mathcal{V}$ via Stokes' theorem; this subsequently vanishes because its integrand is identically zero by virtue of the linearized field equations [to which $h^{\Lambda}_{\mu\nu}(l,m,\omega,P;x)$ are solutions]. The left-hand side of (3.15) thus splits conveniently into just two integrals over $\mathcal{H}^-$ and $\mathcal{I}^-$, each of which can be evaluated explicitly since one knows the form of the radial functions ${}_{-2}R^{\rm in}_{l\omega}$, ${}_{+2}R^{\rm up}_{l\omega}$ at both the horizon and spatial infinity [see (3.7)]. To proceed rigorously one transforms $(t,r)$ to Kruskal null coordinates

$$U = -e^{(r_*-t)/(4M)}, \qquad V = e^{(r_*+t)/(4M)},$$

and employs

$$(d\Sigma^\mu)_{\mathcal{H}^-} = -e_{(2)}{}^{\mu}\frac{4Mr^2}{U}dU\sin\theta\, d\theta\, d\phi,$$
$$(d\Sigma^\mu)_{\mathcal{I}^-} = e_{(1)}{}^{\mu}\frac{2M\triangle}{V}dV\sin\theta\, d\theta\, d\phi,$$

as future-directed surface elements for the null hypersurfaces $\mathcal{I}^-$ and $\mathcal{H}^-$, respectively. Integrations over angular coordinates can be performed using the orthonormality relations (B4) for the spherical harmonics. One discovers that Chrzanowski's mode set satisfies the orthonormality conditions (3.15) provided

$$\left|N^{\rm in}\right|^2 = \frac{1}{4\omega^5}, \tag{3.17a}$$
$$\left|N^{\rm up}\right|^2 = \frac{16}{(2M)^5 p_\omega}, \tag{3.17b}$$

where

$$p_\omega = 2M\omega\left(1+4M^2\omega^2\right)\left(1+16M^2\omega^2\right). \tag{3.18}$$

These normalization factors differ from those given in Ref. [8] only in that we are working in units where $G=1$ while the authors of Ref. [8] chose units where $16\pi G = 1$.

An arbitrary linear perturbation of the Schwarzschild

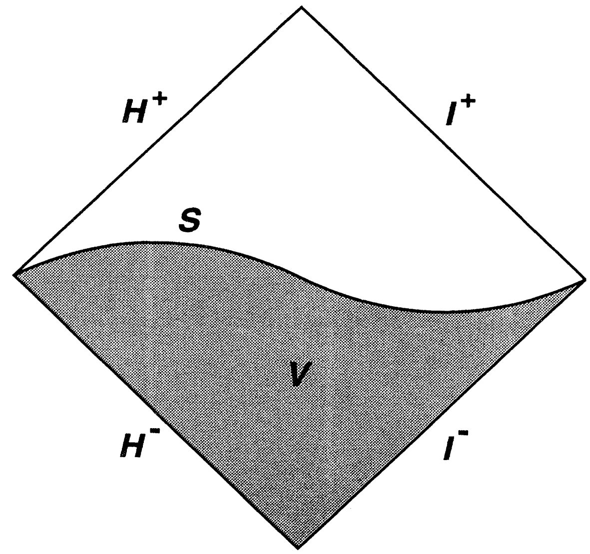


FIG. 2. The space-time region $\mathcal{V}$ of the exterior Schwarzschild space-time.

background can be expanded as follows in terms of the complex mode set (3.5):

$$h_{\mu\nu}(x) = \sum_K \left\{a_K h^K_{\mu\nu}(x) + a^*_K h^{K*}_{\mu\nu}(x)\right\}, \tag{3.19}$$

where $K$ is a shorthand for $(\Lambda, l, m, \omega, P)$. When the theory is quantized $h_{\mu\nu}$, $a_K$, and $a^*_K$ become operators on the Hilbert space of quantum states of the system, subject to canonical commutation relations which (since the mode set is orthonormal) take the simple form

$$\left[a_K, a^\dagger_{K'}\right] = \delta_{KK'}, \qquad [a_K, a_{K'}] = 0 = \left[a^\dagger_K, a^\dagger_{K'}\right]. \tag{3.20}$$

Consider now the expectation value $\langle B||\dot\Psi_A|^2|B\rangle$, where $|B\rangle$ is the vacuum state associated with the mode set (3.5), called the *Boulware vacuum.* Since $\dot\Psi_A$ is linear in $h_{\mu\nu}$ and its derivatives [see (3.3)], one can substitute expansion (3.19) for $h_{\mu\nu}$ and obtain

$$\langle B||\dot\Psi_A|^2|B\rangle \equiv \langle B|\sum_K\left\{a_K\dot\Psi_A\left[h^K\right] + a^\dagger_K\dot\Psi_A\left[h^{K*}\right]\right\}\sum_{K'}\left\{a_{K'}\dot\Psi^*_A\left[h^{K'}\right] + a^\dagger_{K'}\dot\Psi^*_A\left[h^{K'*}\right]\right\}|B\rangle$$
$$= \sum_K\left|\dot\Psi_A\left[h^K\right]\right|^2.$$

Written out fully this is

$$\langle B||\dot\Psi_A|^2|B\rangle = \sum_{l=2}^{\infty}\sum_{m=-l}^{+l}\int_0^\infty d\omega \sum_{P=\pm1}\left\{\left|\dot\Psi^{\rm in}_A(l,m,\omega,P;x)\right|^2 + \left|\dot\Psi^{\rm up}_A(l,m,\omega,P;x)\right|^2\right\}, \tag{3.21}$$

where $\dot{\Psi}_A^\Lambda(l,m,\omega,P;x)$ for $\Lambda = \text{in, up}$ are given explicitly by (3.12).

Although the Boulware state $|B\rangle$ corresponds to the usual notion of a vacuum at large radii, it becomes pathological as one approaches the horizon, in the sense that the components of the renormalized expectation value of the energy-momentum tensor for the scalar and electromagnetic fields in the Boulware vacuum diverge in a freely falling frame as $r \to 2M$ (Ref. [16]; see also [17,18]). It is thus only appropriate for studying a body whose radius exceeds $2M$ such as a neutron star.

The *Unruh vacuum* $|U\rangle$ is the vacuum state of a mode set whose "in" modes are positive frequency with respect to the coordinate $t$ [i.e., they are precisely the $h^{\text{in}}_{\mu\nu}(l,m,\omega,P;x)$ defined by (3.5a)], but whose "up" modes are defined to be positive frequency with respect to the Kruskal null coordinate $U$. The renormalized energy-momentum tensor for the scalar field in this state is regular on $\mathcal{H}^+$ but not on $\mathcal{H}^-$, and corresponds to a flux of black-body radiation as $r\to\infty$. The Unruh vacuum approximates the state of the field long after the gravitational collapse of a massive body.

Finally the *Hartle-Hawking vacuum* $|H\rangle$, defined to be that of a mode set whose "in" modes are positive frequency with respect to $V$ and whose "up" modes are positive frequency with respect to $U$, yields a renormalized energy-momentum tensor which is regular on both horizons, but one then has a bath of thermal "Hawking" radiation at infinity, so that $|H\rangle$ is not a true vacuum state in the usual sense. It corresponds instead to a black hole in unstable equilibrium with an infinite bath of thermal radiation.

If the calculation leading to (3.21) is performed using the Unruh and Hartle-Hawking mode sets instead of the Boulware mode set (3.5), one obtains [8]

$$\langle U||\dot{\Psi}_A|^2|U\rangle = \sum_{l=2}^{\infty}\sum_{m=-l}^{+l}\int_0^\infty d\omega \sum_{P=\pm1}\left\{\left|\dot{\Psi}_A^{\text{in}}(l,m,\omega,P;x)\right|^2 + \coth(4\pi M\omega)\left|\dot{\Psi}_A^{\text{up}}(l,m,\omega,P;x)\right|^2\right\} \tag{3.22}$$

and

$$\langle H||\dot{\Psi}_A|^2|H\rangle = \sum_{l=2}^{\infty}\sum_{m=-l}^{+l}\int_0^\infty d\omega \sum_{P=\pm1}\coth(4\pi M\omega)\left\{\left|\dot{\Psi}_A^{\text{in}}(l,m,\omega,P;x)\right|^2 + \left|\dot{\Psi}_A^{\text{up}}(l,m,\omega,P;x)\right|^2\right\}, \tag{3.23}$$

respectively. On account of their distributional character the product of two field operators $h_{(a)(b)}$ (or their derivatives) evaluated at the same space-time point is *a priori* ill defined; as a result the expectation value of any such product is infinite. Therefore, since $|\dot{\Psi}_A|^2$ is quadratic in $h_{\mu\nu}$ and its derivatives [see (3.3)], expressions (3.21)–(3.23) are all infinite. Rather than become imbrued in the construction of a renormalization scheme for $\langle|\dot{\Psi}_A|^2\rangle$ analogous to that employed in the evaluation of $\langle T^{\mu\nu}\rangle$ for quantum fields of lower spin, we elect instead to work with the *differences*

$$\begin{aligned}\langle|\dot{\Psi}_A|^2\rangle^{H-U} &\equiv \langle H||\dot{\Psi}_A|^2|H\rangle - \langle U||\dot{\Psi}_A|^2|U\rangle \\ &= 2\sum_{l=2}^{\infty}\sum_{m=-l}^{+l}\int_0^\infty d\omega\sum_{P=\pm1}\frac{\left|\dot{\Psi}_A^{\text{in}}(l,m,\omega,P;x)\right|^2}{(e^{8\pi M\omega}-1)}\end{aligned} \tag{3.24a}$$

and

$$\begin{aligned}\langle|\dot{\Psi}_A|^2\rangle^{U-B} &\equiv \langle U||\dot{\Psi}_A|^2|U\rangle - \langle B||\dot{\Psi}_A|^2|B\rangle \\ &= 2\sum_{l=2}^{\infty}\sum_{m=-l}^{+l}\int_0^\infty d\omega\sum_{P=\pm1}\frac{\left|\dot{\Psi}_A^{\text{up}}(l,m,\omega,P;x)\right|^2}{(e^{8\pi M\omega}-1)}\ .\end{aligned} \tag{3.24b}$$

These differences are automatically finite for $r > 2M$, by virtue of the fact that the divergent component of the expectation value of such a two-point product is purely geometrical and hence state independent.

Using these differences we can, of course, work out the difference

$$\begin{aligned}\langle|\dot{\Psi}_A|^2\rangle^{H-B} &\equiv \langle H||\dot{\Psi}_A|^2|H\rangle - \langle B||\dot{\Psi}_A|^2|B\rangle \\ &= \langle|\dot{\Psi}_A|^2\rangle^{H-U} + \langle|\dot{\Psi}_A|^2\rangle^{U-B}.\end{aligned}$$

It is a consequence of the time-reversal invariance of this difference and the transformation properties of the Kinnersley tetrad under time reversal that

$$(r-2M)^4\langle|\dot{\Psi}_0|^2\rangle^{H-B} = 16r^4\langle|\dot{\Psi}_4|^2\rangle^{H-B}. \tag{3.25}$$

Substituting (3.12) for $\dot{\Psi}_A^\Lambda(l,m,\omega,P;x)$ in (3.24) we obtain finally

$$\langle|\dot{\Psi}_0|^2\rangle^{H-U} = \frac{1}{2^8\pi}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty \frac{d\omega}{\omega^5}\frac{\left|{}_{+2}R^{\text{in}}_{l\omega}(r)\right|^2}{(e^{8\pi M\omega}-1)}, \tag{3.26a}$$

$$\langle|\dot{\Psi}_4|^2\rangle^{H-U} = \frac{1}{2^8\pi r^8}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty \frac{d\omega}{\omega^5}\frac{|C_{l\omega}|^2\left|{}_{-2}R^{\text{in}}_{l\omega}(r)\right|^2}{(e^{8\pi M\omega}-1)}, \tag{3.26b}$$

$$\langle|\dot{\Psi}_0|^2\rangle^{U-B} = \frac{1}{4\pi(2M)^5}\sum_{l=2}^{\infty}(2l+1)\int_0^{\infty}\frac{d\omega}{p_\omega}\frac{|C_{l\omega}|^2\,|{}_{+2}R^{\rm up}_{l\omega}(r)|^2}{(e^{8\pi M\omega}-1)}, \tag{3.26c}$$

$$\langle|\dot{\Psi}_4|^2\rangle^{U-B} = \frac{1}{4\pi(2M)^5 r^8}\sum_{l=2}^{\infty}(2l+1)\int_0^{\infty}\frac{d\omega}{p_\omega}\frac{|C_{l\omega}|^4\,|{}_{-2}R^{\rm up}_{l\omega}(r)|^2}{(e^{8\pi M\omega}-1)}, \tag{3.26d}$$

where the addition theorem (B6) for spin-weighted spherical harmonics has been employed to perform the sum over $m$, and (3.17) has been substituted for $|N^\Lambda|^2$.

## IV. POWER SERIES REPRESENTATION FOR ${}_sR_{l\omega}$

In order to evaluate expressions (3.26) an explicit representation of the radial function ${}_sR_{l\omega}$ valid throughout the region exterior to the black hole is required. In what follows we use the methods of Leaver [19] to construct a general solution of the spin $s$ radial equation (2.6) in terms of a convergent power series about $r = 2M$. This procedure has much better stability properties than performing a straight numerical integration of the differential equation (2.6).

In light of (2.12) one particular solution of the spin $s$ radial equation (2.6) can be expanded as

$$ {}_sR_{l\omega}(r) = \Delta^{-s}e^{-i\omega r_*}\,{}_sS_{l\omega}(r), \tag{4.1}$$

where

$$ {}_sS_{l\omega}(r) = \sum_{k=0}^{\infty} a_k(l,\omega,s)\left(1-\frac{2M}{r}\right)^k. \tag{4.2}$$

The coefficients $a_k(l,\omega,s)$ are determined by substituting (4.1) into the radial equation (2.6) and changing variable from $r$ to $x = 1-(2M/r)$. This yields the following differential equation for ${}_sS_{l\omega}$:

$$\left[x(1-x)^2\frac{d^2}{dx^2} + \left\{(1-s)\left(1-x^2\right) - 2x(1-x) - 4iM\omega\right\}\frac{d}{dx} + \left\{(2s-1)\frac{4iM\omega}{(1-x)} + s(s-1) - l(l+1)\right\}\right]{}_sS_{l\omega} = 0. \tag{4.3}$$

Substitution of expansion (4.1) for ${}_sS_{l\omega}$ then yields the following three term recursion relation which determines the $a_k(l,\omega,s)$:

$$k(k-4iM\omega-s)a_k + \left[-3(k-1)^2 + (k-1)(s+4iM\omega) + 4iM\omega(2s-1) + s(s-1) - l(l+1)\right]a_{k-1}$$
$$+\left[3(k-2)^2 + s(k-2) - s(s-1) + l(l+1)\right]a_{k-2} + \left[-(k-3)^2 - s(k-3)\right]a_{k-3} = 0 \tag{4.4}$$

with the initial conditions $a_0 = 1$, $a_{-1} = a_{-2} = 0$.

Another independent solution of the Teukolsky equation is [cf. Eq. (2.12)]

$$ {}_sR_{l\omega}(r) = e^{+i\omega r_*}\,{}_s\tilde{S}_{l\omega}(r) \tag{4.5}$$

with

$$ {}_s\tilde{S}_{l\omega}(r) = \sum_{k=0}^{\infty}\tilde{a}_k(l,\omega,s)\left(1-\frac{2M}{r}\right)^k \tag{4.6}$$

say. Substitution of (4.5) into (2.6) yields the complex conjugate of (4.3) with $s$ replaced by $-s$, so that

$$ {}_s\tilde{S}_{l\omega} = {}_{-s}S^*_{l\omega}. \tag{4.7}$$

One concludes that the general solution of the spin $s$ radial equation (2.6) is

$$ {}_sR_{l\omega}(r) = c_1\Delta^{-s}e^{-i\omega r_*}\,{}_sS_{l\omega}(r) + c_2 e^{+i\omega r_*}\,{}_{-s}S^*_{l\omega}(r), \tag{4.8}$$

where ${}_sS_{l\omega}(r)$ is given by (4.1), the coefficients $a_k(l,\omega,s)$ being determined by the recursion relation (4.4).

The choice of coefficients $c_1$, $c_2$ which yields either ${}_{-2}R^{\rm in}_{l\omega}(r)$ or ${}_{+2}R^{\rm up}_{l\omega}(r)$ is determined by comparing the form of either of these particular solutions as $r \to 2M$ [see Eq. (3.7)] with that of the general solution (4.8). One deduces the representations

$$ {}_{-2}R^{\rm in}_{l\omega} = B^{\rm in}_{l\omega}\Delta^2 e^{-i\omega r_*}\,{}_{-2}S_{l\omega}, \tag{4.9a}$$
$$ {}_{+2}R^{\rm up}_{l\omega} = A^{\rm up}_{l\omega}\Delta^{-2}e^{-i\omega r_*}\,{}_{+2}S_{l\omega} + e^{+i\omega r_*}\,{}_{-2}S^*_{l\omega}, \tag{4.9b}$$

which are valid throughout the exterior region.

The corresponding representations of ${}_{+2}R^{\rm in}_{l\omega}$ and ${}_{-2}R^{\rm up}_{l\omega}$ are obtained by substituting (4.9a) and (4.9b) into (3.10) and (3.14), respectively, and performing the derivative operations explicitly; derivatives of ${}_sS_{l\omega}$ of order $\geq 2$ can be eliminated with the aid of (4.3). The result is

$$ {}_{+2}R^{\rm in}_{l\omega} = 4B^{\rm in}_{l\omega}\Delta^{-2}e^{-i\omega r_*}\left\{\alpha_{l\omega}\,{}_{-2}S_{l\omega} + \beta_{l\omega}\Delta\,{}_{-2}S'_{l\omega}\right\}, \tag{4.9c}$$

$$ 4\,|C_{l\omega}|^2\,{}_{-2}R^{\rm up}_{l\omega} = A^{\rm up}_{l\omega}\Delta^{-1}e^{-i\omega r_*}\left\{\gamma_{l\omega}\,{}_{+2}S'_{l\omega} + \delta_{l\omega}\,{}_{+2}S_{l\omega}\right\} + e^{+i\omega r_*}\left\{\gamma_{l\omega}\Delta\,{}_{-2}S^{*}_{l\omega}{}' + \epsilon_{l\omega}\,{}_{-2}S^{*}_{l\omega}\right\}, \tag{4.9d}$$

where the prime denotes $d/dr$ and

$$\begin{aligned}
\alpha_{l\omega}(r) &= (l-1)l(l+1)(l+2)\Delta^2 - 2i\omega r^2\left\{(l-1)(l+2)r\left(4r^2-10Mr+4M^2\right)\right.\\
&\quad\left.+10Mr^2-40M^2r+24M^3\right\} - 4\omega^2r^4\left\{10Mr-24M^2+3(l-1)(l+2)\Delta\right\} + 8i\omega^3r^6\{r+2M\} + 16\omega^4r^8,\\
\beta_{l\omega}(r) &= -2i\omega r^2\left(4Mr-12M^2+2(l-1)(l+2)\Delta\right) + 8i\omega^3r^6,\\
\gamma_{l\omega}(r) &= 2i\omega r^2\left\{4\left(2r^2-3Mr-3M^2\right)+2(l-2)(l+3)\Delta\right\} - 8i\omega^3r^6,\\
\delta_{l\omega}(r) &= (l-1)l(l+1)(l+2)\Delta - 4i\omega r\left\{8r^2-9Mr-6M^2+(l-2)(l+3)r(2r-3M)\right\}\\
&\quad-4\omega^2r^3\left\{2(2r+3M)+(l-2)(l+3)r\right\} + 24i\omega^3r^5,\\
\epsilon_{l\omega}(r) &= (l-1)l(l+1)(l+2)\Delta^2 + 2i\omega r^2\left\{2\left(8r^3-15Mr^2-12M^2r+12M^3\right)+2(l-2)(l+3)(2r-M)\Delta\right\}\\
&\quad-4\omega^2r^4\left\{2\left(6r^2-7Mr-12M^2\right)+3(l-2)(l+3)\Delta\right\} - 8i\omega^3r^6(r+2M) + 16\omega^4r^8.
\end{aligned}$$

It is reassuring to check that in the limit as $r \to 2M$ these latter representations reduce to

$$ {}_{+2}R^{\rm in}_{l\omega} \sim \frac{16i(2M)^4p_\omega}{(1+2iM\omega)}B^{\rm in}_{l\omega}\Delta^{-2}e^{-i\omega r_*}, \tag{4.10a}$$

$$ {}_{-2}R^{\rm up}_{l\omega} \sim -\frac{i(1+2iM\omega)}{16(2M)^4p_\omega}A^{\rm up}_{l\omega}\Delta^2e^{-i\omega r_*} - \frac{i(2M)^4p_\omega}{|C_{l\omega}|^2(1-2iM\omega)}e^{+i\omega r_*}, \tag{4.10b}$$

which is consistent with their expected behavior (2.12) in this limit. Their behavior as $r\to\infty$ is derived in Appendix C. For completeness we give the leading behavior here:

$$ {}_{+2}R^{\rm in}_{l\omega} \sim 64\omega^4\frac{e^{-i\omega r_*}}{r} + \frac{|C_{l\omega}|^2}{4\omega^4}A^{\rm in}_{l\omega}\frac{e^{+i\omega r_*}}{r^5}, \tag{4.10c}$$

$$ {}_{-2}R^{\rm up}_{l\omega} \sim \frac{4\omega^4}{|C_{l\omega}|^2}B^{\rm up}_{l\omega}r^3e^{+i\omega r_*}. \tag{4.10d}$$

One can now insert the above power series representations (4.9) for the radial functions into the expressions (3.26) for the differences in expectation values of the perturbed Weyl scalars, thereby obtaining

$$\langle|\dot\Psi_0|^2\rangle^{H-U} = \frac{1}{2^4\pi\Delta^4}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty\frac{d\omega}{\omega^5}\frac{|B^{\rm in}_{l\omega}|^2}{\left(e^{8\pi M\omega}-1\right)}\left|\alpha_{l\omega}\,{}_{-2}S_{l\omega}+\beta_{l\omega}\Delta\,{}_{-2}S'_{l\omega}\right|^2, \tag{4.11a}$$

$$\langle|\dot\Psi_4|^2\rangle^{H-U} = \frac{\Delta^4}{2^8\pi r^8}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty\frac{d\omega}{\omega^5}\frac{|B^{\rm in}_{l\omega}|^2}{\left(e^{8\pi M\omega}-1\right)}|C_{l\omega}|^2|\,{}_{-2}S_{l\omega}|^2, \tag{4.11b}$$

$$\begin{aligned}\langle|\dot\Psi_0|^2\rangle^{U-B} &= \frac{1}{4\pi(2M)^5}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty\frac{d\omega|C_{l\omega}|^2}{p_\omega\left(e^{8\pi M\omega}-1\right)}\\
&\quad\times\left\{\left[|A^{\rm up}_{l\omega}|^2\Delta^{-4}|\,{}_{+2}S_{l\omega}|^2+|\,{}_{-2}S_{l\omega}|^2\right]+2\,{\rm Re}\left[A^{\rm up}_{l\omega}e^{-2i\omega r_*}\Delta^{-2}\,{}_{+2}S_{l\omega}\,{}_{-2}S_{l\omega}\right]\right\},\end{aligned} \tag{4.11c}$$

$$\begin{aligned}\langle|\dot\Psi_4|^2\rangle^{U-B} &= \frac{1}{2^6\pi r^8(2M)^5}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty\frac{d\omega}{p_\omega\left(e^{8\pi M\omega}-1\right)}\\
&\quad\times\left\{\left[|A^{\rm up}_{l\omega}|^2\Delta^{-2}\left|\gamma_{l\omega}\,{}_{+2}S'_{l\omega}+\delta_{l\omega}\,{}_{+2}S_{l\omega}\right|^2+\left|\gamma^*_{l\omega}\Delta\,{}_{-2}S'_{l\omega}+\epsilon^*_{l\omega}\,{}_{-2}S_{l\omega}\right|^2\right]\right.\\
&\quad\left.+2\,{\rm Re}\left[A^{\rm up}_{l\omega}e^{-2i\omega r_*}\Delta^{-1}\left(\gamma_{l\omega}\,{}_{+2}S'_{l\omega}+\delta_{l\omega}\,{}_{+2}S_{l\omega}\right)\left(\gamma^*_{l\omega}\Delta\,{}_{-2}S'_{l\omega}+\epsilon^*_{l\omega}\,{}_{-2}S_{l\omega}\right)\right]\right\}.\end{aligned} \tag{4.11d}$$

## V. THE REFLECTION AND TRANSMISSION COEFFICIENTS

It remains to derive formulas from which $B^{\rm in}_{l\omega}$ and $A^{\rm up}_{l\omega}$ can be determined explicitly; expressions (4.11) for the expectation values can then be evaluated numerically.

Such a formula for $B^{\rm in}_{l\omega}$ is derived by comparing the power series representation for ${}_{-2}R^{\rm in}_{l\omega}$ (which is valid for all $r > 2M$) with its asymptotic expansion valid only at large radii (asymptotic expansions of the various radial functions are derived in Appendix C). Thus from (4.9a) and (C1) we have, at large $r$,

$$B^{\rm in}_{l\omega}e^{-i\omega r_*}\,{}_{-2}S_{l\omega}(r)$$
$$= e^{-i\omega r_*}\left[\frac{1}{r^5}+O\left(\frac{1}{r^6}\right)\right] + A^{\rm in}_{l\omega}e^{+i\omega r_*}\left[\frac{1}{r}+\frac{e_1}{r^2}+\frac{e_2}{r^3}+\frac{e_3}{r^4}+\frac{e_4}{r^5}+O\left(\frac{1}{r^6}\right)\right] \quad (5.1)$$

with

$$e_1 = b_1 + 2(2M),$$
$$e_2 = b_2 + 2(2M)b_1 + 3(2M)^2,$$
$$e_3 = b_3 + 2(2M)b_2 + 3(2M)^2 b_1 + 4(2M)^3,$$
$$e_4 = b_4 + 2(2M)b_3 + 3(2M)^2 b_2 + 4(2M)^3 b_1 + 5(2M)^4,$$

where the $b_i$ are given in Appendix C. By ignoring terms of order $r^{-5}$ on the right-hand side of this equation one could write down an approximate formula for the ratio $A^{\rm in}_{l\omega}/B^{\rm in}_{l\omega}$, evaluate it numerically at some suitably large value of $r$, and hence determine $|B^{\rm in}_{l\omega}|^2$ from the Wronskian relation

$$\left|B^{\rm in}_{l\omega}\right|^2 = 4\omega^5\left\{(2M)^5 p_\omega + \frac{|C_{l\omega}|^2}{2^6\omega^3}\left|\frac{A^{\rm in}_{l\omega}}{B^{\rm in}_{l\omega}}\right|^2\right\}^{-1}, \quad (5.2)$$

this last equation being a variant on one of a complete set of Wronskian relations between the reflection and transmission coefficients which are derived in Appendix D. However a more accurate approximation can be obtained by first applying the operator $\mathcal{D}^\dagger$ [recall (3.9)] to both sides of (5.1), so that one need only ignore terms of order $r^{-6}$ to arrive at the approximation

$$\frac{A^{\rm in}_{l\omega}}{B^{\rm in}_{l\omega}} \approx \left[\frac{f_1}{r}+\frac{f_2}{r^2}+\frac{f_3}{r^3}+\frac{f_4}{r^4}+\frac{f_5}{r^5}\right]^{-1} e^{-2i\omega r_*}\frac{d}{dr}\left({}_{-2}S_{l\omega}\right) \quad (5.3)$$

for the ratio of the incoming coefficients, where

$$f_1 = 2i\omega,$$
$$f_2 = 2i\omega\left[e_1 + (2M)\right] - 1,$$
$$f_3 = 2i\omega\left[e_2 + (2M)e_1 + (2M)^2\right] - 2e_1,$$
$$f_4 = 2i\omega\left[e_3 + (2M)e_2 + (2M)^2 e_1 + (2M)^3\right] - 3e_2,$$
$$f_5 = 2i\omega\left[e_4 + (2M)e_3 + (2M)^2 e_2 + (2M)^3 e_1 + (2M)^4\right] - 4e_3.$$

In practice the right-hand side of (5.3) is evaluated for large and increasing values of $r$ until it has converged to the desired accuracy; the result is then inserted in (5.2) to yield $|B^{\rm in}_{l\omega}|^2$.

A slightly different procedure is used to evaluate the reflection amplitude $A^{\rm up}_{l\omega}$. A comparison of the power series expansion (4.9b) for ${}_{+2}R^{\rm up}_{l\omega}$ with its asymptotic form (C3) at large radii yields

$$A^{\rm up}_{l\omega}e^{-i\omega r_*}\,{}_{+2}S_{l\omega} + \triangle^2 e^{+i\omega r_*}\,{}_{-2}S^*_{l\omega}$$
$$= B^{\rm up}_{l\omega}e^{+i\omega r_*}\left[\frac{1}{r}+\frac{g_2}{r^2}+\frac{g_3}{r^3}+\frac{g_4}{r^4}+\frac{g_5}{r^5}+O\left(\frac{1}{r^6}\right)\right] \quad (5.4)$$

with

$$g_2 = c_1 - 4M,$$
$$g_3 = c_2 - 4Mc_1 + 4M^2,$$
$$g_4 = c_3 - 4Mc_2 + 4M^2 c_1,$$
$$g_5 = c_4 - 4Mc_3 + 4M^2 c_2,$$

where the $c_i$ are given in Appendix C, which when operated on by $\mathcal{D}$ gives

$$A^{\rm up}_{l\omega}e^{-i\omega r_*}\left[\frac{d}{dr}\left({}_{+2}S_{l\omega}\right) - \frac{2i\omega r^2}{\triangle}\,{}_{+2}S_{l\omega}\right] + \triangle e^{+i\omega r_*}\left[\triangle\frac{d}{dr}\left({}_{-2}S^*_{l\omega}\right) + 4(r-M)\,{}_{-2}S^*_{l\omega}\right]$$
$$= -B^{\rm up}_{l\omega}e^{+i\omega r_*}\left[\frac{1}{r^2}+\frac{2g_2}{r^3}+\frac{3g_3}{r^4}+\frac{4g_4}{r^5}+\frac{5g_5}{r^6}+O\left(\frac{1}{r^7}\right)\right].$$

To this last equation one now adds (5.4) multiplied by

$$h(r) = \left[\frac{1}{r}+\frac{h_2}{r^2}+\frac{h_3}{r^3}+\frac{h_4}{r^4}+\frac{h_5}{r^5}\right], \quad (5.5)$$

where

$$h_2 = g_2,$$
$$h_3 = 2g_3 - g_2 h_2,$$
$$h_4 = 3g_4 - g_3 h_2 - g_2 h_3,$$
$$h_5 = 4g_5 - g_4 h_2 - g_3 h_3 - g_2 h_4,$$

so that the right-hand side of the resulting equation is

$O\left(r^{-7}\right)$ and can be ignored at large radii. An accurate approximation for $A^{\rm up}_{l\omega}$ is therefore given by

$$A^{\rm up}_{l\omega} \approx -\Delta^2 e^{2i\omega r_*} \times \left[\frac{d}{dr}\left({}_{-2}S^*_{l\omega}\right) + \left\{h(r) + \frac{4(r-M)}{\Delta}\right\} {}_{-2}S^*_{l\omega}\right] \times \left[\frac{d}{dr}\left({}_{+2}S_{l\omega}\right) + \left\{h(r) - \frac{2i\omega r^2}{\Delta}\right\} {}_{+2}S_{l\omega}\right]^{-1} \quad (5.6)$$

which is evaluated in the same way as (5.3).

The values of the reflection and transmission amplitudes, which we have computed numerically, are displayed graphically in Figs. 3–5. We have verified that these values are correct using the following (independent) checks.

(i) The rate of decrease of mass of the black hole due to graviton emission is given by the formula [9]

$$\dot{M} = \int_0^\infty d\omega\, \dot{M}_\omega, \quad (5.7)$$

where

$$\dot{M}_\omega = \sum_{l=2}^{\infty}\sum_{m=-l}^{+l}\sum_{P=\pm 1} \frac{\omega}{2\pi}\frac{1}{\left(e^{8\pi M\omega}-1\right)}\frac{dE^{\rm hor}_\omega/dt}{dE^{\rm in}_\omega/dt}. \quad (5.8)$$

Substituting first (2.15) and (2.16a) for the energy fluxes, and then $\dot{\Psi}^{\rm in}_0(l,m,\omega,P;x)$ [see Eq. (3.12)] for $\dot{\Psi}_0$, one obtains the following expression for the fractional absorption of incoming radiation by the black hole [spherical harmonics are eliminated with the aid of (B4)]:

$$\frac{dE^{\rm hor}_\omega/dt}{dE^{\rm in}_\omega/dt} = \frac{1}{(2M)^4}\frac{4\omega^2}{(16M^2\omega^2+1)}\frac{\left[\Delta^4|{}_{+2}R^{\rm in}_{l\omega}|^2\right]_{r=2M}}{\left[r^2\left|{}_{+2}R^{\rm in}_{l\omega}\right|^2\right]_{r\to\infty}}$$

$$= \frac{(2M)^5}{4\omega^5} p_\omega |B^{\rm in}_{l\omega}|^2,$$

where the last line follows from (4.10a) and (C2), and hence

$$\dot{M}_\omega = \frac{(2M)^5}{4\pi}\sum_{l=2}^{\infty}(2l+1)\frac{p_\omega|B^{\rm in}_{l\omega}|^2}{\omega^4\left(e^{8\pi M\omega}-1\right)}. \quad (5.9)$$

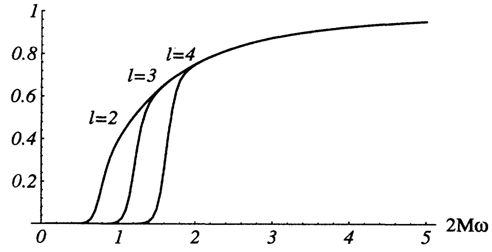


FIG. 3. $(2M)^{10}|B^{\rm in}_{l\omega}|^2$ as a function of $2M\omega$ for $l = 2, 3, 4$.

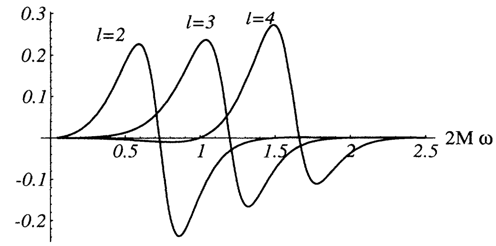


FIG. 4. The real part of $(2M)^{-4}A^{\rm up}_{l\omega}$ as a function of $2M\omega$ for $l = 2, 3, 4$.

Using the values obtained numerically for $|B^{\rm in}_{l\omega}|^2$ in the above equations one obtains the luminosity spectrum shown in Fig. 6 and the value $\dot{M} = (3.84\times 10^{-6})M^{-2}$ for the total luminosity due to graviton emission; both are in close agreement with the previous results of Page [9]. [Page has $\dot{M} = (3.81\times 10^{-6})M^{-2}$; we believe the slight discrepancy to be caused by his choice of an upper limit for the $\omega$ integral which is not high enough to ensure a three significant figure accuracy.]

(ii) When $2M\omega \ll 1$ the spin $s$ radial equation can be solved analytically in two overlapping regions, one of which includes the horizon, and the other of which extends to infinity; solutions are expressed in terms of ordinary and confluent hypergeometric functions in the respective regions (see Ref. [20]). By matching these general solutions in the region of overlap, and then specializing to the particular solutions ${}_{-2}R^{\rm in}_{l\omega}$, ${}_{+2}R^{\rm up}_{l\omega}$ in turn, one can derive analytic approximations for the reflection and transmission amplitudes which are valid in the limit as $2M\omega \to 0$. The derivation is performed in detail in Appendix E. The exact numerical values for the amplitudes are consistent with the analytic approximation in the limit $2M\omega \to 0$.

(iii) A final check on the numerical values we have obtained for $|B^{\rm in}_{l\omega}|^2$ and $|A^{\rm up}_{l\omega}|^2$ is provided by the Wronskian relations derived in Appendix D. From (D4) and (D6) the following relation between the "in" and "up" reflection coefficients may be obtained:

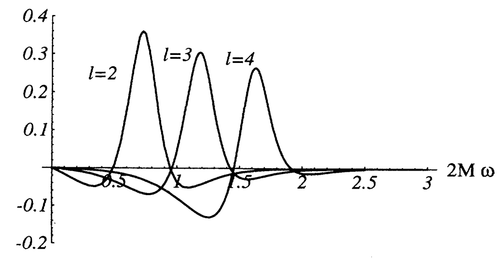


FIG. 5. The imaginary part of $(2M)^{-4}A^{\rm up}_{l\omega}$ as a function of $2M\omega$ for $l = 2, 3, 4$.

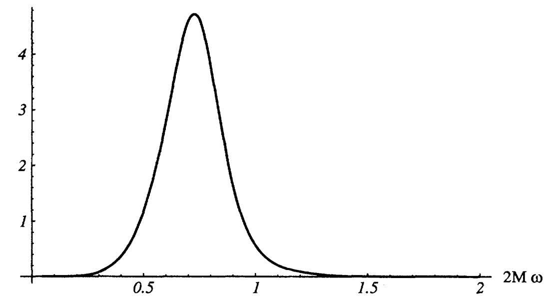


FIG. 6. $\dot{M}_\omega \times 10^5$, the luminosity spectrum due to graviton emission from a Schwarzschild black hole.

$$|A^{\rm in}_{l\omega}|^2 = \frac{16\omega^7}{(2M)^9\,(16M^2\omega^2+1)\,p_\omega}|A^{\rm up}_{l\omega}|^2. \tag{5.10}$$

Substituting this relation into (D5) then yields

$$\frac{(4M^2\omega^2+1)\,|C_{l\omega}|^2}{16(2M)^8 p_\omega^2}|A^{\rm up}_{l\omega}|^2 + \frac{(2M)^5 p_\omega}{4\omega^5}|B^{\rm in}_{l\omega}|^2 = 1. \tag{5.11}$$

We have shown our numerical results to be consistent with the above equation.

## VI. ASYMPTOTIC ANALYSIS

Before we discuss the numerical evaluation of Eqs. (3.26), it is useful to consider their asymptotic behavior in the limits $r \to 2M$ and $r \to \infty$.

Consider first the limit $r \to 2M$. The leading behavior of the various radial functions in the limit as $r \to 2M$ may be deduced from Eqs (4.9), and (4.10); Eqs. (3.26) then give

$$\langle|\dot\Psi_0|^2\rangle^{H-U} \sim \frac{(2M)^9}{\pi\Delta^4}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty d\omega\frac{(1+16M^2\omega^2)\,p_\omega|B^{\rm in}_{l\omega}|^2}{\omega^4\,(e^{8\pi M\omega}-1)}, \tag{6.1a}$$

$$\langle|\dot\Psi_4|^2\rangle^{H-U} \sim \frac{\Delta^4}{2^8\pi(2M)^8}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty d\omega\frac{|C_{l\omega}|^2|B^{\rm in}_{l\omega}|^2}{\omega^5\,(e^{8\pi M\omega}-1)}. \tag{6.1b}$$

The integrals in Eq. (6.1) may be evaluated numerically to give

$$\lim_{r\to 2M}(1-2M/r)^4\langle|\dot\Psi_0|^2\rangle^{H-U} \approx 1.96\times 10^{-4}\,(2M)^{-6}, \tag{6.2a}$$

$$\lim_{r\to 2M}(1-r/2M)^{-4}\langle|\dot\Psi_4|^2\rangle^{H-U} \approx 9.14\times 10^{-5}\,(2M)^{-6}. \tag{6.2b}$$

The same method cannot be used to evaluate the limit of $\langle|\dot\Psi_A|^2\rangle^{U-B}$ as $r \to 2M$. For writing

$$\langle|\dot\Psi_A|^2\rangle = \sum_{l=2}^{\infty}(2l+1)\psi_{A\,l} \tag{6.3}$$

one finds [by substituting (C2) in (3.26)] that

$$\Delta^4\psi^{U-B}_{0\,l} \sim \frac{11(2M)^2}{2^4\times 15\pi} - \Delta^4\psi^{H-U}_{0\,l} \tag{6.4a}$$

and

$$\psi^{U-B}_{4\,l} \sim \frac{11}{2^8\times 15\pi}(2M)^{-6}. \tag{6.4b}$$

In neither case will the sum over $l$ converge since the terms approach a constant times $(2l+1)$ for large $l$ [note from (6.4a) that we have already seen that the corresponding sum over $\psi^{H-U}_{0\,l}$ converges]. The problem is that the limit as $r \to 2M$ and the sum over $l$ do not commute in this case since our asymptotic expansions for the Teukolsky functions are not *uniform* in $l$. Nevertheless Eqs. (6.4) do provide a useful check against the numerical results.

To find the asymptotic behavior of $\langle|\dot\Psi_0|^2\rangle^{U-B}$ and $\langle|\dot\Psi_4|^2\rangle^{U-B}$ as $r \to 2M$ we need asymptotic expressions for the Teukolsky functions which are uniform in $l$. To this end we follow Candelas [17]. Near $r = 2M$ the Teukolsky equation (2.6) may be approximated by the equation

$$\left[(r-2M)\frac{d^2}{dr^2} + (s+1)\frac{d}{dr} + \frac{2M\omega(2M\omega - is)}{(r-2M)} - \frac{(l-s)(l+s+1)}{2M}\right]{}_sR_{l\omega}(r) = 0. \tag{6.5}$$

Defining $\xi = (r/2M-1)^{1/2}$ this becomes

$$\left[\frac{d^2}{d\xi^2} + \frac{(2s+1)}{\xi}\frac{d}{d\xi} + \frac{8M\omega(2M\omega - is)}{\xi^2} - 4(l-s)(l+s+1)\right]{}_sR_{l\omega}(r) = 0, \tag{6.6}$$

which admits solutions in terms of modified Bessel functions. Since it is clear from Eq. (6.4) that the asymptotic forms of $\langle|\dot\Psi_0|^2\rangle^{U-B}$ and $\langle|\dot\Psi_4|^2\rangle^{U-B}$ as $r \to 2M$ are determined by the contribution from large $l$, we may further approximate the constant term in the potential in Eq. (6.6) by $-4l^2$ and then the solutions are given by

$$\xi^{-s}K_{s+4iM\omega}(2l\xi) \qquad \text{and} \qquad \xi^{-s}I_{-s-4iM\omega}(2l\xi).$$

These solutions are uniformly valid for large $l$.

We now concentrate on $\langle|\dot{\Psi}_0|^2\rangle^{U-B}$ and write

$$_{+2}R^{\rm up}_{l\omega} \sim \alpha_l\,\xi^{-2}K_{2+4iM\omega}(2l\xi) + \beta_l\,\xi^{-2}I_{-2-4iM\omega}(2l\xi). \tag{6.7}$$

Following the arguments of Candelas, as $l \to \infty$ for fixed $\xi$, $_{+2}R^{\rm up}_{l\omega}(\xi) \to 0$ and so $\beta_l$ is an exponentially small function of $l$ that will not therefore contribute to the leading asymptotic behavior of $\langle|\dot{\Psi}_0|^2\rangle^{U-B}$ as $r \to 2M$. Furthermore comparison of (6.7) with (3.7b) for fixed $l$ as $\xi \to 0$ yields

$$\alpha_l = \frac{2i\Gamma(3+4iM\omega)\sinh(4\pi M\omega)e^{2iM\omega}}{\pi l^{2+4iM\omega}}. \tag{6.8}$$

The elementary identity

$$|\Gamma(3+4iM\omega)|^2 = \frac{8\pi p_\omega}{\sinh(4\pi M\omega)}$$

then enables us to write

$$|\alpha_l|^2 = \frac{32 p_\omega \sinh(4\pi M\omega)}{\pi l^4}. \tag{6.9}$$

Since for large $l$, $|C_{l\omega}|^2 \sim l^8$, it follows that, to leading order,

$$\sum_{l=2}^{\infty}(2l+1)\,|C_{l\omega}|^2\,|_{+2}R^{\rm up}_{l\omega}(r)|^2$$

$$\sim \frac{64 p_\omega \sinh(4\pi M\omega)}{\xi^4}\int_0^\infty dl\, l^5\,|K_{2+4iM\omega}(2l\xi)|^2 = \frac{64 p_\omega^2}{5\xi^{10}}, \tag{6.10}$$

where we have used Eq. (6.576.3) of [27]. Inserting this into Eq. (3.26c) we obtain the leading asymptotic form as $r \to 2M$ as

$$\Delta^4\langle|\dot{\Psi}_0|^2\rangle^{U-B} \sim \frac{16(2M)^4}{5\pi(r-2M)}\int_0^\infty d\omega\frac{p_\omega}{(e^{8\pi M\omega}-1)} = \frac{191}{2^4\times 315\pi}\frac{(2M)^3}{(r-2M)} \approx 1.21\times 10^{-2}\frac{(2M)^3}{(r-2M)}, \tag{6.11}$$

or, equivalently,

$$\lim_{r\to 2M}(r/2M-1)^5\langle|\dot{\Psi}_0|^2\rangle^{U-B} \approx 1.21\times 10^{-2}\,(2M)^{-6}. \tag{6.12}$$

Since, by (6.1a), $\Delta^4\langle|\dot{\Psi}_0|^2\rangle^{H-U}$ is finite on the horizon it follows that the asymptotic form of $\Delta^4\langle|\dot{\Psi}_0|^2\rangle^{H-B}$ as $r \to 2M$ is also given by (6.11). Then, from (3.25),

$$\langle|\dot{\Psi}_4|^2\rangle^{H-B} \sim \frac{191}{2^8\times 315\pi}\frac{1}{(2M)^5(r-2M)} \approx 7.54\times 10^{-4}\frac{1}{(2M)^5(r-2M)}. \tag{6.13}$$

Since, by (6.1b), $\langle|\dot{\Psi}_4|^2\rangle^{H-U}$ vanishes at the horizon it follows that the asymptotic form of $\langle|\dot{\Psi}_4|^2\rangle^{U-B}$ as $r \to 2M$ is also given by (6.13).

We now turn to the asymptotic forms at infinity. Substituting the asymptotic expansions (C3) and (C4) for $_{+2}R^{\rm up}_{l\omega}$ and $_{-2}R^{\rm up}_{l\omega}$, respectively, in Eqs. (3.26c) and (3.26d) yields the following formulas which describe the leading behavior of $\langle|\dot{\Psi}_A|^2\rangle^{U-B}$ in the limit as $r \to \infty$:

$$\langle|\dot{\Psi}_0|^2\rangle^{U-B} \sim \frac{1}{4\pi(2M)^5 r^{10}}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty d\omega\frac{|C_{l\omega}|^2|B^{\rm up}_{l\omega}|^2}{p_\omega\,(e^{8\pi M\omega}-1)}, \tag{6.14a}$$

$$\langle|\dot{\Psi}_4|^2\rangle^{U-B} \sim \frac{4}{\pi(2M)^5 r^2}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty d\omega\frac{\omega^8|B^{\rm up}_{l\omega}|^2}{p_\omega\,(e^{8\pi M\omega}-1)}. \tag{6.14b}$$

Using the Wronskian relation (D4), the last equation may be rewritten as

$$\langle|\dot{\Psi}_4|^2\rangle^{U-B} \sim \frac{(2M)^5}{4\pi r^2}\sum_{l=2}^{\infty}(2l+1)\int_0^\infty d\omega\frac{p_\omega|B^{\rm in}_{l\omega}|^2}{\omega^2\,(e^{8\pi M\omega}-1)}. \tag{6.14c}$$

Equation (6.14c) is in accord with (5.9) for the luminosity since (2.16b) implies that, for $r \to \infty$,

$$\langle|\dot{\Psi}_{4\,\omega}|^2\rangle^{U-B} \sim \frac{\omega^2\dot{M}_\omega}{r^2}. \tag{6.15}$$

The integrals in Eq. (6.14) may be evaluated numerically to give

$$\lim_{r\to\infty}(r/2M)^{10}\langle|\dot{\Psi}_0|^2\rangle^{U-B} \approx 2.91\times 10^{-2}\,(2M)^{-6}, \tag{6.16a}$$

$$\lim_{r\to\infty}(r/2M)^{2}\langle|\dot{\Psi}_4|^2\rangle^{U-B} \approx 8.42\times 10^{-6}\,(2M)^{-6}. \tag{6.16b}$$

Finally, we consider the limit as $r \to \infty$ of

$$\langle|\dot{\Psi}_A|^2\rangle^{H-B} = \langle|\dot{\Psi}_A|^2\rangle^{H-U} + \langle|\dot{\Psi}_A|^2\rangle^{U-B}.$$

We find that

$$\psi^{H-B}_{0\,l} \sim 16\psi^{H-B}_{4\,l} \sim \frac{1}{2^4(2M)^4 15\pi r^2}, \tag{6.17}$$

where Wronskian relations (D4) and (D5) have been used. As before these forms provide a useful check on our numerical results but are not good enough to yield the asymptotic forms.

The simplest way to obtain the asymptotic forms at infinity is to use the fact that at infinity the calculation reduces to a problem in Minkowski space. Then quantum effects are negligible (except in providing the appropriate temperature) and we may take over the standard classical results given by Eq. (2.16). First, using spherical symmetry, we have

$$\left|\dot\Psi_{0\,\omega}\right|^2 = -64\pi\omega^2 T^{\rm in}_{\omega}{}_r{}^t \quad \text{and} \quad \left|\dot\Psi_{4\,\omega}\right|^2 = 4\pi\omega^2 T^{\rm out}_{\omega}{}_r{}^t,$$

where $T^{\rm in}_{\omega}{}_r{}^t$ and $T^{\rm out}_{\omega}{}_r{}^t$ denote the energy flux components of frequency $\omega$ in the energy-momentum tensors associated with the incoming and outgoing gravitational waves, respectively. Then, since we are dealing with a thermal bath of gravitons at the Hawking temperature, $1/(8\pi M)$, we have

$$\begin{aligned}-\langle T^{\rm in}_{\omega}{}_r{}^t\rangle = \langle T^{\rm out}_{\omega}{}_r{}^t\rangle &= 2\times\int_{k_z>0}\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{k_z\,\delta(k-\omega)}{(e^{8\pi M\omega}-1)}\\ &= 2\times\frac{1}{8\pi^2}\frac{\omega^3}{(e^{8\pi M\omega}-1)},\end{aligned}$$

where the factor of 2 arises from the number of polarization states and multiplies the ingoing and outgoing energy fluxes for a real massless scalar field in a thermal state with the Hawking temperature in Minkowski space. Thus, as $r\to\infty$ we have

$$\begin{aligned}\langle|\dot\Psi_0|^2\rangle^{H-B} &\sim 16\langle|\dot\Psi_4|^2\rangle^{H-B}\\ &\sim \frac{16}{\pi}\int_0^\infty d\omega\frac{\omega^5}{(e^{8\pi M\omega}-1)} \qquad (6.18a)\\ &= \frac{1}{2016\pi}(2M)^{-6} \qquad (6.18b)\\ &\approx 1.58\times10^{-4}(2M)^{-6}. \qquad (6.18c)\end{aligned}$$

Since, by Eq. (6.14) $\langle|\dot\Psi_0|^2\rangle^{U-B}$ and $\langle|\dot\Psi_4|^2\rangle^{U-B}$ vanish at infinity, Eq. (6.18) also gives the asymptotic form for $\langle|\dot\Psi_0|^2\rangle^{H-U}$ and $\langle|\dot\Psi_4|^2\rangle^{H-U}$.

## VII. NUMERICAL RESULTS

In this section we present the results of evaluating expressions (4.11) for the differences between renormalized expectation values of $|\dot\Psi_A|^2$ in the Boulware, Unruh, and Hartle-Hawking vacua, using values for the coefficients $|B^{\rm in}_{l\omega}|^2$ and $A^{\rm up}_{l\omega}$ which were obtained numerically in Sec. V.

All power series and sums over $l$ converge swiftly; likewise the infinite upper limit on the integral presents no difficulty, and may be approximated with sufficient accuracy by $(2M)\omega=2$ in every case. There are nevertheless technical problems associated with the computation, two of which merit some explanation here.

The first problem concerns the evaluation of the integrands of (4.11) at the lower limit of the integral $\omega=0$: although the integrand of $\langle|\dot\Psi_A|^2\rangle^{H-U}$ can be seen to vanish at $\omega=0$ [consider (E7)], the integrand of $\langle|\dot\Psi_A|^2\rangle^{U-B}$ is nonzero at $\omega=0$ and its value there can only be determined by expanding the integrand as a Taylor series in powers of $\omega$ [21]. To this end consider the Taylor expansions

$$\begin{aligned}{}_{-2}S_{l\omega} &= {}_{-2}S_{l0} + [{}_{-2}S_{l0}]'\omega + O(\omega^2),\\ {}_{+2}\hat S_{l\omega} \equiv \omega\,{}_{+2}S_{l\omega} &= {}_{+2}\hat S_{l0} + [{}_{+2}\hat S_{l0}]'\omega + O(\omega^2),\\ \frac{A^{\rm up}_{l\omega}}{\omega} &= [A^{\rm up}_{l0}]' + \frac12[A^{\rm up}_{l0}]''\omega + O(\omega^2),\end{aligned}$$

where we have introduced the notation

$$[f_{l0}]' = \left[\frac{\partial f_{l\omega}}{\partial\omega}\right]_{\omega=0}. \qquad (7.1)$$

We have expanded ${}_{+2}\hat S_{l\omega}$ rather than ${}_{+2}S_{l\omega}$ since the latter diverges like $\omega^{-1}$ as $\omega\to0$ [consider Eq. (4.4) when $k=s=2$]. We have also used $A^{\rm up}_{l0}=0$ which follows from (E7). Inserting these expansions into the power series representation (4.9b) for ${}_{+2}R^{\rm up}_{l\omega}$ one obtains

$$\begin{aligned}{}_{+2}R^{\rm up}_{l\omega} = &\left\{{}_{-2}S^*_{l0} + \Delta^{-2}[A^{\rm up}_{l0}]'{}_{+2}\hat S_{l0}\right\} + \left\{[{}_{-2}S^*_{l0}]' + ir_*\,{}_{-2}S^*_{l0} + \Delta^{-2}[A^{\rm up}_{l0}]'\left([{}_{+2}\hat S_{l0}]' - ir_*\,{}_{+2}\hat S_{l0}\right)\right.\\ &\left.+\tfrac12\Delta^{-2}[A^{\rm up}_{l0}]''{}_{+2}\hat S_{l0}\right\}\omega + O(\omega^2). \qquad (7.2)\end{aligned}$$

In order that Eq. (3.26c) for $\langle|\dot\Psi_0|^2\rangle^{U-B}$ be finite for finite $r>2M$, we must have ${}_{+2}R^{\rm up}_{l\omega}\sim\omega^n$ as $\omega\to0$ where $n\ge1$, i.e.,

$${}_{+2}R^{\rm up}_{l0} = 0 \qquad (7.3)$$

and the value of the integrand in expression (3.26c) for $\langle|\dot\Psi_0|^2\rangle^{U-B}$ at $\omega=0$ is

$$\frac{1}{2^4\pi^2(2M)^7}\sum_{l=2}^{\infty}(2l+1)\left[(l-1)l(l+1)(l+2)\right]^2\left|[{}_{+2}R^{\rm up}_{l0}]'\right|^2, \qquad (7.4)$$

where $[{}_{+2}R^{\rm up}_{l0}]'$ is the coefficient multiplying $\omega$ in (7.2). Equation (7.4) can now be evaluated numerically; in particular the power series are determined straightforwardly from (4.2) and (4.4), and $[A^{\rm up}_{l0}]'$ is obtained from the formula

$$[A_{l0}^{\rm up}]' = \frac{-\triangle^2 \; {}_{-2}S_{l0}^*}{{}_{+2}\hat{S}_{l0}} \tag{7.5}$$

which follows from (7.2) and (7.3). The second derivative $[A_{l0}^{\rm up}]'' = \left[(\partial^2/\partial^2\omega)\,(A_{l\omega}^{\rm up})\right]_{\omega=0}$ is computed using the approximation

$$[A_{l0}^{\rm up}]'' \approx -\frac{2}{{}_{+2}\hat{S}_{l0}}\left[\frac{\partial}{\partial\omega}\,(A_{l\omega}^{\rm up})\,\frac{\partial}{\partial\omega}\left({}_{+2}\hat{S}_{l\omega}\right) + \triangle^2\left\{\frac{\partial}{\partial\omega}\,({}_{-2}S_{l\omega}^*) + 2ir_*\;{}_{-2}S_{l\omega}^*\right\}\right]_{\omega=0} \tag{7.6}$$

[which follows from expanding both sides of (5.6) in powers of $\omega$ and using (7.5)]. Note that (7.6) will only become independent of $r$ at large radii, in contrast with (7.5) which may be computed at any value of $r > 2M$.

The value of the integrand of (3.26d) at $\omega = 0$ is found in a similar way, requiring only that $[(d/dr)_{+2}\hat{S}_{l\omega}]_{\omega=0}$ and $[(d/dr)_{-2}S_{l\omega}^*]_{\omega=0}$ be evaluated numerically in addition to the power series considered above. In both cases the result is finite and nonvanishing.

The second technical difficulty also arises during the computation of $\langle|\dot{\Psi}_A|^2\rangle^{U-B}$, when one attempts to evaluate $|{}_sR_{l\omega}^{\rm up}|^2$ numerically using the power series representations (4.9b) and (4.9d). This latter quantity is composed of two pieces [see, e.g., the expression for $|{}_{+2}R_{l\omega}^{\rm up}|^2$ in braces in (4.9c)], which turn out to be opposite in sign but equal in magnitude to high accuracy; one must therefore work to high precision in order to produce reliable results. The problem becomes more severe as $\omega \to 0$. This difficulty can be understood by once again invoking the analogy between our system and a classical scattering problem; at low frequencies, upcoming radiation from the black hole is unable to surmount the potential barrier ${}_sV_{l\omega}(r)$ and is instead completely reflected back across the event horizon.

The final results are displayed graphically in Figs. 7–11, where the quantities have been scaled to give finite values on the horizon and at infinity. The most physically interesting graphs are those for $\langle|\dot{\Psi}_4|^2\rangle^{U-B}$ (Fig. 10) corresponding to outgoing radiation from an evapo-

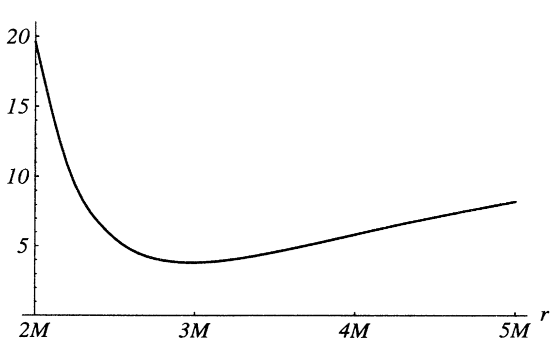


FIG. 7. $(2M)^6(1-2M/r)^4\langle|\dot{\Psi}_0|^2\rangle^{H-U} \times 10^5$. The value of this combination at $r = 2M$ is 19.6 and its asymptotic value at infinity is 15.8.

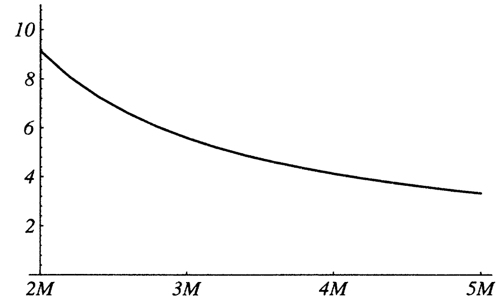


FIG. 8. $(2M)^6(1-2M/r)^{-4}\langle|\dot{\Psi}_4|^2\rangle^{H-U} \times 10^5$. The value of this combination at $r = 2M$ is 9.14 and its asymptotic value at infinity is 0.988.

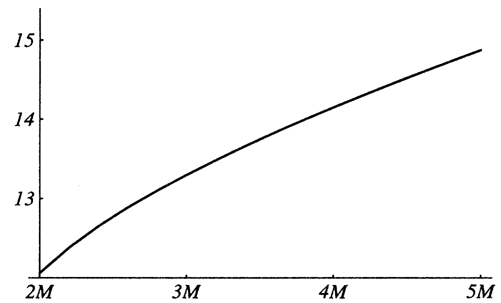


FIG. 9. $(2M)^6(r/2M)^{10}(1-2M/r)^5\langle|\dot{\Psi}_0|^2\rangle^{U-B}\times 10^3$. The value of this combination at $r = 2M$ is 12.1 and its asymptotic value at infinity is 29.1.

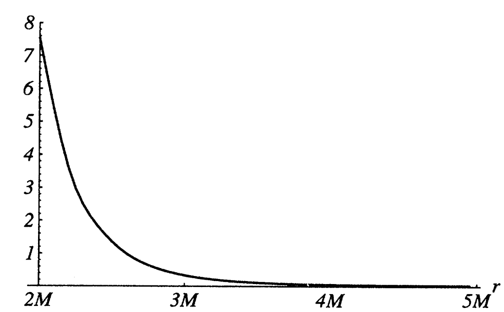


FIG. 10. $(2M)^6(r/2M)^2(1-2M/r)\langle|\dot{\Psi}_4|^2\rangle^{U-B}\times 10^4$. The value of this combination at $r = 2M$ is 7.54 and its asymptotic value at infinity is 0.0842.

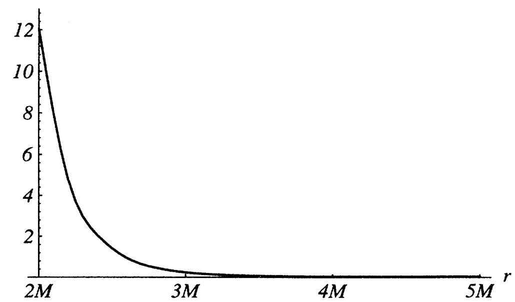


FIG. 11. $(2M)^6(1-2M/r)^5\langle|\dot{\Psi}_0|^2\rangle^{H-B}\times 10^3 = (2M)^6 16(1-2M/r)\langle|\dot{\Psi}_4|^2\rangle^{H-B} \times 10^3$. The value of this combination at $r = 2M$ is 12.1 and its asymptotic value at infinity is 0.158.

rating black hole and $\langle|\dot{\Psi}_A|^2\rangle^{H-B}$ (Fig. 11) corresponding to a black hole in thermal equilibrium at the Hawking temperature. The most striking feature of these graphs is the very rapid decline in vacuum activity with $r$; this is even more pronounced when the asymptotic scaling (which softens the effect) is removed. This suggests that quantum gravitational effects may play a highly significant role in determining the back reaction near the horizon even though the asymptotic flux to infinity measured by $\dot{M}$ is less than that due to lower spin fields.

## VIII. CONCLUSION

In Sec. IV we chose to apply Leaver's method directly to the Teukolsky equation, this has the great virtue of directness. We now briefly mention an alternative procedure which we considered employing but rejected. This alternative is to work with the Regge-Wheeler equation

$$\left[\frac{d^2}{dr_*{}^2} + \omega^2 - U_{l\omega}\right] F_{l\omega} = 0, \tag{8.1}$$

where

$$U_{l\omega} = \frac{r-2M}{r}\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right). \tag{8.2}$$

Equation (8.1) has independent solution $F^{\rm in}$ and $F^{\rm up}$ with the asymptotic behavior

$$F^{\rm in}_{l\omega} \sim \begin{cases} b_{l\omega}e^{-i\omega r_*}, & \text{as } r \to 2M, \\ e^{-i\omega r_*} + a^{\rm in}_{l\omega}e^{+i\omega r_*}, & \text{as } r \to \infty\ , \end{cases} \tag{8.3}$$

and

$$F^{\rm up}_{l\omega} \sim \begin{cases} e^{+i\omega r_*} + a^{\rm up}_{l\omega}e^{-i\omega r_*}, & \text{as } r \to 2M, \\ b_{l\omega}e^{+i\omega r_*}, & \text{as } r \to \infty. \end{cases} \tag{8.4}$$

Defining the complex differential operator

$$\overrightarrow{\mathcal{O}} \equiv 2r(r-3M+i\omega r^2)\left(\frac{d}{dr_*} + i\omega\right) + r^3 U_{l\omega}, \tag{8.5}$$

we can then write the Teukolsky functions in terms of $F^{\rm in}$ and $F^{\rm up}$ as

$${}_{-2}R^{\rm in}_{l\omega} = -\frac{4\omega^2}{C_{l\omega}{}^*}\overrightarrow{\mathcal{O}}F^{\rm in}_{l\omega}, \tag{8.6a}$$

$${}_{-2}R^{\rm up}_{l\omega} = \frac{(2M)^3}{2|C_{l\omega}|^2}(1+2iM\omega)(1+4iM\omega)\overrightarrow{\mathcal{O}}F^{\rm up}_{l\omega}, \tag{8.6b}$$

$${}_{+2}R^{\rm in}_{l\omega} = 16\omega^2\,\Delta^{-2}\overrightarrow{\mathcal{O}}^*F^{\rm in}_{l\omega}, \tag{8.6c}$$

$${}_{+2}R^{\rm up}_{l\omega} = \frac{(2M)^3}{C_{l\omega}{}^*}(1+2iM\omega)(1+4iM\omega)\,\Delta^{-2}\overrightarrow{\mathcal{O}}^*F^{\rm up}_{l\omega}. \tag{8.6d}$$

The apparent advantage of the Regge-Wheeler approach is that it deals with a real equation similar to the spin-0 equation. However, the three term recursion relation obtained through Leaver's approach is still complex so no real advantage is accrued. In addition, the expressions for the Weyl scalars involve derivatives of the Regge-Wheeler functions which considerably complicates and obscures the asymptotic analysis required as compared with that employed in the more direct approach we have followed.

Having argued that there is no advantage in using the Regge-Wheeler formalism for *our* calculation, we should point out that some of the asymptotic formulas involving Teukolsky transmission and reflection coefficients look far neater in terms of Regge-Wheeler transmission and reflection coefficients. For example, it can be shown that

$$|b_{l\omega}|^2 = \frac{(2M)^5 p_\omega}{4\omega^5}|B^{\rm in}_{l\omega}|^2 = \frac{4\omega^5}{(2M)^5 p_\omega}|B^{\rm up}_{l\omega}|^2, \tag{8.7}$$

and so, Eq. (5.9) takes the form

$$\dot{M}_\omega = 2\times\frac{1}{2\pi}\sum_{l=2}^{\infty}(2l+1)\frac{\omega\,|b_{l\omega}|^2}{(e^{8\pi M\omega}-1)}\quad, \tag{8.8}$$

which parallels the scalar result except that the sum starts at $l=2$, the lowest radiative mode of the graviton field, and there is the extra factor of 2 arising from the number of polarization states for the graviton.

One might view the research presented here as a precursor to the numerical evaluation of the renormalized effective energy-momentum tensor for quantized linear gravitational perturbations of a black hole. However, there are severe problems of gauge invariance in defining such an object. One approach to this problem, using the Vilkovisky-DeWitt off-shell gauge-invariant effective action, was suggested in Ref. [22]. Since both the Vilkovisky-DeWitt effective energy-momentum tensor $\langle T_\mu{}^\nu\rangle$ and $\langle|\dot{\Psi}_A|^2\rangle$ consist of terms which are quadratic in the metric perturbation $h_{\mu\nu}$ and its derivatives (compare Eqs. (3.3) of this paper with Eq. (2.6) of Ref. [22]), the evaluation of the *differences* in the renormalized expectation values $\langle T_\mu{}^\nu\rangle^{H-U}$ and $\langle T_\mu{}^\nu\rangle^{U-B}$ will be a straightforward (though laborious) extension of the calculation outlined in this paper. However, if one wishes to compute an individual renormalized expectation value ($\langle H|T_\mu{}^\nu|H\rangle_R$ say) then new ground must be broken, since in this case explicit renormalization is necessary. Problems then arise since the graviton renormalization scheme of Ref. [22] can be only implemented in deDonder gauge $h^\mu{}_{\nu;\mu} - \frac{1}{2}h^\mu{}_{\mu;\nu} = 0$ (where the propagator has Hadamard form), whereas a complete set of solutions to the linearized field equations currently exists only in radiation gauge [see Eqs. (3.5)]. If these technical problems can be resolved, it would be interesting to compare this quantity with its scalar and electromagnetic analogues. To date it has only been possible to make quantitative comparisons of the black hole luminosity of gravitons with that due to radiation of massless particles of lower spin [9]. One finds that the luminosity for gravitons is less than for lower spin fields but one expects that this is due to the increase in the height of the effective potential barrier with spin and so cannot be used to draw conclusions about their importance near the black hole. Indeed, we expect that as in the case of quantum fields propagating near a conical singularity [1], the contribution of gravitons will be seen to dominate the back reaction near a spherical singularity (see also the remarks at the end of Sec. VII).

In this paper we have chosen to sidestep these gauge-invariance problems and instead, following Ref. [8], have concentrated on the gauge-independent Newman-Penrose scalars $\dot{\Psi}_0$ and $\dot{\Psi}_4$. These scalars are in some sense the true gravitational field variables in the Schwarzschild background, for example, it is in principle possible to reformulate the quantum theory at the level of the one-loop effective action in terms of $\dot{\Psi}_0$ and $\dot{\Psi}_4$ (strictly speaking, $\dot{\Psi}_0$ may be regarded as representing the two radiative degrees of freedom and as acting as a "superpotential" from which $\dot{\Psi}_4$ may be obtained). Most importantly, the expectation values we have computed provide important physical measures of the vacuum activity of the quantized gravitational field around a black hole and, in the asymptotic regimes, they directly measure the one-loop quantum gravitational energy flux across the horizon of the black hole and to infinity.

## APPENDIX A: NOTATION AND NP CONVENTIONS

For convenience we repeat here the definition of various symbols used in the text:

$$\triangle = r(r-2M),$$
$$C_{l\omega} = (l-1)l(l+1)(l+2) + 12iM\omega,$$
$$p_\omega = 2M\omega(1+4M^2\omega^2)(1+16M^2\omega^2).$$

The following are the conventions and notation employed in this paper for the NP description of general relativity. In the NP formalism the geometry of a general space-time with metric $g_{\mu\nu}(x)$ is encoded into a null complex tetrad $\{e_{(a)}{}^{\mu}(x) : a = 1,2,3,4\}$ which satisfies the orthonormality conditions

$$g_{\mu\nu} e_{(a)}{}^{\mu} e_{(b)}{}^{\nu} = \eta_{(a)(b)} \quad .$$

$\eta_{(a)(b)}$ is the constant symmetric matrix

$$\eta_{(a)(b)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{(a)(b)}$$

and acts as the NP analogue of the metric tensor; in particular tetrad indices are lowered and raised by $\eta_{(a)(b)}$ and its matrix inverse, denoted $\eta^{(a)(b)}$, respectively. We adopt the convention whereby tensor indices are labeled by Greek letters and tetrad indices by Roman letters enclosed in parentheses.

By expressing tensors $T_{\dots\mu\dots}$ in terms of their *tetrad components* $T_{\dots(a)\dots} = e_{(a)}{}^{\mu} T_{\dots\mu\dots}$ and introducing the *spin coefficients* (NP analogues of Christoffel symbols)

$$\gamma_{(a)(b)(c)} = e_{(a)}{}^{\beta}{}_{;\gamma}\, e_{(b)\beta}\, e_{(c)}{}^{\gamma} \quad ,$$

the fundamental equations of general relativity (e.g., Einstein's equation, the Bianchi identities) can all be expressed as tetrad equations involving only scalar quantities.

We also adopt the notation (see, e.g., Ref. [23])

$$e_{(a)}{}^{\mu}\partial_\mu \equiv (D, \Delta, \delta, \delta^*)$$

for the tetrad operators,

$$\begin{aligned}
&\kappa \equiv \gamma_{(3)(1)(1)}, \quad \rho \equiv \gamma_{(3)(1)(4)}, \quad \epsilon \equiv \tfrac{1}{2}\left(\gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}\right), \\
&\sigma \equiv \gamma_{(3)(1)(3)}, \quad \mu \equiv \gamma_{(2)(4)(3)}, \quad \gamma \equiv \tfrac{1}{2}\left(\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)}\right), \\
&\lambda \equiv \gamma_{(2)(4)(4)}, \quad \tau \equiv \gamma_{(3)(1)(2)}, \quad \alpha \equiv \tfrac{1}{2}\left(\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)}\right), \\
&\nu \equiv \gamma_{(2)(4)(2)}, \quad \pi \equiv \gamma_{(2)(4)(1)}, \quad \beta \equiv \tfrac{1}{2}\left(\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)}\right),
\end{aligned}$$

for the spin coefficients, and

$$\begin{aligned}
&\Psi_0 \equiv -C_{(1)(3)(1)(3)}, \quad \Psi_1 \equiv -C_{(1)(2)(1)(3)}, \\
&\Psi_2 \equiv -C_{(1)(3)(4)(2)}, \\
&\Psi_3 \equiv -C_{(1)(2)(4)(2)}, \quad \Psi_4 \equiv -C_{(2)(4)(2)(4)},
\end{aligned}$$

for the five independent tetrad components of the Weyl tensor, called the *Weyl scalars.*

## APPENDIX B: SPIN-WEIGHTED SPHERICAL HARMONICS

*Spin-weighted spherical harmonics* ${}_sY_l^m(\theta,\phi)$ are defined for $s = -l, -l+1, \dots, l-1, l$ by the set of equations (Refs. [11,24])

$${}_0Y_l^m(\theta,\phi) = Y_l^m(\theta,\phi), \tag{B1a}$$

$${}_{s+1}Y_l^m(\theta,\phi) = [(l-s)(l+s+1)]^{-\frac{1}{2}}\, \eth\, {}_sY_l^m(\theta,\phi), \tag{B1b}$$

$${}_{s-1}Y_l^m(\theta,\phi) = -\,[(l+s)(l-s+1)]^{-\frac{1}{2}}\, \eth'\, {}_sY_l^m(\theta,\phi), \tag{B1c}$$

in terms of the ordinary spherical harmonics

$$Y_l^m(\theta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{\frac{1}{2}} P_l^{|m|}(\cos\theta)\, e^{im\phi}.$$

Here $\eth$ and $\eth'$ are operators which act as follows on a quantity $\eta$ of spin-weight $s$:

$$\eth\,\eta = -\sin^s\theta\left[\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right]\big((\sin^{-s}\theta)\,\eta\big), \quad \text{(B2a)}$$

$$\eth'\eta = -\sin^{-s}\theta\left[\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right]\big((\sin^{s}\theta)\,\eta\big). \quad \text{(B2b)}$$

The application of $\eth$ on a quantity lowers the spin weight by 1: the application of $\eth'$ on a quantity raises the spin weight by 1. It follows immediately from (B1) that

$$\eth'\eth\,{}_sY_l^m(\theta,\phi) = -(l-s)(l+s+1)\,{}_sY_l^m(\theta,\phi) \quad \text{(B3)}$$

or, substituting (B2) for $\eth$ and $\eth'$, that

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}+\frac{2is\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi}\right.$$
$$\left.-s^2\cot^2\theta+s+(l-s)(l+s+1)\right]{}_sY_l^m(\theta,\phi)=0.$$

The spin-weighted spherical harmonics satisfy the orthonormality and completeness relations (Refs. [11,24])

$$\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\,{}_sY_l^m(\theta,\phi)\,{}_sY_{l'}^{m'*}(\theta,\phi)=\delta_{ll'}\delta_{mm'} \quad \text{(B4)}$$

and

$$\sum_{l=s}^{\infty}\sum_{m=-l}^{l}{}_sY_l^m(\theta,\phi)\,{}_sY_l^{m*}(\theta',\phi') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta'), \quad \text{(B5)}$$

respectively; these follow essentially by induction on (B1) and the corresponding relations for the ordinary spherical harmonics. Using the same method one can verify that the "addition theorem"

$$\sum_{m=-l}^{l}|\,{}_sY_l^m(\theta,\phi)|^2=\frac{2l+1}{4\pi} \quad \text{(B6)}$$

holds for $s=\pm1,\pm2$ as well as $s=0$ [25].

## APPENDIX C: ASYMPTOTIC EXPANSIONS OF THE RADIAL FUNCTIONS AT INFINITY

Based on (3.7a), ${}_{-2}R^{\rm in}_{l\omega}$ will have an asymptotic expansion of the form

$${}_{-2}R^{\rm in}_{l\omega}\sim e^{-i\omega r_*}\left\{\frac{1}{r}+O\left(\frac{1}{r^2}\right)\right\} +A^{\rm in}_{l\omega}e^{+i\omega r_*}\left\{r^3+b_1r^2+b_2r+b_3 +\frac{b_4}{r}+O\left(\frac{1}{r^2}\right)\right\}. \quad \text{(C1)}$$

The coefficients $b_n(l,\omega)$ are determined by substituting this expansion into the $s=-2$ radial equation (2.6) and equating the coefficient of each power of $r$ to zero; we find

$$b_1=\frac{(l-1)(l+2)i}{2\omega},\quad b_2=-\frac{C^*_{l\omega}}{8\omega^2},$$

$$b_3=-\frac{l(l+1)iC^*_{l\omega}}{48\omega^3},\quad b_4=\frac{|C_{l\omega}|^2}{384\omega^4}.$$

Next by inserting (C1) into (3.10) we deduce that

$${}_{+2}R^{\rm in}_{l\omega}\sim 64\omega^4\frac{e^{-i\omega r_*}}{r}+\frac{|C_{l\omega}|^2}{4\omega^4}A^{\rm in}_{l\omega}\frac{e^{+i\omega r_*}}{r^5} \quad \text{(C2)}$$

in the limit as $r\to\infty$. A similar argument yields the following asymptotic expansions for the $s=+2$ radial functions:

$${}_{+2}R^{\rm up}_{l\omega}\sim B^{\rm up}_{l\omega}e^{+i\omega r_*}\left\{\frac{1}{r^5}+\frac{c_1}{r^6}+\frac{c_2}{r^7}+\frac{c_3}{r^8}+\frac{c_4}{r^9} +O\left(\frac{1}{r^{10}}\right)\right\}, \quad \text{(C3)}$$

$${}_{-2}R^{\rm up}_{l\omega}\sim B^{\rm up}_{l\omega}e^{+i\omega r_*}\left\{d_1r^3+d_2r^2+d_3r+d_4+O\left(\frac{1}{r}\right)\right\}, \quad \text{(C4)}$$

as $r\to\infty$, where

$$c_1=\frac{i}{2\omega}\left[(l-2)(l+3)\right],$$

$$c_2=\frac{i}{4\omega}\left[\{(l-3)(l+4)-4M\omega i\}c_1+2M\{15-(l-2)(l+3)\}\right],$$

$$c_3=\frac{i}{6\omega}\left[\{(l-4)(l+5)-8M\omega i\}c_2+2M\{30-(l-2)(l+3)\}c_1-15(2M)^2\right],$$

$$c_4=\frac{i}{8\omega}\left[\{(l-5)(l+6)-12M\omega i\}c_3+2M\{49-(l-2)(l+3)\}c_2-24(2M)^2c_1\right],$$

and

$$d_1 = \frac{4\omega^4}{|C_{l\omega}|^2}, \quad d_2 = \frac{2(l-1)(l+2)i\omega^3}{|C_{l\omega}|^2},$$
$$d_3 = -\frac{\omega^2}{2C_{l\omega}}, \quad d_4 = -\frac{l(l+1)i\omega}{12C_{l\omega}}.$$

Note that (C2) and (C4) are consistent with (2.9).

## APPENDIX D: RELATIONS BETWEEN THE REFLECTION AND TRANSMISSION COEFFICIENTS

Equation (2.7) gives the spin $s$ radial equation in Liouville normal form. The Wronskian

$$W[{}_sQ_{l\omega}(r), {}_s\tilde{Q}_{l\omega}(r)]$$

$$\equiv {}_sQ_{l\omega}(r)\frac{d}{dr_*}{}_s\tilde{Q}_{l\omega}(r) - {}_s\tilde{Q}_{l\omega}(r)\frac{d}{dr_*}{}_sQ_{l\omega}(r)$$

of any two particular solutions ${}_sQ_{l\omega}(r), {}_s\tilde{Q}_{l\omega}(r)$ of the equation will be constant. In addition, observe from (2.8) that ${}_{-s}V^*_{l\omega}(r) = {}_sV_{l\omega}(r)$; thus if ${}_sQ_{l\omega}(r)$ is a solution of (2.7), so is ${}_{-s}Q^*_{l\omega}(r)$. Hence

$$W\left[{}_sQ_{l\omega}(r),\ {}_{-s}Q^*_{l\omega}(r)\right]$$

will also be constant. These facts can be used to derive relations between the amplitudes $A^{\Lambda}_{l\omega}$ and $B^{\Lambda}_{l\omega}$.

Consider, for example,

$$W\left[{}_{-2}Q^{\rm in}_{l\omega}(r),\ {}_{-2}Q^{\rm up}_{l\omega}(r)\right],$$

where we are using the obvious notation

$${}_sQ^{\Lambda}_{l\omega}(r) \equiv r\triangle^{\frac{s}{2}}\,{}_sR^{\Lambda}_{l\omega}(r). \tag{D1}$$

This Wronskian is first evaluated on the horizon, by substituting the power series expansions (4.7a) and (4.7d) for ${}_{-2}R^{\rm in}_{l\omega}$ and ${}_{-2}R^{\rm up}_{l\omega}$, respectively, computing the derivatives, and then taking the $r \to 2M$ limit: we find

$$\lim_{r\to 2M} W\left[{}_{-2}Q^{\rm in}_{l\omega}(r),\ {}_{-2}Q^{\rm up}_{l\omega}(r)\right] = \frac{2i(2M)^5 p_\omega B^{\rm in}_{l\omega}}{|C_{l\omega}|^2}, \tag{D2}$$

where $p_\omega$ is given by (3.18). Next we evaluate the same Wronskian in the limit as $r \to \infty$, using the asymptotic expansions (C1) and (C4) derived in Appendix C for ${}_{-2}R^{\rm in}_{l\omega}$, ${}_{-2}R^{\rm up}_{l\omega}$, respectively. The result is

$$\lim_{r\to\infty} W\left[{}_{-2}Q^{\rm in}_{l\omega}(r),\ {}_{-2}Q^{\rm up}_{l\omega}(r)\right] = \frac{8i\omega^5 B^{\rm up}_{l\omega}}{|C_{l\omega}|^2}. \tag{D3}$$

Constancy of the Wronskian therefore yields the following relation between the "up" and "in" transmission amplitudes:

$$B^{\rm up}_{l\omega} = \frac{(2M)^5 p_\omega}{4\omega^5} B^{\rm in}_{l\omega}. \tag{D4}$$

A similar analysis of the Wronskians $W\left[{}_{-2}Q^{\rm in}_{l\omega}(r),\ {}_{+2}Q^{\rm in*}_{l\omega}(r)\right]$ and $W\left[{}_{-2}Q^{\rm in}_{l\omega}(r),\ {}_{+2}Q^{\rm up*}_{l\omega}(r)\right]$ serves to complete the set of independent relations between the reflection and transmission coefficients, yielding

$$\frac{|C_{l\omega}|^2}{(2\omega)^8}\left|A^{\rm in}_{l\omega}\right|^2 + \frac{(2M)^5 p_\omega}{4\omega^5}\left|B^{\rm in}_{l\omega}\right|^2 = 1 \tag{D5}$$

and

$$(-2i\omega)A^{\rm in}_{l\omega}B^{\rm up*}_{l\omega} + 4M\left(1 - 2iM\omega\right)B^{\rm in}_{l\omega}A^{\rm up*}_{l\omega} = 0, \tag{D6}$$

respectively. All other relations between reflection and transmission coefficients follow from the above three.

Equation (D5) may also be derived from conservation of energy flux,

$$\frac{dE^{\rm in}_\omega}{dt} - \frac{dE^{\rm out}_\omega}{dt} = \frac{dE^{\rm hor}_\omega}{dt} \quad . \tag{D7}$$

To see this, first substitute (3.12) for $\dot{\Psi}^{\rm in}_A(l,m,\omega,P;x)$ in (2.15) and (2.16) and integrate over the solid angle; next replace ${}_{\pm 2}R^{\rm in}_{l\omega}$ by formulas describing their leading behavior in the limits $r \to 2M$, $r \to \infty$ as appropriate; finally substitute into the conservation equation above to obtain (D5).

## APPENDIX E: SMALL $\omega$ APPROXIMATIONS FOR $A^{\rm up}_{l\omega}$, $B^{\rm in}_{l\omega}$

In this appendix expressions are derived for the leading behavior of $A^{\rm up}_{l\omega}$ and $B^{\rm in}_{l\omega}$ as $\omega$ tends to zero. Our method is analogous to that employed by Page [9] for his investigation of particle emission rates from a Kerr black hole, and in particular makes use of approximate hypergeometric solutions of the spin $s$ radial equation obtained by Churilov and Starobinskiĭ [20].

In terms of the dimensionless quantities $x = (r/2M - 1)$ and $k = 2M\omega$ , the spin $s$ radial equation (2.6) may be approximated when $k \ll 1$ by

$$\left[x^2(x+1)^2\frac{d^2}{dx^2} + (s+1)x(x+1)(2x+1)\frac{d}{dx} + k^2x^4 + 2iskx^3 - (l-s)(l+s+1)x(x+1) - isk(2x+1) + k^2\right]{}_sR_{l\omega} = 0. \tag{E1}$$

In the region $x \ll (l+1)/k$ (which includes the horizon) the third and fourth terms can be neglected, so that the equation has three regular singular points and its general solution is expressible in terms of hypergeometric functions:

$$ {}_sR_{l\omega} = C_1\, {}_2F_1\left(-l-s, l-s+1; 1-s-2ik; -x\right) x^{-s-ik}(x+1)^{-s+ik} + C_2\, {}_2F_1\left(-l+2ik, l+1+2ik; 1+s+2ik; -x\right)(-1)^s x^{ik}(x+1)^{-s+ik}, \quad \text{(E2)} $$

where $C_1$ and $C_2$ are constants. In the region $x \gg k+1$ (which stretches to infinity) the last two terms of (E1) are ignorable and the following general solution of the approximate equation can be deduced:

$$ {}_sR_{l\omega} = D_1\, {}_1F_1\left(l+1-s, 2l+2; 2ikx\right) e^{-ikx} x^{l-s} + D_2\, {}_1F_1\left(-l-s, -2l; 2ikx\right) e^{-ikx} x^{-l-s-1}. \quad \text{(E3)} $$

One can now match these general solutions in the region of overlap $k+1 \ll x \ll (l+1)/k$. (The hypergeometric functions in (E2) may be approximated when $x \gg k+1$ using Eq. (2) on p. 108 of Ref. [26], and when $x \ll (l+1)/k$ those in (E3) can be simply replaced by 1.) The following matching relations between $C_1, C_2$ and $D_1, D_2$ obtain

$$ D_1 = \frac{\Gamma(2l+1)}{\Gamma(l-s+1)}\frac{\Gamma(1-s-2ik)}{\Gamma(1+l-2ik)}C_1 + (-1)^s \frac{\Gamma(2l+1)}{\Gamma(l+s+1)}\frac{\Gamma(1+s+2ik)}{\Gamma(1+l+2ik)}C_2, \quad \text{(E4a)} $$

$$ D_2 = \frac{\Gamma(-2l-1)}{\Gamma(-l-s)}\frac{\Gamma(1-s-2ik)}{\Gamma(-l-2ik)}C_1 + (-1)^s \frac{\Gamma(-2l-1)}{\Gamma(s-l)}\frac{\Gamma(1+s+2ik)}{\Gamma(-l+2ik)}C_2. \quad \text{(E4b)} $$

Consider now the particular solution ${}_{-2}R^{\rm in}_{l\omega}(r)$ of the $s=-2$ radial equation. First comparing the form of this solution as $r \to 2M$ [see (3.7a)] with that of the general solution (E2) in the same limit, one observes that

$$ C_1 = (2M)^4 B^{\rm in}_{l\omega} \quad \text{and} \quad C_2 = 0 \quad \text{(E5)} $$

for this particular choice. Second comparing the form of ${}_{-2}R^{\rm in}_{l\omega}(r)$ with that of (E3) as $r \to \infty$ (see (3.7a) and Eq. (2), p. 278 of Ref. [26] for the asymptotic expansion of the confluent hypergeometric function ${}_1F_1$) yields

$$ \frac{\Gamma(2l+2)}{\Gamma(l-1)}(-2ik)^{-l-3}D_1 + \frac{\Gamma(-2l)}{\Gamma(-l-2)}(-2ik)^{l-2}D_2 = \frac{1}{2M}, \quad \text{(E6a)} $$

$$ \frac{\Gamma(2l+2)}{\Gamma(l+3)}(2ik)^{-l+1}D_1 + \frac{\Gamma(-2l)}{\Gamma(-l+2)}(2ik)^{l+2}D_2 = (2M)^3 A^{\rm in}_{l\omega}. \quad \text{(E6b)} $$

Eliminating $C_1, C_2, D_1$, and $D_2$ between (E4) (with $s=-2$), (E5) and (E6) results in a pair of equations which determine the incoming reflection and transmission amplitudes; the leading behavior of $B^{\rm in}_{l\omega}$ as $2M\omega \to 0$ is then found to be

$$ B^{\rm in}_{l\omega} \approx (2M)^{-5}\frac{l!}{2!}\frac{(l-2)!(l+2)!}{(2l+1)!(2l)!}(-4iM\omega)^{l+3}. \quad \text{(E7)} $$

A similar comparison of ${}_{+2}R^{\rm up}_{l\omega}$ with the approximate hypergeometric solutions yields

$$ A^{\rm up}_{l\omega} \approx 2(2M)^4\frac{(l-2)!}{(l+2)!}(-4iM\omega) \quad \text{(E8)} $$

for small $2M\omega$.

*Note added in proof.* Since completion of this work, we discovered that this leading behavior and some of the higher order behavior has also been determined by M. Sasaki, Prog. Theor. Phys. **92**, 17 (1994). We have shown our numerical results to be in accord with these higher order terms where they are known.

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