Comparative quantizations of (2+1)-dimensional gravity

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We compare three approaches to the quantization of (2+1)-dimensional gravity with a negative cosmological constant: reduced phase-space quantization with the York time slicing, quantization of the algebra of holonomies, and quantization of the space of classical solutions. The relationships among these quantum theories allow us to define and interpret time-dependent operators in the "frozen time" holonomy formulation.

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I. INTRODUCTION

Over the past few years, there has been a growing interest in (2+1)-dimensional quantum gravity as a simple model for realistic (3+1)-dimensional quantum gravity. As a generally covariant theory of spacetime geometry, general relativity in 2+1 dimensions has the same conceptual foundations as ordinary (3+1)-dimensional gravity. But the reduction in the number of dimensions greatly simplifies the structure of the theory, reducing the infinite number of physical degrees of freedom of ordinary general relativity to a finite number of global degrees of freedom. The model thus allows us to explore the conceptual problems of quantum gravity within the framework of ordinary quantum mechanics, avoiding such issues as nonrenormalizability associated with field degrees of freedom.

A number of different approaches to quantizing (2+1)-dimensional general relativity have been developed recently. These include reduced phase-space quantization with Arnowitt-Deser-Misner (ADM) variables [1–3], quantization of the space of classical solutions of the first-order Chern-Simons theory [4–7], and quantization of the holonomy algebra [8–15]. Each approach has its strengths and weaknesses. ADM quantization, for example, leads to states and operators with clear physical interpretations but depends on an arbitrary classical choice of time slicing, breaking manifest covariance. Quantization of the space of solutions involves no such choice, but requires a detailed understanding of the classical solutions. Quantization of the holonomy algebra is also manifestly covariant, and reveals important underlying algebraic structures, but the physical interpretation of the resulting operators is unclear.

The goal of this paper is to explore the relationships among these three methods of quantization. Such comparisons have been made in the past [6,16–19], but the powerful holonomy algebra approach has not generally been considered. We shall see below that quantization of the space of solutions (sometimes called "covariant canonical quantization") provides a natural bridge between the ADM and holonomy algebra approaches, allowing one to introduce "time"-dependent physical operators into the latter formalism.

The structure of the paper is as follows. In Sec. II, we discuss the first- and second-order formulations of classical general relativity, solving the constraints and introducing the basic physical variables in each approach. In Sec. III, we describe the classical solutions for spacetimes with the topology $\mathbb{R}\times T^2$, focusing on the case of a negative cosmological constant but also discussing the $\Lambda\rightarrow 0$ limit and briefly considering the $\Lambda>0$ case. In Sec. IV, we describe the three methods of quantization and explore their relationships. Our results are summarized in Sec. V.

A preliminary report on aspects of this work has appeared in [20]. Parts of our discussion of classical solutions and our comparison of ADM and Chern-Simons quantization were found independently by Ezawa [18,21], who also discusses the $\Lambda>0$ case in more detail.

II. CLASSICAL THEORIES

To understand the quantization of (2+1)-dimensional gravity, it is first necessary to understand the classical theory. Classical general relativity has two very different formulations: the second-order form, in which the metric is the only fundamental variable, and the first-order form, in which the metric and the connection (or spin connection) are treated independently. As we shall see, these two formulations lead naturally to two different approaches to quantization.

The fundamental feature of classical general relativity in 2+1 dimensions is that the full Riemann curvature tensor depends linearly on the Ricci tensor. As a re-

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sult, the empty space field equations $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ imply that spacetime has constant curvature, that is, that every point has a neighborhood isometric to a neighborhood of de Sitter, Minkowski, or anti–de Sitter space. For a topologically trivial spacetime, this condition eliminates all degrees of freedom. For a spacetime with nontrivial topology, however, there remain a finite number of degrees of freedom that describe the gluing of constant curvature patches around noncontractible curves (see [22] for a more detailed description). It is these degrees of freedom that we shall eventually quantize.

A. ADM formalism

The most traditional approach to classical gravity in 2+1 dimensions begins with the ADM decomposition of the standard second-order form of the Einstein action. This approach has been discussed in some detail by Moncrief [2] and Hosoya and Nakao [1]; in this section, we establish the notation and briefly summarize their results.

Assume that spacetime has the topology $\mathbb{R}\times\Sigma$, where Σ is a closed genus g surface. The Einstein action is then

$$\begin{aligned} I_{\text{Ein}} &= \int d^3x\sqrt{-{}^{(3)}g}({}^{(3)}R - 2\Lambda) \\ &= \int dt \int_\Sigma d^2x(\pi^{ij}\dot{g}_{ij} - N^i\mathcal{H}_i - N\mathcal{H}) \ , \end{aligned} \tag{2.1}$$

where the metric has been decomposed as

$$ds^2 = N^2dt^2 - g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.2}$$

and $\pi^{ij} = \sqrt{g}(K^{ij} - g^{ij}K)$, where K^{ij} is the extrinsic curvature of the surface[1] t =const. The supermomentum and super-Hamiltonian constraints in 2+1 dimensions are

$$\begin{aligned} \mathcal{H}_i &= -2\nabla_j\pi^j{}_i \ , \\ \mathcal{H} &= \frac{1}{\sqrt{g}}g_{ij}g_{kl}(\pi^{ik}\pi^{jl} - \pi^{ij}\pi^{kl}) - \sqrt{g}(R - 2\Lambda) \ . \end{aligned} \tag{2.3}$$

A convenient coordinate choice is the York time slicing [23], in which the mean (extrinsic) curvature is used as a time coordinate, $K = \pi/\sqrt{g} = \tau$. In Ref. [2], Moncrief shows that this is a good global coordinate choice for classical solutions of the field equations.

To solve the constraints, we start with a convenient parametrization of the metric and momentum. Any two-metric on Σ can be written up to a diffeomorphism as [24]

$$g_{ij} = e^{2\lambda}\bar{g}_{ij}(m_\alpha) \ , \tag{2.4}$$

where the $\bar{g}_{ij}(m_\alpha)$ are a finite-dimensional family of metrics of constant curvature, labeled by a set of moduli m_α. For the torus, for instance, we can write $\bar{g}_{ij} = \bar{g}_{ij}(m)$, where $m = m_1 + im_2$ (with $m_2 > 0$) is a complex number, the modulus.[2] Concretely, the spatial metric corresponding to a given m is

$$d\sigma^2 = m_2^{-1}|dx + m\,dy|^2 \ , \tag{2.5}$$

where x and y each have period 1. A similar decomposition of the π^{ij} gives

$$\begin{aligned} \pi^{ij} = e^{-2\lambda}\sqrt{\bar{g}}\Big(p^{ij} &+ \tfrac{1}{2}\bar{g}^{ij}\pi/\sqrt{g} + \bar{\nabla}^iY^j \\ &+ \bar{\nabla}^jY^i - \bar{g}^{ij}\bar{\nabla}_kY^k\Big) \ , \end{aligned} \tag{2.6}$$

where $\bar{\nabla}_i$ is the covariant derivative for the connection compatible with $\bar{g}_{ij}$, indices are now raised and lowered with $\bar{g}_{ij}$, and p^{ij} is a transverse traceless tensor with respect to $\bar{\nabla}_i$, i.e., $\bar{\nabla}_ip^{ij} = 0$. In the language of Riemann surfaces, p^{ij} is a holomorphic quadratic differential; the space of such differentials parametrizes the cotangent space of the moduli space [24].

The momentum constraints now imply that $Y^i = 0$, which the Hamiltonian constraint

$$\begin{aligned} \mathcal{H} = -\tfrac{1}{2}\sqrt{\bar{g}}e^{2\lambda}(\tau^2 - 4\Lambda) &+ \sqrt{\bar{g}}e^{-2\lambda}p^{ij}p_{ij} \\ &+ 2\sqrt{\bar{g}}\Big[\bar{\Delta}\lambda - \tfrac{1}{2}\bar{R}\Big] = 0 \end{aligned} \tag{2.7}$$

uniquely determines λ as a function of $\bar{g}_{ij}$ and p^{ij} [2]. The action (2.1) reduces to

$$\begin{aligned} I_{\text{Ein}} &= \int d\tau\left(p^\alpha\frac{dm_\alpha}{d\tau} - H(m,p,\tau)\right) \ , \\ H &= \int_\Sigma \sqrt{g}\,d^2x = \int_\Sigma e^{2\lambda(m,p,\tau)}\sqrt{\bar{g}}\,d^2x, \end{aligned} \tag{2.8}$$

where the p^α are momenta conjugate to the moduli, i.e.,

$$\{m_\alpha, p^\beta\} = \delta_\alpha^\beta \ , \tag{2.9}$$

and $\lambda(m,p,\tau)$ is determined by (2.7). Three-dimensional gravity is thus reduced to a finite-dimensional system, albeit one with a complicated and time-dependent Hamiltonian.

This system simplifies further when Σ is a torus. The Poisson brackets become

$$\{m,\bar{p}\} = \{\bar{m},p\} = 2, \quad \{m,p\} = \{\bar{m},\bar{p}\} = 0 \tag{2.10}$$

and the Hamiltonian reduces to

$$H = (\tau^2 - 4\Lambda)^{-1/2}[m_2^2p\bar{p}]^{1/2} \ . \tag{2.11}$$

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[1] We use standard ADM notation: g_{ij} and R refer to the induced metric and scalar curvature of a time slice, while the spacetime metric and curvature are denoted ${}^{(3)}g_{\mu\nu}$ and ${}^{(3)}R$.

[2] In the mathematics literature, the modulus is usually denoted by τ. Following Moncrief [2], however, we have already used τ to denote the York time coordinate.

The momentum-dependent term in H may be recognized as the square of the momentum with respect to the Poincaré (constant negative curvature) metric on the torus moduli space:

$$\frac{dm\,d\bar{m}}{m_2^2} \ . \tag{2.12}$$

By construction, the moduli m_α and momenta p^α are invariant under diffeomorphisms that can be obtained by exponentiating infinitesimal transformations. For a spacetime with the topology $\mathbb{R} \times \Sigma$, however, there are also "large" diffeomorphisms generated by Dehn twists, that is, by the operation of cutting open a handle, twisting one end by 2π, and regluing the cut edges. The set of equivalence classes of such large diffeomorphisms (modulo diffeomorphisms that can be deformed to the identity) is known as the mapping class group of Σ, or, for the torus, the modular group.

For the torus, in particular, the mapping class group is generated by two Dehn twists, corresponding to the two independent circumferences γ_1 and γ_2, which we choose to have intersection number $+1$. These act on $\pi_1(T^2)$ by

$$\begin{aligned} S&:\quad \gamma_1 \to \gamma_2^{-1}, \quad \gamma_2 \to \gamma_1 \ , \\ T&:\quad \gamma_1 \to \gamma_1 \cdot \gamma_2, \quad \gamma_2 \to \gamma_2, \end{aligned} \tag{2.13}$$

where the dot in the last line of (2.13) represents composition of curves, or multiplication of homotopy classes. These transformations induce the modular transformations

$$\begin{aligned} S&:\quad m \to -\frac{1}{m}, \quad p \to \bar{m}^2 p \ , \\ T&:\quad m \to m+1, \quad p \to p \ , \end{aligned} \tag{2.14}$$

which may be seen to preserve the Poincaré metric (2.12) and the Poisson brackets (2.10).

Classically, observables should presumably be invariant under all spacetime diffeomorphisms, including those in the mapping class group. Quantum mechanically, this condition may be relaxed, but operators and wave functions should still transform under some unitary representation of the mapping class group. This restriction will be important when we discuss quantization.

B. First-order formalism

Rather than starting with the metric as the fundamental variable, we may instead write the Einstein action in first-order form, treating the triad one-form (or coframe) $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^{ab} = \omega^{ab}{}_\mu dx^\mu$ as independent variables.[3] This leads to the first-order, connection approach to (2+1)-dimensional gravity, inspired by Witten [4] (see also [25]) and developed by Nelson, Regge, and Zertuche [8–13]. The triad e^a is related to the metric of the preceding section through

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \ , \tag{2.15}$$

and the action (2.1) becomes

$$I_{\text{Ein}} = \int \left(d\omega^{ab} - \omega^a{}_d \wedge \omega^{db} + \frac{\Lambda}{3} e^a \wedge e^b \right) \wedge e^c \epsilon_{abc} \ , \qquad a,b,c = 0,1,2 \ . \tag{2.16}$$

For $\Lambda \neq 0$, this action can be written (up to a total derivative) in the Chern-Simons form

$$I_{\text{CS}} = -\frac{\alpha}{4} \int (d\omega^{AB} - \tfrac{2}{3}\omega^A{}_E \wedge \omega^{EB}) \wedge \omega^{CD} \epsilon_{ABCD} \ , \qquad A,B,C = 0,1,2,3 \tag{2.17}$$

with an (anti–)de Sitter spin connection ω^{AB} determined as follows.

Let k denote the sign of Λ, and set $\Lambda = k\alpha^{-2}$. Let $\sqrt{k}$ mean $+1$ for $k=1$ and $+i$ for $k=-1$. Define the tangent space metric as $\eta_{AB} = (-1,1,1,k)$ and the Levi-Civita density as $\epsilon_{abc3} = -\epsilon_{abc}$. Now incorporate the triads by setting $e^a = \alpha\omega^{a3}$, that is,

$$\omega^A{}_B = \begin{pmatrix} \omega^a{}_b & \dfrac{k}{\alpha} e^a \\ -\dfrac{1}{\alpha} e^b & 0 \end{pmatrix} \ . \tag{2.18}$$

The curvature two-form $R^{AB} = d\omega^{AB} - \omega^{AC} \wedge \omega_C{}^B$ has components $R^{ab} + \Lambda e^a \wedge e^b$, $R^{a3} = \frac{1}{\alpha} R^a$, where

$$\begin{aligned} R^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b \ , \\ R^a &= de^a - \omega^{ab} \wedge e_b \end{aligned} \tag{2.19}$$

are the ordinary (2+1)-dimensional curvature and torsion forms. The field equations derived from the action (2.17) are simply $R^{AB} = 0$, implying, as in the formalism of the preceding section, that the torsion vanishes everywhere and that the curvature R^{ab} is constant.

In a (2+1)-dimensional splitting of spacetime, the action (2.17) decomposes as

$$I_{\text{CS}} = \frac{\alpha}{4} \int dt \int d^2x\, \epsilon^{ij} \epsilon_{ABCD} (\omega_j^{CD} \dot{\omega}_i^{AB} - \omega_0^{AB} R_{ij}^{CD}) \tag{2.20}$$

(with $\epsilon^{0ij} = -\epsilon^{ij}$), from which the constraints are

$$R^{AB}{}_{ij} = 0 \ , \tag{2.21}$$

equivalent to the conditions $\mathcal{H} = 0$, $\mathcal{H}^i = 0$ of Eq. (2.3). The Poisson brackets can be read off from (2.20) on a $t = \text{const}$ surface Σ:

$$\{\omega_i^{AB}(x), \omega_j^{CD}(y)\} = \frac{k}{2\alpha} \epsilon_{ij} \epsilon^{ABCD} \delta^2(x-y) \ . \tag{2.22}$$

[3]Note that our $\omega = -\omega$ of [4].

The constraints (2.21) imply that the connection ${\omega^{AB}}_i$ is flat. It can therefore be written locally in terms of an SO(3,1)- or SO(2,2)-valued zero-form ψ^{AB} as $d\psi^{AB} = \omega^{AC}{\psi_C}^B$. It is actually more convenient to use the spinor groups $SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})$ [for SO(2,2)] and $SL(2,\mathbb{C})$ [for SO(3,1)]. (Details of the spinor group decomposition can be found in [13].) Define the one-form

$$\Delta(x) = \Delta_i(x)dx^i = \tfrac{1}{4}\omega^{AB}(x)\gamma_{AB} \tag{2.23}$$

for which (2.21) implies that $d\Delta - \Delta \wedge \Delta = 0$. This form of the constraints can be integrated using multivalued $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$ matrices S, which satisfy

$$dS(x) = \Delta(x)S(x) \ . \tag{2.24}$$

The Poisson brackets corresponding to (2.22) are then

$$\begin{aligned} \{\Delta_i^\pm(x), \Delta_j^\pm(y)\} &= \pm\frac{i}{2\alpha\sqrt{k}}\epsilon_{ij}\sigma^m \otimes \sigma^m \delta^2(x-y) \ , \\ \{\Delta_i^+(x), \Delta_j^-(y)\} &= 0 \ , \end{aligned} \tag{2.25}$$

where the σ^m are Pauli matrices and the $\pm$ refer to the decomposition of the 4×4 representations of $\Delta(x)$, $S(x)$ into 2×2 irreducible parts (see [13]).

In this approach to (2+1)-dimensional gravity, the existence of nontrivial classical solutions arises from the fact that the $S^\pm$ may be multivalued; that is, the connection ω^{AB} and Δ may have nontrivial holonomies. In particular, let $\gamma : [0,1] \to \Sigma$ be a noncontractible closed curve based at $\gamma(0) = x_0$, and take $S^\pm(\gamma(0)) = 1$ as an initial condition for the differential equation (2.24). Then (2.24) can be integrated to obtain a nontrivial value for $S^\pm(\gamma(1)) = S^\pm[\gamma]$, the $SL(2,\mathbb{R})$ or $SL\ (2,\mathbb{C})$ holonomy of Δ. Note that the flatness of the connection Δ implies that $S^\pm[\gamma]$ depends only on the homotopy class of γ.

The Poisson brackets (2.22) now induce brackets between $S^\pm[\sigma]$ and $S^\pm[\gamma]$, where $\sigma, \gamma \in \pi_1(\Sigma, x_0)$. The matrices $S^\pm[\gamma]$ thus furnish a representation of $\pi_1(\Sigma, x_0)$ in $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$. Under a gauge transformation or a change of base point, the $S^\pm$ transform by conjugation, so their traces provide a (overcomplete) set of gauge-invariant variables.

The classical Poisson brackets for these variables were calculated by hand for the genus 1 and genus 2 cases and then generalized and quantized in [9]. For the genus 1 case, which is the focus of this paper, the Poisson algebra is

$$\{R_1^\pm, R_2^\pm\} = \mp\frac{i}{4\alpha\sqrt{k}}(R_{12}^\pm - R_1^\pm R_2^\pm) \quad \text{and cyclical permutations,} \tag{2.26}$$

where $R^\pm = \frac{1}{2}\mathrm{Tr}S^\pm$. Here the subscripts 1 and 2 refer to the two independent intersecting circumferences γ_1, γ_2 on Σ with intersection number[4] $+1$, while the third holonomy $R_{12}^\pm$ corresponds to the path $\gamma_1 \cdot \gamma_2$, which has intersection number -1 with γ_1 and $+1$ with γ_2. Observe that the algebra (2.26) is invariant under the modular transformations (2.13).

Classically, the six holonomies $R_{1,2,12}^\pm$ provide an overcomplete description of the spacetime geometry of $\mathbb{R}\times T^2$, which, as we saw in the preceding section, is completely characterized by four real or two complex parameters m and p. To understand this overcompleteness, consider the cubic polynomials

$$\begin{aligned} F^\pm &= 1 - (R_1^\pm)^2 - (R_2^\pm)^2 - (R_{12}^\pm)^2 + 2R_1^\pm R_2^\pm R_{12}^\pm \\ &= \frac{1}{2}\mathrm{Tr}\left(I - S^\pm[\gamma_1]S^\pm[\gamma_2]S^\pm[\gamma_1^{-1}]S^\pm[\gamma_2^{-1}]\right) \ , \end{aligned} \tag{2.27}$$

where the last equality follows from the identities

$$A + A^{-1} = I\,\mathrm{Tr}A$$

for 2×2 unimodular matrices A. These polynomials have vanishing Poisson brackets with all of the traces $R_a^\pm$, are cyclically symmetric in the $R_a^\pm$, and are invariant under modular transformations. The overcompleteness of our description arises because the $F^\pm$ vanish classically by the $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$ Mandelstam identities, which can be viewed as the application of the group identity

$$\gamma_1 \cdot \gamma_2 \cdot \gamma_1^{-1} \cdot \gamma_2^{-1} = I$$

to the representations $S^\pm$ occurring in the last line of (2.27).

In the first-order approach, the constraints have now been solved exactly. There is no Hamiltonian, however, and no time development. One can think of this formalism as initial data for some (unspecified) choice of time, or alternatively as giving a time-independent description of the entire spacetime geometry.

III. CLASSICAL SOLUTIONS

Before turning to quantization, it is useful to explore the structure of the classical solutions of (2+1)-dimensional gravity in more detail. We shall concentrate on spacetimes with the topology $\mathbb{R} \times T^2$, for which the classical solutions are completely understood, and shall specialize to the case $\Lambda < 0$, briefly discussing the corresponding picture for $\Lambda \geq 0$ at the end of this section. Many of the results presented here have been discovered independently by Ezawa [8,21], and related solutions were found by Fujiwara and Soda [3].

An obvious starting point is the ADM formalism of Sec. II A. Rather than beginning with the York time slicing, however, it is somewhat easier to choose a "time gauge," in which $N = 1$ and $N^i = 0$. Equivalently, in the first-order form we choose $e_i^0 = 0$ and $e_t^0 = 1$. We shall see below that for the topology $\mathbb{R}\times T^2$ this choice is equivalent to the York gauge, although this is no longer the case for spaces of genus $g > 1$.

It is easy to check that the first-order field equations $R^{AB} = 0$ are then solved by

[4]Paths with intersection number $0, \pm 1$ are sufficient to characterize the holonomy algebra for genus 1. For $g > 1$, one must in general consider paths with two or more intersections, for which the brackets (2.26) are more complicated; see [11,12].

$$
\begin{aligned}
e^0 &= dt\ , \\
e^1 &= \frac{\alpha}{2}\left[(r_1^+ - r_1^-)dx + (r_2^+ - r_2^-)dy\right]\sin\frac{t}{\alpha}\ , \\
e^2 &= \frac{\alpha}{2}\left[(r_1^+ + r_1^-)dx + (r_2^+ + r_2^-)dy\right]\cos\frac{t}{\alpha},
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\omega^{12} &= 0\ , \\
\omega^{01} &= -\frac{1}{2}\left[(r_1^+ - r_1^-)dx + (r_2^+ - r_2^-)dy\right]\cos\frac{t}{\alpha}\ , \\
\omega^{02} &= \frac{1}{2}\left[(r_1^+ + r_1^-)dx + (r_2^+ + r_2^-)dy\right]\sin\frac{t}{\alpha}\ ,
\end{aligned}
\tag{3.2}
$$

where $r_1^\pm$ and $r_2^\pm$ are four arbitrary parameters, and the coordinates x and y have period one. The triad (3.1) determines a spacetime metric $g_{\mu\nu} = e^a_\mu e_{a\nu}$, which can be used to compute the moduli and momenta of Sec. II A. In particular, the spatial metric on a slice of constant t describes a torus with modulus

$$
m = \left(r_1^- e^{it/\alpha} + r_1^+ e^{-it/\alpha}\right)\left(r_2^- e^{it/\alpha} + r_2^+ e^{-it/\alpha}\right)^{-1}\ .
\tag{3.3}
$$

The conjugate momentum p can be similarly computed from the extrinsic curvature; it takes the form

$$
p = -\frac{i\alpha}{2\sin(2t/\alpha)}\left(r_2^+ e^{it/\alpha} + r_2^- e^{-it/\alpha}\right)^2\ .
\tag{3.4}
$$

Finally, the York time is

$$
\tau = -\frac{d}{dt}\ln\sqrt{g} = -\frac{2}{\alpha}\cot\frac{2t}{\alpha}\ ,
\tag{3.5}
$$

which ranges from $-\infty$ to ∞ as t varies from 0 to $\pi\alpha/2$. Clearly, τ is a monotonic function of t in this range, so a slicing by surfaces of constant t is equivalent to the York slicing by surfaces of constant K, as claimed.

To check the generality of the solution (3.1)–(3.2), observe first that the four parameters $r_a^\pm$ can be chosen arbitrarily, which in turn implies that the modulus m and momentum p of Eqs. (3.3) and (3.4) can take arbitrary values at an initial surface $t = t_0$. This means that we can specify arbitrary initial data $(m(t_0), p(t_0))$ in the ADM formalism. The results of Moncrief [2] and Mess [26] then guarantee that such data determine a unique maximal spacetime—technically, a maximal domain of dependence of the initial surface—and that any such spacetime can be obtained from suitable initial data.[5]

We can obtain additional information about this solution by calculating the $\mathrm{SL}(2,\mathbb{R})$ holonomies of Eq. (2.24), using the decomposition of the spinor group of SO(2,2) described in [13] and Sec. II B. The computation is again straightforward, and gives

$$
\begin{aligned}
R_1^\pm &= \frac{1}{2}\mathrm{Tr}S^\pm[\gamma_1] = \cosh\frac{r_1^\pm}{2}\ , \\
R_2^\pm &= \frac{1}{2}\mathrm{Tr}S^\pm[\gamma_2] = \cosh\frac{r_2^\pm}{2}\ , \\
R_{12}^\pm &= \frac{1}{2}\mathrm{Tr}S^\pm[\gamma_1\cdot\gamma_2] = \cosh\frac{r_1^\pm + r_2^\pm}{2}\ .
\end{aligned}
\tag{3.6}
$$

Conversely, the metric $g_{\mu\nu}$ can be obtained directly from the holonomies by a quotient space construction. Three-dimensional anti–de Sitter (AdS) space is naturally isometric to the group manifold of $\mathrm{SL}(2,\mathbb{R})$. Indeed, anti–de Sitter space can be represented as the submanifold of flat $\mathbb{R}^{2,2}$ [with coordinates (X_1, X_2, T_1, T_2) and metric $dS^2 = dX_1^2 + dX_2^2 - dT_1^2 - dT_2^2$] on which

$$
\det|\mathbf{X}| = 1,\quad \mathbf{X} = \frac{1}{\alpha}\begin{pmatrix} X_1 + T_1 & X_2 + T^2 \\ -X_2 + T_2 & X_1 - T_1 \end{pmatrix}\ ,
\tag{3.7}
$$

i.e., $\mathbf{X} \in \mathrm{SL}(2,\mathbb{R})$. If one allows the S_a^+ to act on $\mathbf{X}$ by left multiplication and the S_a^- to act by right multiplication, it may be shown that the triad (3.1) represents the geometry of the quotient space $\langle S_1^+, S_2^+\rangle\backslash\mathrm{AdS}/\langle S_1^-, S_2^-\rangle$.

Now, the holonomies (3.6) are *not* the most general possible: an $\mathrm{SL}(2,\mathbb{R})$ matrix can have an arbitrary trace, while our solution requires the holonomies to be hyperbolic.[6] The solution (3.1), (3.2) thus represents only one sector in the space of holonomies, the "hyperbolic-hyperbolic" sector, out of nine possibilities [21]. On the other hand, we argued above that (3.1), (3.2) gave the most general solution to the problem of evolution of initial data on a spacelike surface with the topology T^2. These two statements are not, in fact, inconsistent: for solutions with elliptic or parabolic holonomies, the spacetime still has the topology $\mathbb{R}\times T^2$, but the toroidal slices are not spacelike [21]. In particular, the choice of time gauge is only possible when all holonomies are hyperbolic. A similar phenomenon has been investigated in detail by Louko and Marolf [29] in the case $\Lambda = 0$.

A. Classical time evolution

By construction, we know that the traces (3.6) satisfy the nonlinear classical Poisson brackets algebra (2.26). One may easily verify from (3.6) that the central elements (2.27) are identically zero, and that the $r^\pm$ satisfy

$$
\{r_1^\pm, r_2^\pm\} = \mp\frac{1}{\alpha},\quad \{r_a^+, r_b^-\} = 0\ .
\tag{3.8}
$$

These brackets then induce the correct brackets (2.10) for the modulus and momentum (3.3), (3.4), confirming the consistency of the first- and second-order descriptions. Moreover, the correct time dependence of m and p can be obtained through Hamilton's equations,

[5]This is no longer true in the case $\Lambda > 0$ [26]; the resulting ambiguity is discussed briefly in [27,28]. Moreover, as Louko and Marolf have observed [29], if one starts with a solution that is a domain of dependence, it may be possible to find further extensions to regions containing closed timelike curves.

[6]An $\mathrm{SL}(2,\mathbb{R})$ matrix R is called hyperbolic if $|\mathrm{Tr}R| > 2$, parabolic if $|\mathrm{Tr}R| = 2$, and elliptic if $|\mathrm{Tr}R| < 2$.

$$\frac{dp}{d\tau} = \{p, H\}, \quad \frac{dm}{d\tau} = \{m, H\} , \tag{3.9}$$

where the Hamiltonian (2.11) is computed to be

$$\begin{aligned} H = g^{1/2} &= \frac{\alpha^2}{4} \sin \frac{2t}{\alpha} (r_1^- r_2^+ - r_1^+ r_2^-) \\ &= \frac{\alpha}{2\sqrt{\tau^2 - 4\Lambda}} (r_1^- r_2^+ - r_1^+ r_2^-) . \end{aligned} \tag{3.10}$$

Here we assume that $r_1^- r_2^+ - r_1^+ r_2^- > 0$ so that $g^{1/2} > 0$ in the range $t \in (0, \pi\alpha/2)$. This guarantees that the imaginary part m_2 of the modulus is always positive, as it is in the standard description of torus geometry. The evolution in the parameter t can similarly be obtained from the Hamiltonian $H' = (d\tau/dt)H$.

From this evolution, or equivalently from (3.3), it may be seen that the time-dependent moduli m_1, m_2 lie on a semicircle of radius R:

$$(m_1 - c)^2 + m_2^2 = R^2 , \tag{3.11}$$

where

$$c = \frac{r_1^+ r_2^+ - r_1^- r_2^-}{(r_2^+)^2 - (r_2^-)^2}, \quad R^2 = \left(\frac{r_1^+ r_2^- - r_1^- r_2^+}{(r_2^+)^2 - (r_2^-)^2} \right)^2 , \tag{3.12}$$

while the momenta satisfy

$$p(\bar{m} - c) + \bar{p}(m - c) = 0 . \tag{3.13}$$

This agrees with the results of Ref. [3], and clearly illustrates the nontrivial dynamics of the system, which arises even though the full (2+1)-dimensional curvature tensor is everywhere constant. This is *not* a "gauge" effect, but rather reflects the nontrivial, time-dependent identifications needed to construct a torus from patches of anti–de Sitter space.

The standard action of the modular group on the traces (3.6) suggests that the holonomy parameters transform as

$$\begin{aligned} S&: \quad r_1^\pm \to r_2^\pm, \quad r_2^\pm \to -r_1^\pm , \\ T&: \quad r_1^\pm \to r_1^\pm + r_2^\pm, \quad r_2^\pm \to r_2^\pm . \end{aligned} \tag{3.14}$$

It may be checked that these transformations do indeed leave the brackets (3.8) and the Hamiltonian (3.10) invariant, and that they induce the correct modular transformations (2.14) on the moduli and momenta.

The relationships among the moduli and the holonomy parameters $r^\pm$ allows us to write the reduced phase-space action of (2+1)-dimensional gravity in several equivalent forms,

$$\begin{aligned} I_{\text{Ein}} &= \int dt \int d^2x \, \pi^{ij} \dot{g}_{ij} \\ &= \int dt \int d^2x \, 2\epsilon^{ij} \epsilon_{abc} e^c_j \dot{\omega}^{ab}{}_i \\ &= \int \tfrac{1}{2}(\bar{p}\, dm + p\, d\bar{m}) - H\, d\tau - d(p^1 m_1 + p^2 m_2) \\ &= \alpha \int (r_1^- dr_2^- - r_1^+ dr_2^+) , \end{aligned} \tag{3.15}$$

showing that the holonomy parameters $r_{1,2}^\pm$ are related to the modulus m and momentum p through a (time-dependent) canonical transformation.

B. $\Lambda \to 0^-$

Much of the classical behavior discussed above was studied previously in [1,5] for the case of a vanishing cosmological constant. The $\Lambda = 0$ theory is easy to describe in ADM variables, but the holonomies analogous to $R_a^\pm$ are considerably more complicated, since the relevant gauge group is the nonsemisimple Lie group ISO(2,1), the (2+1)-dimensional Poincaré group. It is therefore useful to describe the relationship between the $\Lambda < 0$ and $\Lambda = 0$ theories. (The corresponding limit for $\Lambda \to 0^+$ has also been studied by Ezawa [18].)

The $\Lambda \to 0$ limit is most easily seen by rescaling the holonomy parameters. Define

$$w_a = \alpha(r_a^+ + r_a^-)/2, \quad u_a = (r_a^+ - r_a^-)/2 \quad (a = 1, 2) , \tag{3.16}$$

where the (time-independent) w_a and u_a remain finite as $\Lambda \to 0$, $\alpha \to \infty$. In this limit, the York time (3.5) becomes

$$\tau = -\frac{1}{t} ,$$

and the solution (3.1), (3.2) reduces to

$$\begin{aligned} e^0 &= dt , \\ e^1 &= t[u_1\, dx + u_2\, dy], \\ e^2 &= [w_1\, dx + w_2\, dy] , \end{aligned} \tag{3.17}$$

$$\begin{aligned} \omega^{12} &= 0 , \\ \omega^{01} &= -[u_1\, dx + u_2\, dy] , \\ \omega^{02} &= 0 . \end{aligned} \tag{3.18}$$

The moduli and momenta of (3.3), (3.4) are now

$$\begin{aligned} m &= (w_1 - itu_1)(w_2 - itu_2)^{-1} , \\ p &= -\frac{i}{t}(w_2 + itu_2)^2 , \end{aligned} \tag{3.19}$$

and the Hamiltonian (3.10) is

$$H = g^{1/2} = t(w_1 u_2 - u_1 w_2) . \tag{3.20}$$

The recollapse of spatial slices now disappears; the tori expand linearly in the range $t \in (0, \infty)$. The new variables u and w may be easily shown to satisfy classical Poisson brackets

$$\{u_1, w_2\} = \{w_1, u_2\} = -\tfrac{1}{2} , \tag{3.21}$$

derivable from the action

$$I = -2 \int (u_1\, dw_2 + w_1\, du_2) . \tag{3.22}$$

They are related to the parameters of Ref. [5] by

$$u_1 = -\lambda, \quad u_2 = -\mu, \quad w_1 = a, \quad w_2 = b \ . \tag{3.23}$$

For the traces (3.6) the limit is more complicated. However, the Poincaré variables and their algebra described in Ref. [10] may be retrieved by using the holonomy parameters

$$r_a^{\pm} = u_a \pm \frac{w_a}{\alpha} \tag{3.24}$$

to expand the traces (3.6) to first order in $1/\alpha$:

$$\begin{aligned} R_a^{\pm} &= \cosh \frac{r_a^{\pm}}{2} \\ &= \cosh \frac{u_a}{2} \pm \frac{w_a}{2\alpha} \sinh \frac{u_a}{2} = q_a \pm \frac{\nu_a}{\alpha} \ . \end{aligned} \tag{3.25}$$

The Poisson brackets (3.21) then imply that, to order $1/\alpha$,

$$\begin{aligned} \{q_1, q_2\} &= -\frac{1}{16\alpha^2}(\nu_{12} - \nu_{12^{-1}}) \ , \\ \{q_1, \nu_2\} &= -\frac{1}{16}(q_{12} - q_{12^{-1}}) + \frac{1}{8\alpha^2}\nu_1\nu_2 \ , \\ \{\nu_1, \nu_2\} &= -\frac{1}{16}(\nu_{12} - \nu_{12^{-1}}) \ . \end{aligned} \tag{3.26}$$

Here

$$\begin{aligned} q_{12} &= \cosh \frac{u_1 + u_2}{2}, \quad q_{12^{-1}} = \cosh \frac{u_1 - u_2}{2} \ , \\ \nu_{12} &= \left(\frac{w_1 + w_2}{2}\right) \sinh \frac{u_1 + u_2}{2} \ , \\ \nu_{12^{-1}} &= \left(\frac{w_1 - w_2}{2}\right) \sinh \frac{u_1 - u_2}{2} \ , \end{aligned} \tag{3.27}$$

which satisfy the identities

$$\nu_{12^{-1}} + \nu_{12} = 2(\nu_1 q_2 + \nu_2 q_1), \quad q_{12^{-1}} + q_{12} = 2 q_1 q_2 \ . \tag{3.28}$$

With these identifications, the algebra (3.26) reproduces that of [10] in the limit $\Lambda \to 0, \alpha \to \infty$. (Note that the ν of [10] is our 8ν.)

C. $\Lambda > 0$

The case of a positive cosmological constant has been studied in detail by Ezawa [18]. For completeness, we point out that the classical solutions for $\Lambda > 0$ can easily be derived from our solutions for $\Lambda < 0$ by substituting hyperbolic sines and cosines for sines and cosines. One finds that the range of

$$\tau = -\frac{2}{\alpha} \coth \frac{2t}{\alpha}$$

is now $-\infty$ to $-2/\alpha$ for $t \in (0, \infty)$ and that the area of the torus expands exponentially from zero to ∞. the $R_a^{\pm}$ are now traces of SL(2,$\mathbb{C}$) holonomies, expressed as the complex conjugates

$$R_a^{\pm}(r_a^+, r_a^-) = \cos \frac{(r_a^+ + r_a^-) \pm i(r_a^+ - r_a^-)}{4} \ , \tag{3.29}$$

which can be written in terms of the parameters u, w of Eq. (3.16) of the previous section as

$$\begin{aligned} R_a^{\pm} &= \cos\left(\frac{w_a}{2\alpha} \pm i\frac{u_a}{2}\right) \\ &= \cos\left(\frac{w_a}{2\alpha}\right)\cosh\left(\frac{u_a}{2}\right) \mp i \sin\left(\frac{w_a}{2\alpha}\right)\sinh\left(\frac{u_a}{2}\right). \end{aligned} \tag{3.30}$$

To first order in $1/\alpha$, we have

$$R_a^{\pm} = q_a \mp i\frac{\nu_a}{\alpha} \quad (a = 1, 2) \tag{3.31}$$

[compare with (3.25) for $\Lambda < 0$], so the limit $\Lambda \to 0$ is again easy to understand. In particular, the q_a and ν_a again satisfy the algebra (3.26) in the limit $\Lambda \to 0^+$.

Note that from (3.29),

$$R_a^{\pm}(r_a^+, r_a^-) = R_a^{\pm}(r_a^+ + 4\pi n_a, r_a^- + 4\pi n_a) \ , \tag{3.32}$$

for any integers n_a. Hence the parameters $r_a^{\pm}$, and therefore the moduli, are not uniquely determined by the SL(2,$\mathbb{C}$) traces, in contrast with the $\Lambda < 0$ case. Such a change corresponds to adding the total derivative

$$4\pi n_1 d(r_2^- - r_2^+)$$

to the action (3.15). This ambiguity was first noted by Mess [26], and was discussed by Witten in Ref. [27]. It suggests that in addition to the traces $R_a^{\pm}$, a new discrete quantum number related to the direct quantization of the parameters $r_a^{\pm}$ may be necessary to describe (2+1)-dimensional gravity with positive cosmological constant.

IV. QUANTUM THEORIES

We now turn to the quantization of the system described above. As we shall see, the different classical descriptions naturally lead to very different approaches to the quantum theory, whose relationship can give us further information about the structure of (2+1)-dimensional quantum gravity.

A. ADM quantization

Let us begin with the second-order formalism of Sec. II A. We saw above that the reduced phase-space action (2.8), the action written in terms of the physical variables m_α and p^α, is equivalent to that of a finite-dimensional mechanical system with a complicated Hamiltonian. We know, at least in principle, how to quantize such a system: we simply replace the Poisson brackets (2.9) with the commutators

$$\left[\hat{m}_\alpha, \hat{p}^\beta\right] = i\hbar\delta^\beta_\alpha \; , \tag{4.1}$$

represent the momenta as derivatives,

$$p^\alpha = \frac{\hbar}{i}\frac{\partial}{\partial m_\alpha} \; , \tag{4.2}$$

and impose the Schrödinger equation

$$i\hbar\frac{\partial\psi(m,\tau)}{\partial\tau} = \hat{H}\psi(m,\tau) \; , \tag{4.3}$$

where the Hamiltonian $\hat{H}$ is obtained from (2.8) by some suitable operator ordering.

One fundamental problem, of course, is hidden in this last step: it is not at all obvious how one should define $\hat{H}$ as a self-adjoint operator on an appropriate Hilbert space. The ambiguity is already evident for the genus 1 Hamiltonian (2.11): $\hat{m}_2$ and $\hat{p}$ do not commute, so the operator ordering is not unique. The simplest choice of ordering is that of Eq. (2.11), for which the Hamiltonian becomes

$$\hat{H} = \frac{\hbar}{\sqrt{\tau^2 - 4\Lambda}}\Delta_0^{1/2} \; , \tag{4.4}$$

where Δ_0 is the ordinary scalar Laplacian or the constant negative curvature moduli space characterized by the metric (2.12). This Laplacian is invariant under the modular transformations (2.14); its invariant eigenfunctions, the weight zero Maass forms, are discussed in considerable detail in the mathematical literature [30].

While this choice of ordering is not unique, the number of possible alternatives is smaller than one might fear. The key restriction is diffeomorphism invariance: the eigenfunctions of the Hamiltonian should transform under a one-dimensional unitary representation of the mapping class group. The representation theory of the modular group (2.14) has been studied extensively [31]; one finds that the possible inequivalent Hamiltonians are all of the form (4.4), but with Δ_0 replaced by[7]

$$\Delta_n = -m_2^2\left(\frac{\partial^2}{\partial m_1^2} + \frac{\partial^2}{\partial m_2^2}\right) + 2in\, m_2\frac{\partial}{\partial m_1} + n(n+1) \qquad 2n \in \mathbb{Z} \; , \tag{4.5}$$

the Maass Laplacian acting on automorphic forms of weight n. (See [7] for details of the required operator orderings.) Note that when written in terms of the momentum p, the operators Δ_n differ from each other by terms of order $\hbar$, as expected for operator ordering ambiguities. Nevertheless, the various choices of ordering can have drastic effects on the physics: the spectra of the various Maass Laplacians are very different.

This ambiguity can be viewed as a consequence of the structure of the classical phase space. The torus moduli space is not a manifold, but rather has orbifold singularities, and quantization on an orbifold is generally not unique. Since the space of solutions of the Einstein equations in 3+1 dimensions has a similar orbifold structure [32], we might expect a similar ambiguity in realistic (3+1)-dimensional quantum gravity.

There is another, potentially more serious, ambiguity in this approach to quantization, coming from the classical treatment of the time slicing. The choice of K as a time variable is rather arbitrary—it greatly simplifies the constraints (2.3), but is otherwise no better than any other classical gauge-fixing technique—and it is not at all clear that a different choice would lead to the same quantum theory. The danger of making a "wrong" choice is illustrated by the classical solution (3.1),(3.2): another standard gauge choice is $\sqrt{g} = t$, but it is evident that when $\Lambda < 0$, $\sqrt{g}$ is not even a single-valued function of τ.

A possible resolution of this problem is to treat the holonomy approach, in which no choice of time slicing is needed, as fundamental. If we can establish a relationship between the $(\hat{m}, \hat{p})$ and suitable operators in the first-order formulation, we can convert the problem of time slicing into one of defining the appropriate physical operators. Different choices of slicing would then merely require different operators to represent moduli, and not different quantum theories.

B. Quantizing traces of holonomies

We next consider an alternative approach to quantization, starting from the first-order formulation of the classical theory. Without assuming *ab initio* any classical relationship between moduli and holonomies, the algebra of the traces $R^\pm$ can be quantized directly for any value of the cosmological constant Λ and any genus g of Σ. For arbitrary genus, one obtains an abstract quantum algebra, the subject of intense study [8,9]. In principle, a representation in terms of some finite set parameters, analogous to the $R^\pm$ of Sec. II B, would determine a quantization of those parameters. For arbitrary g, it is not yet clear exactly how to find such a representation, although for $g = 2$ there has been some recent progress [8].

For the remainder of this section, we shall restrict our attention to the relatively well-understood case of $g = 1$, in order to make contact with the torus moduli quantization of Sec. IV A. We can quantize the classical algebra (2.26) as follows.

(1) We replace the classical Poisson brackets $\{,\}$ by commutators $[,]$, with the rule

$$[x, y] = xy - yx = i\hbar\{x, y\} \; . \tag{4.6}$$

(2) On the right-hand side of (2.26), we replace the product by the symmetrized product

$$xy \to \tfrac{1}{2}(xy + yx) \; . \tag{4.7}$$

The resulting operator algebra is given by

[7] It is argued in [6] that the natural choice of ordering in first-order quantization corresponds to $n = 1/2$.

$$\hat{R}_1^{\pm}\hat{R}_2^{\pm}e^{\pm i\theta} - \hat{R}_2^{\pm}\hat{R}_1^{\pm}e^{\mp i\theta} = \pm 2i\,\sin\theta\,\hat{R}_{12}^{\pm}$$

$$\text{and cyclical permutations} \quad (4.8)$$

with

$$\tan\theta = \frac{i\sqrt{k}\hbar}{8\alpha}\ . \quad (4.9)$$

Note that for $\Lambda < 0$, $k = -1$, and θ is real, while for $\Lambda > 0$, $k = 1$, and θ is pure imaginary.

The algebra (4.8) is not a Lie algebra, but it is related to the Lie algebra of the quantum group $\mathrm{SU}(2)_q$ [13, 33], where $q = \exp(4i\theta)$, and where the cyclically q-Casimir invariant is the quantum analogue of the cubic polynomial (2.27):

$$\hat{F}^{\pm}(\theta) = \cos^2\theta - e^{\pm 2i\theta}[(\hat{R}_1^{\pm})^2 + (\hat{R}_{12}^{\pm})^2] - e^{\mp 2i\theta}(\hat{R}_2^{\pm})^2 + 2e^{\pm i\theta}\cos\theta\,\hat{R}_1^{\pm}\hat{R}_2^{\pm}\hat{R}_{12}^{\pm}. \quad (4.10)$$

The operator algebra (4.8) can be represented by

$$\hat{R}_a^{\pm} = \sec\theta\,\cosh\frac{\hat{r}_a^{\pm}}{2} \quad (a = 1, 2, 12)\ , \quad (4.11)$$

with

$$[\hat{r}_1^{\pm}, \hat{r}_2^{\pm}] = \pm 8i\theta, \quad [\hat{r}_a^{+}, \hat{r}_b^{-}] = 0\ , \quad (4.12)$$

which differ (for Λ small and negative) from the naive expectation

$$[\hat{r}_1^{\pm}, \hat{r}_2^{\pm}] = \mp\frac{i\hbar}{\alpha} \quad (4.13)$$

by terms of order $\hbar^3$.

We must next try to implement the action of the modular group (3.14) on the operators $\hat{R}_a^{\pm}$. The action that preserves the commutators (4.8) is (note the factor ordering)

$$\begin{aligned} S:\ & \hat{R}_1^{\pm} \to \hat{R}_2^{\pm},\ \ \hat{R}_2^{\pm} \to \hat{R}_1^{\pm}, \\ & \hat{R}_{12}^{\pm} \to \hat{R}_1^{\pm}\hat{R}_2^{\pm} + \hat{R}_2^{\pm}\hat{R}_1^{\pm} - \hat{R}_{12}^{\pm} \\ T:\ & \hat{R}_1^{\pm} \to \hat{R}_{12}^{\pm},\ \ \hat{R}_2^{\pm} \to \hat{R}_2^{\pm}, \\ & \hat{R}_{12}^{+} \to \hat{R}_{12}^{+}\hat{R}_2^{\pm} + \hat{R}_2^{\pm}\hat{R}_{12}^{\pm} - \hat{R}_1^{\pm}\ . \end{aligned} \quad (4.14)$$

The second of these can be generated by the unitary operators

$$G^{\pm} = \exp\left(\pm\frac{i(\hat{r}_2^{\pm})^2}{16\theta}\right) \quad (4.15)$$

as

$$y \to G^{\pm} y (G^{\pm})^{-1}\ ,$$

where y is any function of the $\hat{r}_a^{\pm}$.

It is amusing to note that, from (4.12), the operators $\exp\{n\hat{r}_1^{\pm}\}$ and $\exp\{\hat{r}_2^{\pm}\}$ commute when

$$\theta = \pi p/4n$$

with $n, p \in \mathbb{Z}$ and $\Lambda < 0$. This occurs when the parameter q of the quantum group associated with the algebra (4.8) is a root of unity. We see from (4.9) that there are $2n - 1$ solutions α of this equation for any given n. By contrast, in the direct quantization given by equation (4.13), this simplification of the algebra would occur for an infinite number of values of α.

C. Quantizing the space of solutions

A third method of quantization starts with the parameters $r_a^{\pm}$ of the classical solution (3.1), (3.2). This approach can be viewed as a version of covariant canonical quantization, i.e., "quantizing the space of classical solutions" [34,35]. It has the obvious disadvantage of requiring detailed knowledge of the classical solutions, which are completely understood at present only for the simplest topology, $\mathbb{R} \times T^2$. On the other hand, this approach to quantization provides a natural bridge between the ADM and holonomy approaches discussed above, and, in particular, allows us to define a natural set of time-dependent physical operators in the latter theory.

Our starting point is now the set of Poisson brackets (3.8). The natural guess is that these should simply become the commutators (4.13). This leads to a legitimate quantum theory, but we know from the preceding section that the commutators (4.8) of traces of holonomies will not be reproduced. To obtain these traces, we must instead impose the commutators (4.12). With these definitions, the results of the preceding section are all preserved. In particular, it is not hard to show that the quantum modular group action (4.14) is induced by the transformations (3.14) of the $\hat{r}_a^{\pm}$.

We can now make the connection with the ADM quantization of Sec. IV A. The basic idea is to treat wave functions $\psi(r_a)$ as Heisenberg picture states and to define suitable time-dependent operators acting on these states. Now, the *classical* modulus and momentum on a surface $K = \tau$ have already been determined in terms of the $r_a^{\pm}$ and are given by equations (3.3), (3.4). Carrying these definitions over to the quantum theory, we obtain a family of operators $\hat{m}(\tau)$ and $\hat{p}(\tau)$, whose eigenvalues may be interpreted as the ADM modulus and momentum in the York time slicing. Similarly, the operator analog of (3.10) may be interpreted as a Hamiltonian generating the evolution of $\hat{m}$ and $\hat{p}$. Indeed, if we keep the orderings of (3.3), (3.4), and (3.10), defining

$$\begin{aligned} \hat{m} &= (\hat{r}_1^{-}e^{it/\alpha} + \hat{r}_1^{+}e^{-it/\alpha})\,(\hat{r}_2^{-}e^{it/\alpha} + \hat{r}_2^{+}e^{-it/\alpha})^{-1}\ , \\ \hat{p} &= -\frac{i\alpha}{2\sin(2t/\alpha)}(\hat{r}_2^{+}e^{it/\alpha} + \hat{r}_2^{-}e^{-it/\alpha})^2\ , \\ \hat{H} &= \frac{\alpha^2}{4}\sin\frac{2t}{\alpha}(\hat{r}_1^{-}\hat{r}_2^{+} - \hat{r}_1^{+}\hat{r}_2^{-})\ , \end{aligned} \quad (4.16)$$

it follows from the commutators (4.12) that

$$[\hat{m}^{\dagger}, \hat{p}] = [\hat{m}, \hat{p}^{\dagger}] = 16i\alpha\theta, \quad [\hat{m}, \hat{p}] = [\hat{m}^{\dagger}, \hat{p}^{\dagger}] = 0\ , \quad (4.17)$$

$$[\hat{p}, \hat{H}'] = -8i\alpha\theta\frac{d\hat{p}}{dt}, \quad [\hat{m}, \hat{H}'] = -8i\alpha\theta\frac{d\hat{m}}{dt}\ , \quad (4.18)$$

which differ from the corresponding equations in ADM quantization by terms of order $O(\hbar^3)$, small when $|\Lambda| = 1/\alpha^2$ is small. These results depend on operator ordering in $\hat{m}$, $\hat{p}$, and $\hat{H}$, of course, but the orderings of (4.16) are a bit less arbitrary than they might seem: they were chosen to ensure that the modular transformations (3.14) of the $\hat{r}_a^\pm$ induce the correct transformations (2.14) of $\hat{m}$ without any $O(\hbar)$ corrections.

Note that (4.11) can be inverted to give the operators $\hat{r}_a^\pm$ in terms of the traces $\hat{R}_a^\pm$. Equation (4.16) can therefore be viewed as a definition of modulus and momentum operators in the holonomy algebra quantization. These operators are, of course, quite complicated—they involve logarithms of the traces $\hat{R}$—but given a representation of the holonomy algebra, they provide the first known instance of physical observables with clear geometric interpretations.

To further investigate the connection to ADM quantization, we can examine the properties of wave functions that are eigenfunctions of $\hat{m}(r_a^\pm, \tau)$ and its adjoint; that is, we can transform to a "Schrödinger picture." As in the $\Lambda = 0$ case [6], Ezawa has shown that these wave functions transform as Maass forms of weight $\frac{1}{2}$, corresponding to an ordering (4.5) of the ADM Hamiltonian with $n = \frac{1}{2}$ [18]. Also as in the $\Lambda = 0$ case [7], however, this Hamiltonian can be changed by reordering the operators $\hat{m}^\dagger(r_a^\pm, \tau)$ and $\hat{p}^\dagger(r_a^\pm, \tau)$, or equivalently by redefining the inner product.

We can now return to the question of the choice of time slicing raised at the end of Sec. IV A. In the holonomy quantization of Sec. IV B or the approach of this section, no choice of a time coordinate is ever made. A particular time slicing is instead reflected in a choice of time-dependent operators $\hat{m}(r_a^\pm, \tau)$ and $\hat{p}(r_a^\pm, \tau)$ that describe the geometry of the chosen slice. Other choices of classical time coordinate would presumably lead to other operators, which would be used to answer genuinely different physical questions. In some sense, we have thus succeeded in evading the "problem of time" in quantum gravity.

V. CONCLUSION

In most quantum field theories, it is fairly clear from the start what the "right" variables to quantize are. Moreover, we have general theorems that guarantee that local field redefinitions will not change the S matrix. Consequently, we are not used to worrying about how to determine the right quantization of, say, electrodynamics.

Quantum gravity is different. Here, the physical observables are necessarily nonlocal, and there is no reason to believe that quantizations based on different variables should be equivalent. In (3+1)-dimensional gravity, of course, the question is rather premature, since we do not yet have even one complete, consistent quantization. In 2+1 dimensions, though, the problem becomes unavoidable.

It might be hoped, however, that this problem can be turned to our advantage. The various approaches to quantizing (2+1)-dimensional gravity have different strengths, and if their relationship can be understood clearly, we might be able to combine these strengths. In ADM quantization, for instance, the fundamental variables, the moduli and momenta (m, p), have simple geometric interpretations. In the quantization of the traces $R^\pm$, the physical meaning of the observables is much less clear, but the algebraic structures can be directly generalized to arbitrary spatial topologies. A primary goal of this paper has been to demonstrate the relationships between these approaches, thus allowing us to introduce clearly defined physical observables into the algebraic structure of holonomy quantization.

As we have seen, this goal can be achieved for spacetimes of the form $\mathbb{R} \times T^2$. Equations (4.11) and (4.16) give explicit time-dependent operators in the holonomy formalism that represent the moduli and momenta on surfaces of constant York time. These results depend on our knowledge of the exact solutions of the equations of motion, but it may be possible to extend them at least to genus 2: Ref. [8] has developed the description of the quantum algebra of traces of holonomies, while the hyperelliptic nature of genus 2 surfaces is likely to simplify the ADM analysis.

We have also seen hints of a solution of the problem of time in quantum gravity. In ADM quantization, one must choose a classical time slicing, and it is by no means clear that different choices will lead to equivalent theories. In quantization of the holonomy algebra, on the other hand, no such choice need be made; different choices of time show up only as different families of operators describing the spatial geometry of the corresponding slices. The definition of such operators is difficult, of course, and it would be very useful to find a perturbative approach that did not require complete knowledge of the classical solutions, but, in principle, we have found a way to implement Rovelli's approach to "evolving constants of motion" [36] in a theory of quantum gravity.

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