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What is the geometry of superspace?

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The configuration space of general relativity, called superspace or the space of three-geometries, inherits certain geometric structures from the Wheeler-DeWitt metric on the larger space of Riemannian metrics. We analytically investigate the signature properties of the particular geometric structure associated with the choice of constant lapse function. We point out that this metric has rather special properties and generically suffers from signature changes.

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I. INTRODUCTION

As is well known, the dynamics of general relativity can be formulated in terms of a constrained Hamiltonian system, with the configuration space for pure gravity being given by the space of all Riemannian metrics on a three-dimensional manifold Σ of fixed but arbitrary topology. In this article we take Σ to be compact without boundary. We call this space $\mathcal{Q}(\Sigma)$ to indicate its dependence upon the choice of Σ. In this Hamiltonian picture, space-time is looked upon as a history of dynamically evolving geometries on Σ represented by a path $g_{ab}(s)$ in $\mathcal{Q}(\Sigma)$. For example, in the special gauge where the lapse function $N=1$ and the shift vector $N^a=0$, the dynamical part of the vacuum Einstein equations reads (in units where $16\pi G/c^4=1$; a dot means differentiation with respect to the parameter s)

$$\ddot{g}_{ab}+\Gamma^{ijkl}_{ab}\dot{g}_{ij}\dot{g}_{kl}=-2(R_{ab}-\tfrac{1}{4}g_{ab}R)\ , \qquad (1)$$

whereas the constraint part reads

$$G^{ab\,cd}\dot{g}_{ab}\dot{g}_{cd}-4\sqrt{g}R=0 \quad \text{Hamiltonian constraint} \qquad (2)$$

$$G^{ab\,cd}\nabla_b\dot{g}_{cd}=0 \quad \text{momentum constraint.} \qquad (3)$$

R_{ab} and R are the Ricci tensor and Ricci scalar of the metric g_{ab}, respectively. $G^{ab\,cd}$ is the DeWitt metric [1] on the space of symmetric positive-definite matrices (defined below as $G^{ab\,cd}_{\beta}$ for $\beta=1$). The Γ symbols in (1) are the Christoffel symbols for the DeWitt metric. If (2) and (3) are satisfied initially it follows from (1) that they continue to be satisfied throughout the evolution. Equations (1) and (2) have an obvious geometric interpretation, whereas (3) says that the velocity must be orthogonal to the orbits of the diffeomorphism group. This is explained in more detail below.

Because of diffeomorphism invariance, $\mathcal{Q}(\Sigma)$ is endowed with an action of the diffeomorphism group $D(\Sigma)$ of Σ: each point of $\mathcal{Q}(\Sigma)$ is a Riemannian metric on Σ which is acted upon by diffeomorphisms via pullback. Two different metrics which are connected by a diffeomorphism in such a way are considered to be physically indistinguishable. Redundancies of this sort are avoided by going to the quotient $\mathcal{S}(\Sigma)\ :=\ \mathcal{Q}(\Sigma)/D(\Sigma)$, called the superspace associated with Σ. It represents the space of geometries rather than metrics on Σ. Although superspace now faithfully labels physical configurations, paths in superspace do not faithfully represent space-times. Two *different* paths of geometries may be obtained by "waving" Σ differently through the *same* space-time. This redundancy is due to the still existing freedom in the choice of the lapse function. Conversely, we quite obviously (e.g., by counting degrees of freedom) cannot obtain every path in $\mathcal{S}(\Sigma)$ by appropriately "waving" Σ through a *given* space-time.

The existence of some geometric structures of superspace is implicit in many of the investigations into the dynamical structure of general relativity. So, for example, in Wheeler's view of general relativity as geometrodynamics [2] and the associated canonical quantization program, superspace serves as the domain for the quantum mechanical state functional [1]. The equations to be satisfied by this state functional, the Wheeler-DeWitt (WDW) equations,[1] explicitly refer to the inverse metric,

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[1]We use the plural since there is an infinite number of WDW equations to be satisfied. Usually they are written as one six-dimensional Klein-Gordon-like equation per point $x\in\Sigma$ in the six coordinates $\{g_{ab}(x)\}$. But they are more properly interpreted as distributional equations to be integrated against appropriate test fields. The particular WDW equation that results for constant test fields refers to the particular metric we investigate in this article.

just like the classical equation (2) would when expressed in terms of momenta. This formulation is widely believed to provide a qualitative understanding of at least some aspects of quantum gravity (see, e.g., [3] for a recent review on the semiclassical approximation). In these considerations the metric structures on superspace are of central importance and many of the conclusions drawn depend on signature properties of these metrics. Rigorous assertions are usually confined to so-called minisuperspace models where only finitely many of the physical degrees of freedom are considered. The metric restricted to these finitely many directions can then be conveniently studied. The truncation is usually achieved by selecting the allowed degrees of freedom according to imposed symmetry requirements. Another interesting possibility is to consider a fixed triangulation of the three-manifold Σ and let only the finitely many edge lengths determine the geometry. the metric structure of superspace then induces a metric structure on the finite dimensional space of edge length which can then be subjected to numerical as well as analytic investigations [4]. But there are also examples of full perturbation analyses, where to first order all perturbation modes around a fixed and special (maximally symmetric) metric allow us to study the full superspace metric at this particular point [5].

In this article our aim is to make more general statements about the particular WDW metric corresponding to the choice of constant lapse, where we neither want to *a priori* confine ourselves to particular points in superspace nor to finitely many preferred directions in the tangent space. Similar observations to some of ours have already been made in the Appendix of [6].

So let us ask: "What is the geometry of $\mathcal{Q}(\Sigma)$?" Mathematically there are a variety of possibilities to endow $\mathcal{Q}(\Sigma)$ with a geometry. On the other hand, the laws of general relativity select a family of such metrics, one for each choice of the lapse function N. For the particular choice $N = 1$ this is displayed in Eqs. (1)–(3). They define a metric on $\mathcal{Q}(\Sigma)$:

$$\mathcal{G}(h,k) := \int_\Sigma G^{ab\,cd} h_{ab} k_{cd} d^3x \ , \tag{4}$$

which we call the WDW metric. We are interested in the properties of this particular metric connected with its indefinite nature.

Note that, because of the constraint (3), general relativity only uses the WDW metric to calculate inner products on the subspace of tangent vectors satisfying (3), which requires those vectors to be WDW orthogonal to the orbits of the diffeomorphism group. We call the orbit directions *vertical* and the WDW-orthogonal directions *horizontal*. Because of the indefinite nature of the WDW metric, the horizontal subspace might also contain vertical directions. When this is not the case, the WDW metric restricted to the horizontal subspace can be expected to define a metric on the quotient space $\mathcal{S}(\Sigma)$, by first lifting horizontally and then evaluating their scalar product.[2] Clearly, such a lifting is not well defined if the horizontal spaces contain vertical directions. What turns out to generically happen is that in different regions of superspace this metric has different signatures (and is not defined in the transition regions). Such signature changes are precisely signaled by nontrivial intersections of vertical with horizontal subspaces which necessarily occur in the transition regions of superspace. Note that each vector in the intersection must be WDW orthogonal to itself, that is, of zero WDW norm.

To clarify the WDW geometry of superspace would mean to first, characterize the singular set in $\mathcal{Q}(\Sigma)$ which consists of those points where horizontal and vertical subspaces intersect nontrivially, and, second, study the restriction of the WDW metric to the horizontal subspaces. In this article we only derive partial results. Note that we do not consider the constraint equation (2) in the same way as we did with (3). This would select a nonlinear subspace of vectors and thus prevent us from having a pseudo-Riemannian structure at all.

Next to our aim to clarify some general features of the WDW metric, we also wish to show that its properties are rather special. This we do by considering a one-parameter family of ultralocal metrics of which the WDW metric is one member. The parameter will be called β and the WDW metric is obtained for $\beta = 1$. It is in fact easy to see that, up to a trivial overall scale, these exhaust the set of ultralocal metrics.

II. ULTRALOCAL METRICS

In order to do differential geometry on $\mathcal{Q}(\Sigma)$ we heuristically assume that $\mathcal{Q}(\Sigma)$ is a differentiable manifold with tangent space $T_g(\mathcal{Q})$ and cotangent space $T^*_g(\mathcal{Q})$ at the metric $g_{ab} \in \mathcal{Q}$ (we shall sometimes drop the reference to Σ). Elements of $T_g(\mathcal{Q})$ are any symmetric covariant tensor field and elements of $T^*_g(\mathcal{Q})$ are any symmetric contravariant tensor density of weight one on Σ. Suppose we want to define a metric, i.e., a nondegenerate bilinear form in each $T_g(\mathcal{Q})$. Then, up to an overall constant, all ultralocal metrics [i.e., depending locally on $g_{ab}(x)$ but not on its derivatives] are given by the one-parameter family defined as follows: take $h, k \in T_g(\mathcal{Q})$, then

[2]In finite dimensions simple transversality of the two subspaces of course suffices to ensure a direct sum split. In infinite dimensions, however, one also needs to check that the summands are topologically closed subspaces, which typically involves regularity properties of elliptic operators. This is necessary for the projection maps to be continuous. Also, at base points in $\mathcal{S}(\Sigma)$ corresponding to geometries which admit isometries, $\mathcal{S}(\Sigma)$ is not a manifold [7] and a tangent space does not exist in the usual sense. By a metric at such a point we simply mean the induced metric in the horizontal subspaces of the covering points.

$$\mathcal{G}_\beta(h,k) := \int_\Sigma G_\beta^{ab\,cd} h_{ab} k_{cd} d^3x \ , \tag{5}$$

where

$$G_\beta^{ab\,cd} = \frac{\sqrt{g}}{2}(g^{ac}g^{bd} + g^{ad}g^{bc} - 2\beta g^{ab}g^{cd}) \ . \tag{6}$$

The WDW metric, introduced in (4), is just $\mathcal{G}_1$. Given $p, q \in T^*_g(\Sigma)$, the "inverse" metric $\mathcal{G}_\beta^{-1}$ is

$$\mathcal{G}_\beta^{-1}(p,q) := \int_\Sigma G^\beta_{ab\,cd} p^{ab} q^{cd} d^3x \ , \tag{7}$$

where

$$G^\beta_{ab\,cd} = \frac{1}{2\sqrt{g}}(g_{ac}g_{bd} + g_{ad}g_{bc} - 2\alpha g_{ab}g_{cd}) \tag{8}$$

with

$$\alpha + \beta = 3\alpha\beta, \text{ so that } G_\beta^{ab\,nm} G^\beta_{cd\,nm} = \frac{1}{2}(\delta^a_c \delta^b_d + \delta^a_d \delta^d_c) \ . \tag{9}$$

These are nondegenerate bilinear forms for $\beta \neq 1/3$. From now on we exclude $\beta = 1/3$ throughout without further mention. The bilinear form is positive definite for $\beta < 1/3$ and of mixed signature for $\beta > 1/3$ with infinitely many plus as well as minus signs. Because they are ultralocal, they arise from metrics on the space S_3^+ of symmetric positive-definite 3×3 matrices, which is diffeomorphic to the homogeneous space $\mathrm{GL}(3,R)/\mathrm{SO}(3) \cong R^6$, carrying the metric G_β. One has $\mathrm{GL}(3,R)/\mathrm{SO}(3) \cong \mathrm{SL}(3,R)/\mathrm{SO}(3) \times R^+ \cong R^5 \times R^+$ and with respect to this decomposition the metric has a simple warped-product form

$$G_\beta^{ab\,cd} dg_{ab} \otimes dg_{cd} = -\epsilon d\tau \otimes d\tau + \frac{\tau^2}{c^2}\mathrm{tr}(r^{-1}dr \otimes r^{-1}dr) \ , \tag{10}$$

with

$$c^2 = 16|\beta - 1/3|, \quad \tau = cg^{1/4}, \quad r_{ab} = g^{-1/3}g_{ab}, \quad \epsilon = \mathrm{sgn}(\beta - 1/3) \ . \tag{11}$$

The matrices r_{ab} are just the coordinates on $\mathrm{SL}(3,R)/\mathrm{SO}(3)$ and the trace in (10) is just the left-$\mathrm{SL}(3,R)$ invariant metric on this space. This gives rise to eight Killing vectors of G_β. An additional homothety is generated by the multiplicative action of R^+ on the τ coordinate. Geodesics in this metric have been explicitly determined [1]. If we now regard $\mathcal{Q}(\Sigma)$ as a mapping space, i.e., as the space of all smooth mappings from Σ into S_3^+, endowed with the metric (5), then, due to its ultralocal nature, geometric structures such as Killing fields, homotheties, and geodesics of the "target" metric (10) are inherited by the full metric (5). For example, dragging the maps $g_{ab}(x)$ along a Killing flow in S_3^+ produces a Killing flow in $\mathcal{Q}(\Sigma)$. The same can be done for geodesics. In this way, some geometry of the infinite dimensional $\mathcal{Q}(\Sigma)$ can be studied by looking at the six-dimensional S_3^+.

An adapted infinite dimensional split of $T_g(\mathcal{Q})$ into orthogonal subspaces is, e.g., provided by York's decomposition [8]. For $h \in T_g(\mathcal{Q})$ there is a unique transverse-traceless tensor h^{tt}, a unique function Ω, and, up to the addition of Killing fields, a unique vector field ξ, such that

$$h_{ab} = h^{tt}_{ab} + (\nabla_a\xi_b + \nabla_b\xi_a - \tfrac{2}{3}g_{ab}\nabla_c\xi^c) + \tfrac{1}{3}g_{ab}\Omega \ . \tag{12}$$

One easily checks the $\mathcal{G}_\beta$ orthogonality of the three subspaces represented by the three summands in (12) for all β. Orthogonality with respect to the positive-definite metric $\mathcal{G}_{\beta=0}$ ensures transversality of these subspaces. $\mathcal{G}_\beta$ is positive definite on the first two summands and negative (positive) definite on the third for $\beta > 1/3 (< 1/3)$.

Note also that expression (5) is invariant under diffeomorphisms of Σ, that is, diffeomorphisms of Σ act as isometries on $\mathcal{G}_\beta$. An infinitesimal diffeomorphism is represented by a vector field ξ on Σ and gives rise to a vector field X^ξ on $\mathcal{Q}(\Sigma)$:

$$X^\xi_{ab} = \nabla_a\xi_b + \nabla_b\xi_a \ , \tag{13}$$

which is a Killing field of the metric (5). The totality of vectors of the form (13) at $g \in \mathcal{Q}(\Sigma)$ span what we called the vertical vector space $V_g \subset T_g(\mathcal{Q})$. From (12) it appears that they are given by a superposition of modes from the second and third summand by choosing $\Omega = 2\nabla_a\xi^a$. For $\beta > 1/3$ the vertical vector has thus positive, negative, or zero $\mathcal{G}_\beta$ norm, depending on whether the norm of its projection into the second summand is larger, less, or equal to the modulus of the norm of its projection into the third summand.

With respect to $\mathcal{G}_\beta$ we can define the orthogonal complement to V_g which we call the horizontal vector space $H^\beta_g \subset T_g(\mathcal{Q})$. From (5), (6), and (13) we have

$$k_{ab} \in H^\beta_g \Leftrightarrow \nabla^a(k_{ab} - \beta g_{ab}k^c_c) = 0. \tag{14}$$

Under the isometric action of $D(\Sigma)$ on $\mathcal{Q}(\Sigma)$ horizontal spaces are clearly mapped into horizontal spaces.

If we set $\beta = 0$, the metric (5) is positive definite such that orthogonality also implies transversality, i.e., $V_g \cap H^0_g = \{0\}$. It is true that the tangent space splits into the direct sum of closed orthogonal subspaces: $T_g(\Sigma) = V_g \oplus H^0_g$, which allows to define a Riemannian geometry on the quotient space $\mathcal{S}(\Sigma)$ by identifying its tangent spaces with the horizontal spaces in $T(\mathcal{Q})$ [9]. This works for all $\beta < 1/3$. We are, however, interested in the range $1/3 < \beta \leq 1$ with special attention paid to the transition $\beta < 1$ to $\beta = 1$.

For $\beta > 1/3$ the metric (5) is not definite anymore and consequently the intersection $V_g \cap H^\beta_g$ might be nontrivial, depending on the point g. A simple example is the following: Take as Σ a three-manifold that carries a Ricci-flat metric g. In $T_g(\Sigma)$ consider the infinite dimensional vector subspace given by all vectors of the form $k_{ab} = \nabla_a\nabla_b\phi$, where ϕ is a smooth function on Σ. These vectors satisfy (14) for $\beta = 1$ and are therefore in H^1_g. But they are also of the form (13), with $2\xi_a = \nabla_a\phi$, and hence in V_g. Note that such nonzero ξ are never Killing vectors (i.e., X^ξ is nonzero) since the Killing condition would imply ϕ to be harmonic and hence constant. Moreover, suppose the metric is only Ricci flat in an open subset $U \subset \Sigma$. Then we can repeat the argument but this time only using functions ϕ with compact support inside U. Again, these give rise to an infinite dimensional intersection $V_G \cap H^1_g$ for each such partially flat metric g.

III. SOME OBSERVATIONS CONCERNING THE WDW METRIC

It follows from (13) and (14) that a vertical vector X^ξ is horizontal, if and only if

$$D_\beta \xi_a := -\nabla^b(\nabla_b \xi_a - \nabla_a \xi_b) -2(1-\beta)\nabla_a \nabla^b \xi_b - 2R^b_a \xi_b = 0 \ , \tag{15}$$

where R^b_a denote the mixed components of the Ricci tensor. Killing vectors, if existent, are obvious solutions but these do not interest us since they correspond to zero X^ξ. For $0 \le \beta < 1/3$ these are the only solutions since $\mathcal{G}_\beta$ is positive definite. This implies that for $\beta > 1/3$ any non-Killing solution cannot have zero divergence, since for those the β dependence in (15) drops out. A more elegant way to write D_β is, using the exterior derivative d, its adjoint δ (given by minus the divergence on the first index), and $M_{\rm Ric}$ for the map induced by R^b_a:

$$D_\beta = \delta d + 2(1-\beta) d\delta - 2M_{\rm Ric} \ , \tag{16}$$

which also displays its formal self-adjointness. The $\mathcal{G}_\beta$ norm of X^ξ is given by

$$\mathcal{G}_\beta(X^\xi, X^\xi) = 2\int_\Sigma \xi^a D_\beta \xi_a d^3x \ . \tag{17}$$

For $\beta \le 1$ (remember that $\beta \ne 1/3$) and $M_{\rm Ric} < 0$, i.e., strictly negative eigenvalues, this operator is manifestly positive and $\mathcal{G}_\beta$ restricted to V_g is thus positive definite. In particular, we have $V_g \cap H^\beta_g = \{0\}$ with infinitely many negative directions in H^β_g. Since it is known that any three-manifold Σ admits Ricci-negative metrics [10], this tells us that in every superspace there are open regions (the Ricci-negative geometries) with well defined WDW metric, given by the restriction of $\mathcal{G}_1$ to H^1_g, whose signature has infinitely many plus and minus signs.

For a flat metric g and values $\beta < 1$, D_β is nonnegative with kernel given by the covariantly constant ξ. Indeed, from (16) it follows that ξ is curl- and divergence-free on a flat manifold, hence covariantly constant. But this also means that ξ is a Killing vector and therefore X^ξ zero. So for g flat we have $V_g \cap H^\beta_g = \{0\}$ for $\beta < 1$. On the other hand, for $\beta = 1$ and g flat, we can only infer from (16) that ξ must be closed, hence exact or harmonic. but harmonicity implies a Killing vector, as above, so that all horizontal X^ξ are given by gradient fields ξ, as anticipated in the previous section. As stated there, we can localize the construction and obtain an infinite subspace in the intersection $V_g \cap H^1_g$ for metrics g which contain a flat region $U \subset \Sigma$. Clearly, any manifold admits such metrics. In particular, this tells us that in every superspace there are regions where no WDW metric is defined.

It is more difficult to obtain general results for metrics which are neither Ricci negative nor flat. For the very special class of nonflat Einstein metrics[3] it is at least easy to see that for $\beta = 1$, $H^1_g \cap V_g$ is zero. Indeed, for $R_{ab} = \lambda g_{ab}$, where $\lambda \in R - \{0\}$, (16) implies $0 = \delta D_1 \xi = 2\lambda\delta\xi$, so that ξ is divergence free and hence a Killing vector, so that X^ξ must be zero. So we can define WDW metrics at nonflat Einstein geometries in $\mathcal{S}(\Sigma)$ by restricting $\mathcal{G}_1$ to H^1_g. For the study of such metrics it is instructive to look at a particular example in detail to which we now turn.

As nonflat Einstein metric we take the standard round metric on the three-sphere with some unspecified radius. Here $M_{\rm Ric} > 0$ and not much can be directly read off (16) for general β. But taking elements of $T_g(\mathcal{Q})$ as first order perturbations of g, and expanding them in terms of the well known complete set of tensor harmonics, as given in [12], we established the following scenario (we find no need to display the straightforward but lengthy calculation here): For $1/3 < \beta < 1$ the number of negative directions (i.e., the number of linearly independent vectors of negative $\mathcal{G}_\beta$ norm) is finite in V_g and infinite in H^β_g. For the discrete values $\beta = \beta_n$, where

$$\beta_n := \frac{n^2-3}{n^2-1}, \quad n \in \{3,4,5,\ldots\} \ , \tag{18}$$

the intersection $V_g \cap H^\beta_g$ is nontrivial and of some finite dimension $d_n > 0$. At other values of β it is zero. What turns out to happen is that when β passes the value β_n from below, d_n of the negative directions change from H^β_g to V_g. Since the β_n accumulate at 1, this happens infinitely often as we turn up β to 1. At $\beta = 1$ only a single negative direction has remained in H^1_g and infinitely many are now in V_g. The intersection $V_g \cap H^1_g$ is in fact zero, in accordance with the more general argument given above. $\mathcal{G}_1$ restricted to H^1_g is of Lorentzian signature $(-,+,+,+,\ldots)$. This is directly related to the

[3] In three dimensions an Einstein metric implies constant sectional curvature so that Σ is a so-called space form. But not only is the topology of Σ severely restricted (e.g., its second homotopy group must be trivial). If Σ allows for Einstein metrics, they only form a finite dimensional subspace in superspace which is in fact of dimension one if the Einstein constant is nonzero [11]. In these cases the only deformations are the constant rescalings of the metric. In this sense Einstein metrics are very special (rigid).

statement made in quantum cosmology, that the WDW equation for the quantization of perturbations around the Friedmann universe is hyperbolic[4] [5]. It follows from our considerations that this can at best be locally valid in the superspace of the three-sphere, since the WDW metric necessarily suffers from signature changes.[5] Note also how delicately the signature structure of $\mathcal{G}_\beta$ restricted to H_g^β depends on whether $\beta < 1$ or $\beta = 1$.

There are other interesting differences between $\beta < 1$ and $\beta = 1$. Quite striking is the existence of an infinite dimensional intersection $H_g^1 \cap V_g$ for partially flat g. This means that D_1 cannot be an elliptic operator since these have finite dimensional kernels. And, in fact, calculating the principal symbol for D_β from (15), we obtain

$$\sigma_\beta(\zeta)^a_b = \| \zeta \|^2 \left(\delta^a_b + (1-2\beta)\frac{\zeta^a \zeta_b}{\| \zeta \|^2} \right) . \tag{19}$$

This matrix is positive definite for $\beta < 1$, invertible but not positive definite for $\beta > 1$, and singular positive semidefinite for $\beta = 1$. Expressed in standard terminology this says that the operator D_β is strongly elliptic in the first, elliptic but not strongly elliptic in the second, and degenerate elliptic but not elliptic in the third case. This relates to the problem of how one would actually calculate the metric on superspace at the regular points. Throughout we said that it would be obtained by restricting the metric $\mathcal{G}_\beta$ to the horizontal spaces H_g^β. But this means that we have to explicitly calculate the projection $T_g(\mathcal{Q}) \to H_g^\beta$. A general tangent vector $k_{ab} \in T_g(\mathcal{Q})$ is projected by adding a vertical vector X^ξ so that the sum is horizontal, i.e., satisfies (14). This is equivalent to solving

$$D_\beta \xi_b = \nabla^a (k_{ab} - \beta g_{ab} k^c_c) \tag{20}$$

as equation for ξ and given right-hand side. Uniqueness for X^ξ is given at regular geometries, i.e., those for which the kernel of D_β consists of Killing vectors only. Since the right-hand side is orthogonal to Killing vectors, ellipticity (for $\beta < 1$) guarantees existence for any k_{ab}. It is not clear to us whether the failure of manifest ellipticity for $\beta = 1$ does in fact imply any severe problem. For example, in the special cases where g_{ab} is an Einstein metric, we can Hodge decompose ξ and the right-hand side of (20) into exact, coexact, and harmonic forms. The Einstein condition then prevents the Ricci term in D_1 from coupling these components, so that (20) decomposes into three decoupled equations for the Hodge modes, two purely algebraic ones and an elliptic partial differential equation for the coexact mode. In this case we can thus show existence by restricting to appropriate subspaces.

Having seen that $\beta = 1$ is a special value from a mathematical point of view, we might also ask the question of why general relativity picks precisely this value. suppose we just used the metric $G^\beta_{ab\,cd}$ for a value $\beta \neq 1$ in the Hamiltonian:

$$\begin{aligned} H_\beta = & \int N \left(G^\beta_{ab\,cd} \pi^{ab} \pi^{cd} - \sqrt{g} R \right) d^3x \\ & -2 \int N_b \nabla_a \pi^{ab} d^3x \end{aligned} \tag{21}$$

$$= H_{\beta=1} + \int \frac{N}{\sqrt{g}} \left(\frac{\beta - 1}{2(3\beta - 1)} \pi^a_a \pi^b_b \right) d^3x . \tag{22}$$

Would this provide just another dynamics for a general relativistic theory of gravitation? The answer is *no*, due to the well known uniqueness theorems, which state that, up to the cosmological and the gravitational constant, the ordinary gravitational Hamiltonian is uniquely determined by the requirement that the coefficient functions of lapse and shift in the Hamiltonian (i.e., the Hamiltonian and momentum constraint functions) satisfy the standard Poisson bracket relations, which are universally valid for any generally covariant theory [14]. It is in fact rather easy to see how the additional term in (22) alters the Poisson brackets relation between the Hamiltonian constraints. Those involving the momentum constraints are clearly left unchanged. This means that the expression (21) cannot be the Hamiltonian of a generally covariant theory. In other words, if we evolved some initial data with the Hamiltonians (21) corresponding to *all* possible choices of lapse and shift, the resulting family of evolutions could not be interpreted as describing the *same* space-time in which the different motions of three-dimensional hypersurfaces generate the family of evolutions so calculated. In this sense, it is the general covariance of general relativity that picks the value $\beta = 1$.

Finally we wish to point out a geometric similarity of solving (20) with the so-called thin-sandwich problem, of which a local version has recently been proven by Bartnik and Fodor [15]. It consists in the task to calculate lapse and shift for freely specified g_{ab} and $\dot{g}_{ab}$, such that the metric and its conjugate momentum satisfy the Hamiltonian and momentum constraints. For the special

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[4] Since the WDW equations, smeared with test functions, are formulated on the larger space of Riemannian metrics, each of them is clearly ultrahyperbolic. [Choose, e.g., Ω in (12) with support inside a neighborhood where the test function is without zero points. This gives infinitely many negative directions.] Statements on hyperbolicity are always meant with respect to some reduction of the tangent space directions (compare, e.g., chapter 5 of [13]). By the momentum constraint the wave functional does not depend on the vertical directions so that a reduction to the horizontal directions is meaningful only in regions where vertical and horizontal spaces have zero intersection.

[5] In applications, the WDW equations have only been studied in neighborhoods of highly symmetric metrics like the round three-sphere considered here. It would be interesting to know how "far" from such a point one has to go in order to encounter singular regions and signature change. The regions $M_{\rm Ric} < 0$ do not seem "close," and the reason why the WDW equations have not been studied in neighborhoods of those metrics seems to be the fact that $M_{\rm Ric} < 0$ metrics do not allow for any symmetries.

cases in which in this procedure the lapse turns out to be constant (the only case we consider here), the nonlinear thin-sandwich equation for the shift vector simplifies drastically and yields Eq. (20). This is clear from the geometric meaning: both cases ask for the vertical-horizontal decomposition of a given tangent vector. In the general case of nonconstant lapse functions the thin-sandwich equations become, however, much more complicated due to the fact that the lapse function is now a (local) function of g_{ab} and $\dot{g}_{ab}$. This results in nonlinear equations for the general thin-sandwich problem.

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