

Exact hypersurface-homogeneous solutions in cosmology and astrophysics

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A framework is introduced which explains the existence and similarities of most exact solutions of the Einstein equations with a wide range of sources for the class of hypersurface-homogeneous spacetimes which admit a Hamiltonian formulation. This class includes the spatially homogeneous cosmological models and the astrophysically interesting static spherically symmetric models as well as the stationary cylindrically symmetric models. The framework involves methods for finding and exploiting hidden symmetries and invariant submanifolds of the Hamiltonian formulation of the field equations. It unifies, simplifies, and extends most known work on hypersurface-homogeneous exact solutions. It is shown that the same framework is also relevant to gravitational theories with a similar structure, such as Brans-Dicke or higher-dimensional theories.

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I. INTRODUCTION

Exact solutions have always played a central role in the investigation of physical theories whose content is encoded in a complicated set of differential equations. The theory of general relativity is indeed an example of this. A number of exact solutions of Einstein's field equations have been of key importance in the discussion of physical problems. Solutions have been found which describe black holes, stellar interiors, gravitational waves, and even the large scale structure of the Universe itself. Exact solutions have also served as a guide to point out mathematical features of the theory. The Taub-NUT-M (Taub-Newman-Unti-Tamburino-Misner [1,2]) solution, for example, has been of crucial importance for the very definitions one uses in describing the singularities of the full theory. Thus exact solutions may point out features which are not just special to themselves but characterize in some way properties of a wider class of solutions. They may also play a role as "building blocks" for more general solutions. For example, in certain ways the general spatially homogeneous cosmological model near an initial singularity can be understood in terms of very special exact solutions, notably the Kasner and the vacuum Bianchi type-II solutions, which to some extent also describe aspects of general cosmological singularities [3]. Sometimes exactly solvable problems are even used as a guide in developing ideas for the construction of more general theories, e.g., quantum gravity. For example, solvable problems in spatially homogeneous (SH) cosmology have been used to implement a number of different quantization schemes.

Thus there is ample motivation to try to find exact so-

lutions. Indeed, the book by Kramer *et al.* [4] is largely dedicated to the listing of exact solutions. Several chapters of that book deal with hypersurface-homogeneous (HH) solutions, a class for which the Einstein equations reduce to more manageable ordinary differential equations. Within the class of HH solutions there are several subclasses of considerable physical interest, the cosmological SH models and the astrophysical static spherically symmetric spacetimes being the most prominent ones.

Since the birth of general relativity nearly 80 years ago, an overwhelming number of HH solutions have been produced. A look at physics abstracts shows that this production continues even today at a considerable pace. However, often this search is undertaken as an end in itself without attempting to understand how particular successes fit into a larger scheme and without employing any systematic method of attack revealing possible underlying mechanisms. Exceptions do exist though, as illustrated very nicely by the numerous approaches to the problem of finding vacuum solutions for spacetimes with one or two commuting Killing vector fields [4,5]. However, these techniques have not contributed much to the problem of finding HH solutions.

It is the purpose of this paper to give a basic underlying explanation of why exact solutions arise for models with non-null homogeneous hypersurfaces; to provide a set of techniques which makes it possible to obtain these solutions in as simple a form as possible; to show how these techniques are applicable to a wide set of different physical problems; and to unify an otherwise seemingly unrelated zoo of particular results.

Rather than exhaustively treating all possible cases, a wide variety of examples will be discussed which illustrate

the utility of the approach presented here. The choice of examples is a subjective one reflecting our particular tastes. Among the many models and source types considered, the exact SH perfect fluid solutions will be more exhaustively surveyed, updating the work of Kramer *et al.* Scalar fields, not considered in that catalogue but now currently fashionable and physically interesting, will also be examined. Special attention will also be given to the static spherically symmetric perfect fluid models which are important in astrophysical applications.

The present approach will use a combination of Hamiltonian and Lagrangian formulations of the field equations, using the Hamiltonian constraint together with the Lagrangian equations. This may come as a surprise considering the statement by Kramer *et al.* [4] (p. 131) that this formulation “is not well-adapted to searching for exact solutions. . .” but upon second thought, this is quite natural since the Hamiltonian function contains all the dynamical content of the Einstein equations. This enables one to study a single function instead of a whole system of equations, armed with many powerful techniques from classical mechanics. The economy of the Hamiltonian approach also reveals the close mathematical relationships which exist among different types of models. These relationships are often obscured by the particular way in which a particular physical problem suggests expressing the field equations. For example, the usual approaches to static spherically symmetric models and SH cosmological models are quite different, but the Hamiltonian formulation shows their mathematical similarities. Furthermore, the Hamiltonian approach used in this article will also show how most exact solutions are either associated with essentially one-dimensional problems (in terms of degrees of freedom) or with problems which admit a sufficient number of a certain type of Hamiltonian symmetry.

This paper will proceed as follows. Since the Hamiltonian is essential for this discussion, Hamiltonians will be derived for a variety of different problems in Sec. II. This section starts with an outline of the way in which the Hamiltonian function is obtained for the HH models. This is followed by the explicit evaluation of this function, considering separately the diagonal and nondiagonal models. For the more numerous diagonal models, the gravitational Hamiltonian is discussed first and then a number of source contributions to the Hamiltonian are examined. For the fewer nondiagonal examples, each case is studied individually, and only the vacuum and perfect fluid Hamiltonians are derived. In all cases the Hamiltonian is put into a “canonical” form that makes evident the mathematical similarities among different HH models.

In Sec. III the generalized Friedmann equation is reviewed. This equation often arises in the context of one-dimensional invariant submanifolds and as a part of solvable higher-dimensional problems. It therefore frequently plays an important role as a fundamental building block when it comes to finding exact solutions.

To solve a problem with 2 or more degrees of freedom it is necessary to find a sufficient number of symmetries which makes it possible to decompose the problem into

smaller solvable parts. In Sec. IV a particular type of Hamiltonian symmetry is discussed, a so-called Killing tensor symmetry. It turns out that this symmetry, which usually is “hidden,” is responsible for all known solvable Hamiltonian problems with 2 or more degrees of freedom that we are aware of. This symmetry is a generalization of symmetries related to the existence of cyclic variables and Hamilton-Jacobi separability. To exploit the Killing tensor symmetries in order to find explicit solutions, one has to find symmetry compatible dependent and independent variables. A way of doing this is presented.

Section V presents one method for finding invariant Hamiltonian submanifolds, i.e., consistent subsystems of the Hamiltonian equations. This in turn leads to particular solutions. The existence of invariant submanifolds is an important issue in the search for exact solutions since it is usually impossible to solve the most general problems.

Section VI lists problems leading to exact solutions and indicates how to solve them by referencing the relevant parts of this paper. The models in this section are again divided into diagonal and nondiagonal models. However, the diagonal models previously arranged in Sec. II according to how their Hamiltonians are evaluated are now instead collected together according to the dimension of their intrinsic symmetry group, i.e., the group of symmetries of the intrinsic geometry of the individual homogeneous hypersurfaces. This categorization reveals an underlying mathematical unity of entire classes of physically distinct models and allows them to be treated collectively.

The methods developed in this paper are also applicable in contexts beyond four-dimensional classical general relativity. This is discussed in Sec. VII. As examples, some remarks are made about higher-dimensional theories, nonminimally coupled scalar field theories, and quantum cosmology.

In Sec. VIII the present approach is compared with some other exact solution techniques, followed by a concluding discussion which addresses a variety of other issues.

II. A HAMILTONIAN APPROACH TO THE FIELD EQUATIONS

In this section the Hamiltonian will be evaluated for a wide variety of HH models. However, the goal is not just to produce a Hamiltonian, but to obtain it in a certain “canonical” form which reveals the mathematical similarities between different HH models. This is not only useful for the present purpose of finding exact solutions, but may also serve as a starting point for qualitative analysis of the many problems which cannot be solved exactly. Furthermore, one can study a single function rather than be overwhelmed by a whole system of equations which also hides the mathematical connections between different kinds of models. This section concludes with a discussion of the general form of the Hamiltonians which have been obtained in the individual cases.

A. The line element

To explicitly obtain a Hamiltonian for a given model, it is necessary to introduce a line element. In this paper we will consider the Taub-NUT-M model, to be discussed below, and certain models with non-null homogeneous hypersurfaces. For the latter models the line element can be expressed in the form

$$ds^2 = \epsilon N(\lambda)^2 d\lambda^2 + g_{ab}(\lambda) \omega^a \omega^b, \quad (2.1)$$

where the one-forms ω^a ($a = 1, 2, 3$) are associated with the symmetry group acting on the homogeneous hypersurfaces. The quantity $\epsilon = n_\alpha n^\alpha$ is the sign of the norm of the unit normal n^α to the homogeneous hypersurfaces, having the value -1 for the SH models and 1 for the static ones (α assumes the values $0, 1, 2, 3$ with 0 referring to the component associated with the λ direction). For the SH models the single independent variable λ is a time variable t and N is the familiar lapse function. In the static case λ is instead a spatial variable. The gravitational degrees of freedom are associated with the component functions g_{ab} .

For later purposes it is convenient to introduce the function x defined by

$$x = N_{(\text{Taub})}/N, \quad N_{(\text{Taub})} = 12|g|^{1/2}, \quad (2.2)$$

where $g = \det(g_{ab}) = -\epsilon|g|$. The function x is the reciprocal of the relative ‘‘slicing gauge function’’ $\mathcal{N} = N/N_{(\text{Taub})}$ which is sometimes used in $3 + 1$ canonical gravity as a relative lapse. One must fix N or x to determine the parametrization of the family of homogeneous hypersurfaces by the independent variable λ ; this will be referred to as a choice of slicing gauge. The notation $N_{(\text{Taub})}$ comes from the very useful slicing gauge introduced by Taub [6] in the context of SH cosmology.

The HH models admit simply transitive or multiply transitive (MT) homogeneity groups. The models which admit a simply transitive three-dimensional homogeneity group are called Bianchi models, for which the one-forms may be chosen to satisfy

$$d\omega^a = -\frac{1}{2} C^a{}_{bc} \omega^b \wedge \omega^c, \quad (2.3)$$

where $C^a{}_{bc}$ are the components of the structure constant tensor of the Lie algebra of the homogeneity group. The Bianchi models are divided into two classes, class A and class B, according to the vanishing or nonvanishing of the trace $C^b{}_{ab}$. For the class A models one can choose $C^a{}_{bc} = n^{(a)} \epsilon_{abc}$, where the parameters $n^{(a)}$ characterizing the various symmetry types can be chosen to have the values in Table I; explicit coordinate representations of the one-forms can be found in [7]. Class B models which admit a Hamiltonian formulation will be discussed below.

Among the Bianchi models there is a special family of Bianchi type-I, -II, -III, -V, -VII₀, -VII_h, -VIII, and -IX models which admits MT symmetry groups. There are also MT models which do not admit a simply transitive subgroup (i.e., they are not Bianchi models); the most prominent ones are the SH Kantowski-Sachs models and the static spherically symmetric models.

TABLE I. Canonical choices of the symmetry parameters $n^{(a)}$. For nondiagonal type-II models the choice $(n^{(1)}, n^{(2)}, n^{(3)}) = (1, 0, 0)$ is more convenient than the one in the table.

	I	II	VI ₀	VII ₀	VIII	IX
$n^{(1)}$	0	0	1	1	1	1
$n^{(2)}$	0	0	-1	1	1	1
$n^{(3)}$	0	1	0	0	-1	1

B. How to find the Hamiltonian function

Not all HH models admit a Hamiltonian formulation of the field equations. However, of those that do, the Lagrangian and Hamiltonian have the usual simple forms in terms of kinetic and potential terms

$$L = T - U, \quad H = T + U. \quad (2.4)$$

The Hamiltonian must vanish as a consequence of one of the field equations; this is the ‘‘Hamiltonian constraint’’ $H = 0$. From the expression for H in terms of the velocities, one can read off the Lagrangian as well by a simple change of sign. The Hamiltonian constraint and the Lagrangian equations are both needed since a mixture of first and second order differential equations is often required to find explicit solutions. Since the Hamiltonian constraint is of crucial importance in obtaining solutions and is the most economical way of describing the field equations, our method will be referred to as a Hamiltonian approach.

The starting point for obtaining the above Hamiltonian is the Arnowitt-Deser-Misner (ADM) Hamiltonian of the $3 + 1$ or ADM approach described in Misner, Thorne, and Wheeler [8]. This approach reformulates the Einstein equations expressed with respect to a space-like slicing as a parametrized Hamiltonian system with constraints. Since all their formulas involve explicitly the sign ϵ of the norm of the unit normal to the slicing, they continue to hold for a timelike slicing. The ADM Hamiltonian is a linear combination of the super-Hamiltonian and supermomentum constraint functions (whose vanishing is equivalent to the 00 and $0a$ components of the Einstein equations, respectively), with the lapse and shift as the respective Lagrange multiplier coefficients. However, in the present application the supermomentum constraints can be solved and the shift set to zero, reducing the ADM Hamiltonian to the lapse times the super-Hamiltonian. Allowing the lapse function in this Hamiltonian to depend explicitly on the gravitational and possible source variables leads to Hamiltonian equations which differ from each other by terms involving the super-Hamiltonian, required to vanish as the remaining unsolved constraint.

This specialized ADM Hamiltonian is the first step toward obtaining the desired Hamiltonian for HH models and will be referred to simply as the Hamiltonian function. It has the form [8]

$$\begin{aligned}
H &= -2N|g|^{1/2}n^\alpha n^\beta [G_{\alpha\beta} - \kappa T_{\alpha\beta}] \\
&= -24x^{-1}|g|n^\alpha n^\beta [G_{\alpha\beta} - \kappa T_{\alpha\beta}] \\
&= H_{(G)} + H_{(\text{source})}, \tag{2.5}
\end{aligned}$$

where $G_{\alpha\beta}$ is the Einstein tensor and $T_{\alpha\beta}$ is the total energy-momentum tensor, including a possible cosmological constant term, while κ is the appropriate multiple of the gravitational constant. The field equations require $H = 0$.

The gravitational part of H is

$$\begin{aligned}
H_{(G)} &= T_{(G)} + U_{(G)} = -2N|g|^{1/2}n^\alpha n^\beta G_{\alpha\beta} \\
&= N|g|^{1/2}[K^a_b K^b_a - K^a_a K^b_b + \epsilon^{(3)}R] \\
&= 12x^{-1}|g|[K^a_b K^b_a - K^a_a K^b_b + \epsilon^{(3)}R], \tag{2.6}
\end{aligned}$$

where the extrinsic curvature tensor has the usual expression

$$K^a_b = -\frac{1}{2}N^{-1}g^{ac}\dot{g}_{cb} = -\frac{1}{24}x|g|^{-1/2}g^{ac}\dot{g}_{cb} \tag{2.7}$$

(an overdot indicates the λ derivative) and ${}^{(3)}R$ is the scalar curvature of the homogeneous hypersurfaces.

To go further one must discuss how the source enters the Hamiltonian. Rather than attacking the general case, we will confine our attention either to a scalar field which contributes one additional degree of freedom, or to sources whose additional degrees of freedom can be eliminated by integrating the source field equations in such a way that the source variables can be expressed entirely in terms of the gravitational variables and constants of integration. The second step toward obtaining the HH Hamiltonian involves inserting those expressions for the source variables into the Hamiltonian function.

The third step is to solve the supermomentum constraints, if nontrivial. If these latter constraints are holonomic, i.e., can be expressed as the vanishing of a total derivative, they can be integrated. In this case they can be used to reduce the number of gravitational degrees of freedom, and the above Hamiltonian function gives the correct field equations. On the other hand if these constraints are nonholonomic, the reduction of the number of degrees of freedom may not be possible or may not yield a Hamiltonian which gives the correct field equations because of symmetry considerations [9]. Because of these difficulties one should always check that the Hamiltonian function yields the correct field equations.

Recently the relationship between topology and the Hamiltonian formulation of the field equations has attracted some interest [10,11]. If the symmetry type allows a closed topology for the homogeneous hypersurfaces, then a Hamiltonian formulation exists. However, the nonexistence of such a topology does not exclude a Hamiltonian formulation, as exemplified by certain class B type-VI_h models treated below.

However, one does not just want to obtain a Hamiltonian for the problem, but also to express it as simply as possible. This is helpful not only in finding exact solutions but also for a qualitative analysis of dynamics which cannot be described by exact solutions. The fourth and final step toward obtaining the HH Hamiltonians in a use-

ful form is accomplished by choosing new variables which diagonalize the kinetic part T of the Hamiltonian. The way in which this is achieved differs for models which are diagonal or nondiagonal, i.e., models for which the line element can or cannot be expressed in diagonal form. These two cases will be treated separately.

C. Diagonal models

The line element for diagonal HH models can be written in the form

$$\begin{aligned}
ds^2 &= \epsilon N(\lambda)^2 d\lambda^2 + g_{11}(\lambda) (\omega^1)^2 + g_{22}(\lambda) (\omega^2)^2 \\
&\quad + g_{33}(\lambda) (\omega^3)^2. \tag{2.8}
\end{aligned}$$

It is convenient to introduce the variables

$$\begin{aligned}
(|g_{11}|, |g_{22}|, |g_{33}|) &= (R_1^2, R_2^2, R_3^2) \\
&= (e^{2\beta^1}, e^{2\beta^2}, e^{2\beta^3}). \tag{2.9}
\end{aligned}$$

The diagonal models all have a Hamiltonian formulation. The kinetic part $T_{(G)}$ of the gravitational Hamiltonian can not only be diagonalized but even made ‘‘conformally flat’’ by a linear transformation of the β^a variables.

1. Class A models

For these models the momentum constraints are satisfied identically. The parametrization

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{pmatrix} = \begin{pmatrix} \beta^0 + \beta^+ + \sqrt{3}\beta^- \\ \beta^0 + \beta^+ - \sqrt{3}\beta^- \\ \beta^0 - 2\beta^+ \end{pmatrix} \tag{2.10}$$

introduced by Misner [12], leads to the ‘‘conformally flat’’ form

$$T_{(G)} = \frac{1}{2}x\eta_{AB}\dot{\beta}^A\dot{\beta}^B \tag{2.11}$$

of the expression (2.6) for the kinetic part of the gravitational Hamiltonian. Here $\eta_{AB} = \eta^{AB}$ ($A, B = 0, +, -$) are the orthonormal components of the Minkowski metric

$$-\eta_{00} = \eta_{++} = \eta_{--} = 1 \tag{2.12}$$

and the quantity x is the conformal factor, making it more convenient to use than N . Equation (2.6) also yields the gravitational potential

$$\begin{aligned}
U_{(G)} &= \epsilon N|g|^{1/2} {}^{(3)}R = 12\epsilon x^{-1} e^{6\beta^0} {}^{(3)}R \\
&= 6x^{-1} \left[\sum_{a=1}^3 (n^{(a)} g_{aa})^2 - 2n^{(1)}n^{(2)}g_{11}g_{22} \right. \\
&\quad \left. - 2n^{(2)}n^{(3)}g_{22}g_{33} - 2n^{(3)}n^{(1)}g_{33}g_{11} \right]. \tag{2.13}
\end{aligned}$$

In addition to being mathematically convenient, the β^A variables are also adapted to physical quantities. The quantity $|g|^{1/2} = e^{\beta^1 + \beta^2 + \beta^3} = e^{3\beta^0}$ is the volume-element

density. The λ derivatives of the β^0, β^\pm variables are, respectively, proportional to the expansion and the nonzero shear components of the congruence normal to the homogeneous hypersurfaces.

The possibility of choosing the timelike direction along different axes in the static case yields an abundance of models. To avoid being lost in details, the signature freedom will be restricted by requiring ω^1 and ω^2 to be space-like, a condition which will be assumed in the rest of this paper. This leads to the line element

$$ds^2 = \epsilon[N(\lambda)^2 d\lambda^2 - R_3(\lambda)^2 (\omega^3)^2] + R_1(\lambda)^2 (\omega^1)^2 + R_2(\lambda)^2 (\omega^2)^2. \quad (2.14)$$

With this choice of timelike direction, the gravitational Hamiltonian takes the form

$$H_{(G)} = \frac{1}{2}x\eta_{AB}\dot{\beta}^A\dot{\beta}^B + U_{(G)} \\ = \frac{1}{2}x\eta_{AB}\dot{\beta}^A\dot{\beta}^B + 12x^{-1}e^{4\beta^0}V^*(\beta^\pm), \quad (2.15)$$

where

$$V^* = \frac{1}{2}e^{4\beta^+}h_-^2 + \epsilon n^{(3)}e^{-2\beta^+}h_+ + \frac{1}{2}(n^{(3)})^2e^{-8\beta^+}, \\ h_\pm = n^{(1)}e^{2\sqrt{3}\beta^-} \pm n^{(2)}e^{-2\sqrt{3}\beta^-}. \quad (2.16)$$

An interesting subclass of these models is obtained by setting $\beta^- = 0$ (equivalent to $R_1 = R_2$). For the Bianchi types-I, -II, -VII₀, -VIII, and -IX, these models admit multiply transitive symmetry groups which correspond to a local rotational symmetry, for the choice made for the timelike axis in (2.14). The Bianchi types I and -VII₀ models of this type coincide and are plane symmetric. Only the type-I models will be referred to in what follows.

2. Bianchi type-V and -VI_h models

Most class B models do not admit a Hamiltonian formulation, but some particular nondiagonal and all diagonal models (consistent with the field equations, and with the sources studied here) do. In the following we will only consider SH class B models since the static class B models seem to lack physical interest. The SH class B Bianchi types-V and -VI_h admit diagonal cases provided that the structure constants have the form [7,13,14]

$$C^1_{31} = a + q, \quad C^2_{32} = a - q, \quad a^2 = -hq^2. \quad (2.17)$$

The line element has the same form (2.14) as in the class A case, but the variables R_a are related by an algebraic constraint obtained from the single nontrivial holonomic momentum constraint. Canonical choices for the structure constants are

$$\begin{aligned} \text{V:} \quad & q = 0, \quad a = 1, \\ \text{VI}_h: \quad & q = 1, \quad a > 0. \end{aligned} \quad (2.18)$$

Explicit coordinate representations of the one-forms may be found in [7,14].

The following parametrization solves the momentum constraint and diagonalizes $T_{(G)}$:

$$\begin{pmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{pmatrix} = \begin{pmatrix} \beta^0 - c(q - 3a)\beta^\times \\ \beta^0 - c(q + 3a)\beta^\times \\ \beta^0 + 2cq\beta^\times \end{pmatrix}, \quad (2.19)$$

where $c = (q^2 + 3a^2)^{-1/2}$. Note that for the Bianchi type-V case, this reduces to the class A parametrization with $\beta^+ = 0$ and $\beta^- = \beta^\times$. The parameter combination cq takes values in the interval $[0, 1)$ with the end-point value 1 corresponding to the class A limit of type VI₀ which arises from $a \rightarrow 0$, with $q \neq 0$. The parametrization then reduces to the corresponding class A case with $\beta^- = 0$ and $\beta^+ = -\beta^\times$. It is sometimes convenient to choose $a = 1$ in Bianchi type V, leading to $c^{-2} = 3$, since this choice is the one usually used for the open isotropic FRW models which are obtained by setting $\beta^\times = 0$ in type V.

With the above parametrization the gravitational Hamiltonian has the expression

$$H_{(G)} = \frac{1}{2}x(-\dot{\beta}^{02} + \dot{\beta}^{\times 2}) + 24x^{-1}c^{-2}e^{4(\beta^0 - cq\beta^\times)}. \quad (2.20)$$

3. Multiply transitive models

The MT models we have encountered so far are those belonging to class A and the isotropic open type-V model. There are some additional ones described by the same form (2.14) of the line element as in the class A case with the Misner parametrization restricted by the condition $\beta^- = 0$ corresponding to $R_1 = R_2$. For this case, the $(\omega^1)^2 + (\omega^2)^2$ part of the line element represents a two-space of constant curvature $\sigma \in \{1, 0, -1\}$, while $\omega^3 = dx^3$ is an exact differential. Explicit coordinate representations of the other two one-forms may be found in [4].

For this class of models the case $\sigma = 0$ coincides with the locally rotationally symmetric (LRS) Bianchi type-I models already considered. The case $\sigma = -1$ is an LRS class B Bianchi type-III case. The case $\sigma = 1$ is the exceptional case where no three-dimensional simply transitive subgroup exists, not falling under the Bianchi classification. The SH models of this latter case are called Kantowski-Sachs (KS) models, while the static models are the spherically symmetric models.

The three-curvature ${}^{(3)}R$ reduces to the curvature of the constant curvature two-spaces ${}^{(3)}R = R_1^{-2}\sigma = e^{-2\beta^1}\sigma$, leading to the gravitational Hamiltonian

$$H_{(G)} = \frac{1}{2}x(-\dot{\beta}^{02} + \dot{\beta}^{+2}) + 24\epsilon\sigma x^{-1}e^{4\beta^0 - 2\beta^+}. \quad (2.21)$$

The type-III models can be obtained from the previously discussed type-VI_h models. The values $a = 1 = q$, for which $c = \frac{1}{2}$, correspond to the usual Bianchi type-III = VI_{h=-1} structure constants. However, the equivalent choice $a = \frac{1}{2} = q$ leads to $c = 1$ and the same Hamiltonian as (2.21) (with $\sigma = -1$) for this case. In both cases one has $cq = \frac{1}{2}$. The different values of c are related by a translation in the β^A variables associated with a scaling of the corresponding one-forms.

When not otherwise stated a reference to MT models will refer to the LRS type I, III models, and the KS and static spherically symmetric models described by the Hamiltonian (2.21).

4. Source terms

Any combination of potential terms of the following form may be considered, representing different arrangements of sources.

(a) Cosmological constant: A cosmological constant term in Einstein's equations will be considered as an additional term $(-\Lambda/\kappa)g_{\alpha\beta}$ in the total energy-momentum tensor $T_{\alpha\beta}$. Inserting this term in Eq. (2.5) gives rise to the potential

$$U_{(\Lambda)} = -2\epsilon N|g|^{1/2}\Lambda = -24\epsilon x^{-1}e^{6\beta^0}\Lambda. \quad (2.22)$$

(b) Scalar field: A scalar field adds another dependent variable and contributes both to the kinetic part of the Hamiltonian and to the potential [15]

$$T_{(\text{sc})} = \frac{1}{2}x\dot{\beta}^{\dagger 2}, \quad U_{(\text{sc})} = -24\kappa\epsilon x^{-1}e^{6\beta^0}V_{(\text{sc})}(\beta^{\dagger}), \quad (2.23)$$

where $V_{(\text{sc})}(\beta^{\dagger})$ is the scalar field potential and where $\beta^{\dagger} = \sqrt{\kappa/6}\phi$ is the relation between the scaled scalar field variable β^{\dagger} and the usual scalar field ϕ . It will be convenient to introduce $\beta^4 = \beta^{\dagger}$ and an exponential scalar field variable analogous to the gravitational variables R_a by defining

$$R_4 = e^{\beta^{\dagger}}. \quad (2.24)$$

When convenient the index a will take the values in the set $\{1, 2, 3, 4\}$ and the range of the index A may include the value “†.”

(c) Perfect fluids: For a perfect fluid source with energy density ρ , pressure p , and four-velocity u^α , the energy-momentum tensor is

$$T^\alpha{}_\beta = (\rho + p)u^\alpha u_\beta + p\delta^\alpha{}_\beta. \quad (2.25)$$

For the cosmological case $\epsilon = -1$, the four-velocity for a diagonal source must be $u^\alpha = n^\alpha$. For the static case $\epsilon = 1$, the four-velocity must be along the timelike third direction $u_\alpha = u_3\delta^3{}_\alpha$ [recall Eq. (2.14)]. According to Eq. (2.5) the fluid contributes the potential

$$U_{(\text{fluid})} = 2\kappa N|g|^{1/2}n^\alpha n^\beta T_{\alpha\beta} = \begin{cases} 2\kappa N|g|^{1/2}\rho & (\text{SH case}), \\ 2\kappa N|g|^{1/2}p & (\text{static case}), \end{cases} \quad (2.26)$$

to the Hamiltonian but an equation of state is needed to express this entirely in terms of the gravitational variables using the conservation equations.

In the SH case an equation of state $p = p(\rho)$ is imposed. Note that dust models do not exist in the static diagonal case without additional source terms (to prevent the dust from collapsing). Since the fluid potential is proportional to p in this case, an equation of state $\rho = \rho(p)$ is therefore imposed instead.

Following Misner, Thorne, and Wheeler [8], it is convenient to introduce the baryon number density n and the chemical potential $\mu = (\rho + p)/n$ which satisfy

$$d\ln n = (\rho + p)^{-1}d\rho, \quad d\ln \mu = (\rho + p)^{-1}dp. \quad (2.27)$$

The conservation equations then imply

$$n^\alpha T_{\alpha}{}^{\beta}{}_{;\beta} = (\rho + p)\{u_\alpha[u^\beta{}_{;\beta} + (\ln n\mu)_{;\beta}u^\beta] + u_{\alpha;\beta}u^\beta + (\ln \mu)_{;\alpha}\}n^\alpha = 0. \quad (2.28)$$

In the SH case the conservation equation reduces to

$$(nu^\beta)_{;\beta} = 0 \quad \rightarrow \quad ng^{1/2} = ne^{3\beta^0} = \ell = \text{const}. \quad (2.29)$$

This equation, together with (2.27), leads to $\rho = \rho(\beta^0)$ which gives the fluid potential

$$U_{(\text{fluid})} = 24\kappa x^{-1}e^{6\beta^0}\rho(\beta^0). \quad (2.30)$$

For the static case the conservation equation reduces to the simple relation

$$d\ln \mu = \frac{dp}{\rho + p} = -d\beta^3, \quad (2.31)$$

which may be integrated to yield

$$\mu R_3 = \mu e^{\beta^3} = \text{const}, \quad \mu \propto e^{-\beta^3}, \quad (2.32)$$

which means that $p = p(\beta^3) = p(\beta^0 - 2\beta^+)$. This yields the potential

$$U_{(\text{fluid})} = 24\kappa x^{-1}e^{6\beta^0}p(\beta^0 - 2\beta^+). \quad (2.33)$$

For a fluid with the equation of state $p = (\gamma - 1)\rho$, implying $\mu = \gamma\rho/n$, the above relations (2.27) can be integrated to yield

$$\rho/\rho_{(0)} = (n/n_{(0)})^\gamma, \quad \mu/\mu_{(0)} = (n/n_{(0)})^{\gamma-1}, \quad (2.34)$$

and without loss of generality one can set $\rho_{(0)} = n_{(0)}^\gamma$ leading to

$$\rho = n^\gamma, \quad \mu = \gamma n^{\gamma-1}. \quad (2.35)$$

The values $\gamma = 1, \frac{4}{3}, 2$ describe, respectively, dust, radiation, and stiff perfect fluids.

For the SH case, this leads to

$$\rho = \rho_{(0)}e^{-3\gamma\beta^0}, \quad (2.36)$$

where the choice $\rho_{(0)} = \ell^\gamma$ leads to the fluid contribution

$$U_{(\text{fluid})} = 24\kappa x^{-1}\ell^\gamma e^{3(2-\gamma)\beta^0} \quad (2.37)$$

to the Hamiltonian.

In the static case one often assumes $\rho = \rho_{(0)} + (\eta - 1)p$ as an equation of state, where η and $\rho_{(0)}$ are constants. This includes both $p = (\gamma - 1)\rho$ when $\rho_{(0)} = 0$, so that $\eta = \gamma/(\gamma - 1)$, as well as the case of constant energy density obtained by setting $\eta = 1$. Inserting this into Eq. (2.31) yields

$$p = (p_{(0)} + \rho_{(0)})e^{-\eta\beta^3} - \eta^{-1}\rho_{(0)}, \quad (2.38)$$

where $p_{(0)}$ is a constant of integration. This leads to the fluid potential

$$U_{(\text{fluid})} = 24\kappa x^{-1}[(\rho_{(0)} + p_{(0)})e^{(6-\eta)\beta^0+2\eta\beta^+} - \rho_{(0)}e^{6\beta^0}]. \quad (2.39)$$

Occasionally, particularly in the SH case, one considers several noninteracting perfect fluids with a common four-velocity. In this situation one has a perfect fluid potential for each such component fluid.

(d) Electromagnetic fields: An electromagnetic field has the energy-momentum tensor [8]

$$T^\alpha{}_\beta = -\frac{1}{4\pi} (F^\alpha{}_\gamma F^\gamma{}_\beta - \frac{1}{4}\delta^\alpha{}_\beta F^\gamma{}_\delta F^\delta{}_\gamma). \quad (2.40)$$

The simplest electromagnetic fields are aligned with one of the axes, leading to a diagonal energy-momentum tensor. Examining Maxwell's equations for an electric or magnetic field or some combination of the two along a single axis, one finds which directions are allowed (if any). For the class A and MT models the third axis is such a direction, leading to an energy-momentum tensor with nonzero components

$$T^0{}_0 = T^3{}_3 = -T^1{}_1 = -T^2{}_2 = -\rho. \quad (2.41)$$

An examination of the conservation equations similar to that of the perfect fluid case shows that the positive quantity

$$g_{11}g_{22}\rho = \kappa^{-1}e^2 \quad (2.42)$$

is a constant. The electromagnetic contribution to the Hamiltonian then takes the simple form

$$U = -24\epsilon x^{-1}e^2e^{2\beta^3}. \quad (2.43)$$

In the static cylindrically symmetric case the inhomogeneity together with the singular axis of symmetry breaks the duality symmetry, and a magnetic field must lie along one of the spatial directions not associated with the spatial independent variable λ . However, the potential is identical with the previous case once a suitable permutation of the dependent variables R_a is made.

D. Some nondiagonal models

Similar methods can be used to treat some of the nondiagonalizable cases. Some of the simplest nondiagonal cases with a Hamiltonian formulation will be considered to illustrate this procedure. Solving the momentum constraints leads to a problem with an additional nondiagonal gravitational degree of freedom which is associated with a cyclic variable. Using the associated constant momentum, one obtains a reduced Hamiltonian for the diagonal degrees of freedom with an effective potential left behind in the kinetic part of the Hamiltonian. This is

analogous to the centrifugal potential which appears in the central force problem.

1. The Taub-NUT-M model

The Taub-NUT-M model is an excellent example to study in this context for two reasons. First it has a homogeneous slicing whose causal character changes from spacelike to timelike and back to spacelike again. Second it is a nondiagonal model which pieces together diagonal models of each causality type, illustrating the way in which some nondiagonalizable models behave.

Its line element can be put in the form [2]

$$ds^2 = -2z^{-1}d\lambda\omega^1 + g(\omega^1)^2 + e^{2w}[(\omega^2)^2 + (\omega^3)^2], \quad (2.44)$$

where the ω^a are the same one-forms as for the diagonal type-IX models and z , g , and w are functions of λ . The function z is a slicing gauge function. The function g is positive in the SH Taub region, negative in the static NUT region, and zero at the bridging null hypersurfaces between them. The λ coordinate lines are null in contrast with the usual orthogonal coordinate lines for which the line element is diagonalized in the Taub and NUT regions.

Because of the existence of null hypersurfaces, it is perhaps most straightforward to specialize the full curvature scalar Lagrangian to this case, removing a total λ derivative to obtain a Lagrangian valid for the entire spacetime. This is equivalent to the ADM Lagrangian in the separate Taub and NUT regions with the shift and lapse freedom fixed by the null condition on the λ coordinate lines, modulo a conformal factor z representing the freedom remaining in the parametrization of the slicing. The corresponding Hamiltonian is given by

$$H_{(G)} = 2e^{2w}z(\dot{g}\dot{w} + g\dot{w}^2) + z^{-1}(\frac{1}{2}ge^{-2w} - 2) = 0. \quad (2.45)$$

The kinetic part can be easily diagonalized for $g \neq 0$ and $g = 0$ separately, but not for all values of g simultaneously which is needed to describe the full Taub-NUT-M spacetime.

2. Stationary cylindrically symmetric models

The stationary cylindrically symmetric models have a line element which can be written as [4]

$$ds^2 = N^2d\lambda^2 - e^{2\beta^1}(dt + Cd\phi)^2 + e^{2\beta^2}d\phi^2 + e^{2\beta^3}dz^2, \quad (2.46)$$

where β^a , N , and C are functions of the independent variable λ which is interpreted here as a radial coordinate ordinarily denoted by the symbol ρ . As for the diagonal cases, the Hamiltonian is given by

$$H = -2N|g|^{1/2}n^\alpha n^\beta (G_{\alpha\beta} - \kappa T_{\alpha\beta}) = 0. \quad (2.47)$$

Expressing the variables β^a in terms of the Misner parametrization, the vacuum Hamiltonian assumes the explicit form

$$H = \frac{1}{2}x(\eta_{AB}\dot{\beta}^A\dot{\beta}^B + \frac{1}{12}e^{4\sqrt{3}\beta^-}\dot{C}^2) = 0. \quad (2.48)$$

The momentum p_C associated with the cyclic variable C is constant, leading to

$$\dot{C} = 12e^{-\sqrt{3}\beta^-}p_C \quad (2.49)$$

and the reduced Hamiltonian

$$H = \frac{1}{2}x\eta_{AB}\dot{\beta}^A\dot{\beta}^B + 12x^{-1}e^{-4\sqrt{3}\beta^-}p_C^2 = 0. \quad (2.50)$$

3. Spatially homogeneous Bianchi type-VI_{-1/9} models

The orthogonal perfect fluid models ($u^\alpha = n^\alpha$) of this type permit a nondiagonalizable line element of the form

$$ds^2 = -N(t)^2 dt^2 + g_{ab}(t)\omega^a\omega^b, \quad (2.51)$$

where ω^a are the invariant one-forms for this symmetry type, with structure constants

$$\begin{aligned} C^a{}_{bc} &= n^{(a)}\epsilon_{abc} + a\delta^a{}_3 \quad (\text{no sum on } a), \\ n^{(1)} &= -n^{(2)} = 1, \quad n^{(3)} = 0, \quad a = \frac{1}{3}. \end{aligned} \quad (2.52)$$

The three-metric g_{ab} can be conveniently parametrized by $g_{ab} = S^c{}_a S^d{}_b g'_{cd}$, where $g'_{ab} = \text{diag}(e^{2\beta^1}, e^{2\beta^2}, e^{2\beta^3})$ and the matrix $S^a{}_b$ lies in the three-dimensional special automorphism group of the type-VI_{-1/9} Lie algebra

$$(S^a{}_b) = \begin{pmatrix} \cosh \theta^3 & -\sinh \theta^3 & \sqrt{2}\theta^2 \\ -\sinh \theta^3 & \cosh \theta^3 & \sqrt{2}\theta^1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.53)$$

The momentum constraints remain to be satisfied in this nondiagonal case. Using the standard Misner parametrization for the β^a variables and the above θ^a variables in the constraint equations leads to

$$\theta^2 = \theta^1, \quad \theta^3 = -\beta^+. \quad (2.54)$$

These conditions together with the transformation

$$\beta^0 = \tilde{\beta}^0, \quad \beta^+ = -\frac{\sqrt{3}}{2}\tilde{\beta}^+, \quad \theta^1 = \varphi e^{\beta^+}, \quad (2.55)$$

leads to the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}x[-(\tilde{\beta}^0)^2 + (\tilde{\beta}^+)^2 + \frac{2}{3}e^{-4\sqrt{3}\tilde{\beta}^+}\dot{\varphi}^2] \\ &\quad + x^{-1}[32e^{4\tilde{\beta}^0-2\sqrt{3}\tilde{\beta}^+} + 24\ell^\gamma e^{3(2-\gamma)\tilde{\beta}^0}], \end{aligned} \quad (2.56)$$

where ℓ is the same constant as in the diagonal case. Since the fluid is orthogonal, it follows from Eq. (2.5) that it has the same potential as in the diagonal case.

The momentum p_φ associated with the cyclic coordi-

nate φ is constant, leading to

$$\dot{\varphi} = \frac{3}{2}x^{-1}e^{4\sqrt{3}\tilde{\beta}^+}p_\varphi \quad (2.57)$$

and the reduced Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}x[-(\dot{\tilde{\beta}}^0)^2 + (\dot{\tilde{\beta}}^+)^2] + x^{-1}[\frac{3}{4}e^{4\sqrt{3}\tilde{\beta}^+}p_\varphi^2 \\ &\quad + 32e^{4\tilde{\beta}^0-2\sqrt{3}\tilde{\beta}^+} + 24\ell^\gamma e^{3(2-\gamma)\tilde{\beta}^0}] = 0. \end{aligned} \quad (2.58)$$

Note that $\varphi = 0 = p_\varphi$ reduces this case to the corresponding diagonal case.

4. Spatially homogeneous class A models belonging to the symmetric case

The class A perfect fluid models with an equation of state $p = (\gamma - 1)\rho$ admit the special case where the fluid four-velocity has a single nonzero spatial component u_3 and where g_{ab} in the line element

$$ds^2 = -N(t)^2 dt^2 + g_{ab}(t)\omega^a\omega^b \quad (2.59)$$

has only one nonzero offdiagonal coefficient $g_{12} = g_{21}$. This is referred to as a ‘‘symmetric case’’ [7]. For Bianchi type II it is now more convenient to choose the alternative structure constant values $(n^{(1)}, n^{(2)}, n^{(3)}) = (1, 0, 0)$ in place of those of Table I.

As in the previous case, g_{ab} can be conveniently parametrized by $g_{ab} = S^c{}_a S^d{}_b g'_{cd}$, where g'_{ab} is the same but the special automorphism matrix $S^a{}_b$ is now

$$(S^a{}_b) = \begin{pmatrix} c_3 & -\hat{n}^{(1)}s_3 & 0 \\ \hat{n}^{(2)}s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.60)$$

where

$$\begin{aligned} (\hat{n}^{(1)}, \hat{n}^{(2)}) &= e^{-\alpha^3}(n^{(1)}, n^{(2)}), \quad \hat{m}^{(3)} = (-\hat{n}^{(1)}\hat{n}^{(2)})^{1/2}, \\ e^{\alpha^3} &= 2^{-1/2}[(n^{(1)})^2 + (n^{(2)})^2]^{1/2}, \\ c_3 &= \cosh \hat{m}^{(3)}\theta^3, \quad s_3 = (\hat{m}^{(3)})^{-1/2} \sinh \hat{m}^{(3)}\theta^3. \end{aligned} \quad (2.61)$$

For Bianchi types I and II where some of these expressions are undefined, one defines them as the limit in which $n^{(2)} \rightarrow 0$ and in Bianchi type I one then lets $n^{(1)} \rightarrow 0$ [7].

The Hamiltonian constraint (2.5) is then

$$H = T_{(G)} + U_{(G)} + U_{(\text{fluid})} = 0, \quad (2.62)$$

where the kinetic part is given by [7]

$$T_{(G)} = \frac{1}{2}x[\eta_{AB}\dot{\beta}^A\dot{\beta}^B + \frac{1}{12}e^{-2\alpha^3}(h_-)^2(\dot{\theta}^3)^2], \quad (2.63)$$

while the gravitational potential $U_{(G)}$ is the same as for the diagonal case. The fluid potential is given by

$$\begin{aligned} U_{(\text{fluid})} &= 2\kappa N g^{1/2} n^\alpha n^\beta T_{\alpha\beta} \\ &= 2\kappa N g^{1/2} [(\rho + p)(n^\alpha u_\alpha)^2 - p] \\ &= 24\kappa x^{-1} e^{6\beta^0} [(\rho + p)Y - p], \end{aligned} \quad (2.64)$$

where $Y = (n^\alpha u_\alpha)^2$. As for the diagonal case for an

equation of state $p = (\gamma - 1)\rho$ one can introduce $\rho = n^\gamma$. For these models there exists a constant of motion: [7,16]

$$\ell = (-n^\alpha u_\alpha) n g^{1/2} = Y^{1/2} \rho^{1/\gamma} e^{3\beta^0}. \quad (2.65)$$

Solving for ρ leads to

$$\rho = \ell^\gamma e^{-3\gamma\beta^0} Y^{-\gamma/2}. \quad (2.66)$$

This relation is used to eliminate ρ in the fluid potential

$$U_{(\text{fluid})} = 24\kappa\ell^\gamma x^{-1} e^{3(2-\gamma)\beta^0} Y^{-\gamma/2} [\gamma Y - (\gamma - 1)]. \quad (2.67)$$

There is an additional constant of the motion v_3 defined by $v_3 = \mu u_3$ where $\mu = \gamma n^{\gamma-1} = \gamma \rho^{(\gamma-1)/\gamma}$ [7,16]. This constant of the motion can be used to express Y as a function of β^0 and β^+ . Expressing $u^\alpha u_\alpha = -1$ in terms of Y and v_3 yields the implicit relation

$$FY^{\gamma-1} - Y + 1 = 0, \quad (2.68)$$

where

$$F = \gamma^{-2} \ell^{-2(\gamma-1)} v_3^2 e^{2[(3\gamma-4)\beta^0 + 2\beta^+]}. \quad (2.69)$$

For dust ($\gamma = 1$, a case discussed in [17]) and stiff ($\gamma = 2$) perfect fluids this equation is linear and can be explicitly solved. For some other values of γ (namely, $\frac{5}{4}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{7}{4}$), it reduces to at most a fourth degree polynomial equation which can be explicitly solved in principle. However, for certain purposes an explicit expression is not required, as will become apparent later. Note that the stiff perfect fluid potential is the sum of two exponentials.

The variable θ^3 is cyclic and so has a constant conjugate momentum \tilde{P}_3 and the equation of motion

$$\dot{\theta}^3 = 12x^{-1} e^{2\alpha^3} (h_-)^{-2} \tilde{P}_3. \quad (2.70)$$

The single nontrivial supermomentum constraint requires [7]

$$e^{\alpha^3} \tilde{P}_3 = -2\kappa\ell v_3, \quad (2.71)$$

leading to

$$H = \frac{1}{2} x \eta_{AB} \dot{\beta}^A \dot{\beta}^B + U_{(c)} + U_{(G)} + U_{(\text{fluid})}, \quad (2.72)$$

where

$$U_{(c)} = x^{-1} 24\ell^2 \kappa^2 (v_3)^2 (h_-)^{-2}. \quad (2.73)$$

Note that in the type-II case and in the type-VI₀ Taublike symmetric case (a special case defined by $\beta^- = 0$), the ‘‘centrifugal potential’’ $U_{(c)}$ is just an exponential and a constant, respectively.

E. Some remarks

The HH symmetry types include a wide range of models describing quite different physical situations. For ex-

ample, in the diagonal case one may consider any combination of sources by including the corresponding potential terms in the Hamiltonian, which together with the two possible signs of ϵ leads to numerous space-time models that have been considered in the literature. Many people have attacked such problems individually as though they were completely unrelated to the others, each time writing out the field equations and attempting to solve them. However, the Hamiltonian approach reveals the close *mathematical* relationship which exists between them.

For example, all the models considered are characterized by a Hamiltonian which can be reduced to the form

$$H = \frac{1}{2} x \eta_{AB} \dot{\beta}^A \dot{\beta}^B + x^{-1} U_{(\text{Taub})} = 0, \quad (2.74)$$

where the Taub potential, $U_{(\text{Taub})}$, is thus just the value of the total potential in the Taub slicing gauge $x = 1$. The Lorentz character of the kinetic part of the Hamiltonian [18] suggests that Lorentz transformations of the dependent β^A variables play an important role in analyzing the dynamics [19]. Indeed such transformations can be used to reveal how many different problems are mathematically equivalent.

In many cases the Taub potential is a sum of exponentials (SE):

$$U_{(\text{Taub})} = \sum_i A_i e^{\alpha^i \beta^A}. \quad (2.75)$$

Problems of this type are important because the gravitational potential and many source potentials are sums of exponentials when expressed in the Taub slicing gauge.

By inspecting the Hamiltonian one can see that some of these various problems are either equivalent or closely connected. For example, changing the sign of one of the variables $g_{\alpha\alpha}$ in the class A gravitational potential (2.13) to go from the SH case to a corresponding static case either leaves the Hamiltonian unchanged or is equivalent to a change of Bianchi type, e.g., Bianchi types VIII and IX are interchanged by this operation. Thus there is an isomorphism between various static models and the SH ones. Furthermore, the MT cases (2.21) have the same gravitational potential modulo the sign of $\epsilon\sigma$. When it comes to sources the cosmological constant and electromagnetic terms only change sign with the change in sign of ϵ . However, a perfect fluid potential differentiates between the static and homogeneous cases in an essential way since the four-velocity must be along the timelike direction in each case.

The close relationship among many Hamiltonians explains the similarity of the expressions resulting from solving the field equations for the different models.

III. THE GENERALIZED FRIEDMANN EQUATION

The generalized Friedmann equation [20] is an equation of the form

$$\dot{\alpha}^2 = x_\alpha^{-2} \sum_{i=1}^n a_i e^{q_i \alpha} \quad (3.1)$$

in a single dependent variable α , where q_i are a set of distinct constants ordered by increasing value. This generalizes the well-known Friedmann equation which has this form for the scale factor $R = e^\alpha$. Here the arbitrary function x_α allows different choices of the independent variable. For convenience, it will be called the slicing gauge function.

The generalized Friedmann equation may be converted from exponential potentials to power-law potentials by introducing the power variable

$$u = e^{\delta \alpha}, \quad (3.2)$$

where $\delta \neq 0$ is a constant parameter. This leads to

$$\dot{u}^2 = x_\alpha^{-2} \delta^2 \sum_{i=1}^n a_i u^{(2+q_i/\delta)}. \quad (3.3)$$

A. The power-law slicing gauge approach

To solve the generalized Friedmann equation, it is often convenient to introduce a power-law slicing gauge function [21]

$$x_\alpha^2 = e^{\Delta \alpha} = u^{\Delta/\delta}, \quad (3.4)$$

where Δ is a constant parameter. This yields

$$\begin{aligned} \dot{\alpha}^2 &= \sum_{i=1}^n a_i e^{(q_i - \Delta)\alpha}, \\ \dot{u}^2 &= \delta^2 \sum_{i=1}^n a_i u^{r_i}, \\ r_i &= 2 + \frac{q_i}{\delta} - \frac{\Delta}{\delta}. \end{aligned} \quad (3.5)$$

It is convenient to refer to the right-hand sides of either equation as the ‘‘potential’’ for that variable.

When only one potential term is present, one can either choose $\Delta = q_1$ or $\Delta = 2\delta = q_1$ so that α or u , respectively, is affinely related to the independent variable.

In the case of more than one potential term one can always treat one of the terms in the same way one deals with the single term in the case with only one potential term. However, since there are two parameters available (Δ, δ) , one can vary these so that any two power exponents (r_i, r_j) assume any pair of real values. Conversely, given the values of a pair of the original power exponents (q_i, q_j) , the parameters (Δ, δ) are determined by the values of the corresponding new power exponents (r_i, r_j) in the following manner. The scale parameter δ determines the ratio of the power exponent increments

$$\delta = (q_j - q_i)/(r_j - r_i), \quad (3.6)$$

after which the additive parameter Δ is determined by either of the relations

$$\Delta = q_i - \delta(r_i - 2) = q_j - \delta(r_j - 2). \quad (3.7)$$

In the case of more than one potential term the obvious way to fix the parameters (Δ, δ) is to obtain the lowest polynomial power variable potential that exists, if any. However, even transforming the potential to a polynomial may be impossible since this requires that the original power exponents (q_i) be affinely related to a set of non-negative integers. When this is the case for a given set of integers, there are two choices of the pair (Δ, δ) for which a polynomial potential occurs, corresponding to a positive and negative value of the nonzero power variable parameter δ . Thus one always has two different slicing gauges with the same degree polynomial potential, but whose dependent variables are related to each other in an inverse power relationship. One is an increasing function and the other a decreasing function of $R = e^\alpha$.

In general, for a case reducible to a polynomial potential, if $q > 0$ denotes the minimum increment between the ordered coefficients (q_1, \dots, q_n) , then the choice

$$(\Delta, \delta) = (q_1 + 2q, q) \quad (3.8)$$

leads to the polynomial potential with q_1 corresponding to the constant term, while the choice

$$(\Delta, \delta) = (q_n - 2q, -q) \quad (3.9)$$

leads to a polynomial potential where the last term associated with the final value q_n is the constant term. These are the minimal degree polynomial potentials that are possible. The solutions of the generalized Friedmann problem for polynomial potentials up to degree 2 (4) can be expressed in terms of elementary (elliptic) functions.

In the case of two potential terms, one can reduce the potential to polynomials of first or second degree. The parameters (δ, Δ) determining the power variables and choices of gauge function which accomplish this are given in Table II, together with the resulting powers.

If in the case of three potential terms the coefficients (q_1, q_2, q_3) are ‘‘equally spaced,’’ i.e., $q_2 - q_1 = q_3 - q_2$, then there are two values of the pair (Δ, δ) which map them onto either (r_i) equal to $(0, 1, 2)$ or $(2, 1, 0)$. Completing the square leads to a quadratic two potential term problem for a new variable \bar{u} affinely related to u and the

TABLE II. The choices of (Δ, δ) for the two-term potential case which lead to a linear or quadratic potential. (r_1, r_2) are the new powers.

(r_1, r_2)	Δ	δ
$(0, 1)$	$2q_2 - q_1$	$q_2 - q_1$
$(1, 0)$	$2q_1 - q_2$	$q_1 - q_2$
$(0, 2)$	q_2	$\frac{1}{2}(q_2 - q_1)$
$(2, 0)$	q_1	$\frac{1}{2}(q_1 - q_2)$
$(1, 2)$	q_2	$q_2 - q_1$
$(2, 1)$	q_1	$q_1 - q_2$

same kinds of elementary functions for a \bar{u} result that occur for u in two term quadratic case with no first power term. For details regarding all of these cases and those involving more than three potential terms, see [20].

Often the generalized Friedmann equation occurs as a first integral of a second-order equation of the general form

$$\ddot{\theta} + \delta \dot{\theta}^2 + f(e^\theta) = 0, \quad (3.10)$$

where $f(\theta)$ is a sum of exponential terms. This equation has the first integral

$$\mathcal{E} = e^{2\delta\theta}[\dot{\theta}^2 - h(e^\theta)], \quad h(e^\theta) = -2e^{-2\delta\theta} \int e^{2\delta\theta} f(e^\theta) d\theta, \quad (3.11)$$

which may be rewritten in the form of the generalized Friedmann equation as

$$\dot{\theta}^2 = h(e^\theta) + \mathcal{E}e^{-2\delta\theta}, \quad (3.12)$$

or, in power form,

$$\dot{u}^2 = H(u) + \delta^2 \mathcal{E}, \quad H(u) = \delta^2 u^2 h(u^{1/\delta}), \quad (3.13)$$

for the power variable $u = e^{\delta\theta}$ when $\delta \neq 0$. Note that these have one more exponential potential than the number of potential terms in the original second-order equation. As long as $f(e^\theta)$ has no terms such as $e^{-2\theta}$, then $h(e^\theta)$ will also consist of only exponential terms and $H(u)$ of powers of u .

Often the generalized Friedmann equation occurs as a decoupled equation in a larger problem. In this case the slicing gauge function may be fixed by considerations related to the larger problem. When the other part of the problem consists of evaluating quadratures, one tries to find a slicing gauge so that both the generalized Friedmann equation and the additional quadratures become as simple as possible.

B. The intrinsic slicing gauge approach

An intrinsic slicing gauge is defined as a slicing gauge which relates a linear combination of the β^A variables—or an exponential of such a linear combination (a so-called “power variable” discussed in Sec. IV D), affinely to the independent variable, in effect making that dependent variable the new independent variable for the problem. This type of slicing gauge has played a prominent role in the context of static HH models, e.g., for the spherically symmetric models one usually chooses the power variable r as the independent variable. In contrast the power-law slicing gauges have been much more important for the SH models. These natural selection effects seem a bit strange given that the static HH models and SH models are so closely related mathematically.

Recall that the generalized Friedmann equation can either be expressed in its “exponential” form $\dot{\alpha}^2 = x_\alpha^{-2} \sum_{i=1}^n a_i e^{q_i \alpha}$ or its “power” form $\dot{u}^2 = x_\alpha^{-2} \delta^2 \sum_{i=1}^n a_i u^{2+q_i \delta}$ where $u = e^{\delta \alpha}$. In this context an

intrinsic slicing gauge means that one either chooses α or some u variable as the independent variable. Such choices are accomplished by setting

$$\begin{aligned} \alpha = \lambda : \quad x_\alpha &= \left(\sum_{i=1}^n a_i e^{q_i \alpha} \right)^{1/2}, \\ u = \lambda : \quad x_\alpha &= \delta \left(\sum_{i=1}^n a_i u^{2+q_i \delta} \right)^{1/2}. \end{aligned} \quad (3.14)$$

These choices may seem trivial. However, if the generalized Friedmann equation is part of a larger problem, they may not be. For example, one often encounters problems where one has a cyclic variable which is given by a quadrature of the form $\beta^A = \int x_\alpha^{-1} \eta^{AB} \bar{p}_B d\lambda$, where \bar{p}_A is the constant conjugate momenta associated with β^A . Choosing a power variable u leads to the integral $\beta^A = \eta^{AB} \bar{p}_B \delta \int (\sum_{i=1}^n a_i u^{2+q_i \delta})^{-1/2} du$ which gives the dependent variable as a function of the independent one. If on the other hand, one chooses a power-law slicing gauge, this may simplify the equation for the β^A variable while transferring the difficulties to the generalized Friedmann equation. Even if this latter equation is solvable in terms of a quadrature, it gives the independent variable as a function of the dependent variable u , in contrast with the intrinsic approach which yields the dependent variable as a function of the independent one. Thus one is faced with the problem of trying to invert a quadrature in the power-law gauge approach, something which often fails to lead to familiar functions. Thus the intrinsic approach sometimes has advantages. As an example, see [22], where the intrinsic approach was used in the context of SH Bianchi type-V orthogonal perfect fluid models.

IV. KILLING TENSOR SYMMETRIES AND HOW TO USE THEM

To solve a Hamiltonian problem, one needs to find and exploit symmetries which lead to constants of the motion. In the *search* for such symmetries, a particular slicing gauge will be introduced which makes the equations of motion equivalent to the geodesic equations associated with a certain metric (not to be confused with the spacetime metric). This makes it natural to look for a particular type of symmetry, called a Killing tensor symmetry, which generalizes symmetries corresponding to cyclic variables and Hamilton-Jacobi separability.

However, this slicing gauge is not usually well suited to exploiting the symmetry so that *explicit* exact solutions can be obtained. The existence of constants of the motion is not sufficient to obtain such solutions explicitly. This requires the stronger condition of a decoupling of the equations of motion (which in turn leads to constants of the motion), for which other slicing gauges are needed. Even within this latter class of slicing gauges there are choices of gauge which lead to simpler forms of the exact solutions. The situation is similar to the case of a cyclic variable. Any slicing gauge which does not in-

volve this cyclic variable leads to a constant of the motion and decoupling. However, some choices lead to simpler equations and simpler expressions for their solutions.

The present section discusses the symmetry properties of Hamiltonians and how to explicitly find solutions of the equations of motion in as simple a form as possible. It then relates these results to the Hamiltonians of the type encountered when dealing with the HH models described in Sec. II.

A. Why Killing tensor symmetries?

In classical mechanics one usually encounters Hamiltonians of the form

$$H = \frac{1}{2}g^{ab}p_ap_b + U = E, \quad (4.1)$$

where the symmetric matrix (g^{ab}) is positive definite. For such problems there exists an elegant geometric reformulation of the corresponding equations to a geodesic flow on a certain geometry [23–25]. To accomplish such a reformulation one first introduces a new Hamiltonian $\mathcal{H} = H - E = 0$. A Hamiltonian system of this kind can be reparametrized by choosing a new independent variable $\bar{\lambda}$ in place of the original one λ , leading to a new Hamiltonian

$$\mathcal{H}_{\mathcal{N}} = \mathcal{N}\mathcal{H} = \mathcal{N}(H - E), \quad \mathcal{N} = d\lambda/d\bar{\lambda}. \quad (4.2)$$

The final step is to make the Hamiltonian purely kinetic. This is accomplished by a particular choice of parametrization, $\mathcal{N}_J = \frac{1}{2}(E - U)^{-1}$, and by adding a constant to the Hamiltonian:

$$H_J = \mathcal{H}_{\mathcal{N}_J} + \frac{1}{2} = \frac{1}{4(E - U)}g^{ab}p_ap_b = \frac{1}{2}. \quad (4.3)$$

The corresponding Lagrangian equations can then be reinterpreted as those of the geodesic flow of the so-called Jacobi metric $J_{ab} = 2(E - U)g_{ab}$, where g_{ab} is the matrix inverse of g^{ab} , and $\bar{\lambda}$ is an affine parameter along each geodesic.

The geodesic reformulation does not rely on the positive definiteness of g_{ab} . It works locally for any nondegenerate indefinite matrix g_{ab} . For the HH spacetimes of Sec. II, the Hamiltonian can be put in the form

$$H = \frac{1}{2}x\eta^{AB}\dot{\beta}^A\dot{\beta}^B + x^{-1}U_{(\text{Taub})} = 0. \quad (4.4)$$

The Hamiltonian is already parametrized and must vanish, so there is no need for the first steps in the above procedure. The choice

$$x = x_J = 2|U_{(\text{Taub})}| \quad (4.5)$$

leads to a Hamiltonian with a constant value of the potential. Redefining the Hamiltonian by this constant yields

$$H_J = T_J = \frac{1}{2}J_{AB}\dot{\beta}^A\dot{\beta}^B = -\frac{1}{2}\text{sgn}(U_{(\text{Taub})}), \quad (4.6)$$

where $J_{AB} = x_J \eta_{AB}$. In contrast with the usual classical

mechanical problems, the underlying geometry here is Lorentzian rather than Riemannian.

To solve a Hamiltonian problem one needs to find enough symmetries leading to constants of the motion, i.e., variational symmetries, which can be used to reduce the problem sufficiently so that the reduced problem can be solved. The Jacobi formulation is especially suitable for finding such symmetries since they and their associated constants of the motion take a particularly simple form in this formulation as discussed below. Furthermore, the geometric framework makes available a wide range of tools familiar from symmetry investigations on ordinary spacetime.

Variational symmetries are transformations of the phase space which can be represented as transformations on the tangent bundle (velocity phase space) generated by vector fields of the form [26]

$$\mathbf{v} = \phi^a(x, \dot{x}) \frac{\partial}{\partial x^a}. \quad (4.7)$$

The simplest variational symmetries are the point symmetries of the configuration space itself. In the Jacobi formulation, a generator $\phi^a(x)\partial/\partial x^a$ of a point symmetry corresponds to a Killing vector field of the Jacobi metric [27]. All other variational symmetries involve derivatives of the dependent variables and are called generalized symmetries [26].

The simplest generalized symmetries are the ones for which the components of their generating vector fields are linear and homogeneous in the derivatives, i.e., $\phi^a = K^a_b(x)\dot{x}^b$. This corresponds exactly to a second rank Killing tensor K^a_b of the Jacobi metric [27]. Recall that a Killing tensor K_{ab} is a symmetric tensor for which $K_{(ab;c)} = 0$; this includes the trivial case in which K_{ab} is proportional to J_{ab} and the special case in which the Killing tensor is the symmetrized tensor product of Killing vectors. Killing vector symmetries give rise to constants of the motion which are linear and homogeneous in the momenta, while second rank Killing tensor symmetries correspond to homogeneous quadratic constants of the motion ($K^{ab}p_ap_b = \text{const}$) [4,28]. This is another way of understanding how Killing tensor symmetries are the natural generalizations of Killing vector symmetries.

The separability condition for the Hamilton-Jacobi (HJ) equation is equivalent to the existence of a sufficient number of second rank Killing tensor symmetries of the Jacobi metric J_{ab} [28]. Furthermore, in the case of an indefinite Jacobi metric, separability requires the Killing tensors to have non-null eigendirections. However, it turns out that Killing tensors with null eigendirections can also be used to solve the Hamiltonian equations. Thus, in contrast with the positive-definite case, Killing tensor symmetries in the indefinite case are more powerful than Hamilton-Jacobi separability when searching for exact solutions.

The Jacobi slicing gauge is not the most convenient gauge to use when it comes to actually producing exact solutions. There are more suitable symmetry compatible choices of independent variable. However, other

such choices lead to corresponding constants of the motion which are quadratic but not homogeneous in the momenta. Since the constant of the motion takes an especially simple form in the Jacobi slicing gauge, entirely characterized by the Killing tensor alone, it is this gauge which makes *finding* the associated Killing tensor symmetries as simple as possible. Thus it seems clear that a search for these symmetries is a natural first step in attempting to find exactly solvable problems.

Many of the HH field equations have long been known to admit a Hamiltonian formulation very much like the traditional classical mechanical problem except for the signature difference. Therefore it is surprising that no attempt has ever been made to solve the field equations by means of Hamilton-Jacobi separation given its success in the Riemannian case. It is perhaps more understandable that no one has used the more powerful but less familiar Killing tensor techniques.

B. Killing tensor symmetries

For many of the known exact HH solutions, the field equations are ultimately expressible in terms of 2 non-trivial degrees of freedom (although more variables may be involved). It is therefore useful to start with the case of 2 degrees of freedom, reviewing and extending earlier work [29].

When dealing with two-dimensional geometries the group of conformal transformations plays a particularly important role since in this case it is infinite dimensional. For two-dimensional Lorentzian geometries it is useful to use null variables since they are closely related to this group. A Lorentzian two-metric can always be written in the form [15]

$$ds^2 = -2G(w, v)dw dv \quad (4.8)$$

in terms of the null variables w and v . A general conformal transformation to new null variables \tilde{V} and \tilde{W} is of the form $v = F(\tilde{V})$, $w = \tilde{F}(\tilde{W})$, and results in a new conformal factor $\tilde{G} = F'(\tilde{V})\tilde{F}'(\tilde{W})G$. One may then introduce new conformally inertial coordinates which diagonalize the metric

$$T = \frac{1}{2}(\tilde{W} + \tilde{V}), \quad X = \frac{1}{2}(\tilde{W} - \tilde{V}). \quad (4.9)$$

It turns out to be convenient to classify the geometries allowing Killing tensors into three cases characterized by different conditions on the form to which the conformal factor can be transformed. These three cases can be characterized as [29]

- (i) (null) : $\tilde{G} = [A(\tilde{W})V(\tilde{V}) + B(\tilde{W})][dV(\tilde{V})/d\tilde{V}]$,
 - (ii) (non-null HJ) : $\tilde{G} = C(T) + D(X)$,
 - (iii) (non-null harmonic) :
- $$\tilde{G}_{,\tilde{V}\tilde{V}} + \tilde{G}_{,\tilde{W}\tilde{W}} = \frac{1}{2}(\tilde{G}_{,TT} + \tilde{G}_{,XX}) = 0. \quad (4.10)$$

In the above expressions A, B, C, D, V are arbitrary functions and the variables \tilde{V}, \tilde{W} or T, X will be referred to as symmetry-adapted variables.

The Killing tensor in case (i) is characterized by having a degenerate eigenvalue corresponding to a single null eigenvector. The remaining cases (ii) and (iii) have a Killing tensor characterized by nonnull eigenvectors. Case (iii) corresponds to a conformal factor satisfying Laplace's equation which is therefore a harmonic function of \tilde{W}, \tilde{V} or equivalently of T, X . Of the three Killing tensor cases it is only case (ii) that corresponds to Hamilton-Jacobi separability. Cases (i) and (ii) are of special importance. Case (i) will be referred to as the "null case" while, for simplicity, case (ii) will be referred to as the "non-null case" unless explicitly qualified as the "HJ case." Case (iii) will be referred to as the "harmonic case."

The expression for the conformal factor in case (i) can be simplified to

$$(ib) \quad \tilde{G} = A(W)V + B(W) \quad (4.11)$$

by changing the variables to $V = \int V_{,\tilde{V}} d\tilde{V}$ and $\tilde{W} = W$. However, the original expression is more convenient to use as starting point for an analysis of some of the examples to be discussed below. Note that form (i) or (ib) is invariant under the transformations $W \rightarrow W(\tilde{W})$, leading to an equivalence class of symmetry-adapted dependent variables.

When a Killing vector exists rather than a nontrivial Killing tensor, Eq. (4.10) continues to hold with either $AB = 0$ or $CD = 0$. The condition $AB = 0$ is associated with the existence of a null Killing vector, leading to a flat Jacobi geometry. The condition $CD = 0$ corresponds to a non-null Killing vector field, with $C = 0$ describing the timelike case and $D = 0$ the spacelike case.

Case (ii) is also generalizable directly to a two-dimensional subblock of the metric in higher dimension or to the case of complete separability where this factor has the form

$$\tilde{G} = C(T) + \sum_i D_i(X^i). \quad (4.12)$$

The eigenvectors of the Killing tensor in the subblock case are aligned with the conformally inertial coordinate directions T and X . In the case of complete separability one has a set of non-null Killing tensors whose eigenvectors are aligned with T and X^i . Intermediate cases between these two extreme cases are also possible. In the special case of a Killing vector, one can choose symmetry-adapted conformally inertial coordinates such that the conformal factor is independent of one of them.

C. How to use Killing tensor symmetries

We are not just interested in obtaining a sufficient number of constants of the motion to solve the problem implicitly. We want to obtain *explicit* exact solutions. To do this one needs to decouple a sufficient number of equations. In general the Jacobi slicing gauge does not allow this even if one has a sufficient number of Killing tensor symmetries. However, there are other choices of the

independent variable which permits this if one chooses symmetry-adapted dependent variables. Since there is some freedom in the choice of the dependent variables, this freedom may be exploited to help simplify the problem further. A specific choice of dependent variables often suggests which independent variable to use to actually integrate the equations of motion.

To simplify the discussion only two-dimensional examples will be considered. To find useful slicing gauges one must first reexpress the general-slicing-gauge Hamiltonian in terms of the symmetry-adapted dependent variables and the special conformal factor \tilde{G} associated with the existence of one of the three types of Killing symmetries. This then suggests a choice of independent variable in terms of which the equations of motion can be explicitly solved.

Suppose one has a Hamiltonian of the form

$$H = \frac{1}{2}x(-\dot{\alpha}^2 + \dot{\beta}^2) + x^{-1}U_{(\text{Taub})}. \quad (4.13)$$

By introducing the null variables

$$w = \alpha + \beta, \quad v = \alpha - \beta, \quad (4.14)$$

one transforms the Hamiltonian to the form

$$H = -\frac{1}{2}x \dot{w} \dot{v} + x^{-1}U_{(\text{Taub})}. \quad (4.15)$$

This Hamiltonian corresponds to a Jacobi geometry with the conformal factor $G = |U_{(\text{Taub})}|$ (recall that $x_J = 2|U_{(\text{Taub})}|$).

Reexpressing the Hamiltonian in terms of a new set of null variables $w = w(\bar{W})$ and $v = v(\bar{V})$ leads to

$$H = -\frac{1}{2}x (dw/d\bar{W})(dv/d\bar{V}) \dot{\bar{W}} \dot{\bar{V}} + x^{-1}U_{(\text{Taub})}. \quad (4.16)$$

If the Jacobi geometry admits a Killing tensor and these new variables are associated symmetry-adapted variables, then one can make the identification

$$\tilde{G} = (dw/d\bar{W})(dv/d\bar{V}) |U_{(\text{Taub})}|. \quad (4.17)$$

Inserting this relationship into the Hamiltonian leads to

$$H = -\frac{1}{2}x |U_{(\text{Taub})}|^{-1} \tilde{G} \dot{\bar{W}} \dot{\bar{V}} + x^{-1}U_{(\text{Taub})}. \quad (4.18)$$

It is convenient to introduce a new slicing gauge function y defined by

$$y = |U_{(\text{Taub})}|^{-1} \tilde{G} x, \quad (4.19)$$

when discussing symmetry-adapted slicing gauges. Expressing the above Hamiltonian in terms of \tilde{G} and y yields

$$H = -\frac{1}{2}y \dot{\bar{W}} \dot{\bar{V}} + y^{-1} \text{sgn}(U_{(\text{Taub})}) \tilde{G}. \quad (4.20)$$

The two cases (i) and (ii) in Eq. (4.10) will be treated separately when it comes to finding slicing gauges which lead to decoupling and explicit solution of the equations of motion. Case (iii) does not seem to have the same importance as the first two Killing tensor cases and will only be commented on briefly when it does occur.

1. The null Killing tensor case

The Lagrangian equations of motion corresponding to the previous Hamiltonian (4.20) are

$$\ddot{\bar{W}} + y^{-1}(\partial y / \partial \bar{W}) \dot{\bar{W}}^2$$

$$-2y^{-1} \partial(y^{-1} \text{sgn}(U_{(\text{Taub})}) \tilde{G}) / \partial \bar{V} = 0,$$

$$\ddot{\bar{V}} + y^{-1}(\partial y / \partial \bar{V}) \dot{\bar{V}}^2$$

$$-2y^{-1} \partial(y^{-1} \text{sgn}(U_{(\text{Taub})}) \tilde{G}) / \partial \bar{W} = 0. \quad (4.21)$$

Assume that one has chosen symmetry-adapted variables so that the conformal factor takes the simpler form (ib)

$$\text{sgn}(U_{(\text{Taub})}) \tilde{G} = \bar{A}(\bar{W}) \bar{V} + \bar{B}(\bar{W}). \quad (4.22)$$

It is then possible to choose a slicing gauge so that the equation of motion for the variable \bar{W} decouples. Those slicing gauges which allow this decoupling are characterized by the condition

$$y = y(\bar{W}) \quad (4.23)$$

and will be referred to as decoupling slicing gauges. A decoupling of the variable \bar{V} occurs for the analogous conditions with \bar{V} and \bar{W} interchanged.

For a given choice of $y(\bar{W})$ one can choose new dependent variables W and V defined by

$$dW/d\bar{W} = y(\bar{W}), \quad V = \bar{V}. \quad (4.24)$$

This transformation eliminates the quadratic first derivative term from the decoupled equation and leads to the Hamiltonian

$$\begin{aligned} H &= -\frac{1}{2} \dot{W} \dot{V} + [d\bar{W}(W)/dW] \{ \bar{A}[\bar{W}(W)]V + B[\bar{W}(W)] \} \\ &\equiv -\frac{1}{2} \dot{W} \dot{V} + A(W)V + B(W), \end{aligned} \quad (4.25)$$

which has its kinetic energy in the standard Minkowski null form.

A Hamiltonian of this final form leads to the decoupled equation

$$\ddot{W} = 2A(W), \quad (4.26)$$

which has the first integral

$$\dot{W}^2 = 4 \int A(W) dW + \text{const}. \quad (4.27)$$

Rewriting this in the form

$$\frac{1}{2} \dot{W}^2 + \mathcal{U}(W) = E, \quad \mathcal{U}(W) = -2 \int A(W) dW, \quad (4.28)$$

where E is a constant, allows $\mathcal{U}(W)$ to be interpreted as a potential for this one-dimensional problem.

By appropriately choosing the function $y(\bar{W})$, one can make the function $A(W)$ take any desired form. How-

ever, there are preferred choices. For example, choosing $A(W)$ to be proportional to one of the powers 0, 1, or -3 of W or to a sum of terms involving the powers 0 and 1 of W leads to a generalized Friedmann problem with elementary function solutions for W as a function of the independent variable. The first two single term choices lead to a linear and quadratic potential, while in the last single term case the transformation $\bar{W} = W^2$ leads to a linear potential. The combination of 0 and 1 power terms leads to a quadratic polynomial potential. All of these potentials were discussed in Sec. III. After a choice of gauge, leading to a particular form for $A(W)$, the remaining variable V is determined by first solving the Hamiltonian constraint for \dot{V} , yielding

$$\begin{aligned}\dot{V} &= (2/\dot{W})[A(W)V + B(W)] \\ &= \dot{W}V/\dot{W} + 2B(W)/\dot{W},\end{aligned}\quad (4.29)$$

and then integrating this linear first-order equation, leading to

$$V = \dot{W} \left(\text{const} + 2 \int B(\dot{W})^{-2} d\lambda \right). \quad (4.30)$$

Whether or not the integral can be expressed in terms of elementary functions depends on the specific case one is dealing with.

It follows from the above formulas that the expressions for $U_{(\text{Taub})}$ and x then become

$$\begin{aligned}U_{(\text{Taub})} &= [dW(w)/dw][dV(v)/dv][A(W)V + B(W)] \\ &\equiv [g(w)V(v) + h(w)][dV(v)/dv], \\ x &= [dW(w)/dw][dV(v)/dv].\end{aligned}\quad (4.31)$$

As noted above after Eq. (4.11), the Jacobi geometry is flat if $g(w) = 0$ or $h(w) = 0$. One can often see by inspection whether or not a Taub potential is of the above form. For such cases it is convenient to start with the gauge function x and derive Eq. (4.25). Those slicing gauges which allow decoupling are characterized by the condition

$$x = x_1(w) dV(v)/dv, \quad (4.32)$$

where $x_1(w)$ takes the role y previously played. Decoupling in the Taub slicing gauge itself can only occur if V is a linear function of v . If V is not a linear function of v one makes a variable change from v to V to obtain decoupling. This leads to

$$H = -\frac{1}{2}x_1(w)\dot{w}\dot{V} + x_1(w)^{-1}[g(w)V + h(w)] = 0. \quad (4.33)$$

For a given choice of $x_1(w)$, the choice

$$dW/dw = x_1(w), \quad (4.34)$$

of a new dependent variable W leads to

$$\begin{aligned}x &= [dW(w)/dw][dV(v)/dv], \\ H &= -\frac{1}{2}\dot{W}\dot{V} + A(W)V + B(W),\end{aligned}\quad (4.35)$$

that is, the same expressions as the end result of taking the “ y approach.”

Many Hamiltonians correspond to null Killing tensor cases, thereby explaining the existence of a whole host of exact solutions.

2. The non-null Killing tensor case

Making the transformation $T = \frac{1}{2}(\bar{W} + \bar{V})$, $X = \frac{1}{2}(\bar{W} - \bar{V})$ transforms the Hamiltonian in Eq. (4.20) to

$$H = \frac{1}{2}y(-\dot{T}^2 + \dot{X}^2) + y^{-1} \text{sgn}(U_{(\text{Taub})}) \tilde{G}. \quad (4.36)$$

If one has a non-null Killing tensor case expressed in symmetry-adapted variables so that

$$\text{sgn}(U_{(\text{Taub})}) \tilde{G} = C(T) + D(X), \quad (4.37)$$

then the choice

$$y = 1, \quad x = |U_{(\text{Taub})}| \tilde{G}^{-1}, \quad (4.38)$$

leads to the Hamiltonian

$$H = \frac{1}{2}(-\dot{T}^2 + \dot{X}^2) + C(T) + D(X), \quad (4.39)$$

for which both the T and X equations decouple

$$\ddot{T} = dC(T)/dT, \quad \ddot{X} = -dD(X)/dX. \quad (4.40)$$

These equations give the first integrals

$$-\frac{1}{2}\dot{T}^2 + C(T) = E_t, \quad \frac{1}{2}\dot{X}^2 + D(X) = E_x, \quad (4.41)$$

where the integration constants have to satisfy

$$E_t + E_x = 0, \quad (4.42)$$

because of the Hamiltonian constraint. Equation (4.39) corresponds to Hamilton-Jacobi separability and is generalizable to any dimension.

D. Power variables and power-law slicing gauges

Power variables are often the simplest choice of dependent variables adapted to Killing tensor symmetries. Such a variable is defined by an expression of the form

$$u = u_{(0)} \prod_a R_a^{Q_a} = u_{(0)} e^{Q_a \beta^a} = u_{(0)} e^{Q_A \beta^A}. \quad (4.43)$$

Similarly power-law slicing gauges are often the simplest slicing gauges which lead to decoupling. Such gauges are characterized by the following form of the slicing gauge functions [21]:

$$N = N_{(0)} \prod R_a^{Q_a} = N_{(0)} e^{Q_a \beta^a} = N_{(0)} e^{Q_A \beta^A}$$

or

$$x^{-1} = [N_{(0)}/12] e^{(Q_A \beta^A - 3\beta^0)}. \quad (4.44)$$

The conditions $Q_1 = Q_2 = Q_3$, or equivalently $Q_{\pm} = 0$ or $Q_{\times} = 0$ as appropriate (and $Q_4 = 0$ if a scalar field is present), characterize the isotropic power-law gauges which can be useful for SH models in the case of spatially isotropic sources like an orthogonal perfect fluid or a cosmological constant term, the Taub time gauge being an obvious example. However, because of anisotropic spatial curvatures and sources, one needs to exploit anisotropic dependence of the function N on the individual scale factors to make progress in obtaining solutions or simplifying the equations. The condition $Q_1 + Q_2 + Q_3 = 1$ or equivalently $Q_0 = 1$ leads to a scale invariant independent variable. The conformal gauge corresponds to both conditions holding, leading to $N = N_{(0)}e^{\beta^0}$. Finally it is worth noticing that power variables and power-law slicing gauges are not only useful when it comes to finding symmetry-adapted variables but also in the context of qualitative analysis.

E. Killing tensor symmetries in models characterized by SE Hamiltonians

1. Null decoupling

Consider a potential of the form

$$U_{(\text{Taub})} = [g(w)V(v) + h(w)][dV(v)/dv]. \quad (4.45)$$

If $g(w)$, $h(w)$, and V are sums of exponentials including possible constant additive terms, or if $g(w) = 0$ and V is linear in v , then one has a SE Hamiltonian corresponding to a null Killing tensor case. Of particular interest is the case in which V is a single exponential term or linear in v in the “flat” $g(w) = 0$ case. The remainder of the subsection analyzes this situation. The Taub potential for the cases

$$\begin{aligned} V &= e^{c_0 v}/c_0, \quad c_0 \neq 0, \quad g(w) \neq 0 \quad \text{or} \quad g(w) = 0, \\ V &= v, \quad c_0 = 0, \quad g(w) = 0, \end{aligned} \quad (4.46)$$

can then be formally written as

$$U_{(\text{Taub})} = \left[c_0 \left(\sum_{i=0}^p A_i e^{a_i w} \right) e^{c_0 v} + \sum_{i=0}^q B_i e^{b_i w} \right] e^{c_0 v}. \quad (4.47)$$

Note that the three separate cases $c_0 = 0$, or $A_i = 0$ or $B_i = 0$ for all values of i all correspond to flat cases.

Choosing $x = x_1(w)e^{c_0 v}$ and $V = e^{c_0 v}/c_0$ as a new variable in the $c_0 \neq 0$ case leads to the Hamiltonian

$$\begin{aligned} H &= -\frac{1}{2}x_1(w)\dot{w}\dot{V} \\ &+ x_1(w)^{-1} \left[c_0^2 \left(\sum_{i=0}^p A_i e^{a_i w} \right) V + \sum_{i=0}^q B_i e^{b_i w} \right]. \end{aligned} \quad (4.48)$$

In the case $c_0 = 0$, $U_{(\text{Taub})}$ is independent of v , and the Hamiltonian takes this same form with V equal to v .

Usually the simplest class of choices for the remaining gauge function $x_1(w)$ are power-law slicing gauges. For such gauges this function is proportional to an exponential, leading to

$$x = \hat{B}e^{mw}e^{c_0 v}, \quad (4.49)$$

and the Hamiltonian

$$\begin{aligned} H &= -\frac{1}{2}\hat{B}e^{mw}\dot{w}\dot{V} \\ &+ \hat{B}^{-1}e^{-mw} \left[c_0^2 \left(\sum_{i=0}^p A_i e^{a_i w} \right) V + \sum_{i=0}^q B_i e^{b_i w} \right]. \end{aligned} \quad (4.50)$$

Here one may choose m to be either zero or nonzero.

a. The gauge choice $m = 0$. Setting $\hat{B} = 1$ one then has

$$H = -\frac{1}{2}\dot{w}\dot{V} + \left[c_0^2 \left(\sum_{i=0}^p A_i e^{a_i w} \right) V + \sum_{i=0}^q B_i e^{b_i w} \right]. \quad (4.51)$$

This Hamiltonian leads to a decoupled second-order differential equation which in turn can be integrated to yield a first-order differential equation

$$\dot{w}^2 = 4c_0^2 \sum_i (A_i e^{a_i w})/a_i + \text{const} \quad (4.52)$$

if all the coefficients a_i are nonzero. If not, the single $a_i = 0$ term gives rise to a term linear in w which has to be added to this expression in order to give the correct form for the first-order equation.

The decoupled first-order equation for w arising from the $a_i \neq 0$ case is simply the generalized Friedmann equation, while V may be obtained formally by solving the Hamiltonian constraint (which is linear in V) for its derivative and integrating. To simplify the process of integrating this final pair of equations (abandoning further use of the Hamiltonian and/or Lagrangian approach), one can reintroduce the gauge freedom that was fixed above by a specific choice of $x_1(w)$ made to obtain the decoupled first-order equation. However, usually this extra freedom is not needed to find many simple explicit expressions for the solution of the system.

b. The gauge choice $m \neq 0$. If one instead chooses $m \neq 0$, and follows the procedure after Eq. (4.33) and uses the power variable

$$W = e^{mw} \quad (4.53)$$

and sets $\hat{B} = m$, then one obtains

$$\begin{aligned} H &= -\frac{1}{2}\dot{W}\dot{V} + m^{-1} \left[c_0^2 \left(\sum_{i=0}^p A_i W^{a_i/m-1} \right) V \right. \\ &\left. + \sum_{i=0}^q B_i W^{b_i/m-1} \right]. \end{aligned} \quad (4.54)$$

In the case that all a_i are nonzero, the decoupled equation

gives rise to the power form of the generalized Friedmann equation. An $a_i = 0$ term yields a logarithmic term in the integrated expression of the decoupled second-order differential equation. As in the previous case, to integrate this final pair of equations, one can reintroduce the slicing gauge freedom involving rescaling of λ derivatives by functions of W in order to simplify the problem. Without resorting to this additional freedom, one can make the following case by case analysis of some types of SE Hamiltonians which often arise.

c. The case $A_i = 0$ for all i (the flat case). The gauge choice $m = 0$: This leads to $\dot{w} = 0$, so that w is affinely related to the independent variable. Solving the Hamiltonian constraint for \dot{V} leads to an easily integrated expression for V (or v if $c_0 = 0$) which results in a linear combination of exponentials unless one of the exponential coefficients b_i vanishes, leading to a linear term.

The gauge choice $m \neq 0$: This leads to $\dot{W} = 0$ and W is affinely related to λ . Setting the additive constant to zero, again one can integrate the expression for \dot{V} to obtain V , this time resulting in a linear combination of powers of the independent variable

$$V = V_0 + \sum_i C_i \lambda^{b_i/m}, \quad (4.55)$$

unless one of the coefficients b_i vanishes, leading to a logarithmic term. (Note that if $c_0 = 0$ all the above expressions for the $A_i = 0$ case remain valid with V replaced with v .)

d. The case $c_0 \neq 0, A_0 \neq 0, A_i = 0$ ($i \geq 1$). Subcase $a_0 = 0$: Here the choice $m = 0$ is appropriate, leading to $\dot{w} = \text{const}$ and quadratic solutions for w .

Subcase $a_0 \neq 0$: Here the three choices $m = a_0, \pm \frac{1}{2}a_0$ lead to elementary function solutions for W as described after equation (4.28).

e. Two examples. One null exponential potential term in any dimension: When a problem is characterized by a single null potential term, one first uses a Lorentz transformation from the original conformally inertial coordinates to a new set $\bar{\beta}^0, \bar{\beta}, \bar{\beta}^S$ so that the Hamiltonian takes the form

$$H = \frac{1}{2}x \left(-\dot{\bar{\beta}}^0{}^2 + \dot{\bar{\beta}}^2 + \sum_S \dot{\bar{\beta}}^S{}^2 \right) + \frac{1}{2}x^{-1}K e^{\bar{C}(\bar{\beta}^0 + \bar{\beta})}. \quad (4.56)$$

Then one introduces the null variables

$$\bar{w} = \bar{\beta}^0 + \bar{\beta}, \quad \bar{v} = \bar{\beta}^0 - \bar{\beta}. \quad (4.57)$$

In these variables the Hamiltonian then takes the form

$$H = -\frac{1}{2}x\dot{\bar{w}}\dot{\bar{v}} + \frac{1}{2}x^{-1}[\Theta^2 + K e^{\bar{C}\bar{w}}], \quad (4.58)$$

where $\Theta^2 = \sum_S \bar{p}_S^2$ arises from non-null cyclic variables $\bar{\beta}^S$ and S denotes the index labeling the non-null cyclic variables, assuming that x is chosen to be independent of them. The variables $\bar{\beta}^S$ can be evaluated in terms of quadratures arising from the equations

$$\dot{\bar{\beta}}^S = x^{-1}\bar{p}_S. \quad (4.59)$$

The full problem can therefore be decomposed into a set of quadratures and a two-dimensional problem coming from the above reduced Hamiltonian for w and v . This is a flat null Killing tensor case, $A_i = 0$, which has been discussed above. One need only identify the constants $c_0 = 0, b_0 = 0, b_1 = \bar{C}, B_0 = \frac{1}{2}\Theta^2, B_1 = \frac{1}{2}K$ in Eq. (4.48).

One exponential potential term in two dimensions: The one potential term case corresponds to a flat Jacobi geometry. The single potential term can be identified in two ways with the explicit terms in Eq. (4.47). Either $A_0c_0 \neq 0$ or $B_0 \neq 0$. These two different identifications lead to different natural slicing gauges.

If $A_0 = 0$, then $B_0 \neq 0$, then one can choose $m = 0$ and a translation in λ leading to $w \sim \lambda$. Integration of the Hamiltonian constraint yields an exponential expression for V unless $b_0 = 0$ in which case $V \sim \lambda + \text{const}$. In this latter case the Taub slicing gauge potential is null, i.e., it depends only on the null variable v . If one chooses instead $m \neq 0$, then one can make a translation in λ such that $W \sim \lambda$ and $V \sim \lambda^{b_0/m} + \text{const}$ if $b_0 \neq 0$ or $V \sim \ln \lambda + \text{const}$ if $b_0 = 0$. In the first case the Taub potential is non-null and the choice $m = b_0$ makes V affinely related to the independent variable. When both variables are affinely related to λ , the choice of slicing gauge makes the single potential term in the Hamiltonian a constant, i.e., the gauge is the Jacobi slicing gauge.

If $A_0c_0 \neq 0$, then either $a_0 = 0$ or $a_0 \neq 0$. If $a_0 = 0$, then with $m = 0$, the variable w is quadratic in λ and the expression for “ $\ln V$ ” is a standard integral. If $a_0 \neq 0$ then the three choices $m = a_0, \pm \frac{1}{2}a_0$ are relevant according to the discussion following Eq. (4.27). The choice $m = a_0$ leads to a quadratic solution for W . Choosing the zero of the independent variable to eliminate the linear coefficient leads to power-law solutions for V , i.e., $V \sim \lambda^\sigma$, where σ is determined by the quadratic coefficient and B . For the choice $m = \frac{1}{2}a_0$, W is a hyperbolic/trigonometric sine or cosine and the expression for “ $\ln V$ ” is a standard integral. For the choice $m = -\frac{1}{2}a_0$, W^2 is quadratic in λ leading to “ $\ln V$ ” being a quartic expression. An entirely parallel discussion holds with w and v interchanged corresponding to those cases where v decouples. This will lead to additional slicing gauges.

f. Linear decoupling. It is not necessary to choose null variables to solve problems admitting null Killing tensors. The only requirement is that a variable and its derivative occur linearly in the Hamiltonian. As an example, the Hamiltonian problem corresponding to the potential of Eq. (4.47) can be treated as follows. Making a transformation

$$v = (\ln g + w)/c_0 \leftrightarrow g = e^{c_0 v - w}, \quad (4.60)$$

and choosing a new gauge function $y = x/g$ leads to

$$H = -y \left(\frac{1}{2c_0} \right) (\dot{w}g + g\dot{w}^2) + y^{-1} \left[c_0 \left(\sum_{i=0}^p A_i e^{(a_i+1)w} \right) g + \sum_{i=0}^q B_i e^{b_i w} \right] e^w. \quad (4.61)$$

This Hamiltonian leads to the decoupled equation

$$\ddot{w} + \left(y^{-1} \frac{dy}{dw} - 1 \right) \dot{w}^2 - 2c_0^2 y^{-2} \sum_i A_i e^{(a_i+2)w} = 0. \quad (4.62)$$

For the choice $y = De^{dw}$ this leads to

$$\ddot{w} + (d-1)\dot{w}^2 - \frac{2c_0^2}{D^2} \sum_i A_i e^{(a_i-2d+2)w} = 0, \quad (4.63)$$

which is just the second-order form for the generalized Friedmann equation discussed in Sec. III.

Note that this discussion starting from Eq. (4.61) easily generalizes to include all real values of g . The identification $y = -4c_0 e^{2w} z$ shows that the Hamiltonian (2.45) of the Taub-NUT-M spacetime is of this form.

2. Non-null decoupling

A particular type of non-null decoupling occurs for SE-Hamiltonians in any dimension when after a Lorentz transformation to new conformally inertial coordinates $\tilde{\beta}^A$ (where $\tilde{\beta}^0$ denotes the timelike variable), each exponential potential term involves only a single new (non-null) variable. In other words the Taub potential takes the form

$$U_{(\text{Taub})} = \sum_A \sum_i B_{iA} e^{\tilde{\beta}^{iA} \tilde{\beta}^A}. \quad (4.64)$$

Then the total Hamiltonian is just the sum of independent Hamiltonians H_A for each new variable, constrained only by the Hamiltonian constraint on the sum of the individual energies

$$\begin{aligned} H &= \sum_A H_A, \quad \sum_A E_A = 0, \\ H_A &= \frac{1}{2} \eta_{AB} (\tilde{\beta}^B)^2 + \sum_i B_{iA} e^{\tilde{\beta}^{iA} \tilde{\beta}^A} = E_A. \end{aligned} \quad (4.65)$$

Each equation $H_A = E_A$ is a generalized Friedmann equation.

The simplest example of this occurs for a single (non-null) potential term. The Hamiltonian expressed in terms of the new $\tilde{\beta}^A$ variables then has the form

$$\begin{aligned} H &= \frac{1}{2} x \eta_{AB} \tilde{\beta}^A \tilde{\beta}^B + \frac{1}{2} x^{-1} K e^{C^P \tilde{\beta}^P} \\ &= 0 \quad (\text{no sum over } P). \end{aligned} \quad (4.66)$$

In the timelike case the index P will assume the value 0, while in the spacelike case it will assume one of the remaining allowed values.

Letting x only depend on $\tilde{\beta}^P$ results in the equations of motion $(x \tilde{\beta}^Q)' = 0$ for the remaining cyclic variables $\tilde{\beta}^Q$, $Q \neq P$. These lead to constant values of the momenta $\tilde{p}_Q = x \eta_{QR} \tilde{\beta}^R$. Thus the cyclic variables are determined by the equations

$$\dot{\tilde{\beta}}^Q = x^{-1} \eta^{QR} \tilde{p}_R. \quad (4.67)$$

The noncyclic variable may be solved for using the Hamiltonian constraint, which may conveniently be rewritten in the standard form of a generalized Friedmann problem with two exponential terms

$$\begin{aligned} (\dot{\tilde{\beta}}^P)^2 &= \begin{cases} x^{-2} [\Sigma^2 + K e^{C^P \tilde{\beta}^P}], & P = 0 \leftrightarrow \text{timelike case,} \\ x^{-2} [\Gamma - K e^{C^P \tilde{\beta}^P}], & P \neq 0 \leftrightarrow \text{spacelike case,} \end{cases} \end{aligned} \quad (4.68)$$

where $\Sigma^2 = \sum_Q \tilde{p}_Q^2$ and $\Gamma = -\eta^{QR} \tilde{p}_Q \tilde{p}_R$. This problem is easily solved in various choices of slicing gauge and power variables as discussed in Sec. III.

There are two-dimensional SE Hamiltonians corresponding to non-null Killing tensor cases which require nontrivial conformal transformations as well. If one starts with the Hamiltonian in Eq. (4.36) and assumes that the conformal factor is a polynomial in T and X of the form

$$\text{sgn}(U_{(\text{Taub})}) \tilde{G} = \sum_{i=0}^n (A_i T^i + B_i X^i), \quad (4.69)$$

which corresponds to a simply solved separable problem, then the variable transformation

$$T = e^{aw} + e^{bw}, \quad X = e^{aw} - e^{bw}, \quad (4.70)$$

where a and b are arbitrary nonzero constants, and the relation [following from the definition (4.19)]

$$y = (4abe^{aw+bw})^{-1} x \quad (4.71)$$

leads to the Hamiltonian

$$\begin{aligned} H &= -\frac{1}{2} x \dot{w} \dot{v} + x^{-1} (4ab) \sum_{i=0}^n \sum_{k=0}^i (A_i + (-1)^k B_i) \binom{i}{k} \\ &\quad \times e^{(i-k+1)aw + (k+1)bv}. \end{aligned} \quad (4.72)$$

Thus the identification

$$\begin{aligned} U_{(\text{Taub})} &= (4ab) \sum_{i=0}^n \sum_{k=0}^i [A_i + (-1)^k B_i] \binom{i}{k} e^{(i-k+1)aw + (k+1)bv}, \end{aligned} \quad (4.73)$$

can be made. There are several solvable cases appearing in the literature corresponding to polynomials of low degree in Eq. (4.69).

3. Non-null-null decoupling

For those cases whose Jacobi geometry allows both a null Killing tensor and a non-null Killing tensor, one may take yet another approach in solving the field equations. For example, if the Taub potential of a two-dimensional

problem is both of the form (4.47), corresponding to w decoupling, and (4.64), then the Hamiltonian must take the form

$$\begin{aligned} H &= -\frac{1}{2}x\dot{w}\dot{v} + x^{-1}U_{(\text{Taub})}, \\ U_{(\text{Taub})} &= D_1e^{2(bw+cv)} + D_2e^{bw+cv} \\ &\quad + D_3e^{2(-bw+cv)} + D_4e^{-bw+cv}. \end{aligned} \quad (4.74)$$

A similar expression holds for v decoupling with w and v interchanged.

One may then boost $(w, v) = (k\bar{w}, k^{-1}\bar{v})$ to $(\bar{w}, \bar{v}) = (\bar{\alpha} + \bar{\beta}, \bar{\alpha} - \bar{\beta})$, where $k > 0$ is determined so that each exponential term in the potential depends on only one of the new conformally inertial coordinates $\bar{\alpha}$ or $\bar{\beta}$, namely, by the condition

$$\zeta \equiv 2bk = \pm 2ck^{-1}, \quad \rightarrow \quad k = |c/b|^{1/2}. \quad (4.75)$$

Then if $\text{sgn } bc = 1$, the potential becomes

$$U_{(\text{Taub})} = D_1e^{2\zeta\bar{\alpha}} + D_2e^{\zeta\bar{\alpha}} + D_3e^{-2\zeta\bar{\beta}} + D_4e^{-\zeta\bar{\beta}}, \quad (4.76)$$

while if $\text{sgn } bc = -1$, $\bar{\alpha}$ and $\bar{\beta}$ are interchanged in the potential.

Assuming $\text{sgn } bc = 1$, then two power-law slicing gauge choices lead to mutual decoupling of a pair of variables, the null variable \bar{w} and one of the new non-null inertial coordinates. These correspond to making the D_2 and D_4 potential terms constant, respectively. These choices are

$$\begin{aligned} \text{case (a):} \quad &x = e^{\zeta\bar{\alpha}}; \quad \bar{w} \text{ and } \bar{\alpha} \text{ decouple,} \\ \text{case (b):} \quad &x = e^{-\zeta\bar{\beta}}; \quad \bar{w} \text{ and } \bar{\beta} \text{ decouple,} \end{aligned} \quad (4.77)$$

with the respective Hamiltonians

$$\begin{aligned} \text{case (a):} \quad &H = -\frac{1}{2}e^{\zeta\bar{\alpha}}\dot{\bar{w}}(2\dot{\bar{\alpha}} - \dot{\bar{w}}) + D_1e^{\zeta\bar{\alpha}} + D_2 \\ &\quad + D_3e^{\zeta(\bar{\alpha}-2\bar{w})} + D_4e^{-\zeta\bar{w}}, \\ \text{case (b):} \quad &H = -\frac{1}{2}e^{-\zeta\bar{\beta}}\dot{\bar{w}}(\dot{\bar{w}} - 2\dot{\bar{\beta}}) + D_1e^{\zeta(2\bar{w}-\bar{\beta})} \\ &\quad + D_2e^{\zeta\bar{w}} + D_3e^{-\zeta\bar{\beta}} + D_4. \end{aligned} \quad (4.78)$$

Similar results hold with $\bar{\alpha}$ and $\bar{\beta}$ interchanged for $\text{sgn } bc = -1$.

The decoupled equations for case (a) are

$$\begin{aligned} 0 &= \delta L / \delta \bar{\alpha} = e^{\zeta\bar{\alpha}}[-\ddot{\bar{w}} - \frac{1}{2}\zeta\dot{\bar{w}}^2 + \zeta D_1 + \zeta D_3e^{-2\zeta\bar{w}}], \\ 0 &= \delta L / \delta \bar{w} + \delta L / \delta \bar{\alpha} + \zeta H \\ &= e^{\zeta\bar{\alpha}}[-\ddot{\bar{\alpha}} - \zeta\dot{\bar{\alpha}}^2 + 2\zeta D_1 + \zeta D_2e^{-\zeta\bar{\alpha}}], \end{aligned} \quad (4.79)$$

while those for case (b) are

$$\begin{aligned} 0 &= \delta L / \delta \bar{\beta} = -e^{-\zeta\bar{\beta}}[-\ddot{\bar{w}} + \frac{1}{2}\zeta\dot{\bar{w}}^2 + \zeta D_1 + \zeta D_3e^{2\zeta\bar{w}}], \\ 0 &= \delta L / \delta \bar{w} + \delta L / \delta \bar{\beta} - \zeta H \\ &= -e^{-\zeta\bar{\beta}}[-\ddot{\bar{\beta}} + \zeta\dot{\bar{\beta}}^2 + 2\zeta D_3 + \zeta D_4e^{\zeta\bar{\beta}}]. \end{aligned} \quad (4.80)$$

Each of these four decoupled equations is equivalent to the second-order form (3.10) of a generalized Friedmann equation (3.12). They have the following respective values of the potential term $f(e^\theta)$ and the constant

δ appearing in the latter equations:

$$f(e^\theta) = \begin{cases} -\zeta D_1 - \zeta D_3e^{-2\zeta\bar{w}}, & \delta = \frac{1}{2}\zeta, \theta = \bar{w}, \\ 2\zeta D_1 + \zeta D_2e^{-\zeta\bar{\alpha}}, & \delta = \zeta, \theta = \bar{\alpha}, \\ -\zeta D_1 - \zeta D_3e^{2\zeta\bar{w}}, & \delta = -\frac{1}{2}\zeta, \theta = \bar{w}, \\ -2\zeta D_3 - \zeta D_4e^{\zeta\bar{\beta}}, & \delta = -\zeta, \theta = \bar{\beta}. \end{cases} \quad (4.81)$$

The equivalent generalized Friedmann equations are then, respectively,

$$\dot{\theta}^2 = h(e^\theta) + \mathcal{E}e^{-2\delta\theta} = \begin{cases} -D_1 + D_3e^{-2\zeta\bar{w}} + \mathcal{E}e^{-\zeta\bar{w}}, \\ D_1 + D_2e^{-\zeta\bar{\alpha}} + \mathcal{E}e^{-2\zeta\bar{\alpha}}, \\ D_1 - D_3e^{2\zeta\bar{w}} + \mathcal{E}e^{\zeta\bar{w}}, \\ D_3 + D_4e^{\zeta\bar{\beta}} + \mathcal{E}e^{2\zeta\bar{\alpha}}. \end{cases} \quad (4.82)$$

Finally the following power variables convert these equations into quadratic potential problems:

$$\zeta^{-2}u^2 = \begin{cases} -D_1u^2 + D_3 + \mathcal{E}u, & u = e^{\zeta\bar{w}}, \\ D_1u^2 + D_2u + \mathcal{E}, & u = e^{\zeta\bar{\alpha}}, \\ D_1u^2 - D_3 + \mathcal{E}u, & u = e^{-\zeta\bar{w}}, \\ D_3u^2 + D_4u + \mathcal{E}. & u = e^{-\zeta\bar{\beta}}. \end{cases} \quad (4.83)$$

The solutions of these equations describe one-dimensional motion in a quadratic potential and lead to solutions for the dependent variable u which are affinely related to exponential or trigonometric or hyperbolic sines and cosines of an argument affinely related to the independent variable. Of course this method can also be used to treat the single potential term case, but the previous two methods are simpler.

4. Power variables, power-law slicing gauges, and SE Hamiltonians

If one looks at the literature on exact solutions one almost always find the solution expressed in power variables and power-law slicing gauges. Why is this the case? The answer is that practically all solvable problems are described by SE Hamiltonians with a relatively few number of potential terms. As seen above, the simplest symmetry-adapted variables and slicing gauges are usually power variables and power slicing gauges directly related to these exponential terms. On the other hand, if one has many exponential terms one might be forced to use nonpower-law variables. For example, this happens when the function V , in the null Killing tensor case (i), is a sum of exponential terms.

F. The intrinsic approach to null Killing tensor problems

Apart from the general case of the Jacobi slicing gauge, the only specific slicing gauges which have been considered here are the power-law gauges. This subsection will

show how another important class of slicing gauges arises in a natural way for the null Killing tensor cases. These gauges are the so-called “intrinsic” slicing gauges.

Suppose one chooses symmetry-adapted variables and a symmetry compatible slicing gauge in the null Killing tensor case so that one obtains the Hamiltonian (4.54):

$$H = -\frac{1}{2}\dot{W}\dot{V} + A(W)V + B(W) = 0. \quad (4.84)$$

Then one obtains the first-order decoupled equation for W :

$$\dot{W}^2 = 2[E - U(W)] = F(W),$$

where

$$U(W) = -2 \int A(W)dW. \quad (4.85)$$

For most functions $A(W)$, this equation does not admit solutions expressible in terms of elementary functions. For example, consider a function $A(W)$ which consists of more than one term without being linear in W (if it is linear then it is integrable in terms of elementary functions as already discussed). For some cases of this type it is possible to find the solution in terms of elementary functions by use of an intrinsic slicing gauge. As discussed in Sec. III B, such a slicing gauge is characterized by choosing some simple function of the metric components as the independent variable. In the present problem, one can reintroduce the gauge freedom in Eqs. (4.84) and (4.85) by introducing a new independent variable $\bar{\lambda}$ such that $N = N_{(\text{Taub})}x^{-1}z(W)^{-1}$. This choice leads to the decoupled equation

$$\dot{W}^2 = z(W)^{-2}F(W), \quad (4.86)$$

where the overdot refers to the new independent variable $\bar{\lambda}$. Choosing $z = F^{1/2}$ leads to $W = \bar{\lambda}$ as the independent variable (setting the constant of integration to zero), so that the slicing gauge is clearly an intrinsic one. Inserting $W = \bar{\lambda}$ into the Hamiltonian constraint and expressing this in the new slicing gauge yields

$$\frac{dV}{d\bar{\lambda}} = 2F(\bar{\lambda})^{-1}[A(\bar{\lambda})V + B(\bar{\lambda})], \quad (4.87)$$

which is easily solved. However, whether or not the solution can be expressed in terms of elementary functions depends on the explicit expressions for A, B , and F .

To be more explicit, consider the interior Schwarzschild case where the usual Schwarzschild radial coordinate is related to the metric scale factors by the intrinsic slicing condition $r = R_1 = R_2 = e^w$. Using the power variables $W = e^w$, $V = \frac{2}{3}e^{\frac{2}{3}v}$, and the gauge choice $x = e^{w+\frac{2}{3}v}$ leads to

$$H = -\frac{1}{2}\dot{W}\dot{V} + 24[(1 - \kappa\rho_{(0)}W^2)V + \kappa(\rho_{(0)} + p_{(0)})W^{\frac{5}{2}}], \\ \dot{W}^2 = 4[W - \frac{1}{3}\kappa\rho_{(0)}W^3 + \text{const}] = F. \quad (4.88)$$

For the particular case of nonsingular solutions, smoothness conditions at the center (where $W = r = 0$) (see,

e.g., [30]) require that the integration constant occurring in the expression for the decoupled variable must be zero. Reintroducing the gauge freedom and choosing $z = F^{1/2} = [96(W - \frac{1}{3}\kappa\rho_{(0)}W^3)]^{1/2}$ leads to the usual Schwarzschild gauge $r = W = \bar{\lambda}$:

$$\frac{dV}{dr} = \frac{1}{2}[(1 - \kappa\rho_{(0)}r^2)V \\ + \kappa(\rho_{(0)} + p_{(0)})r^{\frac{5}{2}}]/[r - \frac{1}{3}\kappa\rho_{(0)}r^3]. \quad (4.89)$$

This equation is easily integrated and going back to the original metric variables one finds the simple standard expression for the interior Schwarzschild solution [30]. The general case with a nonzero constant has more complicated solutions.

G. Killing tensor symmetries for a subclass of two-dimensional models

By reducing a given problem, either by exploiting symmetries or by specializing to a subcase, one often ends up with a reduced system having only a few degrees of freedom. Apart from the trivial case when there is only a single degree of freedom left, the simplest reduced systems have 2 degrees of freedom. Many problems can be described by a reduced two-dimensional Hamiltonian of the special form

$$H = \frac{1}{2}(-\dot{\alpha}^2 + \dot{\beta}^2) + U = -\frac{1}{2}\dot{w}\dot{v} + U, \quad (4.90)$$

where w, v are the standard null variables of Eq. (4.14), and

$$U = e^{2c\alpha}F(\beta) \quad \text{or} \quad U = e^{2c\beta}F(\alpha), \quad (4.91)$$

where c is a constant. Although c (if nonzero) can always be normalized to unity by a suitable rescaling of α and β we choose not to do so here in order to facilitate comparison with the table below. However, the translational freedom in β will be used to simplify formulas.

When $c \neq 0$ the potential form (4.91) corresponds exactly to the case when the associated Jacobi metric

$$J_{AB} = 2|U|\eta_{AB}, \quad (4.92)$$

admits a homothetic symmetry generated by $\xi = \partial/\partial\alpha$ (timelike homothetic Killing vector (HKV) case), or $\xi = \partial/\partial\beta$ (spacelike HKV case), respectively:

$$\mathcal{L}_\xi J_{AB} = 2cJ_{AB}. \quad (4.93)$$

In the case $c = 0$ the potential depends only on a single variable and ξ reduces to a Killing vector symmetry. It follows from the form of ξ that the above variables are adapted to this symmetry. The problem of classifying the function $F(\beta)$ [or $F(\alpha)$] for which the Jacobi metric (4.92) admits Killing tensor symmetries has been analyzed in [29]. As explained in that reference it is sufficient to consider the timelike HKV case. The spacelike HKV case can then easily be obtained by an appropriate transformation.

In this subsection all potentials will be given for two-dimensional models which admit a second rank Killing tensor of a given weight under the homothetic symmetry, subject to the assumption that the Killing tensor K_{AB} is characterized by a homothetic weight $2b$ through the equation $\mathcal{L}_\xi K_{AB} = 2bcK_{AB}$. This includes some cases which were not stated explicitly in [29].

The classification of potentials admitting such Killing tensors depends on two parameters describing properties of the Killing tensor. The first parameter is the sign of the determinant of the conformal part of the Killing tensor, $\Sigma = \text{sgn det}(P_{AB})$, where $P_{AB} = K_{AB} - \frac{1}{2}KJ_{AB}$ and $K = K^A_A$. The Killing tensor type is related to Σ according to

$$\Sigma = \begin{cases} 0 & \text{(null),} \\ 1 & \text{(non-null HJ),} \\ -1 & \text{(non-null harmonic) ,} \end{cases} \quad (4.94)$$

corresponding, respectively, to the three cases (i), (ii), and (iii) of Eq. (4.10). The second parameter is the homothetic weight factor b . The cases $b = 1$ or $b = 0$ require special treatment compared to $b \neq 0, 1$. With the three values of Σ , this leads to nine different cases altogether.

We now enumerate the potentials of the form (4.91) admitting Killing tensors corresponding to these cases. The three cases corresponding to a null Killing tensor are collectively given by

(A) ($b \neq 1; \Sigma = 0$):
 $U =$ [two exponential term case I in Table III],
 (B) ($b = 1; \Sigma = 0$): $U = [C_1c(w - v) + C_2]e^{2cw}$, (4.95)

where case A also includes the case $b = 0 = \Sigma$. When $b = 1$, the spacelike HKV case is obtained by interchanging w and v in the corresponding expression in (4.95).

For non-null Killing tensors, the function $F(\beta)$ is given by one of the following expressions (modulo a translation of β):

(C) ($b \neq 0, 1; \Sigma = 1$):
 $C_1 \cosh^s[2c\beta/(s + 2)] + C_2 \sinh^s[2c\beta/(s + 2)]$,

(D) ($b \neq 0, 1; \Sigma = -1$):
 $\text{Re} \left\{ D[e^{2c\beta/(s+2)} + ie^{-2c\beta/(s+2)}]^s \right\}$,
 (E) ($b = 1; \Sigma = 1$):
 [two exponential term case III in Table III],
 (F) ($b = 1; \Sigma = -1$):
 $e^{2ck\beta} \{ C_1 \cos[2c(1 - k^2)^{1/2}\beta] + C_2 \sin[2c(1 - k^2)^{1/2}\beta] \}$,
 (G) ($b = 0; \Sigma = 1$):
 $C_1 \ln \coth(c\beta) + C_2$,
 (H) ($b = 0; \Sigma = -1$):
 $C_1 \arctan e^{2c\beta} + C_2$, (4.96)

where $s = -2b/(b - 1)$ and k ($|k| < 1$) are real parameters while D is a complex parameter. In cases (C) and (D) the potential can be expressed explicitly as a sum of exponential terms if s is an integer. The subcases with exactly two exponential terms are given explicitly in Sec. IV H. Case (D) can also be expressed explicitly as a real function in the form

$$F = \cosh^{s/2}[4c\beta/(s + 2)] \{ C_1 \cos[s \arctan e^{4c\beta/(s+2)}] + C_2 \sin[s \arctan e^{4c\beta/(s+2)}] \}. \quad (4.97)$$

Using multiple angle formulas the trigonometric expression inside the curly brackets can be converted to algebraic form provided that s is a rational number m/n . However, since this involves solving a polynomial equation of degree $|n|$, explicit algebraic expressions can only be guaranteed for $|n| \leq 4$. Some of the above Killing tensor cases admit special Killing vector cases, e.g., setting $C_1 = 0$ or $C_2 = 0$ in expression (C) leads to such a case.

H. Killing tensor cases for Taub potentials with two exponential terms

An important special case of two-dimensional systems occurs when the potential is a sum of two exponential terms

TABLE III. Null and non-null parameters for two-dimensional two exponential terms models admitting Killing tensors up to second rank. The index pair (i, j) is a permutation of $(1, 2)$ where appropriate in the table. The parameters b and Σ are undefined in cases II and IV. The last column relates the different cases to the general HKV Killing tensor cases.

	b	Σ	(p_i, q_j)	(c_i, d_j)	HKV KT case
I	$b \neq 1$	0	$p_i = 2p_j$ or $q_i = 2q_j$	$c_i + d_i = 2(c_j + d_j)$ or $c_i - d_i = 2(c_j - d_j)$	(A)
II	-	-	$p_1 = p_2$ or $q_1 = q_2$	$c_1 - c_2 = \pm(d_1 - d_2)$	-
III	1	1	$p_1q_2 + p_2q_1 = 0$	$c_1c_2 - d_1d_2 = 0$	(E)
IV	-	-	$p_1q_2 - p_2q_1 = 0$	$c_1d_2 - c_2d_1 = 0$	-
V	$\frac{1}{2}$	$Z = \pm 1$	$p_i = 3p_j$ and $q_j = 3q_i$	$c_i + d_i = 3(c_j + d_j)$ and $c_j - d_j = 3(c_i - d_i)$	$Z = \begin{cases} 1 & (C) \\ -1 & (D) \end{cases}$

$$\begin{aligned}
U &= C_1 e^{p_1 w + q_1 v} + C_2 e^{p_2 w + q_2 v} \\
&= C_1 e^{c_1 \alpha + d_1 \beta} + C_2 e^{c_2 \alpha + d_2 \beta}.
\end{aligned}
\tag{4.98}$$

Extracting all the two exponential term potential cases from the various types of the previous section and adding the flat case with a null HKV and the non-null Killing vector case corresponding to $c = 0$ in Eq. (4.91) leads to Table III for the corresponding parameter values. In the case (V) of this table, the type of non-null Killing tensor depends on the relative sign $Z = \text{sgn}(C_1 C_2)$ of the two terms in the potential. For $Z = 1$ one has a non-null H-J Killing tensor case ($\Sigma = 1$) while for $Z = -1$ one has a non-null harmonic Killing tensor case ($\Sigma = -1$).

The null (I) and flat (II) cases are easily treated using results from the null decoupling Sec. IV E 1. (Case I admits additional non-null Killing tensor cases for certain parameter values, for such cases one may choose non-null solution techniques.) Referring to that section, the flat case corresponds to $A_i = 0$, while the null cases correspond to $A_i \neq 0$. In case (III) decoupling can be achieved by an appropriate Lorentz transformation leading to two Friedmann equations. Case (IV) is a non-null Killing vector case where decoupling can also be accomplished by a Lorentz transformation leading directly to a single generalized Friedmann equation. In the remaining non-null cases (V), one can introduce power variables leading to an easily solved problem with a potential which is a quadratic form in the new variables. An example of such a case has been dealt with in [31].

V. INVARIANT SUBMANIFOLDS AND HOW TO OBTAIN THEM

Solving the Einstein field equations in general seems to be impossible, particularly in view of recent results that the only generalized local symmetries of these equations are due to scale invariance and the diffeomorphism group [32], and these symmetries are insufficient to lead to a general solution. To find special solutions one imposes spacetime symmetries and/or other restrictions on the dependent variables so that one obtains a more tractable consistent subsystem of differential equations. In other words one tries to find “invariant submanifolds” of the original system of field equations. Even imposing enough spacetime symmetries to reduce the field equations to ordinary differential equations as one does to obtain the HH models still does not lead to such tractable subsystems in general. One must impose further conditions to be able to actually find exact solutions. There is no general systematic method of discovering invariant submanifolds. It is here that creativity and imagination and even plain luck play a role in rooting out these hidden structures. There are many particular ways in which invariant submanifolds have been found, but few of these successes involve a systematic method. Many methods require an arbitrary function such as an unspecified equation of state or an unspecified scalar field potential to produce solutions. In this brief section one systematic method will be presented which does not rely on the existence of ar-

bitrary functions and is relevant to many though not all of the known invariant submanifolds. In particular for Hamiltonian problems this method also yields the class of exact power law (EPL) solutions. EPL solutions have been studied in [33,34].

Hamiltonians which are reducible to the following form play a crucial role in the discussion of HH models:

$$H = \frac{1}{2} \chi \eta_{\mu\nu} \dot{y}^\mu \dot{y}^\nu + \chi^{-1} \mathcal{U}, \tag{5.1}$$

where χ and \mathcal{U} are analogous to the previous slicing gauge function x and the Taub potential. Choosing $\chi = 1$ leads to the equations

$$\ddot{y}^\mu = -\eta^{\mu\nu} \partial \mathcal{U} / \partial y^\nu. \tag{5.2}$$

If $\partial \mathcal{U} / \partial y^\mu = 0$ holds for some value $y_{(0)}^\mu$ of a particular coordinate y^μ independent of the values of the remaining coordinates, and if this condition is compatible with the Hamiltonian constraint, then $y^\mu = y_{(0)}^\mu$ describes an invariant submanifold. The equations for the remaining variables on this submanifold are given by the above Hamiltonian after having inserted the conditions $y^\mu = y_{(0)}^\mu$ and $\dot{y}^\mu = 0$ (for that coordinate alone). Invariant submanifolds within invariant submanifolds are also possible. Reduction down to one dimension automatically leads to a solution since the Hamiltonian constraint only involves a single variable, thus leading to a quadrature.

Lorentz transformations of the β^A variables are symmetry transformations of the Minkowski metric appearing in the expression for the kinetic energy function. They play a crucial role in finding many invariant submanifolds. All HH models of the previous section have Hamiltonians or reduced Hamiltonians with Taub potentials which can be put into the following form by a Lorentz transformation from the variables β^A to new ones $\bar{\beta}^A$:

$$U_{(\text{Taub})} = \sum_i e^{c_i \bar{\beta}^0} F_i(\bar{\beta}^P), \quad P \neq 0. \tag{5.3}$$

If $\partial F_i / \partial \bar{\beta}^P = 0$ holds for all values of i for some particular value $\bar{\beta}_{(0)}^P$ of a particular coordinate $\bar{\beta}^P$ independent of the values of the remaining coordinates, and if this condition is compatible with the Hamiltonian constraint, then $\bar{\beta}^P = \bar{\beta}_{(0)}^P$ describes an invariant submanifold. In many HH cases the index value 0 and some definite value $P \neq 0$ can be interchanged in this discussion, but the Hamiltonian constraint seems to prevent the existence of invariant submanifolds of this type. One-dimensional invariant submanifolds with one exponential term lead directly to EPL solutions.

VI. PROBLEMS LEADING TO EXACT SOLUTIONS AND HOW TO SOLVE THEM

This section will survey the cases which lead to exact solutions. The method of solution which works in each case will be specified by referring to previous sections,

without going through the mechanical details of obtaining and presenting the solution explicitly. In fact, as has been shown, there are often several ways one can solve a given problem and hence more than one representation of the solution exists. Specific examples of how to use the methods of this paper to produce the actual spacetime metrics which correspond to these solutions are given by Ugglá [31] and Ugglá and Rosquist [35].

Except for a few special class B cases and Bianchi type VI₀, all exact solutions arise from spacetimes which admit either additional continuous spacetime symmetries (Killing vectors and/or homothetic Killing vectors) or additional continuous intrinsic symmetries (Killing vectors). The latter are symmetries of the intrinsic geometry of the individual homogeneous hypersurfaces which are not necessarily spacetime symmetries. As in Sec. II, the diagonal and nondiagonal models are treated separately, but the diagonal models are collected according to the dimension of the intrinsic symmetry group. Unless otherwise stated, the only perfect fluid solutions being considered here are those for which $p = (\gamma - 1)\rho$.

A. Diagonal models

The possible dimensions of the intrinsic symmetry group of the geometry of the HH hypersurfaces are 6, 4, and 3. Beginning with dimension 6, models are considered with only β^0 -dependent sources and possible scalar fields. This class of models includes the Bianchi types-I and -V models and the SH constant spatial curvature type-IX models (there are no static models of this latter type). The SH models belonging to this class are intrinsically isotropic.

Next diagonal models with a four-dimensional intrinsic symmetry group and with only β^0, β^+ -dependent sources are treated. These models are all intrinsically LRS and include the Bianchi types-I, -II, and -V models, the LRS Bianchi types-III, -VIII, and -IX models, the SH Kantowski-Sachs models, and the static spherically symmetric models. Note that apart from the SH Bianchi type IX Friedmann-Robertson-Walker (FRW) perfect fluid solutions and the SH LRS Bianchi types-VIII and -IX stiff perfect fluid solutions, there are no other known exact perfect fluid solutions for these two Bianchi types.

Finally the diagonal SH Bianchi type-VI vacuum and perfect fluid models are considered.

1. Models with a six-dimensional intrinsic symmetry group

a. Sources not including a scalar field. The Hamiltonian can be written as [36]

$$H = \frac{1}{2}x(-\dot{\beta}^{02} + \dot{\beta}^{+2} + \dot{\beta}^{-2}) + x^{-1}[-72ke^{4\beta^0} + U_{(\text{TaubS})}(\beta^0)], \quad (6.1)$$

where $U_{(\text{TaubS})}$ is the source potential expressed in the Taub slicing gauge. The variables β^\pm are equal to zero

for the type-IX models while β^+ is equal to zero for the type-V models. One can choose one-forms so that the parameter k has the values 0 for Bianchi type I, 1 for type IX (corresponding to the choice $n^{(1)} = n^{(2)} = n^{(3)} = 2$), and -1 for type V (corresponding to the choice $a = 1$).

Letting x only depend on β^0 results in the equations of motion $(x\dot{\beta}^\pm)' = 0$, which lead to constant values of the momenta $p_\pm = x\dot{\beta}^\pm$. Thus β^\pm are determined by the equations

$$\dot{\beta}^\pm = x^{-1}p_\pm. \quad (6.2)$$

One may solve for β^0 using the Hamiltonian constraint

$$(\dot{\beta}^0)^2 = x^{-2}[\Sigma^2 - 72ke^{4\beta^0} + U_{(\text{TaubS})}], \quad (6.3)$$

where $\Sigma^2 = p_+^2 + p_-^2$. This equation immediately gives a quadrature for β^0 . The most interesting case is when $U_{(\text{TaubS})}$ is a sum of exponential terms and this problem reduces to the the generalized Friedmann problem. References to some of the literature on the most notable SH solutions with a six-dimensional intrinsic symmetry group are given in Table IV. The SH vacuum type-I solution is usually associated with Kasner who found the corresponding static solution [37]. The isotropic vacuum type-V solution is just the Milne form of Minkowski spacetime. Useful references for FRW and FRW- Λ models are Harrison [38], Vajk [39], Anderson [40], and Misner, Thorne, and Wheeler [8]. The book by Kramer *et al.* [4] is also useful in this context as well as for further references on models in Table IV with symmetry groups of dimension 3 and 4.

b. Sources including a scalar field. The Hamiltonian is

$$H = \frac{1}{2}x(-\dot{\beta}^{02} + \dot{\beta}^{+2} + \dot{\beta}^{-2} + \dot{\beta}^{\dagger 2}) - 24x^{-1}[3ke^{4\beta^0} + \epsilon\kappa e^{6\beta^0}V_{(\text{sc})}(\beta^\dagger)]. \quad (6.4)$$

Solvable cases: For most scalar potentials this is not a solvable problem. However, if $V_{(\text{sc})}$ is the sum of exponential terms, then some solutions do exist. An interesting example is the case of scalar field models with a single exponential potential $V_{(\text{sc})} = e^{-2c\beta^\dagger}$ [53]. For such models with $\beta^\pm = 0$, which includes the isotropic models, the Taub potential is given by

$$U_{(\text{Taub})} = -24[3ke^{2w+2v} + \epsilon\kappa e^{(3-c)w+(3+c)v}], \quad (6.5)$$

where $w = \beta^0 + \beta^\dagger$ and $v = \beta^0 - \beta^\dagger$. If $k = 0$ this is a simply solvable one-exponential-term problem. When $k \neq 0$ there are two solvable cases. The first case $c = 1$ corresponds to the $A_i = 0$ flat null decoupling case of Sec. IV E 1. The second case $c = 2$ corresponds to the nonflat null Killing tensor case [29]. The Jacobi metric of the case $k = 0$ and $\beta^\pm = 0$ with an arbitrary scalar potential, $V_{(\text{sc})}$, admits a timelike HKV. Therefore the Killing tensor cases (A) through (H) of Sec. IV G apply and lead to exact solutions. The particular case (C) with $s = 2$ leads to the solutions found in [54,55]. However, one can easily produce many other solutions of comparable physical interest.

TABLE IV. SH models with six-dimensional intrinsic symmetry group not including a scalar field. The abbreviations Mink, deS, anti-deS, Vai-Elt, Ste-Ell, Ell-Mac, and Ein stand, respectively, for Minkowski, de Sitter, anti-de Sitter, Vajk-Eltgroth, Stewart-Ellis, Ellis-MacCallum, and Einstein. The dimension column in this and all subsequent tables refers to the dimension of the spacetime symmetry group.

Bianchi type	Dim	Vacuum	Λ term	Perfect fluid	Λ term plus perfect fluid
I	6	Mink	deS [41]	FRW	FRW- Λ
I	4	Kasner [37], Mink	Saunders [42]	Jacobs [43] Robinson [44] Raychaudhuri [45] Doroshkevich [46] Ste-Ell [47] Vai-Elt [48]	Saunders [42]
I	3	Kasner [37]	Saunders [42]	Jacobs [43] Robinson [44] Raychaudhuri [45]	Saunders [42]
V	6	Mink(Milne [49])	deS [41], anti-deS	FRW	FRW- Λ
V	3	Joseph [50]		Ell-Mac [51]	
IX	6	—	deS [41]	FRW	FRW- Λ (Ein [52])

Invariant submanifolds: If $V_{(\text{sc})}$ has relative extrema, then one has an invariant submanifold corresponding to the corresponding fixed value of the scalar field. The resulting problem yields a generalized Friedmann equation where the scalar potential reduces to an effective cosmological constant.

There are other more interesting invariant submanifolds obtained by a different method [36,56–59]. These correspond to exact solutions describing inflationary models in cosmology as well as static domain walls in an astrophysical context.

2. Models with a four-dimensional intrinsic symmetry group

For the family of intrinsically LRS class A models, which can be chosen to satisfy $n^{(1)} = n^{(2)}$, it is convenient to introduce the notation $\sigma = n^{(1)}n^{(3)}$. One must set $n^{(3)} = 0$ to obtain the remaining LRS models, for which the curvature parameter σ continues to have its previous meaning. The sources considered in this subsection may include a cosmological constant, electromagnetic fields and perfect fluids.

The Hamiltonian for this family of spacetimes is given by

$$\begin{aligned}
H = & \frac{1}{2}x\eta_{AB}\dot{\beta}^A\dot{\beta}^B + 24x^{-1}\left[\frac{1}{4}n^{(3)2}e^{-4(2\beta^+ - \beta^0)}\right. \\
& + \epsilon\sigma e^{2(2\beta^0 - \beta^+)} - \epsilon e^2 e^{-2(2\beta^+ - \beta^0)} \\
& \left. - \epsilon\Lambda e^{6\beta^0}\right] + U_{(\text{fluid})}, \\
& [A, B = 0, +, -],
\end{aligned} \tag{6.6}$$

where β^- must vanish except for the Bianchi types I and II where it is a cyclic variable, provided that x is assumed to be independent of this variable. The system associated with this Hamiltonian provides several interesting examples of null and/or non-null Killing tensor cases which in

turn give rise to many exact solutions.

a. Vacuum, Λ , EM field, SH stiff perfect fluid. The case $\Lambda = 0$: The Hamiltonian for this case can be nicely expressed in terms of a new pair of conformally inertial coordinates obtained by the Lorentz transformation [7]

$$\begin{aligned}
(\bar{\beta}^0, \bar{\beta}^+) &= 3^{-1/2}(2\beta^0 - \beta^+, -\beta^0 + 2\beta^+), \\
(\beta^0, \beta^+) &= 3^{-1/2}(2\bar{\beta}^0 + \bar{\beta}^+, -\bar{\beta}^0 - 2\bar{\beta}^+),
\end{aligned} \tag{6.7}$$

in terms of which the Hamiltonian takes the form

$$\begin{aligned}
H = & \frac{1}{2}x(-\dot{\bar{\beta}}^0{}^2 + \dot{\bar{\beta}}^+{}^2 + \dot{\beta}^-{}^2) + 24x^{-1}\left[\frac{1}{4}n^{(3)2}e^{-4\sqrt{3}\bar{\beta}^+}\right. \\
& \left. + \epsilon\sigma e^{2\sqrt{3}\bar{\beta}^0} - \epsilon e^2 e^{-2\sqrt{3}\bar{\beta}^+} + \kappa\rho_{(0)}\right].
\end{aligned} \tag{6.8}$$

It is easy to see that this Hamiltonian is a non-null H-J Killing tensor case. Furthermore in the Taub slicing gauge $x = 1$, one has a completely decoupled Hamiltonian

$$\begin{aligned}
H_{(\text{Taub})} &= -H_{\bar{\beta}^0} + H_{\bar{\beta}^+} + H_{\beta^-} + H_{(\text{stiff})}, \\
H_{\bar{\beta}^0} &= -\frac{1}{2}\dot{\bar{\beta}}^0{}^2 + 24\epsilon\sigma e^{2\sqrt{3}\bar{\beta}^0} = E_{\bar{\beta}^0}, \\
H_{\bar{\beta}^+} &= \frac{1}{2}\dot{\bar{\beta}}^+{}^2 + \frac{1}{4}n^{(3)2}e^{-4\sqrt{3}\bar{\beta}^+} - \epsilon e^2 e^{-2\sqrt{3}\bar{\beta}^+} = E_{\bar{\beta}^+}, \\
H_{\beta^-} &= \frac{1}{2}\dot{\beta}^-{}^2 = E_{\beta^-}, \\
H_{(\text{stiff})} &= 24\kappa\rho_{(0)} = E_{(\text{stiff})} \quad (\epsilon = -1).
\end{aligned} \tag{6.9}$$

The full problem therefore consists of three generalized Friedmann problems interpretable as one-dimensional problems with exponential potentials and constant energies, restricted only by the constraint [13]

$$E_{\bar{\beta}^0} + E_{\bar{\beta}^+} + E_{\beta^-} + E_{(\text{stiff})} = 0. \tag{6.10}$$

Each of the one-dimensional motion problems is governed by a generalized Friedmann equation, with a potential for the variables β^- (when nonzero), $\bar{\beta}^0$, and $\bar{\beta}^+$ having in general one, two, and three exponential terms re-

spectively, the latter occurring as an “equally spaced” exponential coefficient case (see Sec. III), all of which are equivalent to generalized Friedmann problems with a quadratic potential.

Apart from the trivial cyclic variable β^- which is present for Bianchi types I and II, the natural variables which lead to quadratic potentials for the other 2 degrees of freedom are

$$\begin{aligned} U_{\beta^0} &= e^{-\sqrt{3}\beta^0}, \quad \sigma \neq 0, \\ U_{\beta^+} &= \begin{cases} e^{2\sqrt{3}\beta^+}, & n^{(3)} \neq 0, \\ e^{\sqrt{3}\beta^+}, & n^{(3)} = 0. \end{cases} \end{aligned} \quad (6.11)$$

The case $\beta^- = 0 = U_{(\text{fluid})}$: These models may be re-examined as an example of a two-dimensional null Killing tensor case treated in Sec. IV E 1. Letting

$$w = \beta^0 + \beta^+, \quad v = \beta^0 - \beta^+, \quad (6.12)$$

the Taub time gauge potential takes the form

$$\begin{aligned} U_{(\text{Taub})} &= 24 \left[\frac{1}{4} n^{(3)2} e^{-2w+6v} + \epsilon \sigma e^{w+3v} \right. \\ &\quad \left. - \epsilon e^2 e^{-w+3v} - \epsilon \Lambda e^{3(w+v)} \right]. \end{aligned} \quad (6.13)$$

For the vacuum case this expression can be identified with Eq. (4.47), with $c_0 = 3$ and $A_0 = 2(n^{(3)})^2/3$, allowing w to decouple. This permits a nonzero cosmological constant term since only a single null variable is required to decouple, in contrast with the previous non-null discussion where the cosmological constant had to be zero.

The natural power variables and slicing gauge function are

$$V = e^{3v}, \quad W = e^{mw}, \quad x = 3mVW, \quad (6.14)$$

leading to

$$\begin{aligned} H &= -\frac{1}{2} \dot{W} \dot{V} + \frac{8}{m} \left(\frac{1}{4} n^{(3)2} W^{-\frac{2}{m}-1} V + \epsilon \sigma W^{\frac{1}{m}-1} \right. \\ &\quad \left. - \epsilon e^2 W^{-\frac{1}{m}-1} - \epsilon \Lambda W^{\frac{3}{m}-1} \right). \end{aligned} \quad (6.15)$$

Since $a_0 = -2 \neq 0$, the three choices $m = -2, \pm 1$ lead to elementary function solutions for W . The choice $m = 1$, which makes the ϵ term a constant, was first introduced in the SH context by Misner and Taub [1]. The choice $m = -1$, which makes the e^2 term a constant, was first introduced by Brill [60] in the SH case. A third new slicing gauge arises for the choice $m = -2$ which makes the $n^{(3)}$ term proportional to V . Note that $A_0 = 0$ for the spherically symmetric models. Thus the choice $m = 1$ leads directly to the standard expression for the Reissner-Nordström solution with cosmological constant.

If one also sets $\Lambda = 0$ then one has a non-null-null case which may be solved as in Sec. IV E 3.

If there are several terms equal to zero there are even more slicing gauges and dependent variables one can choose to solve the problem. As an example, consider the LRS Bianchi types II, III, and the KS vacuum models which correspond to a two-dimensional problem with a Taub potential consisting of a single exponential term. This term is a non-null exponential, and the corresponding problem is easily solved using the methods of Secs. IV E 1 or IV E 2.

References to some of the literature on the more prominent solutions are given in Table V. Apart from the solutions indicated in this table, it is worth noting that the general LRS solution with an electromagnetic field and a cosmological constant have been given by Cahen and De-frise [61]. A useful reference and guide to the literature on solutions with electromagnetic fields is the work by MacCallum [62]. MacCallum, together with Siklos, has also made a thorough investigation of HH vacuum mod-

TABLE V. SH models with four-dimensional intrinsic symmetry group not including a scalar field. The abbreviations B, SSS, N-H, Schwar., Nord., and NUT stand, respectively, for Bianchi, static spherically symmetric, Novotný-Horský, Schwarzschild, Nordström, and Newman-Unti-Tamburino.

Sym type	Dim	Vacuum	Λ term	em term	Λ term + em term
B I	4	Kasner [37] ($\epsilon = \pm 1$)	Saunders [42] ($\epsilon = -1$) N-H [65] ($\epsilon = 1$)	Rosen [64] ($\epsilon = -1$) Kar [66] ($\epsilon = 1$) McVittie [67] ($\epsilon = 1$)	
B I	3	Kasner [37] ($\epsilon = \pm 1$)	Saunders [42] ($\epsilon = -1$)	Datta [68] ($\epsilon = -1$) Bonnor [69] ($\epsilon = 1$) Jacobs [70] ($\epsilon = -1$)	
B II	4	Taub [6] ($\epsilon = -1$)		Ruban [71] ($\epsilon = -1$) Barnes [72] ($\epsilon = -1$)	
B II	3	Taub [6] ($\epsilon = -1$)			
B III	4	K-S [73] ($\epsilon = -1$)		Datta [74] ($\epsilon = 1$)	
KS	4	K-S [73] ($\epsilon = -1$)			
SSS	4	Schwar. [75] ($\epsilon = 1$)	Kottler [76] ($\epsilon = 1$)	Reissner [77] ($\epsilon = 1$)	Nord. [78] ($\epsilon = 1$)
B VIII	4	Taub [6] ($\epsilon = -1$)			
B IX	4	Taub [6] ($\epsilon = -1$) NUT [79] ($\epsilon = 1$)	Brill [60] ($\epsilon = -1$)		

els with a cosmological constant [63]. For a discussion on LRS models see [48].

b. Static perfect fluids. The most interesting static models are the astrophysically relevant spherically symmetric ones. The Bianchi type-I models are also of some interest as cylindrically or plane symmetric ($\beta^- = 0$) models. Other static Bianchi models do not seem to be particularly interesting physically and will not be considered here.

Spherically symmetric models: For the astrophysical spherically symmetric models, various equations of state have been considered.

The case $p = (\gamma - 1)\rho$: For the usual equation of state with $1 < \gamma < 2$ one has the Taub potential

$$U_{(\text{Taub})} = 24[e^{4\beta^0 - 2\beta^+} + \kappa p_{(0)} e^{(6-\eta)\beta^0 + 2\eta\beta^+}], \quad (6.16)$$

where $\eta = \gamma/(\gamma - 1)$.

Making the boost

$$\begin{aligned} \beta^0 &= \Gamma(\bar{\beta}^0 + v\bar{\beta}^+), \\ \beta^+ &= \Gamma(v\bar{\beta}^0 + \bar{\beta}^+), \end{aligned}$$

where

$$\Gamma = (1 - v^2)^{-1/2}, \quad (6.17)$$

with the value $v = \frac{1}{2}(\eta - 2)/(\eta + 1) = \frac{1}{2}(2 - \gamma)/(2\gamma - 1)$ of the boost parameter, leads to

$$\begin{aligned} U_{(\text{Taub})} &= 24e^{A\bar{\beta}^0} [e^{B\bar{\beta}^+} + \kappa p_{(0)} e^{C\bar{\beta}^+}], \\ A &= \frac{3\Gamma(\eta+2)}{\eta+1}, \quad B = -\frac{6\Gamma}{\eta+1}, \quad C = \frac{3\Gamma(\eta-2)^2}{2(\eta+1)}. \end{aligned} \quad (6.18)$$

This potential is of the same form as Eq. (5.3) with a single term and has a nonzero minimum value. A one-dimensional invariant submanifold corresponds to this minimum value, leading to a generalized Friedmann equation with one potential term easily solved using the methods of Sec. III. This solution is a special case of solutions found by Tolman [80]. The solution with $\gamma = \frac{4}{3}$ has also been found by Klein [81].

The case $\rho = \rho_{(0)}$: This case has been treated in Sec. IV F. The nonsingular interior solution was first found by Schwarzschild [75]. Solutions which have a singularity are also null cases and have been investigated by Volkoff [82] and Wyman [83].

The case of an unspecified equation of state: The Taub potential is

$$U_{(\text{Taub})} = 24[e^{4\beta^0 - 2\beta^+} + \kappa e^{6\beta^0} p(\beta^0 - 2\beta^+)]. \quad (6.19)$$

Making the boost (6.7) leads to

$$U_{(\text{Taub})} = 24[e^{2\sqrt{3}\bar{\beta}^0} + \kappa e^{4\sqrt{3}\bar{\beta}^0} f(\bar{\beta}^+)], \quad (6.20)$$

where $f = e^{2\sqrt{3}\bar{\beta}^+} p(\bar{\beta}^+)$. If f has a minimum for some value of $\bar{\beta}^+$ then one obtains an invariant submanifold as discussed in Sec. V. Unfortunately such a minimum leads to an unphysical equation of state $p = -\rho/3$.

However, this problem has the same form as the scalar field problem with an unspecified scalar potential mentioned above and dealt with in [36]. Thus the same

method can be applied to the present problem and will produce invariant submanifolds and corresponding exact solutions. Alternatively one can specify $f(\bar{\beta}^+)$ or $p(\bar{\beta}^+)$ to be some function so that one obtains a problem for which one might find an invariant submanifold or a Killing tensor and thus exact solutions. Once a solution is found the equation of state can be derived. Unfortunately, the general solution to the Killing tensor problem is not available at present. However, all of the solvable cases found in the literature can be recovered by a certain ansatz for a conformal transformation relating the standard null variables (w, v) to a set of null variables (W, V) which are adapted to the symmetry [84].

The starting point is to write down the Jacobi metric in standard null variables $w = \beta^0 + \beta^+$, $v = \beta^0 - \beta^+$ leading to (modulo a constant factor)

$$ds_J^2 = -2 \left[e^{w+3v} + \kappa e^{3(w+v)} p\left(\frac{-w+3v}{2}\right) \right] dw dv. \quad (6.21)$$

The ansatz we use for the conformal transformation is

$$e^w = W^r, \quad e^v = V^s, \quad (6.22)$$

where r and s are constants to be determined. Applying this transformation to the Jacobi metric (6.21) yields

$$\begin{aligned} ds_J^2 &= -2\tilde{G}(W, V) dW dV, \\ \tilde{G} &= rs [W^{r-1} V^{3s-1} + \kappa W^{3r-1} V^{3s-1} h(Y)], \end{aligned} \quad (6.23)$$

where $h(Y) = p(\log Y)$ and $Y = e^{\beta^3} = W^{-r} V^{3s/2}$ (recall that $\beta^3 = \beta^0 - 2\beta^+$). We next look for conditions on r , s , and $h(Y)$ which make the variables W and V symmetry adapted with respect to a Killing vector or a Killing tensor. We do this by inserting the expression for \tilde{G} in (6.23) into Eqs. (4.10). Analysis of the resulting set of equations leads to the solvable cases given in Table VI (for how one explicitly solves these cases see [84]).

The cases for which the equation of state is of physical interest are the null case with $s = \frac{2}{3}$ (Schwarzschild's interior solution), the Hamilton-Jacobi case with $r = 1$, $s = \frac{1}{3}$ (the Killing vector case, where $a = b$, corresponds to Buchdahl's generalized polytropic solution of index five [85]) and finally the Hamilton-Jacobi case with $r = 2$, $s = \frac{2}{3}$ (setting $c_+ = 0$ gives Buchdahl's generalized polytrope of index one [86,87], while the general $c_- \neq 0$ case was recently given by Simon [88]).

Bianchi type-I models: Again several equations of state are of interest.

The case $p = (\gamma - 1)\rho$: These models correspond to a problem with a single exponential potential term

$$U_{(\text{Taub})} = 24\kappa p_{(0)} e^{(6-\eta)\beta^0 + 2\eta\beta^+}, \quad (6.24)$$

which is non-null for $1 < \gamma < 2$ and null for $\gamma = 2$ since $\eta = \gamma/(\gamma - 1)$. These two types of problems were dealt with in Secs. IV E 1 and IV E 2.

The case $\rho = \rho_{(0)} + (\eta - 1)p$: The Taub potential is

$$U_{(\text{Taub})} = 24\kappa[(\rho_{(0)} + p_{(0)})e^{(6-\eta)\beta^0 + 2\eta\beta^+} - \rho_{(0)}e^{6\beta^0}]. \quad (6.25)$$

TABLE VI. Spherically symmetric perfect fluids corresponding to Killing tensor cases. In the first two columns a “-” means that the value of r or s is arbitrary. The parameters a, b, c_{\pm} are real constants while c_0 is a complex constant. The shorthand notation $Y = e^{\beta^3}$ is also used. For the last case ($r = 2, s = \frac{2}{3}$), it is not possible to write down an equation of state in closed form. Instead the expression for $\rho(\beta^3)$ is given in the fourth column. The abbreviations “HJ” and “Harm.” stand for Hamilton-Jacobi and harmonic, respectively. The Hamilton-Jacobi Killing tensor case with $(r, s) = (1, \frac{1}{3})$ reduces to a Killing vector case when $a = b$.

r	s	$p(\beta^3)$	Equation of state	Killing tensor type
1	-	$-aY^4 + bY^2$	$a(5p + \rho)^2 = 2b^2(3p + \rho)$	Null
2	-	$-aY^5 + bY^4$	$a^4(6p + \rho)^5 = b^5(5p + \rho)^4$	Null
-	$\frac{1}{3}$	$-aY^2 + b$	$3p + \rho = 2b$	Null
-	$\frac{2}{3}$	$-a + bY^{-1}$	$\rho = a$	Null
1	$\frac{1}{3}$	$aY^4 - 2bY^2 + \Sigma a$	$\frac{(5p + \rho - 4\Sigma a)^2}{2\Sigma a - 3p - \rho} = \frac{8b^2}{a}$	HJ ($\Sigma = 1$) Harm. ($\Sigma = -1$)
2	$\frac{2}{3}$	$(6Y)^{-1} [-c_+(1+Y)^6 + c_-(1-Y)^6]$ $(6Y)^{-1} \Re[c_0(1+iY)^6]$	$\rho = c_+(1+Y)^5 + c_-(1-Y)^5$ $\rho = \Im m[c_0(1+iY)^5]$	HJ Harm.

For $\eta = 6$ (or $\gamma = 6/5$) it is easily seen that one has a non-null Killing tensor case, thus easily solved with the methods of Sec. IV E 2. The solution was first found by Evans [89].

c. Spatially homogeneous Bianchi type-II nonstiff perfect fluid models. These models have the Hamiltonian

$$H = \frac{1}{2}x\eta_{AB}\dot{\beta}^A\dot{\beta}^B + 24x^{-1}\left[\frac{1}{4}e^{-4(2\beta^+ - \beta^0)} + \kappa\rho_{(0)}e^{3(2-\gamma)\beta^0}\right], \quad (6.26)$$

where $[A, B = 0, +, -]$. Making the boost (6.17) with

$$v = \frac{1}{8}(3\gamma - 2) \quad (6.27)$$

leads to the Taub potential

$$U_{(\text{Taub})} = 6e^{3\Gamma(2-\gamma)\bar{\beta}^0} \left[e^{-3\Gamma(6-\gamma)\bar{\beta}^+ / 2} + 4\kappa\rho_{(0)}e^{\Gamma(2-\gamma)(3\gamma-2)\bar{\beta}^+ / 8} \right]. \quad (6.28)$$

The $\bar{\beta}^+$ -dependent factor in this potential has a minimum. As discussed in Sec. V this corresponds to a two-dimensional invariant submanifold, leading to a two-dimensional problem with one non-null exponential term, easily solved using Secs. IV E 2 and IV E 1. When $\bar{\beta}^- \neq 0$ this yields the Collins solution [90], while for $\bar{\beta}^- = 0$ it gives an LRS EPL solution [90–92].

d. Spatially homogeneous KS and Bianchi type-III nonstiff perfect fluid models. Referring to Secs. II C 3 and II C 2, the orthogonal perfect fluid models have a Hamiltonian of the form

$$H = \frac{1}{2}x(-\dot{\beta}^{02} + \dot{\beta}^{+2}) + 24x^{-1}[\sigma e^{4\beta^0 - 2\beta^+} + \kappa\rho_{(0)}e^{3(2-\gamma)\beta^0}]. \quad (6.29)$$

Transforming to the null variables $w = \beta^0 + \beta^+$ and $v = \beta^0 - \beta^+$ allows the potential to be identified with (4.98) of the two term case of Sec. IV H, with the correspondence between the parameters

$$C_1 = 24\sigma, \quad C_2 = 24\kappa\rho_{(0)}, \\ p_1 = 1, q_1 = 3, \quad p_2 = q_2 = 3(2 - \gamma)/2. \quad (6.30)$$

Solvable cases occur for the following parameter values:

The flat null $A_i = 0$ case: Condition (II) in Table III corresponds to the radiation value $\gamma = \frac{4}{3}$.

The null $A_i \neq 0$ case: Condition (I) yields physical solutions for the dust value $\gamma = 1$ and the value $\gamma = \frac{5}{3}$.

The above solutions can be found in [29,46,73,90,93,35].

3. Spatially homogeneous Bianchi type-VI models

a. Vacuum models. All known type-VI vacuum models are Taub symmetric ($\beta^- = 0$) and correspond to a two-dimensional problem with a Taub potential which is the single exponential term in Eq. (2.20), which is non-null except for Bianchi type VI₀ where it is null. These are easily solved using the methods of Secs. IV E 1 and IV E 2. These solutions were first found by [51,89,90,94].

b. Solvable perfect fluid models. Referring to Secs. II C 3 and II C 2, the orthogonal perfect fluid models have a Hamiltonian of the form

$$H = \frac{1}{2}x(-\dot{\beta}^{02} + \dot{\beta}^{\times 2}) + 24x^{-1}[c^{-2}e^{4(\beta^0 - cq\beta^\times)} + \kappa\rho_{(0)}e^{3(2-\gamma)\beta^0}], \quad (6.31)$$

where $c^{-2} = q^2 + 3a^2$.

Transforming to the null variables $w = \beta^0 + \beta^\times$ and $v = \beta^0 - \beta^\times$ allows the potential to be identified with (4.98) of the two term case of Sec. IV H, with the correspondence between the parameters

$$C_1 = 24c^{-2}, \quad C_2 = 24\kappa\rho_{(0)}, p_1 = 2(1 - cq), \\ q_1 = 2(1 + cq), \quad p_2 = q_2 = 3(2 - \gamma)/2. \quad (6.32)$$

The physical cases correspond to

$$0 \leq cq \leq 1, \quad 1 \leq \gamma \leq 2. \quad (6.33)$$

Solvable cases occur for the following parameter values:

The flat null $A_i = 0$ case: Condition (I) in Table III yields $cq = \frac{1}{4}(3\gamma - 2)$. The resulting solution was first found by Collins [90] and has been presented by Wainwright [33] corresponding to the form given in Eq. (4.55) with $m = \frac{3}{2}$.

The null $A_i \neq 0$ case: Conditions (I) of Table III with $a_0 \neq 0$ yield the conditions $cq = \pm \frac{1}{2}(4 - 3\gamma)$ or $cq = \frac{1}{8}(3\gamma + 2)$, respectively, and solutions found by Ugla [31] and Ugla and Rosquist [35].

The non-null case: There also exist two examples of the nonstiff perfect fluid solutions corresponding to non-null cases in which the potential may be reduced to a quadratic expression in the two natural power variables case (V) in Table III. These correspond to the values $cq = \frac{4}{5}$ and $\gamma = \frac{6}{5}$, leading to the solution given by Ugla [31]. These exact solutions together with the above null type VI_h ones were found using the methods developed in this paper and are the first new orthogonal SH non-EPL perfect fluid solutions found in several decades. The type-VI_h ($h \neq 0$) stiff perfect fluid solution can be obtained by making a boost that leads to a generalized Friedmann problem (the type-VI_{h=0} stiff perfect fluid case is contained in the flat $A_i = 0$ case discussed above).

Invariant submanifold perfect fluid models: Making the boost (6.17) with value $v = (3\gamma - 2)/(4cq) < 1$ leads to the Taub potential

$$U_{(\text{Taub})} = 24e^{3\Gamma(2-\gamma)\bar{\beta}^0} [c^{-2}e^{[3\gamma-2-4(cq)^2]\bar{\beta}^\times/(cq)} + \kappa\rho_{(0)}e^{3\Gamma(2-\gamma)(3\gamma-2)\bar{\beta}^\times/(4cq)}]. \quad (6.34)$$

For the type-VI_h models, the $\bar{\beta}^\times$ -dependent factor of this potential has a minimum provided that $(3\gamma - 2)/4 < (cq)^2$, yielding an EPL solution. This solution was discussed by Hsu and Wainwright [49].

Bianchi-type VI₀ models: For nonstiff perfect fluid models the boost (6.17) in the β^+ -direction with the value $v = -\frac{1}{4}(3\gamma - 2)$ leads to the Taub potential

$$U_{(\text{Taub})} = 6e^{3\Gamma(2-\gamma)\bar{\beta}^0} [e^{4\bar{\beta}^+} h_-^2 + 4\kappa\rho_{(0)}e^{B\bar{\beta}^+}], \quad (6.35)$$

$$A = 3\Gamma(2 - \gamma), \quad B = -\frac{3}{4}\Gamma(2 - \gamma)(3\gamma - 2).$$

The $\bar{\beta}^+, \beta^-$ -dependent factor has a minimum, yielding a one-dimensional invariant submanifold which corresponds to the same type VI₀ EPL solution just discussed. However, these new dependent variables are useful for a qualitative discussion of the dynamics for this class of models [95].

B. Nondiagonal models

1. Stationary cylindrically symmetric models

The stationary cylindrically symmetric vacuum models have the reduced Hamiltonian (2.50) which corresponds to the single non-null exponential potential term case and is therefore solvable as discussed in Sec. IVE2. For an explicit representation of this solution see [4].

2. Spatially homogeneous Bianchi type-VI_{-1/9} models

a. Vacuum models. The boost (6.17) with the value $v = 2/(3\sqrt{3})$ transforms the Taub potential from (2.56)

to

$$U_{(\text{Taub})} = e^{8\Gamma\bar{\beta}^0/3} [\frac{3}{2}p_\phi^2 e^{4\sqrt{3}\Gamma\bar{\beta}^+} + 32e^{-10\sqrt{3}\Gamma\bar{\beta}^+/9}]. \quad (6.36)$$

The $\bar{\beta}^+$ -dependent factor has a minimum, which corresponds to a one-dimensional invariant submanifold, leading to an EPL solution [33,96].

b. Perfect fluid models. The same boost as in the previous vacuum case for $\gamma = \frac{10}{9}$ transforms the Taub potential from (2.56) to

$$U_{(\text{Taub})} = e^{8\Gamma\bar{\beta}^0/3} [\frac{3}{2}p_\phi^2 e^{4\sqrt{3}\Gamma\bar{\beta}^+} + 32e^{-10\sqrt{3}\Gamma\bar{\beta}^+/9} + 24\ell^{10/9} e^{8\Gamma\bar{\beta}^+/3}]. \quad (6.37)$$

The $\bar{\beta}^+$ -dependent factor has a minimum, which corresponds to a one-dimensional invariant submanifold, leading to an EPL solution [33]. No known exact solutions exist for other values of γ .

3. Spatially homogeneous class A models belonging to the symmetric case

a. Bianchi type-II perfect fluid models. Here we are going to show how one can obtain tilted type-II EPL solutions by finding invariant submanifolds without explicitly knowing the function Y occurring in the fluid potential. Tilted models are quite complicated and therefore the manipulations become rather cumbersome. However, by using computer algebra they can be done.

Equations (2.16), (2.67), and (2.73) yield the total potential

$$U_{(\text{Taub})} = U_{(c)} + U_{(G)} + U_{(\text{fluid})} \quad (6.38)$$

$$= 24\kappa^2 \ell^2 (v_3)^2 (n^{(1)})^{-2} [e^{-4\sqrt{3}\beta^-} + B e^{4(\beta^0 + \beta^+ + \sqrt{3}\beta^-)} + C e^{3(2-\gamma)\beta^0} Y^{-\gamma/2} (\gamma Y - \gamma + 1)],$$

where B and C are constants defined by

$$B = \frac{1}{4}\kappa^{-2} (n^{(1)})^4 \ell^{-2} (v_3)^{-2}, \quad (6.39)$$

$$C = \kappa^{-1} (n^{(1)})^2 \ell^{-(2-\gamma)} (v_3)^{-2}.$$

It can be simplified by first performing a boost with velocity $v = (4 - 3\gamma)/2$ (excluding the stiff fluid case $\gamma = 2$) in the β^+ direction

$$\beta^0 = \Gamma_v(\tilde{\beta}^0 + v\tilde{\beta}^+), \quad \beta^+ = \Gamma_v(v\tilde{\beta}^0 + \tilde{\beta}^+), \quad \beta^- = \tilde{\beta}^-, \quad (6.40)$$

where

$$\Gamma_v = (1 - v^2)^{-1/2} = 2[3(3\gamma - 2)(2 - \gamma)]^{-1/2}. \quad (6.41)$$

Then the relation (2.69) simplifies to

$$F = A e^{(4/\Gamma_v)\tilde{\beta}^+}, \quad (6.42)$$

where A is a constant given by

$$A = \gamma^{-2} \ell^{-2(\gamma-1)} (v_3)^2, \quad (6.43)$$

thus leading to $Y = Y(\tilde{\beta}^+)$ since $FY^{\gamma-1} - Y + 1 = 0$. Expressed in these variables the potential takes the form

$U_{(\text{Taub})}$

$$\begin{aligned} &= 24\kappa^2 \ell^2 (v_3)^2 (n^{(1)})^{-2} [e^{-4\sqrt{3}\tilde{\beta}^-} \\ &\quad + B e^{6(2-\gamma)\Gamma_v(\tilde{\beta}^0 + \tilde{\beta}^+) + 4\sqrt{3}\tilde{\beta}^-} \\ &\quad + C e^{(3/2)(2-\gamma)\Gamma_v[2\tilde{\beta}^0 - (3\gamma-4)\tilde{\beta}^+]} Y^{-\gamma/2} (\gamma Y - \gamma + 1)]. \end{aligned} \quad (6.44)$$

A further boost with velocity $w = -(\sqrt{3}/4)\Gamma_v(2-\gamma) = -\sqrt{2-\gamma}/[2\sqrt{3\gamma-2}]$ in the $\tilde{\beta}^-$ direction

$$\begin{aligned} \tilde{\beta}^0 &= \Gamma_w(\tilde{\beta}^0 + w\tilde{\beta}^-), \quad \tilde{\beta}^+ = \tilde{\beta}^+, \\ \tilde{\beta}^- &= \Gamma_w(w\tilde{\beta}^0 + \tilde{\beta}^-), \end{aligned} \quad (6.45)$$

where

$$\Gamma_w = (1-w^2)^{-1/2} = \frac{2\sqrt{3\gamma-2}}{\sqrt{13\gamma-10}}, \quad (6.46)$$

leads to

$$U_{(\text{Taub})} = 24\kappa^2 \ell^2 (v_3)^2 (n^{(1)})^{-2} e^{3(2-\gamma)\Gamma_w\tilde{\beta}^0} \Phi(\tilde{\beta}^+, \tilde{\beta}^-), \quad (6.47)$$

where

$$\Phi(\tilde{\beta}^+, \tilde{\beta}^-) = e^{-4\sqrt{3}\Gamma_w\tilde{\beta}^-} + B e^{6(2-\gamma)\Gamma_v\tilde{\beta}^+ - 4\sqrt{3}\Gamma_w(1+2\alpha)\tilde{\beta}^-} + C e^{(3/2)(2-\gamma)(4-3\gamma)\tilde{\beta}^+ - 4\sqrt{3}\Gamma_w(1+\alpha)\tilde{\beta}^-} Y^{-\gamma/2} (\gamma Y - \gamma + 1) \quad (6.48)$$

and $\alpha = -(\Gamma_w)^{-2}$. This form of the potential displays explicitly its homothetic character in the sense that $\partial/\partial\tilde{\beta}^0$ is a homothetic vector of the associated Jacobi geometry.

To locate possible extremal values of the function Φ it is convenient to express it as

$$\begin{aligned} \Phi &= Z^k + B A^{-2b-1} Z^{k+2} Y^{2a+1} (Y-1)^{2b+1} \\ &\quad + C A^{-b} Z^{k+1} Y^a (Y-1)^b (\gamma Y - \gamma + 1), \end{aligned} \quad (6.49)$$

where we have introduced the variable $Z = e^{(4\sqrt{3}/\Gamma_w)\tilde{\beta}^-}$ and the constants

$$\begin{aligned} a &= -\frac{5\gamma-4}{2(3\gamma-2)}, \quad b = \frac{4-3\gamma}{2(3\gamma-2)}, \\ k &= \alpha^{-1} = -(\Gamma_w)^2 = -\frac{4(3\gamma-2)}{13\gamma-10}. \end{aligned} \quad (6.50)$$

The potential can be further simplified by making a suitable translation in $\tilde{\beta}^-$ or equivalently rescaling Z by $Z = \delta\bar{Z}$ where $\delta = A^{b+1/2}B^{-1/2}$. This finally gives $\Phi = \delta^k\bar{\Phi}$ where

$$\bar{\Phi} = Z^k + D(Y)Z^{k+2} + E(Y)Z^{k+1}, \quad (6.51)$$

where

$$\begin{aligned} D(Y) &= Y^{2a+1}(Y-1)^{2b+1}, \\ E(Y) &= 2Y^a(Y-1)^b(Y-1+\gamma^{-1}). \end{aligned} \quad (6.52)$$

To find possible extremal points of $\bar{\Phi}$ we calculate the derivatives

$$\begin{aligned} \bar{\Phi}_{,Y} &= \bar{Z}^{k+1}[D'(Y)\bar{Z} + E'(Y)], \\ \bar{\Phi}_{,\bar{Z}} &= \bar{Z}^{k-1}[k + D(Y)\bar{Z}^2 + E(Y)\bar{Z}], \end{aligned} \quad (6.53)$$

where

$$\begin{aligned} D'(Y) &= Y^{2a}(Y-1)^{2b}[2(a+b+1)Y - 2a - 1], \\ E'(Y) &= 2Y^{a-1}Y^{b-1}\{(a+b+1)Y^2 \\ &\quad + [-2a + b - 1 + (a-b)\gamma^{-1}]Y + a(1-\gamma^{-1})\}. \end{aligned} \quad (6.54)$$

Equating the first of these expressions to zero while noting that $D'(Y) > 0$ gives $\bar{Z} = -E'(Y)/D'(Y)$. Inserting this result in the second equation yields an equation for Y having a single solution given by

$$Y - 1 = \frac{(3\gamma-4)(7\gamma-10)}{2\gamma(17\gamma-18)}, \quad (6.55)$$

leading to

$$\bar{Z} = Y^{-a}(Y-1)^{-b} \frac{4\gamma(17\gamma-18)}{(11\gamma-10)(7\gamma-10)}. \quad (6.56)$$

The conditions $Z > 0$ and $Y > 1$ can only be satisfied if $\gamma > 10/7$.

We conclude that there exists an exact power-law solution with values of $\tilde{\beta}^\pm$ corresponding to the minimum values of Y and \bar{Z} through the relations

$$\begin{aligned} e^{(4/\Gamma_v)\tilde{\beta}^+} &= A^{-1}F = A^{-1}Y^{1-\gamma}(Y-1), \\ e^{(4\sqrt{3}/\Gamma_w)\tilde{\beta}^-} &= A^{b+1/2}B^{-1/2}\bar{Z}. \end{aligned} \quad (6.57)$$

This is exactly Hewitt's EPL solution for the equation of state parameter in the range $10/7 < \gamma < 2$ [97]. (Note that the solution with $\gamma = 10/7$ occurring in [97] is not a tilted one.) Since the Jacobi metric admits a timelike HKV, one can apply the methods developed in [95] to obtain a complete picture of the behavior at late times of these models for $10/7 < \gamma < 2$.

b. Bianchi type- VI_0 stiff perfect fluid models. For stiff perfect fluids and the usual values of the structure constants $n^{(1)} = -n^{(2)} = 1$, the Taub potential takes the form

$$U_{(\text{Taub})} = 6(4 + \kappa v_3^2) e^{4(\beta^0 + \beta^+)} + 24\kappa\ell^2. \quad (6.58)$$

TABLE VII. Hamiltonian spatially homogeneous perfect fluid models. The abbreviations Orth., S-E, C-S, K-C, “Dor.,” K-S, U-R, R-U, E-M, and “Wain.” stand for Orthogonal, Stewart-Ellis, Collins-Stewart, Kompanets-Chernov, Doroshkevich, Kantowski-Sachs, Rosquist-Uggla, Uggla-Rosquist, Ellis-MacCallum, and Wainwright. One can set $q = 1$ in the expression $cq = q/\sqrt{q^2 + 3a^2}$ for type VI_h . Note that when $cq = 1/2$, then $VI_{h=-1}$ =III. A yes in the “Orth.” column implies that the model is orthogonal while a no implies a tilted model.

Bianchi type	Dim	$1 \leq \gamma \leq 2$	Solution	EPL	Orth.	Relevant sections
I	6	γ	Flat FRW	Yes	Yes	VIA 1, III
I	4	1	Robinson [44]	No	Yes	VIA 1, IVE 2, III, IVE 1
		$\frac{4}{3}$	Dor. [46]	No	Yes	VIA 1, IVE 2, III, IVE 1
		γ	Jacobs [43], S-E [47]	No	Yes	VIA 1, IVE 2, III, IVE 1
I	3	1	Robinson [44]	No	Yes	VIA 1, IVE 2, III
		γ	Jacobs [43]	No	Yes	VIA 1, IVE 2, III
II	4	$\gamma < 2$	C-S [91]	Yes	Yes	VIA 2, V, III
		2	Collins [90]	No	Yes	VIA 2, IVE 2, III
II	3	γ	Collins [90]	No	Yes	VIA 2, V, IVE 1
		2	Collins [90]	No	Yes	VIA 2, IVE 2, III
		$\frac{10}{7} < \gamma < 2$	Hewitt [97]	Yes	No	VIB 3, V, III
III	4	$1, \frac{4}{3}$	K-C [93]	No	Yes	VIA 2, IVE 1
		$\frac{5}{3}$	U-R [35]	No	Yes	VIA 2, IVE 1
		2	K-S [73]	No	Yes	VIA 2, IVE 2
KS	4	1	Dor. [46]	No	Yes	VIA 2, IVE 1
		$\frac{4}{3}$	K-C [93]	No	Yes	VIA 2, IVE 1
		$\frac{5}{3}$	R-U [29]	No	Yes	VIA 2, IVE 1
		2	K-S [73]	No	Yes	VIA 2, IVE 2
V	6	γ	Open FRW	No	Yes	VIA 1, III
V	3	γ	E-M [51]	No	Yes	VIA 1, IVE 2, III
VI_h	3	$2(2cq + 1)/3$	Collins [90]	No	Yes	VIA 3, IVE 1
		$\gamma < 2$	Collins [90]	Yes	Yes	VIA 3, V, III
		2	E-M [51] ($h = 0$)	No	Yes	VIA 3, IVE 1
		2	Collins [90] ($h \neq 0$)	No	Yes	VIA 3, IVE 2, III
		$6/5$	Uggla [31] ($h = -3/16$)	No	Yes	VIA 3, IVE 2
		$2(2 \pm cq)/3$	U-R [35]	No	Yes	VIA 3, IVE 1
		$2(4cq - 1)/3$	U-R [35]	No	Yes	VIA 3, IVE 1
		$\frac{10}{9}$	Wain. [100] ($h = -1/9$)	Yes	Yes	VIB 2, V, III
2	Wain. <i>et al.</i> [98] ($h = 0$)	No	No	VIB 3, IVE 1		
VIII	4	2	Jantzen [101]	No	Yes	VIA 2, IVE 2, III
IX	6	γ	Closed FRW	No	Yes	VIA 1, III
IX	4	2	Barrow [102]	No	Yes	VIA 2, IVE 2, III

Transforming to null variables this is seen to correspond to the easily solved flat null case $c_0 = 0$ of Sec. IVE 1. This solution was first found by Wainwright *et al.* [98].

C. Summary of the spatially homogeneous perfect fluid models

All of the exact SH perfect fluid solutions arising from Hamiltonian models are now collected in Table VII. For

the sake of completeness the remaining known “non-Hamiltonian” exact tilted perfect fluid solutions are listed in Table VIII. Whether or not some of the models corresponding to these cases admit a Hamiltonian of the “standard” form $H = T + U$ is not clear. We have not been able to find such a formulation for any of the models using the present approach. Perhaps the “comoving” approach of MacCallum and Taub [99] could be used to produce Hamiltonians in at least the tilted types-V and - VI_0 cases.

TABLE VIII. Non-Hamiltonian spatially homogeneous perfect fluid models. The numerical values 1.0411 and 1.7169 are numerical approximations calculated by Rosquist and Jantzen.

Bianchi type	Dim	$1 \leq \gamma \leq 2$	Solution	EPL	Orth.
V	4	1	Farnsworth [104]	No	No
		2	Maartens and Nel [103]	No	No
V	3	2	Maartens and Wolfaardt [105]	No	No
VI_0	3	$\frac{4}{3}, 1.0411 \leq \gamma \leq 1.7169$	Rosquist [106], Rosquist and Jantzen [107]	Yes, Yes	No, No
VI_h	3	2	Wainwright <i>et al.</i> [98]	No	No
VII_h	3	2	Barrow [102]	No	Yes
		2	Wainwright <i>et al.</i> [98]	No	No

VII. BEYOND FOUR-DIMENSIONAL GENERAL RELATIVITY

The techniques presented in this paper are valuable even outside the realm of four-dimensional general relativity. The only requirement needed to apply them in this larger context is a Hamiltonian theory for which the Hamiltonian has a quadratic form kinetic energy function.

A. Higher-dimensional theories

With the emergence of unified field theories constructed using higher-dimensional spacetimes, the door opened to the search for higher-dimensional analogues of many of the four-dimensional HH solutions of Einstein's equations or related field equations. These occur as solutions of Einstein's equations in higher dimensions and its numerous generalizations, Kaluza Klein and supersymmetric variations of Einstein's equations, with coupling to various matter sources, both in the context of static HH spacetimes and SH cosmological spacetimes.

Although details change, the same general picture applies and again one sees in the literature on this topic the same kinds of similarities that characterize those HH solutions listed by Kramer *et al.* in the four-dimensional case. The simplest models for which solutions can be found are again diagonal, most characterized by the natural generalizations of the four-dimensional Hamiltonians, with the same Lorentz structure of the kinetic part playing a key role in the properties of the dynamics. One may take the mathematical discussion of this paper and apply it with slight modifications directly to the higher-dimensional case [108].

$$\frac{\delta S}{\delta g^{\alpha\beta}} = \frac{1}{2} \sqrt{-^{(4)}g} \{ \kappa^{-1} A(\phi) G_{\alpha\beta} + \kappa^{-1} (\square A(\phi)) g_{\alpha\beta} - \kappa^{-1} A(\phi)_{;\alpha\beta} - B(\phi) \phi_{,\alpha} \phi_{,\beta} + [\frac{1}{2} B(\phi) (\nabla\phi)^2 + V(\phi)] g_{\alpha\beta} \}, \quad (7.3)$$

$$\frac{\delta S}{\delta \phi} = \sqrt{-^{(4)}g} [B(\phi) \square \phi + (2\kappa)^{-1} A'(\phi) {}^{(4)}R - V'(\phi) - \frac{1}{2} B'(\phi) (\nabla\phi)^2],$$

where $(\nabla f)^2 = g^{\alpha\beta} f_{,\alpha} f_{,\beta}$ and $\square f = g^{\alpha\beta} f_{;\alpha\beta}$.

1. Diagonal spatially homogeneous scalar field models

We will assume that the scalar field is SH and therefore a function of t only. Adding an orthogonal perfect fluid with an equation of state $p = (\gamma - 1)\rho$ to the action (7.1) for the diagonal SH models, leads to the Hamiltonian

$$H = -4\kappa N n^\alpha n^\beta \frac{\delta S}{\delta g^{\alpha\beta}} + 2\kappa N g^{1/2} n^\alpha n^\beta T_{(\text{pf})\alpha\beta} \quad (7.4)$$

$$= 6N^{-1} e^{3\beta^0} \left[A(\phi) \eta_{AB} \dot{\beta}^A \dot{\beta}^B - A'(\phi) \dot{\phi} \dot{\beta}^0 + \frac{\kappa}{6} B(\phi) \dot{\phi}^2 \right] + N \left[e^{\beta^0} V^* A(\phi) + 2\kappa e^{3\beta^0} V(\phi) + 2\kappa \rho_{(0)} e^{-3(\gamma-1)\beta^0} \right],$$

where $T_{(\text{pf})\alpha\beta}$ is the energy-momentum tensor of the perfect fluid. Here V^* is given by (2.16) for the class A models for which $A, B = 0, +, -$, by $V^* = 2c^{-2} e^{\beta^0 - 4c\beta^0}$, $A, B = 0, \times$ for the type-V and $-VI_h$ models, by $V^* = -2\sigma e^{\beta^0 - 2\beta^+}$, $A, B = 0, +$ for the MT models, and by $V^* = -6k$ for the FRW models, which are described by

B. Nonminimally coupled scalar fields

Nonminimally coupled scalar field models are described by an action of the form

$$S = \frac{1}{2} \int d^4x \sqrt{-^{(4)}g} \left[\kappa^{-1} A(\phi) {}^{(4)}R - B(\phi) g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - 2V(\phi) \right], \quad (7.1)$$

where ${}^{(4)}g = \det(g_{\alpha\beta})$. If one is so inclined, one may also add other matter fields to this action, and below we will add a perfect fluid contribution. The function $B(\phi)$ can be set equal to 1 without loss of generality by using the freedom to redefine the scalar field by a transformation $\hat{\phi} = f(\phi)$ together with an accompanying redefinition of A . However, it is convenient to retain a general expression $B(\phi)$ for easier comparison with the literature. The coupling between gravity and matter is said to be minimal if $A(\phi)$ is constant. One can then set $A = 1$, $B = 1$. All other couplings are referred to as nonminimal. Often used nonminimal couplings include conformal coupling characterized by $A(\phi) = 1 - \frac{1}{6}\phi^2$, $B(\phi) = 1$, and Brans-Dicke coupling for which $A(\phi) = \kappa\phi$, $B(\phi) = \omega/\phi$, and $V(\phi) = 0$. Some typical scalar potentials are

$$V(\phi) = \begin{cases} \Lambda/\kappa & \text{cosmological constant,} \\ \frac{1}{2} m^2 \phi^2 & \text{mass term,} \\ \frac{1}{4!} \lambda \phi^4 & \lambda \phi^4 \text{ theory,} \\ m^2 \mu^{-2} e^{\mu\phi} & \text{exponential potential,} \end{cases} \quad (7.2)$$

where m , λ , and μ are the physical parameters characterizing the various cases.

Varying the action (7.1) with respect to the metric and the scalar field yields

the line element

$$ds^2 = -N^2 dt^2 + R^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (7.5)$$

with k taking the standard values 1, 0, -1 for the closed,

flat, and open models, respectively, and for which $\ln R = \beta^0, \beta^\pm = 0$. (The isotropic case can of course be obtained from the other models. However, since this case is by far the most discussed when it comes to nonminimally coupled models, it is given explicitly here.) The kinetic energy of a minimally coupled model can be transformed to a manifestly conformally flat form by the choice $B(\phi) = \text{const}$ corresponding to a redefinition of ϕ . For nonminimally coupled models we make the field redefinitions (cf. [109])

$$\begin{aligned}\tilde{\beta}^0 &= \beta^0 + \ln A^{1/2}, & \tilde{\beta}^P &= \beta^P, \\ \tilde{\beta}^\dagger &= 12^{-1/2} \int d\phi [A(\phi)]^{-1} \sqrt{3[A'(\phi)]^2 + 2\kappa A(\phi)B(\phi)},\end{aligned}\quad (7.6)$$

where P takes appropriate values, e.g., \pm for the class A models. This transformation leads to a manifestly conformally flat kinetic energy

$$\begin{aligned}T &= 6N^{-1} e^{3\tilde{\beta}^0} [\tilde{A}(\tilde{\beta}^\dagger)]^{-1/2} (\eta_{AB} \dot{\tilde{\beta}}^A \dot{\tilde{\beta}}^B + \dot{\tilde{\beta}}^{\dagger 2}) \\ &= \frac{1}{2} x_{(nm)} (\eta_{AB} \dot{\tilde{\beta}}^A \dot{\tilde{\beta}}^B + \dot{\tilde{\beta}}^{\dagger 2}),\end{aligned}\quad (7.7)$$

where $\tilde{A}(\tilde{\beta}^\dagger) = A(\phi)$ and $x_{(nm)} = 12N^{-1} e^{3\tilde{\beta}^0} [\tilde{A}(\tilde{\beta}^\dagger)]^{-1/2} = A(\phi)x$. Note also that the redefined scalar field is consistent with the definition made in Sec. II so that $\tilde{\beta}^\dagger = \beta^\dagger$ in the minimal coupling limit $A \rightarrow 1, B \rightarrow 1$. With the above variable choice the potential takes the form

$$\begin{aligned}U &= 24x_{(nm)}^{-1} \left[\frac{1}{2} V^* e^{4\tilde{\beta}^0} + \kappa e^{6\tilde{\beta}^0} [\tilde{A}(\tilde{\beta}^\dagger)]^{-2} \tilde{V}(\tilde{\beta}^\dagger) \right. \\ &\quad \left. + \kappa \rho_{(0)} e^{3(2-\gamma)\tilde{\beta}^0} [\tilde{A}(\tilde{\beta}^\dagger)]^{-(4-3\gamma)/2} \right],\end{aligned}\quad (7.8)$$

where $\tilde{V}(\tilde{\beta}^\dagger) = V(\phi)$. Note that if the potential is given by $V = \tilde{c}A^2$ and if $\gamma = \frac{4}{3}$ (i.e., radiation), then the above Hamiltonian coincides with the one in general relativity describing a massless scalar field and a cosmological constant $\Lambda = \kappa\tilde{c}$. Thus intrinsically isotropic models (i.e., the isotropic models and the type-I and -V models, collectively characterized by $V^* = -6k$ when choosing $a = 1$ for the type-V models) are solvable if $x_{(nm)}$ is chosen to depend only on $\tilde{\beta}^0$ since this leads to a one-dimensional generalized Friedmann problem. Intrinsically isotropic stiff fluid models with the same scalar field potential, $V = \tilde{c}A^2$, are also solvable since they lead to a separable potential. The corresponding Jacobi metric therefore admits a second rank Killing tensor.

a. The conformally coupled case. Conformal coupling corresponds to the choice $A(\phi) = 1 - \phi^2/6, B(\phi) = 1$. Using Eq. (7.6) leads to a redefinition of the scalar field given by the relation $\phi = \sqrt{6} \tanh \tilde{\beta}^\dagger$ implying $\tilde{A}(\tilde{\beta}^\dagger) = \cosh^{-2} \tilde{\beta}^\dagger$. For a model with a cosmological constant, a mass term, and a quartic term the total potential can be written in terms of the redefined fields as

$$\begin{aligned}U &= 24x_{(nm)}^{-1} \left[\frac{1}{2} V^* e^{4\tilde{\beta}^0} + e^{6\tilde{\beta}^0} (\Lambda \cosh^4 \tilde{\beta}^\dagger \right. \\ &\quad \left. + 3\kappa m^2 \cosh^2 \tilde{\beta}^\dagger \sinh^2 \tilde{\beta}^\dagger + 3\kappa \lambda \sinh^4 \tilde{\beta}^\dagger) \right].\end{aligned}\quad (7.9)$$

Obviously this is a SE Hamiltonian. Apart from the ‘‘trivially’’ solvable intrinsically isotropic case with $m =$

0, $\lambda = 0, \Lambda = 0$ (corresponding to setting $\tilde{c} = 0$ in the previous general discussion), there are also a number of other sets of values of the parameters for which the model is solvable [110]. The isotropic case with $\lambda = 0, \Lambda = 0$ was shown to be chaotic by Calzetta and El Hasi [111].

b. A solvable case with nonconformal quadratic coupling. Consider a nonminimally quadratically coupled model with $A(\phi) = 1 - \xi\phi^2$. For these models it follows that the $V(\phi) = \tilde{c}A^2 = \kappa^{-1}\Lambda A^2$ case, discussed above, corresponds to a model with an arbitrary cosmological constant Λ , mass $m = 2\sqrt{-\kappa^{-1}\Lambda\xi}$, and a quartic term with $\lambda = 24\kappa^{-1}\Lambda\xi^2$. To have a physically reasonable mass term we must have $\Lambda\xi < 0$.

c. Brans-Dicke models. In this case the scalar field coupling is defined by the relations $A(\phi) = \kappa\phi, B(\phi) = \omega/\phi$, and $V(\phi) = 0$. Redefining the scalar field by $\phi = \kappa^{-1}e^{2\nu\tilde{\beta}^\dagger} = \kappa^{-1}\tilde{A}(\tilde{\beta}^\dagger)$ where $\nu = \sqrt{3/(3+2\omega)}$ leads to the Hamiltonian

$$\begin{aligned}H &= \frac{1}{2} x_{(nm)} (-\dot{\tilde{\beta}}^0{}^2 + \dot{\tilde{\beta}}^+{}^2 + \dot{\tilde{\beta}}^-{}^2 + \dot{\tilde{\beta}}^{\dagger 2}) \\ &\quad + 24x_{(nm)}^{-1} \left[\frac{1}{2} V^* e^{4\tilde{\beta}^0} + \kappa \rho_{(0)} e^{3(2-\gamma)\tilde{\beta}^0 - (4-3\gamma)\nu\tilde{\beta}^\dagger} \right].\end{aligned}\quad (7.10)$$

Note that this is a SE Hamiltonian and that for radiation ($\gamma = \frac{4}{3}$), it is equivalent to the general relativistic Hamiltonian with two noninteracting perfect fluids, one stiff and the other radiation (provided one chooses $x_{(nm)}$ to be independent of $\tilde{\beta}^\dagger$). Furthermore, if the fluid term in the above Hamiltonian is zero then there is a one to one correspondence between solvable stiff fluid models in general relativity and vacuum solutions in Brans-Dicke theory.

As seen from the above discussion there is a close mathematical relationship between the nonminimally coupled scalar field Hamiltonians (and particularly Brans-Dicke theory) and the SE Hamiltonians occurring in general relativity. This explains the numerous exact solutions one has obtained in these theories and the equally numerous number of papers describing them in the literature (for Brans-Dicke theory see, e.g., [112,113]). Moreover, the above discussion shows how one easily can find new ones if one is so inclined.

C. A note on quantum cosmology

The results of this paper may be used as a first step in quantizing SH models. Ashtekar *et al.* [114] have quantized the intrinsically multiply transitive vacuum models. As can be seen from Secs. VIA 1 and VIA 2, these all have non-null decoupling in the Taub slicing gauge. For any non-null decoupling case the Hamiltonian takes the form

$$H = \sum_{\mu} H_{\mu}(y^{\mu}, p_{\mu})\quad (7.11)$$

when expressed in symmetry adapted dependent variables and in a slicing gauge leading to decoupling. As done in [114], one can make a canonical transformation such that each decoupled Hamiltonian is reduced to the square of a momentum P_{μ} , leading to

$$H = \frac{1}{2} \eta^{\mu\nu} P_{\mu} P_{\nu}.\quad (7.12)$$

The only trace of the original potential is to be found in the ranges of the values of the new variables. At this stage one has a complete set of observables (constants of the motion) and one can follow the quantization procedure used in [114] to quantize these models. However, even for non-null solvable models one obtains a complete set of observables, and the quantization procedure discussed by Torre [115] can be used to quantize them.

VIII. CONCLUDING REMARKS

There are other methods than the ones presented in this paper which exist for producing exact solutions. Their relationship to the present ones is discussed below, but the relationship is not completely understood and deserves further attention. The section is concluded with a general discussion on a number of different issues.

A. Relationship to other solution techniques

1. Comparison with solution generating techniques

Various solution generating techniques have been developed for vacuum, electromagnetic, or stiff perfect fluid spacetimes with one or two commuting Killing vectors (see e.g., [4,116,117]). These techniques rely on the existence of symmetries which allow one to find new solutions from a given solution within the infinite-dimensional space of solutions being considered. For the finite-dimensional Hamiltonian problems studied here, one can also use the Killing tensor symmetries to generate new solutions from a particular one, but in practice this is a moot point since one finds the entire family at once.

There are also solution generating methods which produce new solutions from a particular one but with different source or symmetry characteristics [4,98,101]. Although the present approach analyzes separate Hamiltonian problems, one could also use the variation of parameters idea of [101] to establish relationships between different Hamiltonian problems.

For models with an infinite number of degrees of freedom, one can impose conditions on various geometric quantities or on the functional form of the line element and still obtain a nontrivial problem corresponding to an invariant submanifold. This is in stark contrast to the situation for the finite number of degrees of freedom of the HH models where such conditions usually result in inconsistencies (except in the case when one has an unspecified function, such as an arbitrary scalar field potential). Thus it is critical to have systematic methods for finding invariant submanifolds for such models.

2. Comparison with the exact solution method of Maartens and Wolfaardt

Maartens and Wolfaardt consider a certain class of systems of second-order differential equations and find a constant of the motion linear in the first derivatives [118].

For Hamiltonian systems of this type it therefore seems reasonable that this symmetry corresponds to a Killing vector symmetry since the latter is associated with a constant of the motion which is linear in the momenta. However, their method is also applicable to non-Hamiltonian problems.

They apply their analysis to diagonal SH models. They rederive the Bianchi type-I solutions with either a cosmological constant or a perfect fluid and the orthogonal Bianchi type-II stiff perfect fluid solutions. These Hamiltonian models do indeed admit Killing vector symmetries. However, they also apply their method to a non-Hamiltonian tilted Bianchi type-V stiff model and thereby obtain an exact perfect fluid solution [98,101].

3. Comparison with Hewitt's exact solution method

For polynomial systems of ordinary differential equations one can search for algebraic invariant curves, which then lead to exact solutions. Hewitt has applied such a method to two-dimensional systems arising from the Einstein field equations for certain cosmological models [119], looking for linear and quadratic algebraic invariant curves. For two-dimensional systems the existence of a sufficient number of such curves not only produces the corresponding exact solutions but also makes it possible to solve the full system. The search for these invariant curves is quite complicated and relies on algebraic computing, making it difficult to extend the approach to higher dimensions or to invariant curves of higher degree. Another consideration is the degree of the polynomials occurring in the system of equations, which must be sufficiently low for practical use.

Such two-dimensional polynomial systems can be derived if, for example, the problem is two-dimensional and the Taub potential has at most two exponential terms. All the models of this type which admit Killing tensors are given in Sec. IV H. The Killing tensors give rise to constants of the motion which are linear or quadratic in the momenta and involve exponential factors.

Sometimes these constants of the motion lead to linear or quadratic curves in the polynomial system for certain values of those constants, but not always. Thus there is an overlap with Hewitt's method but it is not clear how large it is. So far all cases which have been found by that method correspond to the existence of Killing tensors although there are many Killing tensor cases which do not lead to linear or quadratic algebraic curves. On the other hand there may exist Hewitt cases which do not correspond to Killing tensors.

The present approach has the advantage that one can immediately see whether or not exact solutions occur by inspection of a *single function*, the Taub potential, by hand, without attacking a *whole system* of differential equations using algebraic computing. Furthermore it is easier to obtain an SE Hamiltonian in "standard" form than by rewriting the field equations in polynomial form. It also has the advantage that higher dimensional problems and those with more complicated Taub potentials are easily handled. On the other hand Hewitt's method

can be applied to two-dimensional problems which do not arise from Hamiltonian systems of the usual type.

B. Discussion

The common practice of obtaining exact HH solutions in gravitational physics is to examine each new scenario as a new problem in isolation without considering its mathematical relationship to other such problems. However, if a problem admits a Hamiltonian formulation, where the kinetic part of the Hamiltonian can be put in the “conformally flat” form, then the present approach can be applied. Since this approach has made it possible to unify, extend, and bring order to many apparently unrelated special results of the existing literature, it should prove to be a useful tool in future studies.

There are many models not explicitly treated in this paper which could be investigated with these methods. Among these are a variety of static [4] and self-similar models [120]. Some timelike self-similar models have already been treated in this way and some new solutions found [121]. Other sources or combinations of sources may also be considered leading to an abundance of models.

All of these examples lie within conventional general relativity. However, the most likely applications will arise in exploring alternative gravitational theories. For example, of the numerous articles which regularly appear in this area, a randomly chosen one [122] analyzes a Bianchi type-I supergravity model, which can be completely explained in terms of the present analysis. This is not untypical. We are not aware of any solvable case in the

literature on HH models which cannot be explained by the existence of rank two Killing tensor and Killing vector symmetries. It would be interesting to find an explicit solvable case solution not admitting such Killing symmetries.

Finally the present framework is not just valuable for the goal of searching for exact solutions but may serve as the starting point for a qualitative analysis of the more general behavior of the field equations. There are various kinds of Hamiltonian symmetries which may not be sufficient to lead to exact solutions. Nevertheless by adapting the variables to these symmetries, one obtains a simpler qualitative description. An example of such a symmetry is the homothetic Killing vector symmetry which many models exhibit [25]. This was exploited to develop an intuitive qualitative picture of the dynamics of a number of models in [95]. This follows in the footsteps of the well-known picture of the mixmaster dynamics of the diagonal Bianchi type-IX models when expressed in Hamiltonian form in terms of the Misner parametrization [19,123,124] and generalized as much as possible to the general case for all Bianchi types in [7]. One can also use adapted variables to attempt a so-called regularization of the field equations [125,126]. Thus it seems clear that the tools presented here may prove useful in many applications involving the rich dynamics of HH models.

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