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Stability of the Cauchy horizon in anti–de Sitter spacetime

T. M. Helliwell*
Department of Physics, Harvey Mudd College, Claremont, California 91711

D. A. Konkowski†
Department of Mathematics, U.S. Naval Academy, Annapolis, Maryland 21402
and School of Mathematical Sciences, Queen Mary and Westfield College,
Mile End Road, London E14NS, United Kingdom
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A previously developed Cauchy horizon stability conjecture is used to investigate the stability of the Cauchy horizon in the covering space of anti–de Sitter spacetime when infalling and outgoing null dust is introduced. With infalling null dust, the Cauchy horizon is found to be stable, except for the single point $r = 0$ to which the dust collapses. An exact solution of the field equations is presented containing infalling null dust, which reduces to anti–de Sitter spacetime when the dust is removed and which has a nonsingular Cauchy horizon except for a shell-focusing singularity at $r = 0$. The conjecture predicts a nonscalar curvature singularity at $r = 0$, but in fact the shell-focusing singularity is a scalar curvature singularity. When both infalling and outgoing null dust test fields are introduced, the Cauchy horizon is predicted to be stable except for a scalar curvature singularity at $r = 0$.

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I. INTRODUCTION

Anti–de Sitter spacetime is a spacetime of constant negative curvature. It has the topology $S^1 \times R^3$, and contains closed timelike lines. The universal covering space of anti–de Sitter (AdS) spacetime has the topology R^4 and contains no closed timelike lines. Throughout this paper we will mean the universal covering space when referring to anti–de Sitter spacetime [1]. Null infinity is timelike in AdS spacetime, so the spacetime contains no global Cauchy surfaces [2–4]. If one places initial data on a spacelike surface, one cannot predict beyond its Cauchy development; new information could always arrive from timelike null infinity. The Cauchy horizons are the boundary of the Cauchy development of a maximal Cauchy surface.

In this paper we investigate the stability of the Cauchy horizons in AdS space. Are they stable if finite-density matter or fields are introduced on the initial spacelike surface? Or is there a buildup of the material along the Cauchy horizons, as has been found for Cauchy horizons in the Reissner-Nordström and Kerr spacetimes [5–11]?

There are two reasons for our interest in the stability of AdS Cauchy horizons (CH's). First, anti–de Sitter spacetime has been used in the construction of supersymmetric supergravity domain-wall spacetimes, and the interior of vacuum bubbles [4,12,13]. If the CH's are unstable, then singularities may form in these models. Second, these CH's serve as a testing ground for a stability conjecture we have developed.

In a number of papers [9,10,14–20], we have developed stability conjectures for the investigation of mild singularities and CH's in solutions of Einstein's equations. We look at the behavior of test fields in the vicinity of the singularity or CH, and based upon this behavior we predict what should become of the singularity or CH if the fields are allowed to influence the geometry through back reaction calculations using Einstein's equations. In a few cases these back reaction calculations have actually been carried out [9,16,18]; in each of them the results agree with the predictions of the conjectures.

In this paper we use the CH conjecture [9,10] to investigate the stability of the CH's in AdS space when null dust is added. In Sec. II we define singularity types and review our stability conjectures and their tests. In Sec. III we begin by reviewing properties of AdS space. Then in Sec. III A we derive a prediction in the case of infalling null dust. In Sec. III B we present an exact solution of Einstein's equations corresponding to null dust falling inward in the AdS spacetime. In Sec. III C we derive a prediction in the case of both infalling and outgoing null dust. In Sec. IV we summarize our conclusions.

II. SINGULARITY CLASSIFICATION AND STABILITY CONJECTURES

We use a singularity classification scheme based on one devised by Ellis and Schmidt [21]. They classified singularities in maximal spacetimes into three basic types: quasiregular, nonscalar curvature, and scalar curvature. The mildest singularity is quasiregular and the strongest is scalar curvature. At a scalar curvature singularity, physical quantities such as energy density and tidal forces diverge in the frames of all observers who approach the singularity. At a nonscalar curvature singularity, there exist curves through each point arbitrarily close to the singularity such that observers moving on

*Electronic address: helliwell@thuban.ac.hmc.edu
†Electronic address: dak@sma.usna.navy.mil

these curves experience perfectly regular tidal forces [21]. For a quasiregular singularity, no observers see physical quantities diverge, even though their world lines end at the singularity in a finite proper time.

Our version of the Ellis and Schmidt classification scheme can be expressed mathematically. We define singular points simply as the end points of incomplete geodesics in maximal spacetimes; Ellis and Schmidt use instead a b-boundary construction to define the singular points. In our scheme a singular point q is a quasiregular singularity if all components of the Riemann tensor R_{abcd} evaluated in an orthonormal frame parallel propagated along an incomplete geodesic ending at q are C^0 (or C^{0-}). In other words, the Riemann tensor components tend to finite limits (or are bounded). On the other hand, a singular point q is a curvature singularity if some components are not bounded in this way. If all scalars in g_{ab}, the antisymmetric tensor η_{abcd}, and R_{abcd} nevertheless tend to a finite limit (or are bounded), the singularity is nonscalar, but if any scalar is unbounded, the point q is a scalar curvature singularity.

We have previously used stability conjectures [9,10,14–20] to test the stability of quasiregular singularities, nonscalar curvature singularities, and Cauchy horizons. For singularities our conjecture states the following.

Conjecture 1. If a test field stress-energy tensor evaluated in a parallel-propagated orthonormal (PPON) frame mimics the behavior of the Riemann tensor components which indicate a particular type of singularity, then a complete nonlinear back reaction calculation would show that this type of singularity occurs.

For Cauchy horizons, the conjecture is slightly modified [9,10] to state the following.

Conjecture 2. For all maximally extended spacetimes with CH's, the back reaction due to a field (whose test-field stress-energy tensor is $T_{\mu\nu}$) will affect the horizon in the following manner: (1) If both $T^\mu{}_\mu$ and $T_{\mu\nu}T^{\mu\nu}$ are finite and if the stress-energy tensor $T_{(\alpha\beta)}$ in all PPON frames is finite, then the CH will remain nonsingular; (2) if both $T^\mu{}_\mu$ and $T_{\mu\nu}T^{\mu\nu}$ are finite but $T_{(\alpha\beta)}$ diverges in some PPON frame, then a nonscalar curvature singularity will be formed at the CH; (3) if either $T^\mu{}_\mu$ or $T_{\mu\nu}T^{\mu\nu}$ diverges, then a scalar curvature singularity will be formed at the CH.

Conjecture 1 has been tested in several cases, as reviewed in a previous paper [9]. Conjecture 2 has been tested so far only in Reissner-Nordström spacetime [9] and Kerr spacetime [10]. In Reissner-Nordström spacetime the conjecture predicts that the addition of infalling null dust with a power-law tail produces a nonscalar curvature singularity at the CH in the Reissner-Nordström-Vaidya spacetime studied by Hiscock [5], in agreement with his exact result. The conjecture also predicts that a combination of infalling and outgoing null dust produces a scalar curvature singularity at the CH. This prediction was verified using the mass inflation results of Poisson and Israel [6–8]. Finally, the conjecture predicts that the addition of infalling scalar or electromagnetic waves produces a scalar curvature singularity at the CH; we have found no exact solutions with which to verify the conjecture in these cases.

III. STABILITY TESTS OF THE ANTI–DE SITTER CAUCHY HORIZON

Anti–de Sitter spacetime is described clearly by Hawking and Ellis [2] and Cvetic *et al.* [12]. It is a maximally symmetric solution to Einstein's equations

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 3\alpha^2 g_{\mu\nu} \tag{1}$$

corresponding to an empty universe with a negative cosmological constant $\Lambda = -3\alpha^2$. In Einstein universe coordinates, the metric is

$$ds^2 = (\alpha \cos\psi)^{-2}(-dt^2 + d\psi^2 + \sin^2\psi d\Omega^2) \ , \tag{2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $0 \leq \psi < \pi/2$, $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$. In ordinary anti–de Sitter space, t is restricted to $-\pi \leq t \leq \pi$, the topology is $S^1 \times R^3$, and there are closed timelike lines. In the universal AdS covering space, the space is unwrapped S^1 to R, giving an R^4 topology with $-\infty < t < \infty$ and no closed timelike lines, as shown in Fig. 1.

The surface $\psi = \pi/2$ represents both null and spacelike infinity. The surface is timelike, so AdS spacetime cannot have a global Cauchy surface. Field data placed on an initial spacetime slice, e.g., $t = -\pi/2$ in Fig. 1, cannot uniquely determine the evolution of the field beyond the null diamonds indicated, since new data could flow in from $\psi = \pi/2$. The null diamonds are Cauchy horizons. Note that if a different constant-t slice were chosen, different Cauchy horizons would be formed: This is a consequence of the maximal symmetry of the spacetime.

Timelike geodesics have an interesting focusing property. A spray of timelike geodesics beginning at $\psi =$

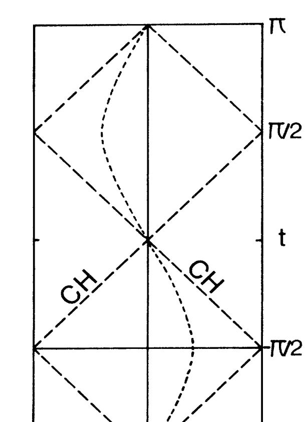

FIG. 1. Universal covering space of anti–de Sitter spacetime. Spatial and null infinity at $\psi = \pi/2$ is timelike. One timelike geodesic is shown; null geodesics are at 45°. The spacelike hypersurface at $t = -\pi/2$ has the Cauchy horizon indicated. A different choice of spacelike hypersurface would have a different Cauchy horizon. (The figure is taken from Cvetic *et al.* [12] and Avis *et al.* [3].)

$0, t = -\pi$ will reconverge again at $\psi = 0, t = \pi$, then diverge and converge, again and again. Future timelike geodesics from $\psi = 0, t = -\pi$ never reach $\psi = \pi/2$; they can only reach the interior of the infinite sequence of diamondlike regions, the Cauchy development of $t = -\pi/2$. Interestingly, a general point outside the Cauchy development cannot be reached by any geodesic normal to $t = -\pi/2$.

A useful alternative AdS metric with one null coordinate v is

$$ds^2 = -\alpha^{-2}(1+\alpha^2 r^2)dv^2 + 2\alpha^{-1}dvdr + r^2 d\Omega^2 \ , \quad (3)$$

where $v = t + \psi$ and $r = \alpha^{-1}\tan\psi$. In v, r, θ, φ coordinates, radial null geodesics have four-velocities $u^\mu_{\rm out} = [2\alpha/(1+\alpha^2 r^2), 1, 0, 0]$ for outgoing rays and $u^\mu_{\rm in} = (0, -1, 0, 0)$ for infalling rays, using r as affine parameter. The corresponding four-velocities for radial timelike geodesics, using proper time as the affine parameter, are

$$u^\mu = \left[\frac{E \pm S(r)}{1+\alpha^2 r^2}, \pm\alpha^{-1}S(r), 0, 0\right] \ , \quad (4)$$

where E is the particle's energy and $S(r) = [E^2 - \alpha^2(1+\alpha^2 r^2)]^{1/2}$.

A. Infalling null-dust test field

A test field of radially infalling null dust has stress-energy

$$T^{\mu\nu}_{\rm in} = \rho_{\rm in}(v,r)u^\mu_{\rm in}u^\nu_{\rm in} = \rho_{\rm in}(v,r){\rm diag}(0,1,0,0) \ . \quad (5)$$

The continuity equation $T^{\mu\nu}_{{\rm in};\nu} = 0$ shows that the density has the form $\rho_{\rm in}(v,r) = F(v)/4\pi r^2$, where $F(v)$ is any function of v which is zero at $r = 0$, so that $\rho_{\rm in}(v,r)$ is finite on the initial Cauchy surface. The scalars $T^\mu{}_\mu$ and $T^{\mu\nu}T_{\mu\nu}$ both vanish everywhere, so our conjecture predicts that the addition of infalling null dust will not cause a scalar curvature singularity to form at the CH. To see whether a nonscalar curvature singularity (NSCS) should form instead, we need the stress-energy tensor in a PPON frame. The frame follows a timelike geodesic which penetrates the CH, the geodesic of a radially moving observer who is not at rest on the initial Cauchy surface at $t = -(\pi/2)$. The frame vectors are

$$\begin{aligned} E^\mu_{(0)} &= [A(r), \alpha^{-1}S(r), 0, 0] \ , \\ E^\mu_{(1)} &= [A(r), \alpha^{-1}E, 0, 0] \ , \\ E^\mu_{(2)} &= (0, 0, r^{-1}, 0) \ , \\ E^\mu_{(3)} &= [0, 0, 0, (r^{\sin}\theta)^{-1}] \ , \end{aligned} \quad (6)$$

where $A(r) = [E + S(r)]/(1+\alpha^2 r^2)$.

In a PPON frame, the stress-energy tensor is

$$T_{(ab)} = E^\mu_{(a)}E^\nu_{(b)}T_{\mu\nu} = \frac{A^2(r)F(v)}{4\pi\alpha r^2}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \ , \quad (7)$$

where $\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Except for the $r = 0$ world line to which the dust collapses, $T_{(ab)}$ is finite if $F(v)$ is finite, as it will be if $F(v)$ is set to be finite on the initial Cauchy surface. Therefore according to the conjecture, no NSCS will form along the CH, so the CH will remain nonsingular. This is in contrast with the CH's in Reissner-Nordström and Kerr black holes, each of which is converted into a NSCS by the addition of infalling null dust [9,10]. A nonscalar curvature singularity *is* predicted to form at the single point $r = 0$ on the Cauchy horizon, because $T_{(ab)} \to \infty$ there, while $T^\mu{}_\mu$ and $T^{\mu\nu}T_{\mu\nu}$ are zero. If $F(v)$ is nonzero for all v, there is a similar focusing of null dust shells at $r = 0$ for all times between the initial surface and the Cauchy horizon, so an extended conjecture would predict that a nonscalar curvature singularity should form at $r = 0$ for all such times, if radially infalling null dust is added to an anti–de Sitter background spacetime. However, a finite-density shell of such null dust achieves *infinite* density when it reaches the $r = 0$ world line, which may signal the formation of a scalar curvature singularity in spite of the conjecture.

B. Exact solution with infalling null dust

We now present an exact solution of the field equations with infalling null dust, which reduces to the anti–de Sitter spacetime when the dust is turned off. The solution is analogous to Vaidya's generalization of the Schwarzschild geometry to include radiating null dust [22], and to the "Reissner-Nordström-Vaidya" geometry with infalling or outgoing null dust in the geometry of an electrically charged black hole [5]. In these cases the constant mass M of the black hole becomes a function of the null coordinate v (or the other null coordinate $u = t - \psi$), related to the null-dust density. Consider a metric of the form

$$ds^2 = f(v,r)dv^2 + 2\alpha^{-1}dvdr + r^2 d\Omega^2 \ , \quad (8)$$

generalizing the AdS metric by allowing f to depend upon v as well as r. The nonzero components of the Einstein tensor are then

$$\begin{aligned} G^0{}_0 &= G^1{}_1 = -(1 + \alpha^2 f + \alpha^2 r f_{,r})/r^2 \ , \\ G^2{}_2 &= G^3{}_3 = -(\alpha^2/2r^2)(r^2 f_{,r})_{,r} \ , \\ G^1{}_0 &= \alpha^2 f_{,v}/r \ . \end{aligned} \quad (9)$$

The choice $f = -\alpha^{-2}(1+\alpha^2 r^2)$ gives $G^0{}_0 = G^1{}_1 = G^2{}_2 = G^3{}_3 = 3\alpha^2$, corresponding to the AdS geometry, an empty spacetime with a cosmological constant $\Lambda = -3\alpha^2$.

Now if $f(v,r)$ has the form

$$f(v,r) = -\alpha^{-2}(1+\alpha^2 r^2) + h(v,r) \ , \quad (10)$$

the diagonal elements of the Einstein tensor $G^\mu{}_\nu$ are unchanged if $h = 2H(v)/\alpha^2 r$, where $H(v)$ is arbitrary. The off-diagonal elements remain zero except for $G^1{}_0 = 2H'(v)/r^2$, where $H' \equiv dH/dv$. Therefore the null dust has stress-energy tensor ${T^1}_0 = H'(v)/4\pi r^2$, the same form as the stress-energy tensor ${T^1}_0 = F(v)/4\pi\alpha r^2$ for a null-dust test field if we identify $H'(v) = \alpha^{-1}F(v)$. The solution is therefore an exact solution of the field equations with infalling null dust, an "anti–de Sitter–Vaidya" solution

$$ds^2 = \left[-\alpha^{-2}(1+\alpha^2 r^2) + \frac{2H(v)}{\alpha^2 r}\right] dv^2 + 2\alpha^{-1} dv dr + r^2 d\Omega^2 , \quad (11)$$

which reduces to AdS spacetime if the dust is removed. A similar solution is known for outgoing radiation in de Sitter spaces [23].

Except at $r = 0$, the Cauchy horizon remains nonsingular in the exact solution, in agreement with the stability conjecture. However, examination of the Riemann scalar $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ shows that the point $r = 0$ in the exact solution is a scalar curvature singularity. The scalars $R = -12\alpha^2$ and $R^{\mu\nu}R_{\mu\nu} = 36\alpha^4$ are constant and unaffected by the null dust, but the scalar

$$R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = 24\left(\alpha^4 + \frac{2H^2}{r^6}\right) \quad (12)$$

diverges as $r \to 0$. This world line is therefore a scalar curvature singularity in the exact solution, in disagreement with the stability conjecture. The conjecture correctly predicts a curvature singularity but fails to predict the correct type. The reason for the failure is that the divergence of $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ is due to a divergence of the Weyl tensor $C_{\mu\nu\lambda\sigma}$. The Weyl tensor is zero if $H = 0$, but if $H \neq 0$ the Weyl scalar

$$C^{\mu\nu\lambda\sigma}C_{\mu\nu\lambda\sigma} = 48H^2/r^6 \quad (13)$$

diverges as $r \to 0$. Because the conjecture inspects test-field stress-energy tensors, which can be related to the Ricci tensor through the field equations, it is not surprising that the conjecture can only predict divergences in the Ricci tensor and not the Weyl tensor portion of the curvature. Although this is the first case we have met in which the conjecture fails, we would not be surprised to find that it fails in other situations in which the singularity occurs purely in the Weyl tensor part of the curvature.

The singularity at $r = 0$ has been termed a "shell-focusing" singularity in other contexts [24]. Of course here the singularity occurs at $r = 0$ for all times between the initial surface and the Cauchy horizon, as long as $F(v)$ is nonzero on the initial surface for all v.

C. Infalling and outgoing null-dust test fields

Now add an outgoing spherically symmetric null-dust test field to the infalling null-dust test field of Sec. III A. We assume the beams do not interact, so each is separately conserved. In the Reissner-Nordström and Kerr black hole solutions, such a combination is sufficient to convert the Cauchy horizons into scalar curvature singularities [9,10].

The infalling null dust has stress-energy

$$T^{\mu\nu}_{\text{in}} = \rho_{\text{in}}(v,r)u^{\mu}_{\text{in}}u^{\nu}_{\text{in}} , \quad (14)$$

where $\rho_{\text{in}}(v,r) = F(v)/4\pi r^2$ and $(u^{\mu}_{\text{in}}) = (0,-1,0,0)$, as described in Sec. III A. Outgoing null dust has stress-energy

$$T^{\mu\nu}_{\text{out}} = \rho_{\text{out}}(u,r)u^{\mu}_{\text{out}}u^{\nu}_{\text{out}} , \quad (15)$$

where $\rho_{\text{out}}(u,r) = G(u)/4\pi r^2$. The function $G(u)$ is arbitrary except that $G(u) = 0$ at $r = 0$ on the initial hypersurface, so that $\rho_{\text{out}}(u,r)$ is finite there. We take an initial surface for the radiation to be the $t =$ constant hypersurface corresponding to the CH in question. If the initial data is finite, both $T^{\mu\nu}_{\text{in}}$ and $T^{\mu\nu}_{\text{out}}$ are finite at the CH. Scalars constructed from either $T^{\mu\nu}_{\text{in}}$ or $T^{\mu\nu}_{\text{out}}$ are zero. The total null-dust stress energy for both infalling and outgoing null dust is

$$T^{\mu\nu}_{\text{tot}} = \rho_{\text{in}}(v,r)u^{\mu}_{\text{in}}u^{\nu}_{\text{in}} + \rho_{\text{out}}(u,r)u^{\mu}_{\text{out}}u^{\nu}_{\text{out}} \quad (16)$$

for which ${T^{\mu}}_{\mu} = 0$. Because of cross-product terms, however, the scalar

$$T^{\mu\nu}T_{\mu\nu} = \rho_{\text{in}}(v,r)\rho_{\text{out}}(u,r)[(u^{\mu}_{\text{in}}u_{\text{out}\mu})^2 + (u^{\mu}_{\text{out}}u_{\text{in}\mu})^2] = \frac{F(v)G(u)}{2\pi^2 r^4(1+\alpha^2 r^2)^2} . \quad (17)$$

This scalar remains finite on the CH, except as $r \to 0$, where finite-density spherically symmetric null dust becomes infinite as it collapses to the origin. A scalar curvature singularity is therefore predicted to form at $r = 0$ but nowhere else on the Cauchy horizon. In fact, a scalar curvature singularity should form all along the $r = 0$ world line if $G(u)$ and $F(v)$ are nonzero for all u and v.

IV. CONCLUSION

Our study of the behavior of null dust in anti–de Sitter spacetime leads to two interesting conclusions. The first is that our Cauchy horizon stability conjecture is incomplete, in that a spacetime whose curvature singularity arises from the behavior of the Weyl tensor cannot be reliably characterized by the behavior of test-field stress-energy tensors. The second conclusion is that domain walls and vacuum bubbles, which use anti–de Sitter spacetime, may contain scalar curvature singularities when matter or radiation is added.

The Cauchy horizon stability conjecture predicts that anti–de Sitter spacetime with radially infalling null dust has a stable Cauchy horizon except at $r = 0$, where a nonscalar curvature singularity should develop. Nevertheless, the development of infinite dust density at $r = 0$ leads one to suspect that a scalar curvature singularity should be formed instead. An exact "anti–de Sitter Vaidya" spacetime with infalling null dust shows that the spacetime does have a stable Cauchy horizon except at $r = 0$, where there is a scalar curvature singularity, caused by focusing of the infalling shells. Thus even though the stability conjecture predicts that a curvature singularity should form, it predicts the wrong type. We believe this failure is due to the fact that test field stress-energy tensors are related only to the Ricci tensor portion of the curvature, so they are unable to predict divergences in scalars formed from the Weyl tensor portion of the curvature.

When both infalling and outgoing null-dust test fields are added to the anti–de Sitter spacetime, the stability conjecture does predict a scalar curvature singularity at $r = 0$, and an otherwise stable Cauchy horizon.

It therefore appears that the AdS domain wall spacetimes and vacuum bubbles considered by Cvetic *et al.* [4,12] and Gibbons [13] have generally stable Cauchy horizons when infalling and/or outgoing null dust is added. A shell-focusing scalar curvature singularity is formed along the $r = 0$ worldline, so that $r = 0$ is the only singularity formed on the Cauchy horizon under the conditions studied here.

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[1] A paper regarding the instability of the ordinary anti–de Sitter spacetime with closed timelike lines recently appeared [J.-M. Xu, L.-X. Li, and L. Liu, Phys. Rev. D **50**, 4886 (1994)]. We do not consider that spacetime.

[2] S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973).

[3] S. J. Avis, C. J. Isham, and D. Storey, Phys. Rev. D **18**, 3565 (1978).

[4] M. Cvetič, S. Griffies, and H. H. Soleng, Phys. Rev. Lett. **71**, 670 (1993).

[5] W. A. Hiscock, Phys. Lett. **83A**, 110 (1981).

[6] E. Poisson and W. Israel, Phys. Rev. D **41**, 1796 (1990).

[7] E. Poisson and W. Israel, Phys. Rev. Lett. **63**, 1663 (1989).

[8] E. Poisson and W. Israel, Phys. Lett. B **233**, 74 (1989).

[9] T. M. Helliwell and D. A. Konkowski, Phys. Rev. D **47**, 4322 (1993).

[10] D. A. Konkowski and T. M. Helliwell, Phys. Rev. D **50**, 841 (1994).

[11] For a review of earlier results, see, for example, F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980), Vol. 2.

[12] M. Cvetič, S. Griffies, and H. H. Soleng, Phys. Rev. D **48**, 2613 (1993).

[13] G. W. Gibbons, Nucl. Phys. **B394**, 3 (1993).

[14] D. A. Konkowski and T. M. Helliwell, Phys. Lett. **91A**, 149 (1982).

[15] D. A. Konkowski, T. M. Helliwell, and L. C. Shepley, Phys. Rev. D **31**, 1178 (1985).

[16] D. A. Konkowski and T. M. Helliwell, Phys. Rev. D **31**, 1195 (1985).

[17] D. A. Konkowski and T. M. Helliwell, Phys. Lett. A **129**, 305 (1988).

[18] T. M. Helliwell and D. A. Konkonski, Phys. Rev. D **41**, 2507 (1990).

[19] D. A. Konkowski and T. M. Helliwell, Phys. Rev. D **43**, 609 (1991).

[20] T. M. Helliwell and D. A. Konkowski, Phys. Rev. D **46**, 1424 (1992).

[21] G. F. R. Ellis and B. G. Schmidt, Gen. Relativ. Gravit. **8**, 915 (1977).

[22] P. C. Vaidya, Proc. Indian Acad. Sci. **A33**, 264 (1951).

[23] R. L. Mallett, Phys. Rev. D **31**, 416 (1985); D. Vick, Lett. Nuovo Cimento **44**, 127 (1985); D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980), p. 157.

[24] W. A. Hiscock, L. G. Williams, and D. M. Eardley, Phys. Rev. D **26**, 751 (1982); Y. Kuroda, Prog. Theor. Phys. **72**, 63 (1984).