Real Ashtekar variables for Lorentzian signature space-times

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I suggest in this paper a new strategy to attack the problem of the reality conditions in the Ashtekar approach to classical and quantum general relativity. By writing a modified Hamiltonian constraint in the usual SO(3) Yang-Mills phase space I show that it is possible to describe space-times with a Lorentzian signature without the introduction of complex variables. All the features of the Ashtekar formalism related to the geometrical nature of the new variables are retained; in particular, it is still possible, in principle, to use the loop variables approach in the passage to the quantum theory. The key issue in the new formulation is how to deal with the more complicated Hamiltonian constraint that must be used in order to avoid the introduction of complex fields.

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The purpose of this paper is to suggest a new strategy to deal with the problem of the reality conditions in the Ashtekar approach to classical and quantum gravity. At the present moment there is some consensus about the reasons behind the success of the Ashtekar variables program [1]. One of them is the geometrical nature of the new variables. In particular, the fact that the configuration variable is a connection is especially interesting because this allows us to use loop variables both at the classical and quantum level [2]. Another advantage of the formalism is the simplicity of the constraints, especially the Hamiltonian constraint, that have been very helpful in finding solutions to all of them. There are, however, some difficulties in the formalism that must be solved and are not present in the traditional Arnowitt-Deser-Misner (ADM) scheme [3]. The most conspicuous one is the fact that complex variables must be used in order to describe Lorentzian signature space-times. This is often put in relation with the fact that the definition of self-duality in these space-times demands the introduction of imaginary coefficients. The now accepted way to deal with this issue is the introduction of reality conditions. They impose some consistency requirements on the scalar product in the Hilbert space of physical states. In fact, the hope is that this scalar product can be selected by the reality conditions. There are, however some difficulties with this approach too. Specifically it is very difficult to implement the reality conditions in the loop variables scheme. Only recently some positive results in this direction have been reported [6]. The main point of this paper is to consider the geometrical nature of the Ashtekar variables as the most important asset of the formalism. With this idea in mind, it is easy to see that the introduction of complex variables is necessary only if one wants to have an especially simple form for the Hamiltonian constraint. If we accept a more complicated Hamiltonian constraint in the Ashtekar phase space we can use real variables.

An interesting consequence of this, as emphasized by Rovelli and Smolin, is that all the results obtained within the loop variables approach (the existence of volume and area observables, weave states, and so on [4,5]) whose derivation is independent of the particular form of the scalar constraint scalar can be maintained, even for Lorentzian signature space-times, because it is possible to describe Lorentzian gravity with real fields in the Ashtekar phase space. More specifically, the issue is not the implementation of the reality conditions (at least at the kinematical level) but rather the construction of a scalar product, normalizability of the quantum physical states, and so on. The proposal presented in this paper does not address this problem. It must also be said that the construction of area and volume observables referred to above must still be put in a completely sound and rigorous mathematical basis that may very well be provided by the approach presented in [6] to incorporate the reality conditions in the loop variables approach by using a generalization of the Bargmann-Siegel transform to spaces of connections. This paper has nothing new to add concerning this issue.

In the following, tangent space indices and SO(3) indices will be represented by lower case Latin letters from the beginning and the middle of the alphabet, respectively. The three-dimensional Levi-Civita tensor density and its inverse will be denoted[1] by $\tilde{\eta}^{abc}$ and $\underset{\sim}{\eta}_{abc}$ and the internal SO(3) Levi-Civita tensor by ϵ_{ijk}. The variables in the SO(3) ADM phase space (ADM formalism with internal SO(3) symmetry as discussed in [7]) are a densitized triad $\tilde{E}^a_i$ (with determinant denoted by $\tilde{\tilde{E}}$) and its canonically conjugate object K^i_a (closely related to the extrinsic curvature). The (densitized) three-dimensional metric built from the triad will be denoted $\tilde{\tilde{q}}^{ab} \equiv \tilde{E}^a_i \tilde{E}^{bi}$ and its determinant $\tilde{\tilde{q}}$ so that $q^{ab} = \tilde{\tilde{q}}^{ab}/\tilde{\tilde{q}}$. I will use also the SO(3) connection Γ^i_a compatible with the triad.

[1] I represent the density weights by the usual convention of using tildes above and below the fields.

The variables in the Ashtekar phase space are $\tilde{E}_i^a$, again, and the SO(3) connection A_a^i. The curvatures of A_a^i and Γ_a^i are, respectively, given by $F_{ab}^i \equiv 2\partial_{[a}A_{b]}^i + \epsilon^i{}_{jk}A_a^j A_b^k$ and $R_{ab}^i \equiv 2\partial_{[a}\Gamma_{b]}^i + \epsilon^i{}_{jk}\Gamma_a^j\Gamma_b^k$. Finally, the action of the covariant derivatives defined by these connections on internal indices are $\nabla_a\lambda_i = \partial_a\lambda_i + \epsilon_{ijk}A_a^j\lambda^k$ and $\mathcal{D}_a\lambda_i = \partial_a\lambda_i + \epsilon_{ijk}\Gamma_a^j\lambda^k$. They may be extended to act on tangent indices, if necessary, by introducing a space-time torsion-free connection; for example, the Christoffel connection Γ_{ab}^c built from q^{ab}. All the results presented in the paper will be independent of such extension. The compatibility of Γ_a^i and $\tilde{E}_i^a$ thus means

$$\mathcal{D}_a\tilde{E}_i^b \equiv \partial_a\tilde{E}_i^b + \epsilon_i{}^{jk}\Gamma_{aj}\tilde{E}_k^b + \Gamma_{ac}^b\tilde{E}_i^c - \Gamma_{ac}^c\tilde{E}_i^b = 0.$$

I will start from the SO(3) ADM constraints

$$\begin{aligned} \epsilon_{ijk}K_a^j\tilde{E}^{ak} &= 0, \\ \mathcal{D}_a[\tilde{E}_k^a K_b^k - \delta_b^a\tilde{E}_k^c K_c^k] &= 0, \\ -\zeta\sqrt{\tilde{\tilde{q}}}R + \frac{2}{\sqrt{\tilde{\tilde{q}}}}\tilde{E}_k^{[c}\tilde{E}_l^{d]}K_c^k K_d^l &= 0, \end{aligned} \tag{1}$$

where R is the scalar curvature of the three-metric q_{ab} (the inverse of q^{ab}). The variables $K_{ai}(x)$ and $\tilde{E}_j^b(y)$ are canonical; i.e., they satisfy

$$\begin{aligned} \{K_a^i(x), K_b^j(y)\} &= 0, \\ \{\tilde{E}_i^a(x), K_b^j(y)\} &= \delta_j^i\delta_b^a\delta^3(x,y), \\ \{\tilde{E}_i^a(x), \tilde{E}_j^b(y)\} &= 0. \end{aligned} \tag{2}$$

The parameter ζ is used to control the space-time signature. For Lorentzian signatures we have $\zeta = -1$ whereas in the Euclidean case we have $\zeta = +1$. The constraints (1) generate internal SO(3) rotations, diffeomorphisms, and time evolution. I write now the usual canonical transformation to the Ashtekar phase space:

$$\tilde{E}_i^a = \tilde{E}_i^a, \tag{3}$$

$$A_a^i = \Gamma_a^i + \beta K_a^i, \tag{4}$$

where β is a free parameter that I will adjust later. The Poisson brackets between the new variables A_a^i and $\tilde{E}_i^a$ are

$$\begin{aligned} \{A_a^i(x), A_b^j(y)\} &= 0, \\ \{A_a^i(x), \tilde{E}_j^b(y)\} &= \beta\delta_j^i\delta_a^b\delta^3(x,y), \\ \{\tilde{E}_i^a(x), \tilde{E}_j^b(y)\} &= 0, \end{aligned} \tag{5}$$

and thus, the transformation is canonical. Introducing (3) and (4) in (1) we get immediately the following constraints in the Ashtekar phase space:

$$\tilde{G}_i \equiv \nabla_a\tilde{E}_i^a = 0, \tag{6}$$

$$\tilde{V}_a \equiv F_{ab}^i\tilde{E}_i^b = 0, \tag{7}$$

$$\begin{aligned} \tilde{\tilde{S}} \equiv& -\zeta\epsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b F_{abk} \\ &+ \frac{2(\beta^2\zeta - 1)}{\beta^2}\tilde{E}_{[i}^a\tilde{E}_{j]}^b(A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j) \\ =& \ 0. \end{aligned} \tag{8}$$

They are the Gauss law, vector and scalar constraints of the Ashtekar formulation. The traditional attitude with regard to (8) has been to consider that the last term introduces unnecessary complications in the formalism. For this reason it has always been canceled by choosing β such that $\beta^2\zeta - 1 = 0$. For Euclidean signatures we can take $\beta^2 = 1$ and remain within the limits of the real theory. For Lorentzian signatures, however, we are forced to take $\beta^2 = -1$ and then the variables (specifically the connection) cease to be real.[2] It must be emphasized that this is true only if we insist in canceling the last term in (8). If we keep it, there is no reason to introduce complex objects in the theory. The value of β (as long as it is different from zero) is also irrelevant so we can choose $\beta = -1$ and have the following Hamiltonian constraint in the Lorentzian case:

$$\epsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b F_{abk} - 4\tilde{E}_{[i}^a\tilde{E}_{j]}^b(A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j) = 0. \tag{9}$$

The relevant Poisson brackets in (5) become

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \delta_j^i\delta_a^b\delta^3(x,y). \tag{10}$$

Since we have obtained this result by performing a canonical transformation, the Poisson algebra of the constraints is preserved. If we define the functionals

$$\begin{aligned} G[N^i] &\equiv \int d^3x N^i\tilde{G}_i, \\ V[N^a] &\equiv \int d^3x N^a\tilde{V}_a, \\ S[\underset{\sim}{N}] &\equiv \int d^3x \ \underset{\sim}{N} \ \tilde{\tilde{S}}, \end{aligned} \tag{11}$$

we have the usual Poisson algebra; in particular the Poisson brackets of $S[\underset{\sim}{N}]$ and $S[\underset{\sim}{M}]$ are given by

$$\{S[\underset{\sim}{N}], S[\underset{\sim}{M}]\} = +V[\tilde{E}_i^a\tilde{E}^{bi}(\underset{\sim}{N}\ \partial_b\ \underset{\sim}{M} - \underset{\sim}{M}\ \partial_b\ \underset{\sim}{N})]. \tag{12}$$

The plus sign on the right-hand side of (12) shows that we have, indeed, a Lorentzian signature. It is possible to rewrite (9) in a more appealing form. The second term, in particular, can be expressed in terms of covariant derivatives of $\tilde{E}_i^a$. To this end I introduce e_{ai}, the inverse of

$$\tilde{E}_i^a/\sqrt{\tilde{\tilde{E}}}, e_{ai} \equiv \frac{1}{2\sqrt{\tilde{\tilde{E}}}}\ \underset{\sim}{\eta}_{abc}\epsilon^{ijk}\tilde{E}_j^b\tilde{E}_k^c, \tag{13}$$

where $\tilde{\tilde{E}} \equiv \det\tilde{E}_i^a$ and rewrite (9) in the form

$$\begin{aligned} \epsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b F_{abk} - \tilde{\eta}^{a_1a_2a_3}\tilde{\eta}^{b_1b_2b_3}[(e_{a_1}^i\nabla_{a_2}e_{a_3i})(e_{b_1}^j\nabla_{b_2}e_{b_3j}) \\ -2(e_{a_1}^j\nabla_{a_2}e_{a_3i})(e_{b_1}^i\nabla_{b_2}e_{b_3j})] = 0. \end{aligned} \tag{14}$$

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[2]Notice that, when this choice is made, the configuration variable A_a^i can be understood as the pullback of a self-dual Lorentz connection.

The last term in the previous formula is still quadratic in the connections but its dependence on $\tilde{E}^a_i$ is complicated. It must be noted also that if we restrict ourselves to nondegenerate triads it can be cast in polynomial form (of degree 8 in $\tilde{E}^a_i$) by multiplying it by the square of $\tilde{\tilde{E}}$. If one is interested in checking explicitly the Poisson algebra and use the Hamiltonian constraint (14), it is useful to notice that $e_{ai}[\tilde{E}]$ and $-2\tilde{\eta}^{abc}\nabla_b e_{ci}$ are canonically conjugate objects. This may eventually be useful in order to write the new Hamiltonian constraint in terms of loop variables, maybe by allowing us to extend the set of T variables with objects built out of $e_{ai}[\tilde{E}]$ and $-2\tilde{\eta}^{abc}\nabla_b e_{ci}$.

There is another appealing way to write a Hamiltonian constraint for Lorentzian general relativity in terms of real Ashtekar variables. One starts by writing the Hamiltonian constraint in the SO(3) ADM formalism in the form

$$-2\zeta\sqrt{\tilde{\tilde{q}}}R + \zeta\sqrt{\tilde{\tilde{q}}}R + \frac{2}{\sqrt{\tilde{\tilde{q}}}}\tilde{E}^{[c}_k\tilde{E}^{d]}_l K^k_c K^l_d$$
$$= -2\zeta\sqrt{\tilde{\tilde{q}}}R - \frac{1}{\sqrt{\tilde{\tilde{q}}}}\left[\zeta\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j F_{abk} - \frac{2(\beta^2\zeta+1)}{\beta^2}\tilde{E}^a_{[i}\tilde{E}^b_{j]}(A^i_a - \Gamma^i_a)(A^j_b - \Gamma^j_b)\right] = 0, \quad (15)$$

where the first equality is to be understood modulo the Gauss law. Now, in the Lorentzian case we can choose $\beta^2 = 1$ and cancel the last term to give

$$2\tilde{\tilde{q}}R + \epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j F_{abk} = 0 \quad (16)$$

remembering that

$$\tilde{\tilde{q}}R = -\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j R_{abk} \quad (17)$$

we can finally write the Hamiltonian constraint as[3]

$$\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j(F_{abk} - 2R_{abk}) = 0. \quad (18)$$

The geometrical interpretation of the term that we must add to the familiar Hamiltonian constraint in the Ashtekar formulation in order to describe Lorentzian gravity in the Ashtekar phase space is simpler than in (14); it is just the curvature of the SO(3) connection compatible with the triad $\tilde{E}^a_i$. Some comments are now in order.

First, the presence of a potential term in (14) and (18) certainly makes them more complicated than the familiar Ashtekar Hamiltonian constraint. Taking into account that one of the sources of difficulties in the ADM formalism is precisely the presence of a potential term in the Hamiltonian constraint (see [9] and references therein for examples on how the quantization of ADM gravity would simplify in the absence of such a term) it is fair to expect some difficulties in the treatment of the theory with this new Hamiltonian constraint. The simplification brought about by removing the reality conditions has been traded for a more complicated Hamiltonian constraint.

The way the difference between the Euclidean and Lorentzian cases arises is rather interesting; there is a potential term in the Lorentzian case that is absent in the Euclidean formulation. This asymmetry between the Euclidean and Lorentzian cases (not apparent in the ADM formalism) is somehow puzzling. Why is it that the "complicated formulation" is found for the Lorentzian case or rather, would it be possible to find a real canonical transformation such that the formulation that becomes simple is the Lorentzian one?

The fact that the theory is written in an SO(3) Yang-Mills phase space makes it possible to attempt its quantization by using loop variables. This can be achieved in principle because we know [10] that loop variables are good coordinates (modulo sets of measure zero) in the (Gauss law reduced) constraint hypersurface. The key problem is now how to write the potential term in terms of the familiar loop variables. The obvious solution would be to add additional objects built with traces of holonomies of the connection Γ^i_a, notice, however, that it is not straightforward to add them to the set of elementary variables T^0 and T^1 because this would spoil the closure under the Poisson brackets. It is worthwhile noting that the possibility of writing the Hamiltonian constraint for a real Lorentzian general relativity in the two alternative forms (14) and (18) may be useful when trying to write them in terms of loop variables. It is conceivable that one form may be simpler to deal with than the other.

The form of the constraints of the theory makes it possible to use an approach similar to that of Capovilla, Dell, and Jacobson in [11] to solve both the vector and scalar constraints. We define, for nondegenerate triads, the matrix ψ_{ij} as

$$\psi_{ij}\tilde{E}^a_j = \tilde{B}^a_i - 2\tilde{R}^a_i, \quad (19)$$

where

$$\tilde{B}^c_i \equiv \tilde{\eta}^{abc}F_{abi}, \quad (20)$$

$$\tilde{R}^c_i \equiv \tilde{\eta}^{abc}R_{abi}; \quad (21)$$

the scalar constraint is then

$$\epsilon^{ijk}\underset{\sim}{\eta}_{abc}\tilde{E}^a_i\tilde{E}^b_j\psi_{kl}\tilde{E}^c_l = 2\tilde{\tilde{E}}\,\mathrm{tr}\psi = 0 \Longrightarrow \mathrm{tr}\psi = 0. \quad (22)$$

[3]It is my understanding that this formulation was independently considered by Ashtekar [8] before the loop variables formalism had been introduced, and discarded due to the presence of the potential term.

The vector constraint can be rewritten now as

$$\tilde{E}_i^a(F_{ab}^i - 2R_{ab}^i) = 0 \Longleftrightarrow \tilde{E}_i^{[a}(\tilde{B}^{b]i} - 2\tilde{R}^{b]i}) = 0 \tag{23}$$

because the relation $R_{ab}^i = -\frac{1}{2}\epsilon^{ijk}R_{abc}{}^d e_c^j e_d^k$ and the Bianchi identity $R_{[abc]}{}^d = 0$ (the three-dimensional Riemann tensor built with q^{ab}) imply that $\tilde{E}_i^a R_{ab}^i = 0$. We have then

$$\tilde{E}_i^{[a}\tilde{E}_j^{b]}\psi_{ij} = 0 \Longrightarrow \psi_{[ij]} = 0 \tag{24}$$

so that a symmetric and traceless ψ_{ij} solves both the vector and scalar constraints. As in the usual case we are left with one last equation: the Gauss law. Here is where the main difference between the usual Hamiltonian constraint and (18) arises. Without the potential term of (18) we could very easily write the remaining equation in terms of A_{ai} and ψ_{ij}:

$$\nabla_a[\psi_{ij}^{-1}\tilde{B}_j^a] = 0. \tag{25}$$

Now the situation is more complicated because we are forced to consider a system of coupled particle differential equations (PDE's)

$$\nabla_a[\psi_{ij}^{-1}(\tilde{B}_j^a - 2\tilde{R}_j^a)] = 0, \tag{26}$$

$$\psi_{ij}\tilde{E}_j^a = \tilde{B}_i^a - 2\tilde{R}_i^a. \tag{27}$$

The second equation could be solved, in principle, for $\tilde{E}_i^a$ and then the first would become an equation for ψ_{ij} and A_a^i only as in (25).

The main result presented in this paper has been the introduction of several alternative forms for the Hamiltonian constraint for Lorentzian space-times in the Ashtekar formalism with real variables. The problem of implementing the reality conditions in the theory has been transformed into the problem of working with the new Hamiltonian constraints introduced here.

The previous results strongly suggest that Lorentzian general relativity is a theory of two SO(3) connections [in the sense that both the curvatures of A_a^i and Γ_a^i seem to be playing a role as is apparent in (18)]. A completely different two-connection formulation for both Euclidean and Lorentzian general relativity has been reported elsewhere [12]. In that formulation the main difference between the Euclidean and Lorentzian cases is the appearance of terms depending on the difference of the curvatures for the Lorentzian signature case. The fact that, even for Lorentzian signatures, the Hamiltonian constraint of that formulation is a low-order polynomial of the curvatures makes it suitable to be written in terms of loop variables built with the two connections. My hope is that the comparison of the several different approaches discussed above may provide useful information about the way to proceed with the quantization program for general relativity and the role of complex fields in it.

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