

Reality conditions and Ashtekar variables: A different perspective

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We give in this paper a modified self-dual action that leads to the $SO(3)$ Arnowitt-Deser-Misner formalism without having to face the difficult second-class constraints present in other approaches (for example, if the starting point is the Hilbert-Palatini action). We use the new action principle to gain some new insight into the problem of the reality conditions that must be imposed in order to get real formulations from complex general relativity. We derive also a real formulation for Lorentzian general relativity in the Ashtekar phase space by using the modified action presented in the paper.

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I. INTRODUCTION

The purpose of this paper is to present a modified form of the self-dual action and use it to discuss the problem of reality conditions in the Ashtekar description of general relativity. By now, the Ashtekar formulation [1] has provided us with a new way to study gravity from a non-perturbative point of view. The success of the program can be judged from the literature available about it [2]. In our opinion there are two main technical points that have contributed to this success. The first one is the fact that the configuration variable is an $SO(3)$ connection. This allows us to formulate general relativity in the familiar phase space of the Yang-Mills theory for this group. We can then take advantage of the many results about connections available in the mathematical physical literature. In particular, it proves to be very useful to have the possibility of using loop variables [3] (essentially Wilson loops of the Ashtekar connection and related objects) in both the classical and the quantum descriptions of the theory. A second important feature of the Ashtekar formalism is the fact that the constraints (in particular the Hamiltonian constraint) have a very simple structure when written in terms of the new variables. This has been very helpful in order to find solutions to all the constraints of the theory and is in marked contrast with the situation in the Arnowitt-Deser-Misner (ADM) formalism [4], where the scalar constraint is very difficult to work with because of its rather complicated structure.

In spite of all the success of the formulation, there are still several problems that the Ashtekar program has to face. The one that we will be mostly concerned with in this paper is the issue of the reality conditions. As is well known, the so-called reality conditions must be imposed on the complex Ashtekar variables in order to recover the usual real formulation of general relativity for space-times with Lorentzian signatures. Their role is to guarantee that both the three-dimensional metric and its time derivative (evolution under the action of the Hamiltonian constraint) are real. This introduces key difficulties in the formulation, especially when one tries to work

with loop variables (although some progress on this issue has been recently reported [5]).

The main purpose of this paper is to clarify some issues related with the real formulations of general relativity that can be obtained from a given complex theory. We will see, for example, that both in the¹ $SO(3)$ ADM and in the Ashtekar phase space it is possible to find Hamiltonian constraints that trivialize the reality conditions to be imposed on the complex theory (regardless of the signature of the space-time). Conversely, any of these alternative forms for the constraints in a given phase space can be used to describe Euclidean or Lorentzian space-times, provided that we impose suitable reality conditions. Though this fact is, somehow, obvious in the ADM framework, it is not so in the Ashtekar formalism. In doing this we will find a real formulation for Lorentzian general relativity in the Ashtekar phase space. The main difference between this formulation and the more familiar one is the form of the scalar constraint. We will need a complicated expression in order to describe Lorentzian signature space-times. In our approach, the problem of the reality conditions is, in fact, transformed into the problem of writing the new Hamiltonian constraint in terms of loop variables and, in the Dirac quantization scheme, imposing its quantum version on the wave functionals (issues that will not be addressed in this paper). Of course one must also face the difficult problems of finding a scalar product in the space of physical states, etc.

A rather convenient way of obtaining the new Hamiltonian constraint is by starting with a modified version of the usual self-dual action [7] that leads to the $SO(3)$ ADM formalism in such a way that the transition to the Ashtekar formulation is very transparent. We will

¹In the following we mean by $SO(3)$ ADM formalism the version of the ADM formalism in which an internal $SO(3)$ symmetry group has been introduced as in [6].

take advantage of this fact in order to obtain the real Lorentzian formulation and to discuss the issue of reality conditions.

The layout of the paper is as follows. After this introduction we review, in Sec. II, the self-dual action and rewrite it as the Husain-Kuchař [8] action coupled to an additional field. This will be useful in the rest of the paper. Section III will be devoted to the modified self-dual action that leads to the SO(3) ADM formalism. We discuss the issue of reality conditions in Sec. IV. We will show that, although multiplying the usual self-dual action by a purely imaginary constant factor does not change anything (both at the level of the field equations and the Hamiltonian formulation), the same procedure, when used with the modified self-dual action, changes the form of the ADM Hamiltonian constraint (in fact it changes the relative sign between the kinetic and potential terms that in a real formulation controls the signature of the space-time). In Sec. V we derive the real Ashtekar formulation for Lorentzian signature, and we end the paper with our conclusions and comments in Sec. VI.

II. THE SELF-DUAL ACTION AND ASHTEKAR VARIABLES

We will start by introducing our conventions and notation. Tangent space indices and SO(3) indices are represented by lower case latin letters from the beginning and the middle of the alphabet, respectively. No distinction will be made between three-dimensional and four-dimensional tangent space indices (the relevant dimensionality will be clear from the context). Internal SO(4) indices are represented by capital latin letters from the middle of the alphabet. The three-dimensional and four-dimensional Levi-Civita tensor densities will be denoted² by $\tilde{\eta}^{abc}$ and $\tilde{\eta}^{abcd}$ and the internal Levi-Civita tensors for both SO(3) and SO(4) represented by ϵ_{ijk} and ϵ_{IJKL} . The tetrads e_{aI} will be written in components as $e_{aI} \equiv (v_a, e_{ai})$ [although at this point the i index only serves the purpose of labeling the last three internal indices of the tetrad, we will show later that it can be taken as an SO(3) index]. SO(4) and SO(3) connections will be denoted by A_{aIJ} and A_{ai} , respectively, with corresponding curvatures F_{abIJ} and F_{abi} given by $F_{abIJ} \equiv 2\partial_{[a}A_{b]IJ} + A_{aI}{}^K A_{aKJ}$ and $F_{ab}^i \equiv 2\partial_{[a}A_{b]}^i + \epsilon^i{}_{jk} A_a^j A_b^k$. The actions of the covariant derivatives defined by these connections on internal indices are $\nabla_a \lambda_I = \partial_a \lambda_I + A_{aI}{}^K \lambda_K$ and $\nabla_a \lambda_i = \partial_a \lambda_i + \epsilon_{ijk} A_{aj} \lambda_k$. They can be extended to act on tangent space indices by introducing a torsion-free connection (for example the Christoffel connection Γ_{ab}^c built with the four-metric $q_{ab} \equiv e_{aI} e_b^I$). All the results in the paper will be independent of such an extension. We will work with self-dual and anti-self-dual objects satisfying $B_{IJ}^\pm = \pm \frac{1}{2} \epsilon_{IJ}{}^{KL} B_{KL}^\pm$, where we raise and lower SO(4) in-

indices with the internal Euclidean metric $\text{Diag}(++++)$. In particular, A_{IJ}^- will be an anti-self-dual SO(4) connection [taking values in the anti-self-dual part of the complexified Lie algebra of SO(4)] and F_{abIJ}^- its curvature. In space-times with a Lorentzian signature a factor i must be included in the definition of self-duality if we impose the usual requirement that the duality operation be such that its square is the identity and raise and lower internal indices with the Minkowski metric $\text{Diag}(-+++)$. In this paper we will consider complex actions invariant under complexified SO(4). For the purpose of performing the 3 + 1 decomposition, the space-time manifold is restricted to having the form $\mathcal{M} = \mathbb{R} \times \Sigma$ with Σ a compact three-manifold with no boundary.

The Samuel-Jacobson-Smolin [7] action is

$$S = \int_{\mathcal{M}} d^4x \tilde{\eta}^{abcd} F_{ab}^{-IJ} e_{cI} e_{dJ}. \quad (1)$$

It is useful to rewrite it in a slightly modified manner [9]. We start by writing the anti-self-dual connection and the tetrad in matrix form as

$$A_{aIJ}^- \equiv \frac{1}{2} \begin{bmatrix} 0 & A_a^1 & A_a^2 & A_a^3 \\ -A_a^1 & 0 & -A_a^3 & A_a^2 \\ -A_a^2 & A_a^3 & 0 & -A_a^1 \\ -A_a^3 & -A_a^2 & A_a^1 & 0 \end{bmatrix}, \quad e_a^I \equiv \begin{bmatrix} v_a \\ e_a^1 \\ e_a^2 \\ e_a^3 \end{bmatrix}. \quad (2)$$

Under anti-self-dual and self-dual SO(4) infinitesimal transformations generated by

$$\Lambda_{IJ}^- = \begin{bmatrix} 0 & \Lambda^1 & \Lambda^2 & \Lambda^3 \\ -\Lambda^1 & 0 & -\Lambda^3 & \Lambda^2 \\ -\Lambda^2 & \Lambda^3 & 0 & -\Lambda^1 \\ -\Lambda^3 & -\Lambda^2 & \Lambda^1 & 0 \end{bmatrix}, \quad \Lambda_{IJ}^+ = \begin{bmatrix} 0 & L^1 & L^2 & L^3 \\ -L^1 & 0 & L^3 & -L^2 \\ -L^2 & -L^3 & 0 & L^1 \\ -L^3 & L^2 & -L^1 & 0 \end{bmatrix}, \quad (3)$$

the fields transform as

$$\begin{aligned} \delta^-(\Lambda) A_{aIJ}^- &= -\partial_a \Lambda_{IJ} - A_{aI}{}^K \Lambda_{KJ} + A_{aJ}{}^K \Lambda_{KI}, \\ \delta^-(\Lambda) v_a &= \Lambda_i e_a^i, \\ \delta^-(\Lambda) e_{ai} &= -\Lambda_i v_a - \epsilon_{ijk} e_a^j \Lambda^k, \end{aligned} \quad (4)$$

$$\begin{aligned} \delta^+(L) A_{aIJ}^- &= 0, \\ \delta^+(L) v_a &= L_i e_a^i, \\ \delta^+(L) e_{ai} &= -L_i v_a + \epsilon_{ijk} e_a^j L^k. \end{aligned}$$

The transformations of the connections can be written also as

$$\delta^-(\Lambda) A_{ai} = -2(\partial_a \Lambda_i + \epsilon_i{}^{jk} A_{aj} \Lambda_k), \quad \delta^+(L) A_{ai} = 0. \quad (5)$$

It is easy to show that δ^- and δ^+ are two sets of commuting SO(3) transformations corresponding to the factors in $\text{SO}(4) = \text{SO}(3) \otimes \text{SO}(3)$. The transformation law of A_{ai} under anti-self-dual SO(4) transformations is that

²We represent the density weights by the usual convention of using tildes above and below the fields.

of an $SO(3)$ connection³ but that of the rest of the fields is not [i.e., we cannot take i, j, k, \dots as $SO(3)$ indices at this stage]. However, by considering simple combinations of self-dual and anti-self-dual transformations $\delta^1(M) \equiv \delta^-(M/2) - \delta^+(M/2)$, we have

$$\begin{aligned}\delta^1(M)A_{ai} &= -(\partial_a M_i + \epsilon_i^{jk} A_{aj} M_k), \\ \delta^1(M)v_a &= 0, \\ \delta^1(M)e_{ai} &= -\epsilon_{ijk} e_a^j M^k.\end{aligned}\quad (6)$$

As we can see, A_{ai}, e_{ai}, v_a do transform as $SO(3)$ objects under the action of δ^1 if we consider the indices i, j, k, \dots as $SO(3)$ indices. The invariance of v_a under these transformations makes it very natural to consider the gauge fixing condition $v_a = 0$ that we will use later. In terms of A_{ai}, v_a , and e_{ai} the action (1) reads

$$S = \int_{\mathcal{M}} d^4x \tilde{\eta}^{abcd} [v_a e_{bi} F_{cdi} - \frac{1}{2} \epsilon^{ijk} e_{ai} e_{bj} F_{cdk}]. \quad (7)$$

This form of the Samuel-Jacobson-Smolin action has some nice features. It shows, for example, that general relativity can be obtained from the Husain-Kuchař [8] model action by introducing a vector field v_a and a suitable interaction term. This is useful in order to study

the dynamics of degenerate solutions given by the action (in contrast with the usual approach of extending the validity of the Ashtekar constraints to the degenerate sector of the theory). The action (7) will also be the starting point of the next section in which we show that a certain modification of it gives rise to the $SO(3)$ ADM formalism and provides a very natural way of linking it to the Ashtekar formulation. Notice that the number of degrees of freedom per space point is reduced, in spite of the presence of the new field v_a because the action now has an $SO(4)$ symmetry instead of $SO(3)$.

The fact that complexified $SO(4)$ and $SO(1,3)$ coincide means that we can start from (1), raise and lower indices with the Minkowski metric $\text{Diag}(-+++)$ and define self-dual and anti-self-dual fields by $B_{IJ}^\pm = \pm(i/2)\epsilon_{IJ}{}^{KL} B_{KL}^\pm$. It is straightforward to show that the resulting action is equivalent to (1) because they can be related by simple redefinitions of the fields.

In the passage to the Hamiltonian formulation⁴ corresponding to (7), we introduce a foliation of the space-time manifold \mathcal{M} defined by hypersurfaces of constant value of a scalar function t . We need also a congruence of curves with tangent vector t^a satisfying $t^a \partial_a t = 1$ (with this last requirement time derivatives can be interpreted as Lie derivatives \mathcal{L}_t along the vector field t^a). Performing the $3+1$ decomposition, we have

$$\begin{aligned}S &= \int dt \int_{\Sigma} d^3x \{ (\mathcal{L}_t A_a^i) \tilde{\eta}^{abc} [2v_b e_{ci} - \epsilon_i^{jk} e_{bj} e_{ck}] + A_0^i \nabla_a [\tilde{\eta}^{abc} (2v_b e_{ci} - \epsilon_i^{jk} e_{bj} e_{ck})] \\ &\quad + v_0 \tilde{\eta}^{abc} e_a^i F_{bci} - e_0^i \tilde{\eta}^{abc} [v_a F_{bci} + \epsilon_i^{jk} e_{aj} F_{bck}] \} \equiv \int dt \mathcal{L}(t),\end{aligned}\quad (8)$$

where $A_0^i \equiv t^a A_a^i, e_0^i \equiv t^a e_a^i$, and $v_0 \equiv t^a v_a$. All the objects in (8) are effectively three dimensional (they can be taken as tensors in the spatial hypersurfaces Σ). Denoting as $\tilde{\pi}_i^a(x), \tilde{\pi}_i(x), \tilde{\sigma}_i^a(x), \tilde{\sigma}_i(x), \tilde{p}^a(x)$, and $\tilde{p}(x)$ the momenta canonically conjugate to $A_a^i(x), A_0^i(x), e_a^i(x), e_0^i(x), v_a(x)$, and $v_0(x)$, respectively [$\{A_a^i(x), \pi_j^b(y)\} = \delta_a^b \delta_j^i \delta^3(x, y)$, and so on], we get from (8) the primary constraints

$$\begin{aligned}\tilde{\pi}_i &= 0, \\ \tilde{\sigma}_i &= 0, \\ \tilde{p} &= 0,\end{aligned}\quad (9)$$

$$\begin{aligned}\tilde{\pi}_i^a - \tilde{\eta}^{abc} (2v_b e_{ci} - \epsilon_i^{jk} e_{bj} e_{ck}) &= 0, \\ \tilde{\sigma}_i^a &= 0, \\ \tilde{p}^a &= 0.\end{aligned}\quad (10)$$

The constraints (9) are first class, whereas (10) are second class. The conservation in time of these constraints gives the secondary constraints

$$\begin{aligned}\nabla_a [\tilde{\eta}^{abc} (2v_b e_{ci} - \epsilon_i^{jk} e_{bj} e_{ck})] &= 0, \\ \tilde{\eta}^{abc} [v_a F_{bci} + \epsilon_i^{jk} e_{aj} F_{bck}] &= 0, \\ \tilde{\eta}^{abc} e_a^i F_{bci} &= 0\end{aligned}\quad (11)$$

that added to the set of primary constraints are second class. We must find now all the independent linear combinations of (10) and (11) that are first class and solve the remaining second class constraints. The result is

$$\begin{aligned}\nabla_a \tilde{\pi}_i^a &= 0, \\ e_a^i \tilde{p}^a &= 0, \\ \tilde{\pi}_i^a F_{ab}^i &= 0, \\ \epsilon^{ijk} \tilde{\pi}_i^a \tilde{\pi}_j^b F_{abk} &= 0,\end{aligned}\quad (12)$$

where A_a^j and $\tilde{\pi}_i^a$ are a canonically conjugate pair of variables and e_a^i is the (complicated) expression involving $\tilde{\pi}_i^a$ and v_a obtained by solving the second-class constraint

$$\tilde{\pi}_i^a = \tilde{\eta}^{abc} (2v_b e_{ci} - \epsilon_i^{jk} e_{bj} e_{ck}) = 0$$

³This is the reason why we introduced anti-self-dual connections in the action (1).

⁴We include this short discussion for further reference; the details can be found in [7].

(see also [10]). The first two constraints in (12) generate the full $SO(4)$ symmetry; the remaining ones are the usual vector and scalar constraints in the Ashtekar formulation. At this point it is clear that we can perform a partial gauge reduction by removing \tilde{p}^a and v_a from the formulation.⁵ It is also possible to simplify the algebra involved by imposing the gauge-fixing condition $v_a = 0$ once the first-class constraints have been identified and before solving the second-class ones. Notice, by the way, that the more usual gauge-fixing condition $v_a = n_a$ (where n_a is the normal to the foliation) is equally good and leads to the same result although the algebra involved is more complicated. The price that we pay if we make the previous gauge reduction or gauge fixing is that we will not find the generator of the full $SO(4)$ in the final Hamiltonian formulation (given by the familiar Ashtekar constraints) but only one of the $SO(3)$ factors.

III. THE MODIFIED SELF-DUAL ACTION

We show in this section that a simple modification of the action (7) gives a theory with Hamiltonian formulation given by the $SO(3)$ ADM formalism (see [11] for a proposal somewhat related to ours). The derivation of this result is easier than in the case of starting from the Palatini action as in [6] because the second-class constraints are much simpler to deal with. This result is interesting for several reasons. It will be used in the next section to discuss the reality conditions of the theory. It leads also in a very natural way to some of the real Hamiltonian formulations for Lorentzian general relativity in the Ashtekar phase space discussed in [12]. Throughout this section all the fields will be taken as complex.

The key idea to get the modified action is realizing that

$$\tilde{\eta}^{abcd}\epsilon^{ijk}e_{ai}e_{bj}F_{cdk} = -2\tilde{\eta}^{abcd}e_a^i\nabla_b\nabla_c e_{di} = -2\tilde{\eta}^{abcd}[\nabla_b(e_a^i\nabla_c e_{di}) + (\nabla_a e_b^i)(\nabla_c e_{di})].$$

By adding, then, a total derivative to (7) we get

$$S = \int_{\mathcal{M}} d^4x \tilde{\eta}^{abcd}[(\nabla_a e_b^i)(\nabla_c e_{di}) + v_a e_b^i F_{cdi}]. \quad (13)$$

In doing this we are, in fact, using the familiar procedure to generate canonical transformations by adding a divergence to the Lagrangian. The term introduced in order to get (13) can be found in [13] and is given by

$$\int d^3x \tilde{\eta}^{abc}\epsilon_{ijk}(A_a^i - \Gamma_a^i)e_b^j e_c^k = \int d^3x \tilde{\eta}^{abc}e_a^i\nabla_b e_{ci} \quad (14)$$

in this last expression we have used the compatibility of Γ_a^i and e_a^i (that allows us to write $\partial_{[a}e_{b]}^i = -\epsilon^i{}_{jk}\Gamma_{[a}^j e_{b]}^k$). With this in mind, and taking into account that (14) generates the canonical transformations from $SO(3)$ ADM to the Ashtekar formalism we expect that the action (13) leads to $SO(3)$ ADM (as it turns out to be the case). We follow now the usual procedure to get the Hamiltonian formulation. The 3 + 1 decomposition gives

$$S = \int dt \int_{\Sigma} d^3x \{ (2\mathcal{L}_t e_a^i)\tilde{\eta}^{abc}\nabla_b e_{ci} - 2(\mathcal{L}_t A_a^i)\tilde{\eta}^{abc}e_{bi}v_c + v_0\tilde{\eta}^{abc}e_a^i F_{bci} - A_0^i\nabla_a[\tilde{\eta}^{abc}(\epsilon_i{}^{jk}e_{bj}e_{ck} - 2v_b e_{ci})] + e_0^i\tilde{\eta}^{abc}[\epsilon_i{}^{jk}F_{abj}e_{ck} - v_a F_{bci}] \}. \quad (15)$$

From (15) we get the primary constraints

$$\begin{aligned} \tilde{\pi}_i &= 0, \\ \tilde{\sigma}_i &= 0, \\ \tilde{p} &= 0, \end{aligned} \quad (16)$$

$$\begin{aligned} \tilde{\pi}_i^a + 2\tilde{\eta}^{abc}e_{bi}v_c &= 0, \\ \tilde{\sigma}_i^a - 2\tilde{\eta}^{abc}\nabla_b e_{ci} &= 0, \\ \tilde{p}^a &= 0. \end{aligned} \quad (17)$$

We define now a total Hamiltonian H_T by adding the primary constraints (multiplied by Lagrange multipliers u^i , u_a^i , v^i , v_a^i , w , and w_a) to the Hamiltonian derived from (15):

⁵Notice that this gauge reduction does not change the number of degrees of freedom.

$$H_T = \int_{\Sigma} d^3x \{ A_0^i \nabla_a [\tilde{\eta}^{abc} (\epsilon_i^{jk} e_{bj} e_{ck} - 2v_b e_{ci})] - e_0^i \tilde{\eta}^{abc} [\epsilon_i^{jk} F_{abj} e_{ck} - v_a F_{bci}] \\ - v_0 \tilde{\eta}^{abc} F_{abi} e_c^i + u^i \tilde{\pi}_i + u_{ai} [\tilde{\pi}_i^a + 2\tilde{\eta}^{abc} e_{bi} v_c] + v^i \tilde{\sigma}_i + v_a^i [\tilde{\sigma}_i^a - 2\tilde{\eta}^{abc} \nabla_b e_{ci}] + w\tilde{p} + w_a \tilde{p}^a \}. \quad (18)$$

The conservation in time of the primary constraints under the evolution given by H_T gives the secondary constraints

$$\begin{aligned} \nabla_a [\tilde{\eta}^{abc} (\epsilon_i^{jk} e_{bj} e_{ck} - 2v_b e_{ci})] &= 0, \\ \tilde{\eta}^{abc} [\epsilon_i^{jk} F_{abj} e_{ck} - v_a F_{bci}] &= 0, \\ \tilde{\eta}^{abc} F_{abi} e_c^i &= 0. \end{aligned} \quad (19)$$

When added to the set of primary constraints they are second class. As usual, it is possible to find linear combinations of the second-class constraints that are first class by solving some consistency equations for the Lagrange multipliers introduced in H_T . For example, we can show that each of the secondary constraints (19) will give rise to a first class constraint in the final formulation. In addition to these, there is an additional first class constraint [responsible for generating the SO(3) factor in SO(4) that is usually gauged away] given by

$$(v_a \delta_{ik} + \epsilon_{ijk} e_a^j) [\tilde{\sigma}_k^a - 2\tilde{\eta}^{abc} \nabla_b e_{ck}] - e_{ck} \tilde{p}^c = 0. \quad (20)$$

As commented above it is possible to gauge away v_a and thus, remove both v_a and \tilde{p}^a from the final canonical formulation (this can be done also by imposing the gauge fixing condition $v_a = 0$ and solving $\tilde{p}^a = 0$). After doing this we are left with the second class constraints

$$\begin{aligned} \tilde{\pi}_i^a &= 0, \\ \tilde{\sigma}_k^a - 2\tilde{\eta}^{abc} \nabla_b e_{ck} &= 0, \end{aligned} \quad (21) \quad (22)$$

that must be solved, and the constraints

$$\nabla_a [\tilde{\eta}^{abc} \epsilon_{ijk} e_b^j e_c^k] = 0, \quad (23)$$

$$\tilde{\eta}^{abc} \epsilon_{ijk} F_{ab}^j e_c^k = 0, \quad (24)$$

$$\tilde{\eta}^{abc} F_{abi} e_c^i = 0. \quad (25)$$

Introducing the solution to (21) and (22) in (23)–(25), they will become first class. Equation (21) suggests that the best thing to do, at this point, is to write the connection A_a^i in terms of $\tilde{\sigma}_k^a$ and e_{ai} by using $\tilde{\sigma}_k^a - 2\tilde{\eta}^{abc} \nabla_b e_{ck} = 0$. In this way we can rewrite all the constraints (23)–(25) in terms of the canonically conjugate pair of variables $\tilde{\sigma}_k^a$ and e_{ai} . Notice that the only place in which the condition $\tilde{\pi}_i^a = 0$ must be taken into account is in the symplectic structure⁶

$$\Omega = \int_{\Sigma} d^3x [d\tilde{\pi}_i^a(x) \wedge dA_a^i(x) + d\tilde{\sigma}_i^a(x) \wedge de_a^i(x)], \quad (26)$$

⁶ d represents the generalized exterior differential in the infinite-dimensional phase space spanned by $A_a^i(x)$, $e_a^i(x)$, $\tilde{\pi}_i^a(x)$, and $\tilde{\sigma}_i^a(x)$.

where it cancels the first term. This means that $\tilde{\sigma}_k^a$ and e_{ai} are indeed a canonical pair of variables in the final phase space. The solution to $\tilde{\sigma}_k^a - 2\tilde{\eta}^{abc} \nabla_b e_{ck} = 0$ is

$$A_a^i = \Gamma_a^i + K_a^i, \quad (27)$$

where Γ_a^i and K_a^i are given by

$$\Gamma_a^i = -\frac{1}{2\tilde{e}} (e_a^i e_b^j - 2e_a^j e_b^i) \tilde{\eta}^{bcd} \partial_c e_{dj}, \quad (28)$$

$$K_a^i = \frac{1}{4\tilde{e}} (e_a^i e_b^j - 2e_a^j e_b^i) \tilde{\sigma}_j^b \quad (29)$$

($\tilde{e} \equiv \frac{1}{6} \tilde{\eta}^{abc} \epsilon_{ijk} e_a^i e_b^j e_c^k$ is the determinant of the triad). It is straightforward to show that the previous Γ_a^i is compatible with e_a^i (i.e.,

$$\mathcal{D}_a e_b^i \equiv \partial_a e_b^i - \Gamma_{ab}^c e_c^i + \epsilon^i{}_{jk} \Gamma_a^j e_b^k = 0,$$

where Γ_{ab}^c are the Christoffel symbols built with the three-dimensional metric $q_{ab} \equiv e_a^i e_{bi}$).

Equation (23) gives immediately (just substituting $2\tilde{\eta}^{abc} \nabla_b e_{ck} = \tilde{\sigma}_k^a$)

$$\epsilon_{ijk} e_a^j \tilde{\sigma}^{ak} = 0. \quad (30)$$

This is the generator of SO(3) rotations. Differentiating now in Eq. (22) we get $\nabla_a \tilde{\sigma}_i^a = 2\tilde{\eta}^{abc} \nabla_a \nabla_b e_{ci} = \tilde{\eta}^{abc} \epsilon_{ijk} F_{ab}^j e_c^k = 0$, where we have made use of (24). In order to eliminate the A_a^i from $\nabla_a \tilde{\sigma}_i^a$ we add and subtract $\epsilon_i{}^{jk} \Gamma_{aj} \tilde{\sigma}_k^a$ to get $\mathcal{D}_a \tilde{\sigma}_i^a = -\epsilon_i{}^{jk} K_a^j \tilde{\sigma}_k^a$. It is straightforward to show that the right-hand side of this last expression is zero by using the definition of K_a^i and the ‘‘Gauss law’’ (30). We have then

$$\mathcal{D}_a \tilde{\sigma}_i^a = 0. \quad (31)$$

Finally, the scalar constraint is obtained by introducing $F_{ab}^i = R_{ab}^i + 2\mathcal{D}_{[a} K_{b]}^i + \epsilon^i{}_{jk} K_a^j K_b^k$ (where $R_{ab}^i \equiv 2\partial_{[a} \Gamma_{b]}^i + \epsilon^i{}_{jk} \Gamma_a^j \Gamma_b^k$ is the curvature of Γ_a^i) in (25) and using the Gauss law (30). The final result is

$$\tilde{\eta}^{abc} R_{ab}^i e_{ci} + \frac{1}{8\tilde{e}} [e_a^i e_b^j - 2e_a^j e_b^i] \tilde{\sigma}_i^a \tilde{\sigma}_j^b = 0. \quad (32)$$

In order to connect this to the usual ADM and SO(3) ADM formalisms we first write

$$q_{ab} = e_a^i e_{bi}, \quad (33)$$

$$K_{ab} = \frac{1}{4\tilde{e}} [2q_{c(a} e_{b)}^c \tilde{\sigma}_i^c - q_{ab} e_c^i \tilde{\sigma}_i^c].$$

Taking into account that $\tilde{p}^{ab} = \sqrt{\tilde{q}} (K^{ab} - Kq^{ab}) + (\tilde{q})$ is the determinant of the three-metric $q_{ab} \equiv e_a^i e_{bi}$ we find

that (33) implies⁷

$$\tilde{p}^{ab} = \frac{1}{2} e_i^{(a} \tilde{\sigma}^{b)i}. \quad (34)$$

These expressions allow us to immediately check that q_{ab} and \tilde{p}^{ab} are a pair of canonically conjugate variables. By using the ‘‘Gauss law’’ (30) we can remove the symmetrizations in (34) and write $\tilde{p}^{ab} = \frac{1}{2} e_i^b \tilde{\sigma}^{ai}$. With this last expression it is straightforward to show that the constraint (31) gives $\mathcal{D}_a \tilde{p}^{ab} = 0$ (i.e., the familiar vector constraint in the ADM formalism). Because there is no internal symmetry in the usual ADM formalism the only thing we are left to compute is the scalar constraint. From (32) and using the fact that

$$R = -\epsilon^{ijk} R_{abi} e_j^a e_k^b = -\frac{1}{\tilde{\epsilon}} \tilde{\eta}^{abc} R_{ab}^i e_{ci}$$

and $\tilde{\sigma}_i^a = 2\tilde{p}^{ab} e_{bi}$, modulo (30), we get

$$-\sqrt{\tilde{q}} R + \frac{1}{\sqrt{\tilde{\epsilon}}} \left(\frac{1}{2} \tilde{p}^2 - \tilde{p}^{ab} \tilde{p}_{ab} \right) = 0. \quad (35)$$

The relative signs between the potential and kinetic terms in the previous expression correspond to Euclidean signature if we take real fields.

In order to see how our result gives the SO(3) ADM formalism of Ref. [6] we write⁸

$$\tilde{\pi}_i^a = \frac{\mu}{2} \tilde{\eta}^{abc} \epsilon_{ijk} e_b^j e_c^k, \quad (36)$$

$$K_a^i = \frac{1}{2\mu\tilde{\epsilon}} (e_a^i e_b^j - 2e_a^j e_b^i) \tilde{\sigma}_j^b \quad (37)$$

and their inverses

$$e_{ai} = \frac{1}{2\sqrt{\mu\tilde{\epsilon}}} \tilde{\eta}_{abc} \epsilon^{ijk} \tilde{\pi}_j^b \tilde{\pi}_k^c \quad (38)$$

$$\tilde{\sigma}_i^a = 2\sqrt{\mu/\tilde{\epsilon}} \tilde{\pi}_i^{[a} \tilde{\pi}_k^{c]} K_c^k. \quad (39)$$

It is straightforward to check that these equations define a canonical transformation for every value of the arbitrary constant μ ; the relevant Poisson brackets are

$$\{\tilde{\pi}_i^a(x), K_b^j(y)\} = \delta_b^a \delta_i^j \delta^3(x, y). \quad (40)$$

In the following I will use the inverse of $\tilde{\pi}_i^a/\tilde{\pi}$ that I will denote E_a^i . Substituting (38), (39) in the constraints (30)–(31) we easily get the Gauss law and the vector constraint

$$\epsilon_{ijk} K_a^j \tilde{\pi}^{ak} = 0, \quad (41)$$

$$\mathcal{D}_a [\tilde{\pi}_k^a K_b^k - \delta_b^a \tilde{\pi}_k^c K_c^k] = 0.$$

In order to get the Hamiltonian constraint we need

$$\begin{aligned} \tilde{\eta}^{abc} R_{ab}^i(e) e_{ci} &= \frac{1}{2\sqrt{\mu\tilde{\epsilon}}} \eta^{abc} R_{ab}^i(e) \tilde{\eta}_{cde} \epsilon_i^{jk} \tilde{\pi}_j^d \tilde{\pi}_k^e \\ &= \frac{1}{\sqrt{\mu}} R_{ab}^i(E) \tilde{\eta}^{abc} E_{ci} = -\sqrt{\tilde{q}/\mu} R, \end{aligned} \quad (42)$$

where $\tilde{q}^{ab} = \tilde{\pi}_i^a \tilde{\pi}^{bi}$ and we have used the fact that $R_{ab}^i(e) = R_{ab}^i(E)$. We also need

$$\begin{aligned} \frac{1}{8\tilde{\epsilon}} [e_a^i e_b^j - 2e_a^j e_b^i] \tilde{\sigma}_i^a \tilde{\sigma}_j^b \\ = \frac{\mu^{3/2}}{4\tilde{\epsilon}} [(\tilde{\pi}_i^a K_a^i)^2 - (\tilde{\pi}_i^a K_a^j)(\tilde{\pi}_j^b K_b^i)] \end{aligned} \quad (43)$$

to finally get the Hamiltonian constraint

$$\sqrt{\tilde{q}} R + \frac{\mu^2}{2\sqrt{\tilde{q}}} \tilde{\pi}_i^{[b} \tilde{\pi}_j^{a]} K_a^i K_b^j = 0. \quad (44)$$

By choosing $\mu = 2$ we find the result of [6].

The action introduced in this section has some nice characteristics. It shows, for example, that it is possible to get an action for the ‘‘geometrodynamical’’ Husain-Kuchař model simply by removing the term with v_a . It can also be used to discuss the issue of reality conditions and to get one of the real Ashtekar formulations for Lorentzian gravity presented in [12]. This will be the scope of the next two sections.

IV. REALITY CONDITIONS

In this section, I will show how the action (13) can be used to discuss the reality conditions of the theory and the signature of the space-time. The starting point is realizing that, since we are working with complex fields, multiplying (13) by a purely imaginary factor (say i) cannot have any effect on the theory (because the field equations will remain unchanged). However, the same factor produces some changes in the Hamiltonian formulation. Following the derivation presented in the preceding section, we find now that the primary constraints are

$$\begin{aligned} \tilde{\pi}_i &= 0, \\ \tilde{\sigma}_i &= 0, \\ \tilde{p} &= 0, \end{aligned} \quad (45)$$

$$\begin{aligned} \tilde{\pi}_i^a + 2i\tilde{\eta}^{abc} e_{bi} v_c &= 0, \\ \tilde{\sigma}_i^a - 2i\tilde{\eta}^{abc} \nabla_b e_{ci} &= 0, \\ \tilde{p}^a &= 0, \end{aligned} \quad (46)$$

whereas the secondary constraints are still given by (19). Fixing the gauge $v_a = 0$, we have

$$\tilde{\pi}_i^a = 0, \quad (47)$$

$$\tilde{\sigma}_k^a - 2i\tilde{\eta}^{abc} \nabla_b e_{ck} = 0, \quad (48)$$

together with (23)–(25). As we did before, we must solve the second-class constraints (47) and (48). The connec-

⁷ e_i^a is the inverse of e_{ai} .

⁸ $\tilde{\pi} \equiv \det \tilde{\pi}_i^a$.

tion A_a^i is now given by

$$A_a^i = \Gamma_a^i - \frac{i}{4\tilde{e}} (e_a^i e_b^j - 2e_a^j e_b^i) \tilde{\sigma}_j^b. \quad (49)$$

Although the solution for A_a^i has some imaginary factors, it is clear that nothing will change in the symplectic structure because we have $\tilde{\pi}_i^a = 0$. In order to get the expressions for the constraints, we can simply make the change $\tilde{\sigma}_i^a \rightarrow -i\tilde{\sigma}_i^a$ in (30)–(32). The Gauss law and the vector constraint will be unaffected because they are linear and homogeneous in $\tilde{\sigma}_i^a$. There is, however, a change in the Hamiltonian constraint precisely in the relative sign between the kinetic and potential terms (due to the fact that it is not a homogeneous polynomial in $\tilde{\sigma}_i^a$). Actually we get

$$\tilde{\eta}^{abc} R_{ab}^i e_{ci} - \frac{1}{8\tilde{e}} [e_a^i e_b^j - 2e_a^j e_b^i] \tilde{\sigma}_i^a \tilde{\sigma}_j^b = 0. \quad (50)$$

If the fields in (50) are taken to be real, the relative sign between the two terms in this Hamiltonian constraint corresponds to Lorentzian general relativity. Notice, however, that, as long as we remain within the realm of the complex theory the signature of the space-time is not defined. It is only when we add the reality conditions that we pick a signature or the other. The conclusion that we draw from this fact is that, when we use the modified self-dual action, the signature of the space-time in the real formulation is controlled both by the form of the Hamiltonian constraint and the reality conditions. If we start from (32) the reality conditions

$$e_a^i \text{ real}, \quad (51)$$

$$\tilde{\sigma}_i^a \text{ real}$$

and

$$e_a^i \text{ real}, \quad (52)$$

$$\tilde{\sigma}_i^a \text{ purely imaginary}$$

lead to Euclidean and Lorentzian general relativity, respectively, whereas if we use (50) then the reality conditions (51) give Lorentzian signature and (52) Euclidean signature.

We could do the same thing starting from the self-dual action written in the form (7), we would find, then, that the primary constraints (10) become

$$\begin{aligned} \tilde{\pi}_i^a - i\tilde{\eta}^{abc}(2v_b e_{ci} - \epsilon_i^{jk} e_{bj} e_{ck}) &= 0, \\ \tilde{\sigma}_i^a &= 0, \\ \tilde{p}^a &= 0, \end{aligned} \quad (53)$$

whereas the rest of the primary and secondary constraints are left unchanged.⁹ As we have $\tilde{\sigma}_i^a = 0$ nothing

will change in the symplectic structure after writing e_a^i in terms of $\tilde{\pi}_i^a$. The new constraints can be obtained now by making the transformation $\tilde{\pi}_i^a \rightarrow -i\tilde{\pi}_i^a$ in (12). As they are homogeneous polynomials in the momenta we get exactly the result that we found in Sec. II. We have then an interesting situation. With the modified self-dual action we have two alternative expressions for the constraints and two different sets of reality conditions that we can use to control the signature of the space-time in the final real formulation. If we use the self-dual action, however, it seems that we can only control the signature by using reality conditions. It turns out that this is not strictly true. In fact, as happens in the previous example, there are actually several possible ways to write the Hamiltonian constraint in terms of Ashtekar variables with trivial reality conditions [12], some of them for Lorentzian signatures and some for Euclidean signatures. This is the subject of the next section.

V. A REAL LORENTZIAN FORMULATION

I show in this section that it is possible to use the constraint analysis of the modified self-dual action to obtain a real formulation with Ashtekar variables for Lorentzian signature space-times. As already pointed out in the Introduction, there are two key issues that lead to the success of the Ashtekar approach to classical and quantum gravity. One of them is the geometrical nature of the new variables, the other the simple polynomiality of the constraints (especially the scalar constraint). Many of the insights about quantum gravity gained with the new formalism have to do with the use of loop variables. Among them the introduction of the area and volume observables [14,15] and the construction of weave states are very interesting because they give physical predictions about the structure of space-time at the Planck scale. Unfortunately, the implementation of the reality conditions in the loop variables framework is rather difficult in the absence of an explicit real formulation and then the results obtained had to be accepted only modulo the reality conditions. For this reason it is very desirable to formulate the theory in terms of a real Ashtekar connections and triads. If this can be achieved, then it is possible to argue that all the results obtained within the loop variables framework, which do not require the use of the scalar constraint are true without having to worry about the issue of reality conditions. This applies, for example to the area and volume observables mentioned before.

In retrospect, getting a real formulation for Lorentzian signature space-times in terms of Ashtekar variables is very easy once we accept living with a more complicated Hamiltonian constraint. In fact, one can just take the scalar constraint (50) (corresponding to Lorentzian signature if written in terms of real fields) and substitute $\tilde{\sigma}_i^a$ and e_a^i for their values in terms of A_a^i and $\tilde{\pi}_i^a$ given by (38), (39): namely,

$$e_{ai} = \frac{1}{2\sqrt{2\tilde{\pi}}} \tilde{\eta}_{abc} \epsilon^{ijk} \tilde{\pi}_j^b \tilde{\pi}_k^c, \quad (54)$$

⁹As above, we will fix the gauge $v_a = 0$.

$$\tilde{\sigma}_i^a = 2\sqrt{2/\tilde{\pi}}\tilde{\pi}_i^c[\tilde{\pi}_k^a](A_c^k - \Gamma_c^k). \quad (55)$$

The Gauss law and the vector constraints are the usual ones, whereas the Hamiltonian constraint becomes

$$\epsilon^{ijk}\tilde{\pi}_i^a\tilde{\pi}_j^bF_{abk} - \tilde{\eta}^{a_1a_2a_3}\tilde{\eta}^{b_1b_2b_3}[(e_{a_1}^i\nabla_{a_2}e_{a_3i})(e_{b_1}^j\nabla_{b_2}e_{b_3j}) - 2(e_{a_1}^j\nabla_{a_2}e_{a_3i})(e_{b_1}^i\nabla_{b_2}e_{b_3j})] = 0, \quad (56)$$

where e_{ai} must be written in terms of $\tilde{\pi}_i^a$ from (54). As we can see, in addition to the usual term there is another involving covariant derivatives of $\tilde{\pi}_i^a$. The same result is obtained if we do not use any imaginary unit in the canonical transformations that connects the SO(3) ADM and Ashtekar formulations and we start from the Lorentzian ADM scalar constraint [12]. The fact that the previous formulation can be obtained by using a canonical transformation proves that we will recover the familiar Poisson algebra of general relativity. This can also be shown by an explicit computation made easier by the observation that $e_{ai}[\tilde{\pi}]$ and $-2\tilde{\eta}^{abc}\nabla_b e_{ci}[\tilde{\pi}]$ are canonically conjugate objects.¹⁰

As we discussed before, we can use both the form of the constraints and the reality conditions to control the signature of the space-time. This means that both the familiar Hamiltonian constraint in the Ashtekar formulation and (56) can be used to describe any space-time signature by choosing appropriate reality conditions. Of course, if these two forms of the Hamiltonian constraint are used for Euclidean and Lorentzian signatures, respectively, the reality conditions will be very simple (just the condition that the fields be real). If, instead, the usual Hamiltonian constraint is used for Lorentzian signatures or the new one for Euclidean ones then the reality conditions will be more complicated.

VI. CONCLUSIONS AND OUTLOOK

By using a modified form of the self-dual action that leads to the SO(3) ADM formalism without the appearance of difficult second-class constraints, we have studied the reality conditions of the theory and obtained a real formulation in terms of Ashtekar variables for Lorentzian signature space-times. We have been able to show that, both in the ADM and Ashtekar phase spaces, it is possible to find different forms for the Hamiltonian constraints for complexified general relativity. In order to pass to a real formulation we need to impose reality conditions that can be chosen to pick the desired space-time signature. In a sense, it is no longer necessary to talk about reality conditions because we can impose the trivial ones (real fields) and control the signature of the space time by choosing appropriate Hamiltonian constraints.

The fact that a real formulation in the Ashtekar phase

¹⁰This also suggests the possibility of extending the loop variables with the addition of objects depending on $\tilde{\eta}^{abc}\nabla_b e_{ci}[\tilde{\pi}]$ and may be useful in order to write the new Hamiltonian constraint in terms of them.

space is available means that all the results obtained by using loop variables that are independent of the detailed form of the Hamiltonian constraint are true without having to worry about reality conditions. On the other hand, there is a price to be paid, namely, that the Hamiltonian constraint is no longer a simple quadratic expression in both the densitized triad and the Ashtekar connection. This makes it more difficult to discuss all those issues that depend critically on having the theory formulated in terms of simple constraints; in particular, solving the constraints will be harder now.

In this respect one can honestly say that the structure of the Hamiltonian constraint presented above (or the alternative forms discussed in [10]) is, at least, as complicated as the one of the familiar ADM constraint. In spite of that, some interesting and basic features of the Ashtekar formulation are retained. The phase space still corresponds to that of a Yang-Mills theory, so we can continue to use loop variables in the passage to the quantum theory. The “problem” of reality conditions has now been transformed into that of writing the new (and complicated) Hamiltonian constraint in terms of loop variables and solving the quantum version of the constraints acting on the wave functional. The final success of this approach will depend on the possibility of achieving this goal.

One interesting point of discussion suggested by the results presented in the paper has to do with the obvious asymmetry between the formulations of gravity in a real Ashtekar phase space for Lorentzian and Euclidean signatures. In the geometrodynamical approach, there is little difference, both in the Lagrangian and Hamiltonian formulations, between them (at least at the superficial level of the complication of the expressions involved). In fact it all boils down to the relative signs between the potential and kinetic terms in the scalar constraint. In our case, however, the formulations that we get are, indeed, rather different.

The existence of the formulation presented in this paper also suggests that the origin of the signature at the Lagrangian level is also rather obscure. From the (real) self-dual action it seems quite natural to associate, for example, the Euclidean signature with the fact that the gauge group is SO(4) and the metric is $e_a I e_b^I$. However, the observation that one of the SO(3) factors “disappears” from the theory may be telling us that, perhaps, it is not necessary to start with an SO(4) internal symmetry. In fact, the Capovilla-Dell-Jacobson [16] Lagrangian leads to the same formulation using only SO(3) as the internal symmetry.

Although we still do not have a four-dimensional Lagrangian formulation of the theory, the marked asymmetry between the real Hamiltonian formulations for different space-time signatures strongly suggests that it would differ very much from the usual self-dual action (a fact also supported by the lack of success of all the attempts to get Lorentzian general relativity by introducing simple modifications in the known actions). This could have intriguing consequences in a perturbative setting because the UV behavior (controlled to a great extent by the functional form of the Lagrangian) of the Euclidean and the

Lorentzian theories could be very different. It is worthwhile remembering at this point that the Einstein-Hilbert action and the so-called higher derivative theories, that differ in some terms quadratic in the curvatures, have very different UV behaviors. The first one is nonrenormalizable, whereas the second one is renormalizable but nonunitary. In my opinion, this is an issue that deserves further investigation.

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- [1] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. D **36**, 1587 (1987); *Non Perturbative Canonical Gravity* (notes prepared in collaboration with R. S. Tate) (World Scientific, Singapore, 1991).
 - [2] Bibliography of publications related to Classical and Quantum Gravity in terms of the Ashtekar Variables, updated by T. Schilling, gr-qc 9409031.
 - [3] C. Rovelli and L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nucl. Phys. **B331**, 80 (1990).
 - [4] R. Arnowitt, S. Deser, and C. W. Misner, *The Dynamics of General Relativity in Gravitation, An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
 - [5] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann, *Funct. Anal.* (to be published).
 - [6] A. Ashtekar, A. P. Balachandran, and S. G. Jo, *Int. J. Mod. Phys. A* **4**, 1493 (1989).
 - [7] J. Samuel, *Pramana J. Phys.* **28**, L429 (1987); T. Jacobson and L. Smolin, *Phys. Lett. B* **196**, 39 (1987); *Class. Quantum Grav.* **5**, 583 (1988).
 - [8] V. Husain and K.uchař, *Phys. Rev. D* **42**, 4070 (1990).
 - [9] Chopin Soo (private communication).
 - [10] M. Henneaux, J. E. Nelson, and C. Schomblond, *Phys. Rev. D* **39**, 434 (1989).
 - [11] R. P. Wallner, *Phys. Rev. D* **46**, 4263 (1992).
 - [12] J. F. Barbero G., this issue, *Phys. Rev. D* **51**, 5507 (1995).
 - [13] B. P. Dolan, *Phys. Lett. B* **233**, 89 (1989).
 - [14] C. Rovelli and L. Smolin, *Nucl. Phys. B* (to be published).
 - [15] A. Ashtekar, C. Rovelli, and L. Smolin, *Phys. Rev. Lett.* **69**, 237 (1992).
 - [16] R. Capovilla, J. Dell, and T. Jacobson, *Phys. Rev. Lett.* **63**, 2325 (1989).