Cosmological histories for the new variables of Ashtekar

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Histories and measures for quantum cosmology are investigated through a quantization of the Bianchi type IX cosmology using path integral techniques. The result, derived in the context of Ashtekar variables, is compared with earlier work.

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I. INTRODUCTION

Attempts to apply quantum theory to the Universe have been for some time divided into two main schools focused on either the canonical or the path integral approach. While these two approaches are driven by different conceptions of how a quantum theory of the Universe is to be constructed and interpreted, progress in these directions has been circumscribed by the particular strengths and weaknesses of the two formalisms. These strengths and weaknesses allow each formalism to address, in rather different ways, key issues in quantum cosmology associated with the time reparametrization invariance. This reflects a lack of clocks and observers "outside" the Universe, the essential problem of quantum cosmology.

In the first approach, the Dirac procedure of the canonical formalism gives us a prescription to construct physical states, define physical observables, and propose (perhaps through the reality conditions of the theory) a physical inner product. The strength of the canonical approach arises in the way time reparametrization invariance, as well as other invariances of the theory, can be treated directly, yielding a gauge invariant quantization. The existence, for quantum gravity as well as for $(2+1)$ gravity and one or two Killing field models, of exact results concerning physical states and operators speaks of this strength.

The great weakness of the canonical approach is that physical observables are very difficult to construct explicitly because, both classically and quantum mechanically, observables must commute with the Hamiltonian constraint and necessarily are independent of parameter time. This difficulty is real; it reflects the fact that physical operators which describe time evolution ought to be constructed as correlations between the degrees of freedom, one of which one would like to take as a clock [1]. For example, suppose we pick a condition which picks out a slicing of spacetime into spacelike leaves (assuming this is possible) according to some degrees of freedom of the theory. We may define some observables which measure spatially diffeomorphism invariant information on these leaves. For example, let $A(q,p)$ be a measurement of geometry (where q and p are coordinates on phase space) and let $T(q,p)$ be another diffeomorphism invariant quantity which we will take as measuring time. Then for every possible value τ of this time observable there is a *physical* observable which measures the value of $A(q,p)$ on the $T(q,p)=\tau$ leaf. As the condition which picks out the leaves is expressed in terms of the degrees of freedom, this procedure is completely gauge invariant; within any gauge one can specify these leaves and evaluate the variables $A(q,p)$ and $T(q,p)$.

While simple to specify, to express such correlations explicitly in terms of functions on phase space or operators on the physical states one must finish the hard task of solving the dynamics of the theory. For, to find the value of $A(q,p)$ on the $T(q,p)=\tau$ leaf, one must write A in terms of the physical degrees of freedom. Thus, the construction of time reparametrization invariant observables in the canonical theory is a dynamical problem, which one cannot expect to solve without approximation procedures for theories outside of integrable systems.

In the second approach, the path integral formalism, we find that the problem of taking expectation values of physical observables can be easily realized as soon as one has a measure and a set of histories which represent physically meaningful, gauge invariant amplitudes. The expectation value of the observable $A(T=\tau)$ is simply given by summing (with the appropriate measure) paths weighted by the classical action and the value of the classical observable $A(q,p)$ on the leaf $T(q,p)=\tau$,

$$\left\langle \psi|\hat{A}(T=\tau)|\psi\right\rangle = \frac{\int [d\mu(q,p)A(T=\tau)]\, e^{\frac{i}{\hbar}I}}{\int [d\mu(q,p)]\, e^{\frac{i}{\hbar}I}}. \qquad (1)$$

Varying τ, we describe the evolution of the system in terms of time reparametrization invariant quantities. Though the path integral formalism steps by the difficulties of the canonical approach, the path integral has complimentary difficulties. Setting interpretational issues aside, we lack a prescription which allows us to unambiguously find the set of histories, appropriate contours, and a measure $\mu(q,p)$ which implement gauge invariances and reality conditions.

We would like to suggest that this situation points to a mixed approach in which physical, diffeomorphism invariant quantum states of the canonical theory define a path integral and measure, after which dynamics of physical observables are computed with path integral techniques. This program, if it can be concretely realized, offers a possibility of an unambiguous formalism for quantum cosmology.

To investigate this possibility, it would be very useful

to have a working model of a quantum cosmology which has dynamics complicated enough so that problems of constructing physical observables, an inner product, and a path integral measure are nontrivial. However, the results ought to be simple enough that the path integrals for physically meaningful quantities could be computed by relatively simple numerical or approximation techniques. This is the first of two papers which aim to lay the groundwork to construct such a model of quantum cosmology based on the Bianchi type IX spatially homogeneous spacetimes. In this paper we show that a gauge invariant measure can be constructed in this model, following the Faddeev-Popov procedure [2] and using the new variables [3]. In a companion paper we consider a physical canonical quantization procedure for Bianchi type IX and show how, and under what conditions, quantities defined through this canonical formalism can be expressed in terms of path integral expressions of the kind derived here.

The Bianchi type IX model describes a family of cosmologies in which space is homogeneous, but the geometry has two dynamical degrees of freedom — measures of anisotropy. It has been studied extensively, especially since the late 1960s when Misner found that the dynamics can be expressed in terms of the motion of a particle in a time-dependent potential [4]. Though it is a simple system with only two degrees of freedom, this model displays a surprisingly rich behavior even at the classical level. For instance, it has been shown that the Lyaponov exponent is greater than 1 for certain choices of time, meaning that the model is chaotic [5]. (However, the Lyapunov exponent, a measure of the exponential separation of nearby trajectories in time, is not time reparametrization invariant [6].) In the face of this it is unlikely that the theory can be exactly solved, making it an ideal candidate to test the program.

At the quantum level, although there does not exist, to our knowledge, a complete quantization of the Bianchi type IX model in either a path integral or canonical formalism, a number of results have been found previously. An exact physical state has been found by Kodama [7] using the new Hamiltonian variables of Ashtekar, which can even be transformed into the metric representation [8]. Graham has constructed a supersymmetric solution to this Bianchi model [9]. Numerical work, following the methods of Euclidean quantum cosmology [10], shows qualitative agreement with the exact solutions — at early times the wave function is spread over the anisotropy space while at later times the wave function peaks at the isotropic model (the closed Friedmann-Robertson-Walker model).

We present the derivation in "geometrized units" in which $G = c = 1$.

II. BIANCHI TYPE IX IN THE NEW VARIABLES

The new variables provide a complex chart on the phase space of general relativity with configuration variables, connections A_a^I, and conjugate momenta, densities $\tilde{E}_I^a$. Our notational convention denotes spatial indices as lowercase latin letters, e.g., $a, b, c, \ldots$, and denotes internal indices as uppercase latin letters, e.g., $I, J, \ldots$. Densities of weight 1, such as the conjugate momenta $\tilde{E}_I^a$, claim a tilde. The phase space is endowed with a structure given by

$$\{A_a^I(x), \tilde{E}_J^b(y)\} = i\,\delta_a^b\,\delta_J^I\,\delta^3(x,y). \tag{2}$$

The more familiar metric is obtained from the frame fields $\tilde{E}_I^a$ by defining triads on a three-manifold Σ, $E_i^I = \frac{1}{\sqrt{h}}\tilde{E}_i^I$, and by letting $h_{ij} = E_i^I E_{Ij}$. As this chart is a complex one, to regain general relativity we must choose a section of the phase space in which reality conditions, such as

$$(h_{ij})^* = h_{ij} \tag{3}$$

and

$$\left(\dot{h_{ij}}\right)^* = \dot{h_{ij}}, \tag{4}$$

hold.

In the 3+1 decomposition, with σ chosen as the time parameter, the classical action is

$$I[A,E,N] = \int_{\sigma_i}^{\sigma_f} d\sigma \int_\Sigma d^3x \left(-i\tilde{E}_a^I \dot{A}_I^a - N_* C^*\right). \tag{5}$$

The asterisk represents an index which runs from 0 to 6; it has one value for each constraint:

$$S := \varepsilon_{IJK} F_{ab}^I \tilde{E}^{aJ} \tilde{E}^{bK} = 0, \tag{6}$$

$$G_I := D_a \tilde{E}_I^a = 0, \tag{7}$$

$$V_a := F_{ab}^I \tilde{E}_I^b = 0, \tag{8}$$

which are known as the scalar or Hamiltonian, Gauss, and vector or diffeomorphism constraints, respectively. The covariant derivative is associated with the connection $D_a f^I := \partial_a f^I + \varepsilon^{IJK} A_{aJ} f_K$ and the curvature $F_{ab}^I := \partial_{[a} A_{b]}^I + \varepsilon^{IJK} A_{aJ} A_{bK}$. We investigate class A Bianchi type IX models.[1] Homogeneity provides us with a foliation of spacetime into homogeneous spacelike surfaces and gives each leaf a left invariant vector and one-form basis (v, ω) in which to expand the new variables [11]. On each leaf we can write

$$A_a^I = a_S^I(\sigma)\omega_a^S(x) \tag{9}$$

and

$$E_I^a = e_I^S(\sigma) v_S^a(x). \tag{10}$$

These expansion coefficients may be viewed as 3×3 matrices. Homogeneity reduces field theory to mechanics, from 9×9 degrees of freedom per spacetime event to 9×9 for each spatial section. The action simplifies to

[1]The classification of Bianchi models involves splitting the structure constants of the Lie group of isometries into two irreducible pieces. Denoting these by S^{LI} and V_K, the structure constants may be written as $C_{JK}^I = \varepsilon_{JKL} S^{LI} + \delta_{[J}^I V_{K]}$. Class A models are those for which $V_I = 0$.

$$I[A,E,N] = \int_{\sigma_i}^{\sigma_f} d\sigma \left(-i\Omega e_L^I \dot{a}_I^L - N_* C^*\right). \tag{11}$$

Here, $\Omega = \int_\Sigma \omega \wedge \omega \wedge \omega = 16\pi^2$ is the volume element on SU(2). (The Lagrange multipliers have been rescaled.) Henceforth, we will work in the unusual units $G = c = \Omega = 1$, meaning we measure fields in terms of this volume element and use a conversion factor of $c^4/G(\Omega)^{1/3}$ for energy. In terms of the expansion coefficients defined in Eqs. (9) and (10), the constraints Eqs. (6), (7), and (8) become

$$S = \varepsilon_A^{BC}\left(-\varepsilon_{GF}^D a_D^A + \varepsilon_{DE}^A a_G^D a_F^E\right) e_B^G e_C^F, \tag{12}$$

$$G_I = \varepsilon_{IJ}^K a_L^J e_K^L, \tag{13}$$

and

$$V_J = \varepsilon_{JK}^I a_I^L e_L^K. \tag{14}$$

We choose to fix the six gauge and diffeomorphism constraints by a "diagonal gauge" [11] to yield the form of Misner in which the cosmology may be seen as a particle moving in (2+1)-dimensional spacetime. This choice parallels Misner's β_+, β_- diagonalization in the geometrodynamic framework (we will later translate our result into Misner's notation for comparison).

We define

$$\epsilon_1 := e_1^1, \quad \epsilon_2 := e_2^2, \quad \epsilon_3 := e_3^3 \tag{15}$$

and choose

$$\chi_J^I \equiv e_J^I = 0 \quad \text{for } I \neq J. \tag{16}$$

Upon imposing these conditions the three Gauss constraints vanish, while the three remaining vector constraints require that the off-diagonal components of a_J^I vanish as well. As above, we define

$$a_1 := a_1^1, \quad a_2 := a_2^2, \quad a_3 := a_3^3. \tag{17}$$

At the end of the kinematical gauge fixing, we are left with six canonical degrees of freedom per leaf.

At this kinematical level in which the Gauss and vector constraints have been solved, but the Hamiltonian constraint has not, the model is not difficult to quantize. The six canonical degrees of freedom can be taken to be diagonal components of the frame fields and (imaginary parts of) diagonal components of the connections. States, in the diagonal metric representation, may be expressed as functions of the ϵ's. The reality conditions, Eq. (3) and Eq. (4), are realized by the inner product

$$\langle\psi(\epsilon)|\phi(\epsilon)\rangle = \int d^3\epsilon\, e^{-F(\epsilon)} \bar{\psi}(\epsilon)\phi(\epsilon), \tag{18}$$

where

$$F(\epsilon) := \frac{\epsilon_1\epsilon_2}{\epsilon_3} + \frac{\epsilon_2\epsilon_3}{\epsilon_1} + \frac{\epsilon_3\epsilon_1}{\epsilon_2}. \tag{19}$$

Unfortunately this quantization cannot be used to compute physical quantities; reparametrization invariance remains. The Hamiltonian constraint must be solved to reduce to the physical phase space of four degrees of freedom.

To pull out dynamics, we fix reparametrization invariance by choosing a gauge condition. We have two forms of this gauge condition.[2] The first fixes σ to be proportional to the volume of each spatial surface,

$$\chi^0(\sigma) := \ln(\sqrt{h}) - \sigma = 0. \tag{21}$$

This choice of 0 is monotonic on *half* the history of any given classical Bianchi type IX cosmology [6]. (Bianchi type IX expands from an initial singularity and then collapses in a final singularity [12].) The second choice which we will discuss is monotonic on the whole of the evolution. In this choice the parameter time is proportional to the momentum conjugate to $\ln(\sqrt{h})$, the trace of the extrinsic curvature of the leaves. We will exhibit the path integral in this gauge in Sec. IV.

To proceed to completely specify the canonical quantization in this gauge we should find a complete set of physical coordinates and momenta on the subspaces labeled by σ. We do not know how to do this. Fortunately, the Faddeev-Poppov ansatz allows us to compute the path integral.

III. CONSTRUCTION OF THE PATH INTEGRAL

Ideally, we would provide a chart for physical phase space, use this chart as the groundwork of an operator algebra, endow the space of states with an inner product, and produce dynamics through a Hamiltonian composed of these operators. We denote the canonical coordinates as $\bar{q}^\iota(\sigma)$ and $\bar{p}_\iota(\sigma)$ (where $\iota = 1, 2$) and denote the set of eigenstates by $\cdot|\bar{q}^\iota(\sigma)\rangle$ and $\cdot|\bar{p}_\iota(\sigma)\rangle$. As classically $\bar{q}^\iota(\sigma)$ and $\bar{p}_\iota(\sigma)$ are canonically conjugate, we would have $\langle\bar{q}^\iota(\sigma)|\bar{p}_I(\sigma)\rangle = \exp[i\bar{q}^\iota(\sigma)\bar{p}_\iota(\sigma)]$. If the Hamiltonian which realizes evolution from a fixed volume slice to a fixed volume slice is $h(\bar{p},\bar{q},\sigma)$, then we would have

$$\begin{aligned}\left\langle \bar{q}_f^\iota(\sigma_f)|\bar{q}_i^\iota(\sigma_i)\right\rangle &= K\left(q^\iota,\sigma_f;q^\iota,\sigma_i\right)\\ \left\langle \bar{q}_f^\iota(\sigma_f)|\bar{q}_i^\iota(\sigma_i)\right\rangle &= K\left(q^\iota,\sigma_f;q^\iota,\sigma_i\right)\\ &= \int [d\bar{p}\,d\bar{q}] \exp\left\{ i\int [\bar{p}\dot{\bar{q}} \right.\\ &\qquad \left. -h(\bar{p},\bar{q},\sigma)]d\sigma\right\},\end{aligned} \tag{22}$$

where the integral is over all possible physical trajectories passing through initial point q_i^ι, at volume σ_i, and final point q_f^ι, at volume σ_f. Brackets indicate one such factor on each time slice of the skeletonization of the path integral and indicate a factor of $1/\sqrt{2\pi}$ for each differential. This notation will be used for the remainder of this

[2]There is yet another choice of time which produces a measure which has no momentum dependence. Unfortunately, this time choice is complex, *even with the reality conditions satisfied.* This gauge, in diagonal form, is written

$$\chi^0(\sigma) = \sigma + i\left[a_I\epsilon_I + \frac{1}{2}F(\epsilon)\right]. \tag{20}$$

paper.

This construction is only useful through its link to a path integral over the whole (unphysical) phase space. Denoting coordinates and momenta of the whole phase space by q and p [2],

$$K\left(q_f^i,\sigma_f;q_i^i,\sigma_i\right)=\int\left[dp\,dq\,dN\prod_*\delta(\chi^*)\left|\{C_*,\chi^*\}\right|\right]\times\exp\left\{i\int\left[p\dot{q}-h(p,q)-N^*C_*(p,q)\right]d\sigma\right\},\tag{23}$$

where the C are the constraints of the theory and the $\chi^*(p,q)$ are the gauge choices. The key element of the link, the determinant, involves Poisson brackets between constraints and gauge fixing conditions. The time coordinate σ is an arbitrary parametrization of phase space trajectories. Initial and final conditions of the path integral are chosen to agree with those in Eq. (22). (The standard procedure for gauge theories described in [2] generalizes to the case in which time reparametrization invariance is one of the gauge invariances, as long as the choice is consistent [13].)

In this Bianchi type IX model, the kinematical gauge symmetries are fixed by the choice of a diagonal gauge and the time reparametrization invariance is fixed by associating time with volume. Writing physical coordinates ($\bar{q}$ and $\bar{p}$ above) as the anisotropies β, a transition element from σ_i to σ_f is generated by integrating the initial state with the kernel:

$$\langle\beta|\psi(\sigma_f)\rangle=\int[d\beta]\,K\left(\beta,\sigma_f;\beta,\sigma_i\right)\langle\beta|\psi(\sigma_i)\rangle\,.\tag{24}$$

The kernel, the object we shall be concerned with from now on, may be written in the new variables as

$$K(\epsilon_f,\sigma_f;\epsilon_i,\sigma_i)=\int\left[d^3\epsilon\,d^3a\,\left|\{C_*,\chi^*\}\right|\delta[\chi^0(\epsilon,\sigma)]\delta(S)\right]\exp\left\{i\int_{\sigma_i}^{\sigma_f}d\sigma\left(-i\epsilon^T\dot{a}\right)\right\}.\tag{25}$$

Expressions for this propagator will always be up to the normalization constant $K(\epsilon_i,\sigma_i;\epsilon_i,\sigma_i)$. In Eq. (25) we performed the trivial integration over off-diagonal pieces of a_J^I and e_I^J, eliminating vector and diffeomorphism constraints and their associated δ functions. However, the measure contains contributions from both kinematical gauge fixing and time parametrization (still explicitly indicated). The remaining constraint, the Hamiltonian constraint, is written as

$$S=\left(-a_1+a_2a_3\right)\epsilon_2\epsilon_3+\left(-a_2+a_3a_1\right)\epsilon_3\epsilon_1+\left(-a_3+a_1a_2\right)\epsilon_1\epsilon_2.\tag{26}$$

Exponentiating this constraint as

$$\delta(S)=\int_{-\infty}^{\infty}\delta_\sigma dN e^{-i\delta_\sigma NS},\tag{27}$$

in which δ_σ is the step size of the skeletonization of the path integral, we can rewrite the kernel of Eq. (25),

$$K(\epsilon_f,\sigma_f;\epsilon_i,\sigma_i)=\int\left[d^3\epsilon\,d^3a\,dN\,\left|\{C_*,\chi^*\}\right|\delta[\chi^0(\epsilon,\sigma)]\right]\exp\left\{i\int_{\sigma_i}^{\sigma_f}d\sigma\left(-i\epsilon^T\dot{a}-NS\right)\right\}.\tag{28}$$

The action is in the form of the (gauge-fixed) action of Eq. (11). This propagator has yet to be fully defined, for it is a path integral over six complex dimensions.

The effects of our gauge choices can be computed explicitly. With the Poisson brackets with the scalar constraint and the time choice of

$$\left|\{C_0,\chi^0\}\right|=2a_I\epsilon_I-F(\epsilon)\tag{29}$$

[$F(\epsilon)$ is defined in Eq. (19)], the whole determinant is

$$\left|\{C_*,\chi^*\}\right|=\begin{vmatrix}0&0&0&0&-\epsilon_2&0&\epsilon_3\\0&0&\epsilon_1&0&0&-\epsilon_3&0\\0&-\epsilon_1&0&\epsilon_2&0&0&0\\0&0&0&0&\epsilon_3&0&-\epsilon_2\\0&0&-\epsilon_3&0&0&\epsilon_1&0\\0&\epsilon_2&0&-\epsilon_1&0&0&0\\\{C_0,\chi^0\}&0&0&0&0&0&0\end{vmatrix}$$

$$=\left|\frac{\epsilon_2^3\epsilon_3^3}{\epsilon_1}\left(\epsilon_3^2-\epsilon_2^2\right)+\frac{\epsilon_1^3\epsilon_3^3}{\epsilon_2}\left(\epsilon_1^2-\epsilon_3^2\right)+\frac{\epsilon_1^3\epsilon_2^3}{\epsilon_3}\left(\epsilon_2^2-\epsilon_1^2\right)-\left(2a_I\epsilon_I\right)\left[\epsilon_3^2\epsilon_2^2\left(\epsilon_2^2-\epsilon_3^2\right)+\epsilon_3^2\epsilon_1^2\left(\epsilon_3^2-\epsilon_1^2\right)+\epsilon_2^2\epsilon_1^2\left(\epsilon_1^2-\epsilon_2^2\right)\right]\right|.$$

This measure has dependence on both coordinates and momenta. Fortunately, when the path integral is defined through the reality conditions the measure $|\{C_*, \chi^*\}|$ factors into a product of momenta and a function of coordinates. Imposition of the scalar constraint is enforced by integrating over N, which has a range from $-\infty$ to $+\infty$. The parametrization, Eq. (21), restricts the range of the ϵ integration to the positive real axis $(0, +\infty)$. Meanwhile, restricting the ϵ integral to the real axis also satisfies the reality conditions of Eq. (3) requiring the three-metric to be real.

To implement the other reality conditions, Eq. (4), we may choose a contour for the a_I integral which reflects these conditions. Recall that the A^I_a's are complex variables which depend on the original canonical variables of relativity via $A^I_a = \Gamma^I_a(E) + iK^I_a(E, \Pi)$ with SU(2) connection Γ^I_a and extrinsic curvature K^I_a. The momenta Π is canonically conjugate to E. We have a similar relation for the diagonalized expansion coefficients:

$$a_I = \gamma_I(\epsilon) + i\kappa_I. \tag{30}$$

The reality condition Eq. (4) is satisfied if

$$(\bar{a}_I + a_I)\,\epsilon_I\,(\epsilon_J^2 + \epsilon_K^2) = 2\epsilon_1\,\epsilon_2\,\epsilon_3\ , \tag{31}$$

in which $I, J, K = 1, 2, 3, 6$ are summed such that $I \neq J \neq K$. This suggests that the κ_I's may be taken as independent variables in the path integral. The contours in Eq. (28) may then be taken along the imaginary a axes or, equivalently, performed for real κ. At each σ we have

$$\int d^3\epsilon\, d^3\kappa = \int d^3\epsilon\, d^3a. \tag{32}$$

After integrating by parts and discarding a complex boundary term $(-\epsilon^T\kappa|_{\sigma_f} + \epsilon^T\kappa|_{\sigma_i})$ the propagator of Eq. (28) becomes

$$K(\epsilon_f, \sigma_f; \epsilon_i, \sigma_i) = \int_0^\infty [d^3\epsilon] \int_{-\infty}^\infty [d^3\kappa] \int_{-\infty}^\infty [dN\,|\{C_*, \chi^*\}|\,\delta\,[\chi_0(\epsilon, \sigma)]\,] \exp\left(i \int_{\sigma_i}^{\sigma_f} d\sigma \kappa^T Q \kappa + b^T \kappa\right)\ , \tag{33}$$

where the coefficient of the quadratic term is

$$Q = \frac{N}{2}\begin{pmatrix} 0 & \epsilon_1\epsilon_2 & \epsilon_1\epsilon_3 \\ \epsilon_1\epsilon_2 & 0 & \epsilon_2\epsilon_3 \\ \epsilon_1\epsilon_3 & \epsilon_2\epsilon_3 & 0 \end{pmatrix} \tag{34}$$

and the coefficient of the linear term is

$$b = \begin{pmatrix} \dot{\epsilon_1} \\ \dot{\epsilon_2} \\ \dot{\epsilon_3} \end{pmatrix} - iN \begin{pmatrix} \epsilon_2\epsilon_3 \\ \epsilon_1\epsilon_3 \\ \epsilon_1\epsilon_2 \end{pmatrix}. \tag{35}$$

The integration over κ may be done; it is Gaussian. However, as we are integrating three-momenta in a space with only two independent momenta, the matrix Q fails to provide convergence for all momenta κ. We can diagonalize Q by reexpressing our result in terms of physical degrees of freedom — the anisotropies. To accomplish this and to compare with previous studies of the Bianchi cosmology, we translate Eq. (33) back into Misner's chart on the phase space of Bianchi type IX using

$$h_{ij} = e^{2\alpha}\left(e^{2\beta}\right)_{ij} = \epsilon_i^2\delta_{ij}\ , \tag{36}$$

where β is the traceless matrix $\mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$ and

$$\begin{aligned} \kappa_1 &= \exp\left(-\alpha - \beta_+ - \sqrt{3}\beta_-\right)\left(\frac{p_\alpha}{3} + \frac{p_+}{6} + \frac{p_-}{2\sqrt{3}}\right), \\ \kappa_2 &= \exp\left(-\alpha - \beta_+ + \sqrt{3}\beta_-\right)\left(\frac{p_\alpha}{3} + \frac{p_+}{6} - \frac{p_-}{2\sqrt{3}}\right), \\ \kappa_3 &= \exp\left(-\alpha + 2\beta_+\right)\left(\frac{p_\alpha}{3} - \frac{p_+}{3}\right). \end{aligned} \tag{37}$$

The Jacobian of this transformation from the (ϵ, κ) chart into the (β, p) chart is 1. The integral of Eq. (33) may then be written, rescaling N and adjusting the zero of α, as

$$\begin{aligned} K(\beta_f, \sigma_f; \beta_i, \sigma_i) = \int_{-\infty}^\infty &[d^2\beta_\pm\, d\alpha\, d^2p_\pm\, dp_\alpha\, dN\,|p_\alpha\,\mu(\beta_\pm)\,e^{6\alpha}|\delta(\alpha - 3\sigma)] \\ &\times \exp\left[i\left(\int_{\sigma_i}^{\sigma_f} d\sigma\, p_\alpha\dot{\alpha} + p_+\dot{\beta_+} + p_-\dot{\beta_-} - N\mathcal{H}(\beta_\pm, p_\pm, \alpha, p_\alpha)\right)\right]. \end{aligned} \tag{38}$$

The action is the expected Hamiltonian form with

$$\mathcal{H}(\beta_\pm, p_\pm, \alpha, p_\alpha) = -p_\alpha^2 + p_+^2 + p_-^2 + e^{2\alpha}\mathcal{U}(\beta_\pm)\ , \tag{39}$$

where the potential,[3] $V(\beta_\pm) := \mathcal{U}(\beta_\pm) + 1$ is the familiar triangular potential of Bianchi type IX [4]:

$$\mathcal{U}(\beta) = \frac{1}{3}e^{-8\beta_+} - \frac{4}{3}e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) + \frac{2}{3}e^{4\beta_+}\left[\cosh(4\sqrt{3}\beta_-) - 1\right].$$

[3]Note that in the literature one often finds written $V(\beta_\pm) := \mathcal{U}(\beta_\pm) + 1$. This is convenient because $V(\beta_\pm)$ is bounded from below; however, what is important to remember is that the actual potential $\mathcal{U}(\beta_\pm)$ is bounded from below by -1.

A contour plot of this potential appears in Fig. 1. The action is that of Misner's Bianchi type IX formulation of a particle moving in a time-dependent potential. The evolution of the cosmology is seen as dynamics of a free particle reflecting off roughly triangular, exponentially steep "walls" shown in Fig. 1. The measure factor $\mu(\beta_\pm)$ contains (along with the factor $|p_\alpha|$) remains of the gauge fixing procedure. It may be written as

$$\mu(\beta_\pm) = |\sinh(4\sqrt{3}\beta_-) - \sinh(6\beta_+ + 2\sqrt{3}\beta_-) + \sinh(6\beta_+ - \sqrt{3}\beta_-)|. \tag{40}$$

Graphed in Fig. 2, this measure has sixfold symmetry. The points the of the "star" are the minima, lending little ($\approx 10^{-4}$) support to wave functions with maxima in the corners or near the center of the triangular walls of the Bianchi type IX potential.

Letting the δ function consume the α integration we find

$$K(\beta_f,\sigma_f;\beta_i,\sigma_i) = \int_{-\infty}^{\infty} \left[d^2\beta_\pm d^2 p_\pm dp_\sigma dN |p_\sigma| \mu(\beta_\pm)\, |e^{18\sigma}|\right] \exp\left(i\int_{\sigma_i}^{\sigma_f} d\sigma 3p_\sigma + p_+\dot{\beta}_+ + p_-\dot{\beta}_- - N\mathcal{H}(\beta_\pm,p_\pm,\sigma)\right). \tag{41}$$

As indicated earlier, the propagator may be integrated to a configuration-space propagator. The $p_\pm$ integrals converge in the usual sense; the coefficient of the quadratic piece, after analytic continuation, is real and negative. Convergence is ensured in the p_σ integral by the same continuation. Before analytic continuation, one-half of the history must be selected. To secure convergence in the propagator of Eq. (41) we must rotate to $\sigma \to \sigma_N$, where $\sigma_N = \sigma \exp\left(i\frac{\pi}{2}\right)$ when $N > 0$ and $\sigma \to \sigma_N = \sigma \exp\left(-i\frac{\pi}{2}\right)$ when $N < 0$, effectively excluding "backwards evolving" histories. Alternately, it is possible to restrict the N integration at the onset by using a θ function in the exponentiation, Eq. (27), as may be done in the path integral for the relativistic particle [14]. When the parameter σ is continued the measure factor $|\exp(18\sigma)|$ becomes 1 for all values of σ.

The integration of p_σ involves two integrals of the form

$$I_\pm = \int_0^\infty dp_\sigma\, p_\sigma\, e^{-ap_\sigma \pm bp_\sigma}. \tag{42}$$

As the integral for p_σ is $I_+ + I_-$, any terms linear in b cancel on account of the absolute value in the measure of Eq. (41). The value of one of these integrals is a sum of three terms, two of which are linear in b and one of which is only dependent on a. Thus the only term which survives is the a-dependent one. The other two momenta integrations are standard quadratics. Upon integration, we find

$$K(\beta_f,\sigma_f;\beta_i,\sigma_i) = \int_{-\infty}^{\infty} \left[d^2\beta_\pm \frac{dN}{N^2}\,\mu(\beta_\pm)\right] \times \exp\left(i\int d\sigma_N \mathcal{L}(\beta_\pm,\sigma_N,N)\right), \tag{43}$$

with the Lagrangian

$$\mathcal{L}(\beta_\pm,\sigma_N,N) = \frac{1}{4N}\left(\dot{\beta}_+^{\,2} + \dot{\beta}_-^{\,2}\right) - Ne^{6\sigma}\mathcal{U}(\beta_\pm). \tag{44}$$

This is the Lagrangian form of the path integral. The "lapse" plays a dual role of enforcing the constraint and generating time evolution. We may integrate over the "lapse" function. To do this we must write explicitly what we mean by the formal path integral expressions we have written previously. The Lagrangian form is

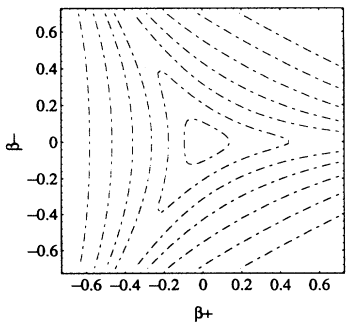

FIG. 1. The classical potential for Bianchi type IX with contours spaced by powers of e with the first line at $U(\beta_\pm) = -0.9$. Three "channels," one along the positive β_- axis and the other two sloping diagonally in the negative β_- axis, indicate the minima of the potential. The contours form a rough triangle with steep walls.

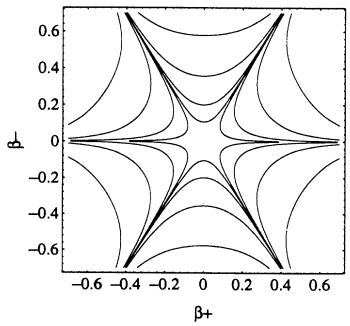

FIG. 2. The measure $\mu(\beta_\pm)$ for the path integral of Bianchi type IX with the parametrization of Eq. (21). In the center and in the six channels the measure is small, lending little support to histories there. The contours are separated by powers of e with the first contour at $\mu(\beta_\pm) = e^{-5}$ and ending at $\mu(\beta_\pm) = e^3$.

$$K(\beta_f,\sigma_f;\beta_i,\sigma_i)=\lim_{n\to\infty}\left(\frac{1}{2\pi i\delta_\sigma}\right)^{n/2}\prod_{j=1}^{n-1}\int_{-\infty}^{\infty}d^2\beta_{\pm_j}\,\mu(\beta_{\pm_j})\int_0^\infty\frac{dN_j}{N_j^2}$$
$$\times\exp\left\{\frac{i}{4N}\delta_\sigma\left[-\left(\frac{\beta_{+_{j+1}}-\beta_{+_j}}{\delta_\sigma}\right)^2-\left(\frac{\beta_{-_{j+1}}-\beta_{-_j}}{\delta_\sigma}\right)^2\right]-iNe^{6j\delta_\sigma}\mathcal{U}(\beta_{\pm_j})\right\}.$$

The "lapse" may be integrated out, yielding a modified Bessel function

$$K(\beta_f,\sigma_f;\beta_i,\sigma_i)=\left(\frac{1}{2\pi i\delta_\sigma}\right)^{n/2}\prod_{j=1}^{n-1}\int_{-\infty}^{\infty}d^2\beta_{\pm_j}\,\mu(\beta_{\pm_j})\left(\frac{e^{-6j\delta_\sigma}\bar\beta_{\pm_j}^2}{4\delta_\sigma^3\mathcal{U}(\beta_{\pm_j})}\right)^{1/2}\mathcal{K}_1\left[2\left(\frac{\bar\beta_{\pm_j}^2}{4\delta_\sigma}e^{6j\delta_\sigma}\mathcal{U}(\beta_{\pm_j})\right)^{1/2}\right],\qquad(45)$$

where we have rotated the parameter σ back to real values. We have written the average of β_{j+1} and β_j as $\bar\beta_{\pm_j}$. The expression Eq. (45), as far as we believe, can be gone in the evaluation of the path integral of the Bianchi type IX cosmology, without turning to numerical calculations. It is reassuring that the N-dependent rotations necessary to make the $p_{\pm,\alpha}$ integrals converge serve again to make the N integral well defined.

IV. TRACE K PARAMETRIZATION OF TIME

Alternatively, we can perform the path integral quantization choosing the trace of the extrinsic curvature as the time parameter. This choice has the advantage of being monotonic on the whole cosmological history. However, it comes with other difficulties. In the diagonal new variables of Sec. II the gauge condition is written as

$$\chi_0=\sigma-i\left(F(\epsilon)-a_I\epsilon_I\right).\qquad(46)$$

The measure with this choice of time, once translated into the (β,p) chart, mixes momenta. Hence the analogous expression to Eq. (38) is

$$K(\beta_f,\sigma_f;\beta_i,\sigma_i)=\int_{-\infty}^{\infty}\left[d^2\beta_\pm\,d\alpha\,d^2p_\pm\,dp_\alpha\,dN\,\mu(\beta_\pm,p_\pm,\alpha,p_\alpha)\,\delta(\sigma-p_\alpha)\right]$$
$$\times\exp i\left(\int_{\sigma_i}^{\sigma_f}d\sigma\,p_\alpha\dot\alpha+p_+\dot\beta_++p_-\dot\beta_--N\mathcal{H}(\beta_\pm,p_\pm,\alpha,p_\alpha)\right),\qquad(47)$$

in which

$$\mu(\beta_\pm,p_\pm,\alpha,p_\alpha)=\Bigg|2e^{2\alpha+2\beta_+}\cosh2\sqrt3\beta_--e^{2\alpha-4\beta_+}\left(\cosh4\sqrt3\beta_-+\frac12\right)$$
$$+2ie^\alpha\left[\frac{p_\alpha}{3}\left(e^{4\beta_+}+2e^{-2\beta_+}\cosh2\sqrt3\beta_-\right)+\frac{p_+}{3}\left(-e^{4\beta_+}+2e^{-2\beta_+}\cosh2\sqrt3\beta_-\right)\right.$$
$$\left.-\frac{p_-}{\sqrt3}\left(e^{-2\beta_+}\sinh2\sqrt3\beta_-\right)\right]\Bigg|.\qquad(48)$$

This choice of time is not "diagonal" in the measure. As this choice mixes momenta and coordinates the momentum integration is more difficult than before. We will compare this form of the propagator with the propagator obtained from a physical quantization in the companion paper.

V. CONCLUSION

We used the Faddeev-Popov prescription to construct a path integral for the Bianchi type IX quantum cosmology. Our strategy began with a classical dynamical system on a (9+9)-dimensional complex phase space defined by the constraints Eqs. (6), (8), and (7). The Faddeev-Popov ansatz defined the measure of the path integral. We chose contours of integration which ensure that phase space histories satisfy the reality conditions, corresponding to the condition that the metric of spacetime is real and of Minkowskian signature.

We find, as a result, the anisotropy space path integral Eq. (43) or Eq. (45) for the Bianchi type IX cosmology in the gauge in which time is 0 by the volume of space. It is interesting to note that the effect of the measure is independent of time and, hence, the volume of the Universe. To further understand the effects of the measure in the path integral, it is necessary to finish the evaluation of the path integral. As we have seen that there is an analytic continuation which makes the integral in Eq. (45) real and convergent, this should be possible with defining the integral through standard Monte Carlo techniques, or by semiclassical techniques.

In particular, given this kernel one can proceed di-

rectly to the evaluation of the expectation values of gauge invariant, and hence physically meaningful, quantities. For example, any quantity of the form $F(\beta_\pm, (\sigma))$ which measures correlations of anisotropies, defined on leaves with particular volumes, is gauge invariant and meaningful [1]. Quantities like this have been evaluated successfully in a variety of cosmological models including $(2+1)$ gravity [15], Gowdy models [16], and the Bianchi type I model [17]. These models were all exactly solvable, so that expectation values of some physical quantities could be computed exactly. We believe that with path integrals of the form of Eq. (43), in which gauge invariance is guaranteed by the construction, it is possible to extend the calculations of physically meaningful quantities in quantum cosmology.

While the Faddeev-Popov technique guarantees that the resulting path integral represents a physical amplitude, quantum cosmology is more demanding. We must consider the fact that universes described by the Bianchi cosmological model live for a finite time. Any quantization of cosmological models can give sensible answers to physical questions about evolution in time only if the answers are formulated in a manner that is both diffeomorphism invariant and takes into account that any given classical or quantum universe may no longer exist after a certain span of time. We take up such issues in the companion paper.

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