

Universe reheating after inflation

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We study the problem of scalar particle production after inflation by an inflaton field which is oscillating rapidly relative to the expansion of the universe. We use the framework of the chaotic inflation scenario with quartic and quadratic inflaton potentials. Particles produced are described by a quantum scalar field χ , which is coupled to the inflaton via linear and quadratic couplings. The particle production effect is studied using the standard technique of Bogolyubov transformations. Particular attention is paid to parametric resonance phenomena which take place in the presence of the quickly oscillating inflaton field. We have found that in the region of applicability of perturbation theory the effects of parametric resonance are crucial, and estimates based on first-order Born approximation often underestimate the particle production. In the case of the quartic inflaton potential $V(\varphi) = \lambda\varphi^4$, the particle production process is very efficient for either type of coupling between the inflaton field and the scalar field χ even for small values of coupling constants. The energy density of the universe after the decay of the inflaton oscillations is in this case a factor $[\lambda \ln(1/\lambda)]^{-1}$ times larger than the corresponding estimates based on first-order Born approximation. In the case of the quadratic inflaton potential the reheating process depends crucially on the type of coupling between the inflaton and the scalar field χ and on the magnitudes of the coupling constants. If the inflaton coupling to fermions and its linear (in inflaton field) coupling to scalar fields are suppressed, then, as previously discussed by Kofman, Linde, and Starobinsky, the inflaton field will eventually decouple from the rest of the matter, and the residual inflaton oscillations may provide the (cold) dark matter of the universe. In the case of the quadratic inflaton potential we obtain the lowest and the highest possible bounds on the effective energy density of the inflaton field when it freezes out.

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I. INTRODUCTION

According to the simplest version of the inflationary scenario (see [1]) the universe in the past expands almost exponentially with time (such an expansion is called “inflation”) while its energy density is dominated by the effective potential energy density of a special scalar field φ , called the inflaton. Sooner or later inflation terminates and the inflaton field starts quasiperiodic motion with slowly decreasing amplitude. Right after inflation the universe is empty of particles, i.e., “cold.” Quasiperiodic evolution of the inflaton field leads to creation of particles of various kinds, after thermalization of which due to collisions and decays the universe becomes “hot.” The role of the inflaton field in the inflationary scenario is not necessarily played by a fundamental scalar field. It can be the expectation value of a scalar operator constructed of fields of other type. It can also be the effective scalar introduced in the inflationary models based on a higher derivative theory of gravity [2]. Almost all such scenarios have an important feature: the inflationary stage is followed by quasiperiodic evolution of the effective inflaton

field that leads to particle creation.

In this work we are going to study the transition of the universe from an inflationary to a hot stage sketched above. The problem is not new, and there are many papers devoted to its study using different methods in the context of various inflationary scenarios [3–8] (see also Refs. [1] and [9]). There is, nevertheless, a drawback common to most of them—the calculations were based on ordinary perturbation theory. The rates of particle production by the oscillating inflaton field were calculated in first-order Born approximation. However, as has been shown in Ref. [10] this approach disregards possible parametric resonance effects which can enhance the rate of boson particle production. To detect such effects one has to go beyond standard perturbation theory. Such resonance amplification effects have been studied in [10] in the context of the new inflationary scenario. It was shown that typically the oscillating inflaton field produces many particles due to the parametric resonance effect even in the case of extremely small couplings, and that estimates based on ordinary perturbation theory miss the largest effect. Parametric resonance effects as relevant to reheating of the universe were also studied in Ref. [11], where

it was also demonstrated that there is no resonance enhancement of fermionic particle production.

In the present paper we are going to study the problem in a systematic way. We will consider a model with inflation based on a power law potential for the inflaton field. The inflaton field φ throughout this paper will be regarded as classical. In Sec. II we will describe the evolution of the inflaton field after inflation without taking into account its couplings to other fields. Couplings of the inflaton field φ to fields describing fermionic spin- $\frac{1}{2}$ and bosonic scalar particles will be considered in the following sections. In particular, the process of production of these particles by the oscillating inflaton field will be studied. In all the cases we will assume the couplings to be sufficiently small so that perturbation theory is valid. In Sec. III we will derive the expressions for particle production rates in first-order Born approximation of ordinary perturbation theory. The material of Secs. II and III will closely follow a previous work [12]. In Sec. IV we study general aspects of the parametric resonance effect, and in Secs. V and VI we apply the results obtained to the concrete cases of quartic and quadratic inflaton potentials. In Sec. VII we present our general conclusions.

It will be shown that in many important cases it is not legitimate to neglect parametric resonance effects. These effects lead to a great enhancement of the particle production as compared to the estimates based on ordinary perturbation theory. The number of particles N_k in a mode with wave number k that has experienced the resonance turns out to be much larger than unity. The universe reheating temperature therefore can turn out to be sufficiently large even for small values of the couplings between inflaton and other fields.

The opinion of the authors of Ref. [11] is that parametric resonance effects of scalar particle production are likely to be suppressed by the processes of scattering and decay of the particles produced. The reason is that particle scattering and decay will tend to reduce the mean occupation numbers N_k in each particular mode so that these numbers will grow not so fast as to be able to become large during the resonance period. This question is very important and deserves thorough study. We do not aim to touch on it in this work, but hope to consider it in future publications.

In parallel to our own work, Kofman, Linde, and Starobinsky have also been working on these issues for several years. Their results were recently summarized in Ref. [13]. In Sec. VIII we will compare our results with those of Ref. [13].

II. INFLATON FIELD DYNAMICS

In this section we will derive approximate analytic expressions for the evolution of the inflaton field as it undergoes quasiperiodic motion in the expanding universe, and compute the resulting effective equation of state of the inflaton stress-energy.

Consider the chaotic inflation scenario based on the scalar (inflaton) field dynamics. The Lagrangian of the model is

$$L = \frac{1}{2}(\nabla\varphi)^2 - V(\varphi) - \frac{M_P^2}{16\pi}R, \quad (1)$$

with the inflaton field potential

$$V(\varphi) = \lambda\mu^{4-q}|\varphi|^q, \quad (2)$$

where q is an arbitrary positive power. We are working in the standard system of units in which $\hbar = c = 1$, and M_P denotes the Planck mass.

The equations of motion for a homogeneous isotropic universe and for a homogeneous scalar field look like

$$H^2 + \frac{\kappa}{a^2} = \frac{8\pi}{3M_P^2}[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)], \quad (3)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \quad (4)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, $\kappa = 0, \pm 1$ corresponds to different signs of spatial curvature, overdots denote derivatives with respect to the cosmological time t , and the prime denotes a derivative with respect to φ .

From Eq. (4) it follows that

$$\dot{\varphi} = a^{-3} \left(\text{const} - \int V'(\varphi) a^3 dt \right). \quad (5)$$

This equation describes both the rapid evolution of φ , for which $\varphi \propto a^{-3}$, and the regime of slow rolling down during inflation, when $\dot{\varphi} \approx -(1/3H)V'(\varphi)$.

According to the inflation scenario, during inflation the scalar field φ is rolling down the slope of its potential from its relatively large value towards its minimum. The conditions for slow rolling and for inflation are, respectively,

$$|\ddot{\varphi}| \ll 3H|\dot{\varphi}| \quad (6)$$

and

$$|\dot{H}| \ll H^2. \quad (7)$$

During inflation the term κ/a^2 in the left-hand side of Eq. (3) soon becomes insignificant, and the conditions (6) and (7) read, respectively,

$$[\sqrt{V(\varphi)}]'' \ll \frac{12\pi}{M_P^2} \sqrt{V(\varphi)}, \quad (8)$$

$$[V'(\varphi)]^2 \ll \frac{24\pi}{M_P^2} V^2(\varphi). \quad (9)$$

For the scalar field potential (2) these two conditions become, respectively,

$$\varphi^2 \gg \frac{q|q-2|}{48\pi} M_P^2, \quad (10)$$

$$\varphi^2 \gg \frac{q^2}{24\pi} M_P^2. \quad (11)$$

Note that conditions (10) and (11) are essentially the same.

As the scalar field evolves towards its smaller values,

the conditions (10) and (11) cease to be valid, and a new regime for the scalar field begins, namely, the regime of quasiperiodic evolution with decaying amplitude. To describe this new cosmological period, it is convenient to rewrite the system of Eqs. (3) and (4) (with $\kappa=0$) as follows:

$$H^2 = \frac{8\pi}{3M_P^2} \rho, \quad (12)$$

$$\dot{\rho} = -3H\dot{\varphi}^2, \quad (13)$$

where

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad (14)$$

is the scalar field energy density. Introducing the positive value $\varphi_0(t)$ by the relation

$$V(\varphi_0(t)) = \rho(t), \quad (15)$$

we can present the evolution of the scalar field in the form

$$\varphi(t) = \varphi_0(t) \cos \int W(t) dt, \quad (16)$$

where $W(t)$ is some unknown function of time. Note that the representation (16) for the scalar field evolution with φ_0 defined by (15) is always possible. Using Eqs. (2) and (12)–(15), one can then derive the following exact expression for the function $W(t)$:

$$W = \sqrt{\frac{2[\rho - V(\varphi)]}{\varphi_0^2 - \varphi^2}} \left(1 \pm \frac{6H\varphi}{q\sqrt{2\rho}} \sqrt{1 - \frac{V(\varphi)}{\rho}} \right). \quad (17)$$

The signs “+” and “−” correspond, respectively, to the cases $\dot{\varphi} > 0$ and $\dot{\varphi} < 0$. The second term in the large parentheses in (17) is much less than unity if

$$\varphi_0^2 \ll \frac{q}{48\pi} \left(\frac{q+2}{2} \right)^{(q+2)/q} M_P^2. \quad (18)$$

This condition is just the opposite of (10) and (11). When (18) is valid, the expression (17) for $W(t)$ acquires the simple approximate form

$$W \approx \sqrt{\frac{2[\rho - V(\varphi)]}{\varphi_0^2 - \varphi^2}}, \quad (19)$$

and, in addition, the following condition becomes valid:

$$|\dot{\varphi}_0/\varphi_0| \ll W, \quad (20)$$

which allows us to regard the evolution of $\varphi(t)$ as quasiperiodic with slowly decaying amplitude $\varphi_0(t)$. Henceforth we assume that the condition (18) [hence also (19) and (20)] is satisfied.

Our aim now will be to derive approximate evolution equations for the values $\rho(t)$ and $\varphi_0(t)$ under the condition (18). First we average the equation of motion

(13) over the time period T of quasiperiodic motion of φ . Rewriting (13) in accord with (14) as

$$\dot{\rho} = -6H[\rho - V(\varphi)], \quad (21)$$

and taking the average of both sides, we get

$$\frac{\Delta\rho(T)}{T} = -\frac{6}{T} \int_0^T H[\rho - V(\varphi)] dt, \quad (22)$$

where $\Delta\rho(T)$ is the change of ρ over the time period T . Now take into account that according to (20) and to the equations (15) and (12) the values $\varphi_0(t)$, $\rho(t)$, and $H(t)$ change only insignificantly during one period of oscillation of the φ field. This enables us to replace all variables except φ under the integral in (22) by their averaged values. We will have then, to a good approximation,

$$\frac{\Delta\rho(T)}{T} \approx -\frac{6}{T} H \int_0^T [\rho - V(\varphi)] dt, \quad (23)$$

where H and ρ in (23) and below denote the corresponding averaged values. The integral in (23) can be evaluated as

$$\begin{aligned} \frac{1}{T} \int_0^T [\rho - V(\varphi)] dt &\approx \frac{\int_{-\varphi_0}^{\varphi_0} \sqrt{\rho - V(\varphi)} d\varphi}{\int_{-\varphi_0}^{\varphi_0} d\varphi / \sqrt{\rho - V(\varphi)}} \\ &\approx \frac{\frac{1}{2} \int_{-\varphi_0}^{\varphi_0} \varphi V'(\varphi) d\varphi / \sqrt{\rho - V(\varphi)}}{\int_{-\varphi_0}^{\varphi_0} d\varphi / \sqrt{\rho - V(\varphi)}} \\ &\approx \frac{q}{q+2} \rho. \end{aligned} \quad (24)$$

In (24) we took into account the relation (14), integrated in the numerator by parts, and made use of Eq. (2). In the approximation considered we can replace the finite difference expression on the left-hand side of (23) by the derivative. Equation (23) then becomes

$$\dot{\rho} \approx -\frac{6q}{q+2} H\rho. \quad (25)$$

All the values in (25) are now to be regarded as averaged in the sense described above. Equation (25) is valid on time scales large compared to the period of quasioscillations of $\varphi(t)$. This implies the following effective equation of state for the matter described by the scalar field φ :

$$p = \frac{q-2}{q+2} \rho. \quad (26)$$

In particular, for the most interesting cases $q = 2$ (quadratic potential) and $q = 4$ (quartic potential) we obtain, respectively, $p = 0$ (dust) and $p = \rho/3$ (radiation).

Using the definition (15) we immediately obtain the approximate equation for the evolution of the value $\varphi_0(t)$,

$$\dot{\varphi}_0 \approx -\frac{6}{q+2} H\varphi_0. \quad (27)$$

III. PARTICLE PRODUCTION: BORN APPROXIMATION

In this section we will study the process of particle production by the rapidly oscillating inflaton field. We take the interaction Lagrangian to be

$$L_{\text{int}} = -f\varphi\bar{\psi}\psi - (\sigma\varphi + h\varphi^2)\chi^2, \quad (28)$$

where $\bar{\psi}$ and ψ describe spinor particles and χ describes scalar particles with corresponding masses m_ψ and m_χ , f and h are dimensionless coupling constants, and σ is a coupling constant of dimension of mass. We will treat the scalar field φ as a classical external field. In this section we will employ ordinary perturbation theory in coupling constants, that is, we will work in the Born approximation. To calculate the rate of particle production we first develop the quasiperiodic evolution of the scalar field φ into harmonics

$$\varphi(t) = \sum_{n=1}^{\infty} \varphi_n \cos(n\omega t), \quad (29)$$

$$\varphi^2(t) = \bar{\varphi}^2 + \sum_{n=1}^{\infty} \zeta_n \cos(2n\omega t), \quad (30)$$

where the value $\bar{\varphi}^2 \approx \varphi_0^2/2$ which is slowly varying with time is φ^2 averaged over the rapid oscillations of the scalar field φ . φ_n and ζ_n are amplitudes which are slowly varying with time, and

$$\omega = \frac{2\pi}{T} = c \frac{\sqrt{\rho}}{\varphi_0} \quad (31)$$

is the leading frequency, also slowly varying with time, related to the period T of the oscillations of the field φ . The constant c in (31) is of order 1, and is given by

$$c = \pi\sqrt{2} \left/ \int_{-1}^1 \frac{dx}{\sqrt{1-x^q}} \right. \quad (32)$$

in the case of a scalar field potential given by (2). To derive the estimate (31) we made use of the relation (14).

Assuming the condition (18), it follows that the oscillation period of φ is small compared to the Hubble time,

$$H \ll \omega. \quad (33)$$

The last condition allows us to disregard the effect of cosmological expansion in evaluating particle production rates. We will also assume that particle masses and the coupling constant h are sufficiently small, so that

$$m_\psi^2, m_\chi^2 + h\bar{\varphi}^2 \ll \omega^2. \quad (34)$$

The rates of particle production, that is, the total number of pairs produced per unit volume and unit time are then given in first-order perturbation theory in coupling constants f , σ , and h by the following relations:

$$w_{\bar{\psi}\psi} = \frac{f^2}{16\pi} \sum_{n=1}^{\infty} (\varphi_n n\omega)^2, \quad (35)$$

$$w_{2\chi} = \frac{1}{16\pi} \left(\sigma^2 \sum_{n=1}^{\infty} \varphi_n^2 + h^2 \sum_{n=1}^{\infty} \zeta_n^2 \right). \quad (36)$$

Remembering the expansions (29) and (30) we can put the sums in the last expressions into the following form:

$$\sum_{n=1}^{\infty} (\varphi_n n\omega)^2 = 2\langle \dot{\varphi}^2 \rangle, \quad (37)$$

$$\sum_{n=1}^{\infty} \varphi_n^2 = 2\langle \varphi^2 \rangle, \quad (38)$$

$$\sum_{n=1}^{\infty} \zeta_n^2 = 2\langle (\varphi^2 - \bar{\varphi}^2)^2 \rangle, \quad (39)$$

where the brackets $\langle \dots \rangle$ denote the average over quasioscillations of the scalar field $\varphi(t)$. These average values in the expressions (37)–(39) can be estimated in exactly the same manner in which Eq. (24) is derived. We obtain, finally, the following relations for the particle production rates:

$$w_{\bar{\psi}\psi} = \frac{f^2}{4\pi} \frac{q}{q+2} \rho, \quad (40)$$

$$w_{2\chi} = \frac{1}{8\pi} (c_\sigma \sigma^2 \varphi_0^2 + c_h h^2 \varphi_0^4), \quad (41)$$

where

$$c_\sigma = \int_{-1}^1 \frac{x^2 dx}{\sqrt{1-x^q}} \left/ \int_{-1}^1 \frac{dx}{\sqrt{1-x^q}} \right., \quad (42)$$

$$c_h = \int_{-1}^1 \frac{(x^2 - c_\sigma^2)^2 dx}{\sqrt{1-x^q}} \left/ \int_{-1}^1 \frac{dx}{\sqrt{1-x^q}} \right., \quad (43)$$

are constants of order unity.

Higher order contributions to the particle production rates (40) and (41) will be negligible if

$$f^2, h, \frac{\sigma}{\varphi_0} \ll \left(\frac{\omega}{\varphi_0} \right)^2. \quad (44)$$

Hence, our assumption (34) for the smallness of the coupling constant h is justified since $\bar{\varphi}^2 \approx \varphi_0^2/2$. We will see below what the conditions (44) imply for the concrete inflaton potentials.

Equations (40) and (41) enable us to estimate the energy density production in the form of $\bar{\psi}$, ψ , and χ particles created. To do this, note that pairs of particles produced have average energy of the order ω , if we reasonably assume that the sums on the right-hand sides of (35) and (36) rapidly converge. Then the equation for ρ_p , the particle energy density produced, can be written as follows:

$$\dot{\rho}_p(\text{production}) = (\Gamma_\psi + \Gamma_\chi)\rho, \quad (45)$$

where

$$\Gamma_\psi = O(1) \frac{f^2}{4\pi} \quad (46)$$

is the contribution corresponding to the production of fermionic particles, and

$$\Gamma_\chi = O(1) \frac{\sigma^2}{4\pi\omega} + O(1) \frac{(h\varphi_0)^2}{4\pi\omega} \quad (47)$$

is the contribution corresponding to the production of scalar particles χ . Here and below we denote by the symbol $O(1)$ a constant of order unity. The values Γ_ψ and Γ_χ can be interpreted as the “ φ -particle” decay rates in corresponding decay channels.

IV. PARTICLE PRODUCTION: PARAMETRIC RESONANCE

The expressions (45)–(47) for the particle energy density production rates have been obtained in first-order Born approximation, and in this form they have been used in all the seminal papers on reheating. However, as has been shown in [10], this approximation does not take appropriate account of the parametric resonance phenomena which take place during the oscillation period of the scalar field φ . Parametric resonance effects can be shown to be insignificant for fermionic fields [11]. The estimate (46) for fermion production made in the preceding section thus remains valid. However, the question of the parametric resonance effect for the scalar field χ has to be considered and this will be the aim of this section. We will show that the resonance occurs when the frequency ω_k of the quantum field mode is equal to half-integer multiples of the inflation frequency, $\omega_k^2 \approx [(n/2)\omega]^2$. This results in exponentially enhanced production of particles in narrow resonance bands, at rates which are computed below.

The evolution of a particular mode χ_k of the quantum scalar field χ in the presence of the quickly oscillating classical scalar field φ with the interaction (28) is described by the following equation:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + (\underline{k}^2 + m_\chi^2 + 2\sigma\varphi + 2h\varphi^2)\chi_k = 0, \quad (48)$$

where $\underline{k} = k/a$ is the physical wave number of the mode under consideration and k is the comoving number.

Performing the transformation

$$\chi_k = \frac{Y_k}{a^{3/2}}, \quad (49)$$

one obtains from Eq. (48) the following equation for the function Y_k :

$$\ddot{Y}_k + [\omega_k^2(t) + g(\omega t)]Y_k = 0, \quad (50)$$

where

$$\omega_k^2(t) = \underline{k}^2 + m_\chi^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} + 2h\varphi^2 \quad (51)$$

and

$$g(\omega t) = 2\sigma\varphi + 2h(\varphi^2 - \bar{\varphi}^2). \quad (52)$$

The function $g(\omega t)$ is to a good approximation a $2\pi/\omega$ -periodic function of time t with two leading frequencies ω and 2ω , as can be seen from (52) and the expansions (29) and (30). We will regard this function as a small perturbation in Eq. (50), and to be able to do this we require the condition

$$2\sigma\varphi_0 + h\varphi_0^2 \ll \omega_k^2, \quad (53)$$

which we will assume to be valid.

The time dependence of the coefficients in Eqs. (48) and (50) will lead to χ -particle production which can be described by the standard Bogolyubov transformation technique. The problem consists in finding the solution to Eq. (50) with initial conditions corresponding to the initial vacuum state and then evaluating the Bogolyubov transformation coefficients. We expect that for Eq. (50) the parametric resonance effect will be dominant; hence we are going to study this effect in detail.

From the general theory of parametric resonance (see Appendix A for details¹) one knows that parametric resonance occurs for Eq. (50) for certain values of the frequencies ω_k . Namely, the resonance in the lowest frequency resonance band occurs for those values of ω_k for which

$$\omega_k^2 - \left(\frac{n}{2}\omega\right)^2 \equiv \Delta_n < |g_n|, \quad (54)$$

where n is an integer and g_n is the amplitude of the n th Fourier harmonic of the function $g(\omega t)$. Of course, for such a conclusion to be valid in our case when the frequencies ω_k and ω depend on time we must be sure that these frequencies change with time slowly as compared to their own values, that is,

$$\left| \frac{\dot{\omega}_k}{\omega_k} \right|, \left| \frac{\dot{\omega}}{\omega} \right| \ll \omega_k, \omega. \quad (55)$$

But this condition immediately follows from the estimates (20), (33), and from (53) if we also take the relation (54) into account. Note that condition (53) for ω_k in the resonance band coincides with the conditions (44) on the validity of Born approximation.

In this paper we will consider only resonance in the lowest resonance frequency band for each Fourier harmonic of the function g as given by Eq. (54). The effects of higher resonance bands, as is well known, are of higher order in the amplitudes $|g_n|$ which we assumed to be small. Hence we expect these effects to be less efficient.

¹In Appendix A we explicitly introduced a small dimensionless parameter ϵ . In the main text we have absorbed ϵ into the definition of g .

From the conditions (20), (33), (34), and (53) it also follows that part of the expression (51) satisfies

$$\omega_k^2 - \underline{k}^2 \equiv m_\chi^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} + 2h\bar{\varphi}^2 \ll \omega^2. \quad (56)$$

It then follows from Eq. (54) that

$$\underline{k}^2 \gg m_\chi^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} + 2h\bar{\varphi}^2 \quad (57)$$

for the resonance values of \underline{k}^2 . This means that the redshift of the frequency ω_k due to the expansion of the universe is, to a good approximation,

$$\omega_k(t) \propto \underline{k} \propto a^{-1}(t). \quad (58)$$

Due to the condition (33) we can neglect this cosmological redshift on the time scale of the scalar field φ oscillations. The mean occupation numbers for the generated χ particles are then given by the Bogolyubov coefficients β_k as $N_k = |\beta_k|^2$ and the expression for them can be approximately written as follows (see Appendix B for derivation²)

$$N_k \simeq \sinh^2 \left(\int \mu_+ dt \right), \quad (59)$$

where

$$\mu_+ = \frac{1}{n\omega} \sqrt{|g_n|^2 - \Delta_n^2} \quad (60)$$

is the eigenvalue of the growing resonance mode, and Δ_n is given by (54).

The approximation (59) is valid only if we can treat the value μ_+ as adiabatic. For our purposes it is sufficient to consider values of μ_+ close to the resonance (when $\Delta_n = 0$) and the variation of μ_+ with time is mostly due to the expansion of the universe. Then the adiabaticity condition [see Eq. (B31)] will be as follows:

$$\mu_+ \gg H. \quad (61)$$

As time goes on, the mode with the particular wave number k gets out of the resonance band, and the value N_k ceases to grow. The reason for the frequency ω_k (initially in resonance) to get out of the resonance band is that the redshift evolution (58) of the frequency ω_k is different from the evolution of the leading frequency ω and hence the resonance frequency $n\omega/2$ is continuously passing through the spectrum of frequencies ω_k of the modes of the scalar field χ . (In the case of the $\lambda\varphi^4$ potential, the value ω does redshift at the same rate, and one has to take into account the back reaction of the created particles to determine this frequency shift; see Sec. V.)

Denote by ω_{res} the resonance frequency $n\omega/2$ and by $\Delta\omega_k$ denote the difference $\omega_k - \omega_{\text{res}}$. Both frequencies ω_{res} and ω_k change with time. In a small time δt the shift between these frequencies will be $\delta\omega = |d\Delta\omega_k/dt|_{\Delta\omega_k=0} \delta t$. Hence a new region in phase space of volume $\omega_{\text{res}}^2 \delta\omega/2\pi^2$ will be filled with particles. The average number of particles in every state in this phase space region will be given by N_k and each particle will have energy ω_{res} . Then taking the sum over all parametric resonance bands [with various numbers n in (54)] we can write down the equation for the energy density ρ_χ of the scalar particles produced in the following form:

$$\begin{aligned} \dot{\rho}_\chi(\text{resonance production}) \\ = \sum_{\text{all resonance bands}} \frac{1}{2\pi^2} N_{\text{res}} \omega_{\text{res}}^3 \left| \frac{d\Delta\omega_k}{dt} \right|_{\Delta\omega_k=0}, \end{aligned} \quad (62)$$

where N_{res} is the maximal value of N_k , for the current resonance value of k , achieved after the corresponding mode has passed through the resonance band and has been amplified.

Under condition (34) particles produced will be ultra-relativistic and hence their contribution to the energy density will decrease due to the cosmological expansion. Taking this process also into account we are able to write down the following complete equation for the evolution of the energy density ρ_p of the particles produced (including the contribution from fermions):

$$\dot{\rho}_p = -4H\rho_p + (\Gamma_\psi + \Gamma_\chi^{(\text{res})})\rho, \quad (63)$$

where

$$\Gamma_\chi^{(\text{res})} = \frac{1}{2\pi^2\rho} \sum_{\text{all resonance bands}} N_{\text{res}} \omega_{\text{res}}^3 \left| \frac{d\Delta\omega_k}{dt} \right|_{\Delta\omega_k=0} \quad (64)$$

plays the role analogous to that of Γ_χ in (45). The values of $\Gamma_\chi^{(\text{res})}$ given by (64) and Γ_χ given by (47) are to be compared in order to make conclusions about the significance of the parametric resonance for the process of scalar particle production. In what follows we consider two important cases of the inflaton potential $V(\varphi)$, namely, quartic and quadratic potentials.

V. CASE OF POTENTIAL $V(\varphi) = \lambda\varphi^4$

In this case we have from (31)

$$\omega = c\sqrt{\lambda}\varphi_0, \quad (65)$$

where the constant c is given by (32) with $q = 4$. The resonance frequency ω_{res} is thus proportional to φ_0 and by virtue of (27) in the absence of back reaction of the particle production process on the scalar field φ we would have

$$\omega_{\text{res}} \propto a^{-1}. \quad (66)$$

²We must use the expression (B32) for the number of particles produced, because in our case the value μ_+ varies with time.

As we also have the condition (58) this would mean that the resonance amplification takes place for all times in the same wave bands, which satisfy the condition (54). However, due to the energy loss due to particle production, the scalar field amplitude φ_0 , and hence the frequency ω_{res} , will decrease more rapidly than given by (66). We see that in order to make correct estimates of particle production one has to take into account back reaction on the inflaton evolution.

Let us then consider back reaction of the particle production process on the evolution of the scalar field φ . Since particle production is a random quantum mechanical phenomenon, the change in the evolution of the scalar field φ will be random. We can picture the back reaction as leading to abrupt changes in the state of the scalar field at those random moments of time at which particles are produced. On sufficiently large time scales, on which many particles are produced, this change will exhibit regular features, and it is these features which we are to determine.

The evolution of the scalar field φ can be described by two parameters, namely, the scalar field energy density ρ and the frequency ω of its quasioscillations. In the absence of back reaction both these parameters are changing slowly with time, when compared to the time scale $T = 2\pi/\omega$ of the quasioscillations. The value ω is directly related to the energy density through the relation (31). We find it possible to describe the state of the scalar field in terms of these two parameters also if we take the back reaction mentioned above into account. Indeed, it is always possible to speak of the scalar field energy density. As far as its quasioscillation frequency is concerned, we note that its very existence is necessary for particle production, with rates given by (40) and (41), or with average numbers (59) in the case of parametric resonance.

Also note that on the quasioscillation time scale there can be but few particles produced (this fairly general statement follows from the standard analysis of transition amplitudes). Hence random moments of the abrupt changes in the scalar field state will be time separated on average by more than one scalar field quasioscillation period. Between these moments of abrupt changes the scalar field is evolving unperturbed. From this it follows that we can assume the validity of the relation (31) between the quasioscillation frequency ω and the energy density ρ .

To determine in the most simple way the changes in the evolution of the scalar field energy density ρ we can apply the energy conservation law. The energy is transferred from the scalar field φ to the particles produced. Hence, given the energy production rate (63) of the latter, we thereby know the energy loss of the scalar field φ . The equation for the inflaton energy density will then be

$$\dot{\rho} = -(4H + \Gamma_\psi + \Gamma_\chi^{(\text{res})})\rho, \quad (67)$$

where $\Gamma_\chi^{(\text{res})}$ given by (64). From Eq. (2) with $q = 4$ and from (15) it then follows

$$\dot{\varphi}_0 = -[H + \frac{1}{4}(\Gamma_\psi + \Gamma_\chi^{(\text{res})})]\varphi_0. \quad (68)$$

Now taking (58) and (65) into account we will obtain the following expression for the time derivative factor in (62):

$$\left| \frac{d\Delta\omega_k}{dt} \right|_{\Delta\omega_k=0} = \frac{1}{4}(\Gamma_\psi + \Gamma_\chi^{(\text{res})})\omega_{\text{res}}, \quad (69)$$

and from Eq. (64) we then obtain

$$\sum_{\text{all resonance bands}} N_{\text{res}}\omega_{\text{res}}^4 = 8\pi^2\rho \left(1 + \frac{\Gamma_\psi}{\Gamma_\chi^{(\text{res})}} \right)^{-1}. \quad (70)$$

It is very remarkable that in spite of all the details of the interaction between the fields φ and χ the maximal occupation number mean values depend through (70) only on the resonance frequencies ω_{res} in different resonance bands. This significant feature of the quartic inflaton potential makes the total analysis rather simple.

It remains to estimate the value $\Gamma_\chi^{(\text{res})}$. The expression for N_k is given by (59) where the integration in the argument of the hyperbolic sine is taken over the time during which resonance is taking place for the mode with the wave number k . In terms of the value $\Delta\omega_k = \omega_k - \omega_{\text{res}}$, the expression (60) for μ_+ can be rewritten as follows:

$$\mu_+ = \frac{1}{2\omega_{\text{res}}} \sqrt{|g_n|^2 - \Delta\omega_k^2(\Delta\omega_k + 2\omega_{\text{res}})^2}. \quad (71)$$

Then using (69) and taking into account the smallness of the value $|g_n|$ as compared to ω_{res}^2 [this follows from the relations (52) and (53)] we can estimate the integral in the exponent of (59) as follows:

$$\begin{aligned} \int \mu_+ dt &\approx \int_{-|g_n|/2\omega_{\text{res}}}^{|g_n|/2\omega_{\text{res}}} \frac{1}{2\omega_{\text{res}}} \sqrt{|g_n|^2 - \Delta\omega_k^2(2\omega_{\text{res}})^2} \\ &\quad \times \frac{4}{(\Gamma_\psi + \Gamma_\chi^{(\text{res})})\omega_{\text{res}}} d\Delta\omega_k \\ &\approx \frac{|g_n|^2}{(\Gamma_\psi + \Gamma_\chi^{(\text{res})})\omega_{\text{res}}^3} \int_{-1}^1 \sqrt{1-x^2} dx \\ &= \frac{\pi}{2} \frac{|g_n|^2}{(\Gamma_\psi + \Gamma_\chi^{(\text{res})})\omega_{\text{res}}^3}. \end{aligned} \quad (72)$$

In deriving (72) we considered slowly changing variables as constants. Then the expression (59) for N_{res} becomes

$$N_{\text{res}} \simeq \sinh^2 \left(\frac{\pi|g_n|^2}{2(\Gamma_\psi + \Gamma_\chi^{(\text{res})})\omega_{\text{res}}^3} \right). \quad (73)$$

Substituting this expression for N_{res} into (70) we obtain the following implicit expression for the value $\Gamma_\chi^{(\text{res})}$:

$$\begin{aligned} \sum_{\text{all resonance bands}} \omega_{\text{res}}^3 \sinh^2 \left(\frac{\pi|g_n|^2}{2(\Gamma_\psi + \Gamma_\chi^{(\text{res})})\omega_{\text{res}}^3} \right) \\ \simeq 8\pi^2\rho \left(1 + \frac{\Gamma_\psi}{\Gamma_\chi^{(\text{res})}} \right)^{-1}. \end{aligned} \quad (74)$$

Consider first the case when energy production of scalar particles dominates over that of fermions so that

$$\Gamma_\psi < \Gamma_\chi^{(\text{res})}. \quad (75)$$

Typically one of the terms, say of the n th resonance band dominates in the sum in the left-hand side of (74). In the case (75) $\Gamma_\chi^{(\text{res})}$ is determined by this dominant term as follows:

$$\Gamma_\chi^{(\text{res})} \simeq \frac{\pi |g_n|^2}{\omega_{\text{res}}^3} \ln^{-1} \left(\frac{32\pi^2 \rho}{\omega_{\text{res}}^4} \right). \quad (76)$$

Two of the Fourier harmonics of the function g given by (52) will be of appreciable value, namely,

$$|g_1| \simeq \sigma \varphi_0 \quad \left(\text{with } \omega_{\text{res}} = \frac{\omega}{2} \right) \quad (77)$$

and

$$|g_2| \simeq \frac{h\varphi_0^2}{2} \quad \left(\text{with } \omega_{\text{res}} = \omega \right). \quad (78)$$

If the contribution from $n=1$ is dominant in (74) (that is, if $\varphi_0 < \sigma/h$) then from (76) and (77) and using the expression (65) for ω and the explicit expression for the inflaton energy density $\rho = \lambda\varphi_0^4$, we get

$$\Gamma_\chi^{(\text{res})} \simeq \frac{8\pi}{c^3\lambda^{3/2}} \frac{\sigma^2}{\varphi_0} \ln^{-1} \left(\frac{512\pi^2}{c^4\lambda} \right) \quad \text{for } \varphi_0 < \frac{\sigma}{h}. \quad (79)$$

In this case the adiabaticity condition (61) reads

$$\varphi_0^2 \ll \frac{M_P \sigma}{\lambda}. \quad (80)$$

If the contribution from $n=2$ is dominant in (74) then from (76) and (78) we in a similar way obtain

$$\Gamma_\chi^{(\text{res})} \simeq \frac{\pi h^2}{4c^3\lambda^{3/2}} \varphi_0 \ln^{-1} \left(\frac{32\pi^2}{c^4\lambda} \right) \quad \text{for } \varphi_0 > \frac{\sigma}{h}. \quad (81)$$

In this case the adiabaticity condition (61) implies

$$\varphi_0 \ll M_P \frac{h}{\lambda}. \quad (82)$$

According to the note made at the end of the previous section, the value (79) and (81) for $\Gamma_\chi^{(\text{res})}$ are to be compared to the values of Γ_χ in the corresponding cases. The expression for Γ_χ is given by (47). Calculating the ratios of $\Gamma_\chi^{(\text{res})}$ to Γ_χ in the two cases discussed just above we obtain

$$\frac{\Gamma_\chi^{(\text{res})}}{\Gamma_\chi} \simeq \frac{32\pi^2}{c^2\lambda} \ln^{-1} \left(\frac{512\pi^2}{c^4\lambda} \right) \quad \text{for } \varphi_0 < \frac{\sigma}{h}, \quad (83)$$

$$\frac{\Gamma_\chi^{(\text{res})}}{\Gamma_\chi} \simeq \frac{\pi^2}{c^2\lambda} \ln^{-1} \left(\frac{32\pi^2}{c^4\lambda} \right) \quad \text{for } \varphi_0 > \frac{\sigma}{h}. \quad (84)$$

We see that these ratios look similar to two cases considered, and for typical values of the inflaton coupling constant [1]

$$\lambda \sim 10^{-12} \quad (85)$$

they are extremely large. Hence, for the inflaton field with the potential $\lambda\varphi^4$ with small λ , in the domain of applicability of perturbation theory [given by (44)], parametric resonance effects strongly dominate over the usual Born approximation estimates, so that the latter represent a serious underestimation of the particle production rate.

Let us proceed further and estimate the possible reheating temperature for the model considered. The equation for the energy density of the particles produced is (63). We should take into account also Eq. (67), modify Eq. (12) as follows:

$$H^2 = \frac{8\pi}{3M_P^2} (\rho + \rho_p), \quad (86)$$

and then solve the system of equations (63), (67), and (86) with initial condition $\rho_p = 0$ at the moment of the end of inflation. Then the maximal value achieved in the course of evolution by ρ_p would determine the reheating temperature.³ The system of equations mentioned can be solved exactly if we use the condition (75) to neglect Γ_ψ and the expressions (79) or (81) for $\Gamma_\chi^{(\text{res})}$. In the case (79) the solution will be

$$\rho = \rho_0 \left(1 - \frac{\Gamma_0}{12H_0} (e^{3\tau} - 1) \right)^4 e^{-4\tau}, \quad (87)$$

and in the case (81) the solution is

$$\rho = \rho_0 \left(1 + \frac{\Gamma_0}{4H_0} (e^\tau - 1) \right)^{-4} e^{-4\tau}, \quad (88)$$

where Γ_0 and H_0 are the initial values of $\Gamma_\chi^{(\text{res})}$ and H , respectively, ρ_0 and a_0 are the initial values of the total energy density and the scale factor, and $\tau = \ln(a/a_0)$. In both cases (87) and (88) the solution for ρ_p is given by

$$\rho_p = \rho_0 e^{-4\tau} - \rho. \quad (89)$$

Until the condition $\rho_p \lesssim \rho$ holds, the solutions (87) and (88) can be approximately written as

$$\rho \approx \rho_0 \left[1 - \frac{1}{12} \left(\frac{\Gamma_\chi^{(\text{res})}}{H} - \frac{\Gamma_0}{H_0} \right) \right]^4 e^{-4\tau}, \quad (90)$$

³The value of the reheating temperature T_R will depend also on the rates of particle decay and scattering, the processes which lead to eventual thermalization. If all scattering processes are strongly suppressed (this need not be the case since the self-coupling constant of the χ field is not necessarily small), then T_R could be many orders of magnitude smaller than what would be inferred from the maximal value of the energy density ρ_p . In this paper we do not discuss the theory of thermalization.

and

$$\rho \approx \rho_0 \left[1 + \frac{1}{4} \left(\frac{\Gamma_\chi^{(\text{res})}}{H} - \frac{\Gamma_0}{H_0} \right) \right]^{-4} e^{-4\tau}, \quad (91)$$

respectively.

The solutions obtained determine the maximal value achieved by ρ_p . Using the expressions (89)–(91) it is easy to see that this value is achieved as soon as the condition

$$\Gamma_\chi^{(\text{res})} \gtrsim 4H \quad (92)$$

becomes valid. After this particle creation proceeds very effectively and the inflaton field loses all its energy in less than one Hubble time. We shall therefore use the condition (92) to estimate the maximal value of ρ_p in various cases. First, assume that this value is achieved in the region of $\varphi_0 > \sigma/h$. Then using the expression (81) for $\Gamma_\chi^{(\text{res})}$ we obtain from (92)

$$\varphi_0 \lesssim 10^{-1} M_P \left(\frac{h}{\lambda} \right)^2 \ln^{-1} \frac{1}{\lambda}, \quad (93)$$

where we also used that λ is small to simplify the logarithmic factor. Now, the condition $\varphi_0 > \sigma/h$ assumed above will read

$$\sigma \lesssim 10^{-1} M_P h \left(\frac{h}{\lambda} \right)^2 \ln^{-1} \frac{1}{\lambda}. \quad (94)$$

Hence, if the condition (94) is valid, the value of the inflaton field amplitude φ_0 at the moment of maximal ρ_p is given by the right-hand side of (93). Note that the adiabaticity condition (82) is valid for the values of φ_0 given by (93) since the ratio h/λ has to be small according to the conditions (44).

If the condition (94) is not valid then the maximal value of ρ_p is achieved when φ_0 is in the region $\varphi_0 < \sigma/h$ and we must use the expression (79) for $\Gamma_\chi \varphi$. In this case from (92) we obtain the estimate

$$\varphi_0 \lesssim \frac{M_P^{1/3} \sigma^{2/3}}{\lambda^{2/3}} \ln^{-1/3} \frac{1}{\lambda}, \quad (95)$$

and the condition $\varphi_0 < \sigma/h$ would imply, as it should, the opposite of (94), namely,

$$\sigma \gtrsim 10^{-1} M_P h \left(\frac{h}{\lambda} \right)^2 \ln^{-1} \frac{1}{\lambda}. \quad (96)$$

In this case the last inequality together with (95) leads to the estimate

$$\varphi_0 \gtrsim 10^{-1} M_P \left(\frac{h}{\lambda} \right)^2 \ln^{-1} \frac{1}{\lambda} \quad (97)$$

for the boundary value of φ_0 at which effective particle creation starts.

Due to the conditions (44) the adiabaticity condition (80) can again be seen to be valid for the values (95) of φ_0 .

We see that the expression in the right-hand side of

(93) gives the minimum possible value for the inflaton field amplitude φ_0 at the moment of maximal ρ_p . We stress again that for the values of φ_0 obeying (93) the condition $4H \lesssim \Gamma_\chi^{(\text{res})}$ holds and reheating is extremely effective. To estimate the numerical value of the threshold (93), note that according to the conditions (44) (which we remember to be necessary in order that perturbation theory is valid) we have the condition $h \ll \lambda$. So taking for example

$$\frac{h}{\lambda} \simeq 10^{-1} \quad (98)$$

and taking into account (85) we will get from (93)

$$\varphi_0 \lesssim 10^{-4} M_P. \quad (99)$$

The maximal value of ρ_p is estimated from the condition that it is of the order of ρ at the moment when the condition (92) is first achieved. If we assume that thermalization of the particles created occurs soon after their energy density ρ_p achieves its maximum, then the reheating temperature T_{rh} is estimated using the relation

$$T_{\text{rh}} \simeq \rho_p^{1/4}. \quad (100)$$

Finally we obtain

$$T_{\text{rh}} \sim \lambda^{1/4} \varphi_0 \sim 10^{-1} M_P \lambda^{1/4} \left(\frac{h}{\lambda} \right)^2 \ln^{-1} \frac{1}{\lambda}. \quad (101)$$

In case of the values (85) and (98) for the couplings we get

$$T_{\text{rh}} \sim 10^{-7} M_P \simeq 10^{12} \text{ GeV}. \quad (102)$$

Actually, in the case of slow thermalization the reheating temperature is lower than given by (102).

Now let us see what our condition (75) means. In the region $\varphi_0 > \sigma/h$ using the expressions (46) and (81) we obtain the condition for the coupling f (again simplifying the logarithmic factor by using the smallness of the coupling constant λ),

$$f^2 < O(10) \left(\frac{h}{\lambda} \right)^2 \ln^{-1} \frac{1}{\lambda}, \quad (103)$$

which does not depend on the value of φ_0 . For the values (98) of the ratio h/λ the condition (103) is not very restrictive.

In the region $\varphi_0 < \sigma/h$ using expression (79) we obtain from (75) the estimate

$$f^2 < O(10^2) \frac{1}{\lambda^2} \left(\frac{\sigma}{\varphi_0} \right)^2 \ln^{-1} \frac{1}{\lambda}, \quad (104)$$

which depends on the value of φ_0 but certainly is satisfied for φ_0 sufficiently small. For the condition (104) to be valid for our estimate (95) of the inflaton field amplitude at the moment of maximal ρ_p , the coupling f is to be

$$f^2 < O(10^2) \left(\frac{\sigma}{\lambda M_P} \right)^{2/3} \ln^{-1/3} \frac{1}{\lambda}. \quad (105)$$

Thus, to conclude this part, if the value σ satisfies (94) and the coupling f satisfies (103) then the value of the inflaton field amplitude φ_0 at the moment of maximal value of ρ_p is given by the right-hand side of (93). If the value σ satisfies (96) and the coupling f satisfies (105) then the value of the inflaton field amplitude φ_0 at the same moment is given by the right-hand side of (95) and satisfies (97). In both cases the maximal reheating temperature is given by (101), with the numerical estimates (102) for the coupling values (85) and (98).

If neither of the two mutual conditions just mentioned holds then at the moment of maximal ρ_p the condition (75) is not valid, and the reheating temperature is determined by fermionic particle production. In this latter case at the moment of maximal ρ_p we have

$$\Gamma_\chi^{(\text{res})} < \Gamma_\psi, \quad (106)$$

and in the condition (92) we should replace the value $\Gamma_\chi^{(\text{res})}$ by Γ_ψ . Then using expression (46) for Γ_ψ we obtain for the value of the inflaton field amplitude φ_0 at the moment of effective particle production the well-known expression

$$\varphi_0 \lesssim 10^{-2} f^2 M_P, \quad (107)$$

and for the possible reheating temperature we get

$$T_{\text{rh}} \sim 10^{-2} \lambda^{1/4} f^2 M_P. \quad (108)$$

In the case (106) from Eq. (74) we obtain the following expression for $\Gamma_\chi^{(\text{res})}$:

$$\Gamma_\chi^{(\text{res})} \simeq \frac{\Gamma_\psi}{8\pi^2 \rho} \sum_{\text{all resonance bands}} \omega_{\text{res}}^4 \sinh^2 \left(\frac{\pi |g_n|^2}{2\Gamma_\psi \omega_{\text{res}}^3} \right). \quad (109)$$

It is useful to compare this value of $\Gamma_\chi^{(\text{res})}$ to the value of Γ_χ given by (47). To do this, note that mainly two resonance bands contribute to the sum (109), namely, those given by (77) and (78). Therefore we can write

$$\Gamma_\chi^{(\text{res})} = \Gamma_1^{(\text{res})}(\sigma) + \Gamma_2^{(\text{res})}(h), \quad (110)$$

where $\Gamma_1^{(\text{res})}$ depends only on the coupling σ and $\Gamma_2^{(\text{res})}$ only on the coupling h . These two terms in the sum (110) correspond to the two terms in the expression (47) for Γ_χ . We then rewrite the expression for Γ_χ in a similar way:

$$\Gamma_\chi = \Gamma_1(\sigma) + \Gamma_2(h). \quad (111)$$

Now using the expressions (46), (47), (65), (77), and (78) we can easily estimate the ratio of each of the two terms in (110) to its corresponding term in (111) as

$$\frac{\Gamma_n^{(\text{res})}}{\Gamma_n} \simeq \frac{1}{x_n} \sinh^2 x_n, \quad (112)$$

where

$$x_n = \frac{\pi |g_n|^2}{2\Gamma_\psi \omega_{\text{res}}^3}. \quad (113)$$

We can see that parametric resonance effects for the scalar field χ would be significant in the case (106) if the values of x_n given by (113) were larger than unity. This last condition, however, can be shown to be incompatible with the assumption that *fermionic* particle production dominates at the moment of efficient particle production. Hence at this moment the value Γ_χ given by (47) is larger than the resonance value $\Gamma_\chi^{(\text{res})}$ given under the conditions considered by (109).

All the results obtained in this section for the inflaton field amplitude during effective particle production can be written in a single equation

$$\varphi_0 \lesssim M_P \max \left\{ 10^{-2} f^2, 10^{-1} \left(\frac{h}{\lambda} \right)^2 \ln^{-1} \frac{1}{\lambda}, \left(\frac{\sigma}{M_P \lambda} \right)^{2/3} \ln^{-1/3} \frac{1}{\lambda} \right\}, \quad (114)$$

from which also all the necessary inequalities between coupling constants can be easily derived. We stress once again that for the values of φ_0 given by (114) the condition $4H < \Gamma_\psi + \Gamma_\chi^{(\text{res})}$ is valid so the particle production process is very effective.

The expression for the possible temperature of reheating is then

$$T_{\text{rh}} \sim \lambda^{1/4} \varphi_0 \sim M_P \lambda^{1/4} \max \left\{ 10^{-2} f^2, 10^{-1} \left(\frac{h}{\lambda} \right)^2 \ln^{-1} \frac{1}{\lambda}, \left(\frac{\sigma}{M_P \lambda} \right)^{2/3} \ln^{-1/3} \frac{1}{\lambda} \right\}. \quad (115)$$

It is very instructive to compare our results (114) and (115) to those which would be obtained by making use of the Born approximation (described in Sec. III) for the scalar particle χ production rates. If instead of $\Gamma_\chi^{(\text{res})}$ we used the value Γ_χ given by (47) then we would arrive at the following expressions:

$$\varphi_0 \lesssim M_P \max \left\{ 10^{-2} f^2, 10^{-2} \lambda \left(\frac{h}{\lambda} \right)^2, \lambda \left(\frac{\sigma}{M_P \lambda} \right)^{2/3} \right\} \quad (116)$$

and

$$T_{\text{rh}} \sim M_P \lambda^{1/4} \max \left\{ 10^{-2} f^2, 10^{-2} \lambda \left(\frac{h}{\lambda} \right)^2, \lambda \left(\frac{\sigma}{M_P \lambda} \right)^{2/3} \right\}. \quad (117)$$

One can see that for the scalar particle couplings the values (116) and (117) are smaller by an enormous factor $[\lambda \ln(1/\lambda)]^{-1}$ than the corresponding correct estimates (114) and (115).

VI. CASE OF POTENTIAL $V(\varphi) = \frac{1}{2} m^2 \varphi^2$

In this section the analysis will proceed along the same lines as in the previous one. In the case of a quadratic inflation potential we get from (31) and (32)

$$\omega = m, \quad (118)$$

so that inflaton field oscillates with constant frequency. The resonance frequency $\omega_{\text{res}} = n\omega/2$ will thus be constant in time. For the time derivative factor in (62) we then obtain, taking into account (58),

$$\left| \frac{d\Delta\omega_k}{dt} \right|_{\Delta\omega_k=0} = H\omega_{\text{res}}, \quad (119)$$

and from (64) we will have

$$\Gamma_{\chi}^{(\text{res})} = \frac{H}{2\pi^2 \rho} \sum_{\text{all resonance bands}} N_{\text{res}} \omega_{\text{res}}^4, \quad (120)$$

where all the relevant variables have been explained in Sec. IV. The expressions (59) and (71) remain valid in the case considered but the estimate analogous to (72) will be modified as follows:

$$\begin{aligned} \int \mu_+ dt &\approx \int_{-|g_n|/2\omega_{\text{res}}}^{|g_n|/2\omega_{\text{res}}} \frac{1}{2\omega_{\text{res}}} \sqrt{|g_n|^2 - \Delta\omega_k^2 (2\omega_{\text{res}})^2} \\ &\times \frac{1}{H\omega_{\text{res}}} d\Delta\omega_k \\ &\approx \frac{|g_n|^2}{4H\omega_{\text{res}}^3} \int_{-1}^1 \sqrt{1-x^2} dx = \frac{\pi}{8} \frac{|g_n|^2}{H\omega_{\text{res}}^3}, \end{aligned} \quad (121)$$

so that the occupation numbers N_{res} will be given by

$$N_{\text{res}} \simeq \sinh^2 \left(\frac{\pi}{8} \frac{|g_n|^2}{H\omega_{\text{res}}^3} \right), \quad (122)$$

and Eq. (120) for $\Gamma_{\chi}^{(\text{res})}$ will read

$$\Gamma_{\chi}^{(\text{res})} \simeq \frac{H}{2\pi^2 \rho} \sum_{\text{all resonance bands}} \omega_{\text{res}}^4 \sinh^2 \left(\frac{\pi}{8} \frac{|g_n|^2}{H\omega_{\text{res}}^3} \right). \quad (123)$$

We must compare this value to the value of Γ_{χ} obtained in the Born approximation. Writing $\Gamma_{\chi}^{(\text{res})}$ and Γ_{χ} in the form (110) and (111) and using the expressions (47), (77),

and (78), we will arrive at an estimate of the same form as (112) but with different x_n :

$$x_n = \frac{\pi}{8} \frac{|g_n|^2}{H\omega_{\text{res}}^3}. \quad (124)$$

Again we see that parametric resonance effects for the scalar field χ are significant if the values x_n are large.

As usual, one of the terms in the sum (123) will dominate. In the case $\varphi_0 < \sigma/h$ the harmonic with $n = 1$ dominates, whereas for $\varphi_0 > \sigma/h$ the harmonic corresponding to $n = 2$ is more important. The relevant expressions for the amplitudes in these two cases are given by (77) and (78).

In the case $\varphi_0 < \sigma/h$ we get

$$\Gamma_{\chi}^{(\text{res})} \simeq \frac{Hm^2}{16\pi^2 \varphi_0^2} \sinh^2 \left(\frac{\pi \sigma^2 \varphi_0^2}{Hm^3} \right), \quad (125)$$

and the condition of effective particle production (92) in this case will read

$$\left(\frac{\varphi_0}{m} \right)^2 \gtrsim \frac{Hm}{\pi \sigma^2} \ln \frac{Hm}{\sigma^2}. \quad (126)$$

As soon as particle creation begins the Hubble parameter depends on the total energy density of the inflaton field and of the particles created. So the value of the Hubble parameter definitely exceeds the expression given by (12) with only inflaton energy density taken into account. This enables us to obtain the lowest possible bound on the region of the effective particle production. If we substitute in (126) the lowest possible value for the Hubble parameter given by (12) we will get the result, up to a numerical factor of order 1,

$$\frac{\varphi_0}{m} \gtrsim \frac{m^3}{\sigma^2 M_P} \ln \frac{m^3}{\sigma^2 M_P}. \quad (127)$$

This estimate is valid only if the coupling constant σ is in the range given by

$$\left(\frac{m}{M_P} \right)^2 \ll \left(\frac{\sigma}{m} \right)^2 \ll \frac{m}{M_P}. \quad (128)$$

The first inequality in (128) is the condition for perturbation theory to be valid; it stems from (44). The second inequality in (128) is the criterion of large occupation numbers of the particles produced, that is, of effectiveness of the resonance particle production [in other terms, it is the condition for the value x_1 given by (124) to be larger than unity]. The validity of the adiabaticity condition (61) in this case follows from (127) and from the first inequality in (128).

In any case there is also another region in which the condition (92) holds so that particle production can be

effective. This region is given by

$$\frac{\varphi_0}{m} \lesssim \frac{\sigma^2 M_P}{m^3}, \quad (129)$$

and it can be obtained by using the first term in the expression (47) for the value Γ_χ since parametric resonance in this region is not effective.

In the case opposite to the one just considered, namely, when $\varphi_0 > \sigma/h$, we obtain

$$\Gamma_\chi^{(\text{res})} \simeq \frac{Hm^2}{\pi^2 \varphi_0^2} \sinh^2 \left(\frac{\pi h^2 \varphi_0^4}{32Hm^3} \right), \quad (130)$$

and the condition (92) will give the estimate

$$\left(\frac{\varphi_0}{m} \right)^4 \gtrsim \frac{32H}{\pi h^2 m} \ln \frac{H}{h^2 m}. \quad (131)$$

The lowest possible bound on the effective particle production region in this case will be obtained if we substitute the lowest possible value for H , given by (12), into the right-hand side. We will get

$$\left(\frac{\varphi_0}{m} \right)^3 \gtrsim \frac{65m}{h^2 M_P} \ln \frac{m}{h^2 M_P}. \quad (132)$$

The theory which led to the result (132) will be self-consistent if the coupling h is in the range

$$\left(\frac{m}{M_P} \right)^4 \ll h^2 \ll \frac{m}{M_P}. \quad (133)$$

Again, the first inequality in (133) is the requirement that perturbation theory be valid and the second one is the requirement of large occupation numbers of the particles produced. For the values of φ_0 (132) the adiabaticity condition (61) can be shown to hold.

We emphasize that the condition (92) in the case considered determines not the possible reheating temperature but rather the ‘‘freeze-out’’ boundary (127) or (132) for the inflaton field φ : below the values of φ_0 given by these expressions the particle creation process is ineffective and the inflaton field energy density decrease is dominated by the cosmological redshift rather than by particle production. The energy density of ultrarelativistic particles created decreases more rapidly than that of the inflaton field and the universe can well become again dominated by the inflaton field φ until the condition (129) is satisfied. If the couplings σ and h are extremely small so that

$$\left(\frac{\sigma}{m} \right)^2, h \ll \left(\frac{m}{M_P} \right)^2, \quad (134)$$

then the effective reheating condition (92) in the model with quadratic inflaton potential takes place only for the values of φ_0 given by (129). It is easy to see that in all the cases considered above the minimal possible ‘‘freeze-out’’ energy density ρ_{freeze} of the inflaton field is

$$\rho_{\text{freeze}} \sim m^4. \quad (135)$$

If during the particle creation process the energy density of the universe becomes dominated by the particles produced, then the estimates for the actual values of the inflaton field energy density at which it freezes out will be higher than those given by (127) and (132). Such a condition is likely to take place in the range of coupling constants given by (128) and (133) since, as discussed in [13], at the onset of the oscillation period of the inflaton, the particle production process may be very efficient. In this case one must use Eqs. (126) and (131), in which it should be taken into account that the Hubble parameter H is dominated by the contribution from the particles created and, hence, is much higher than given by Eq. (18). We note that in the range of couplings (128) and (133), at the moment of the onset of the inflaton oscillations, the perturbation theory developed in this paper breaks down. It becomes valid again when the amplitude φ_0 decreases sufficiently and the conditions (44) start to hold. The value of the Hubble parameter in Eqs. (126) and (131), however, cannot be higher than its value $H \sim m$ at the end of inflation. This allows one to estimate also the maximal possible boundaries for the inflaton field freezing out which will be [13]

$$\left(\frac{\varphi_0}{m} \right)^2 \sim \left(\frac{m}{\sigma} \right)^2 \ln \frac{m}{\sigma} \quad (136)$$

and

$$\left(\frac{\varphi_0}{m} \right)^4 \sim h^{-2} \ln h^{-2}. \quad (137)$$

These estimates are just on the boundary of the applicability of perturbation theory.

If interactions of the inflaton φ with fermions take place, we obtain, using Eq. (46), the estimate

$$\varphi_0 \leq 10^{-2} f^2 M_P \quad (138)$$

for the condition of effective particle production, which coincides with the analogous estimate (107) in the case of the potential $V(\varphi) = \lambda\varphi^4$ which we considered in the previous section. It can be shown, using the expression (31) for the value ω and the expression (46) for the value of Γ_ψ , that the estimate (138) does not depend on the power q of the inflaton potential $V(\varphi)$.

VII. DISCUSSION

In this paper we considered the problem of scalar particle production after inflation by a quickly oscillating inflaton field. We were using the framework of the chaotic inflation scenario with quartic and quadratic inflaton potentials, and we considered linear and quadratic coupling of the inflaton field φ to a quantum scalar field χ [see formula (28) for the interaction Lagrangian]. The particle production effect has been studied using the standard technique of Bogolyubov transformations. Specific attention has been paid to parametric resonance phenomena nomena which take place in the presence of a quickly oscillating inflaton field.

We have found that in the region of applicability of

perturbation theory [when inequalities (44) are valid] the effects of parametric resonance are crucial, and estimates based on the first-order Born approximation are often not correct.

In the case of the quartic inflaton potential $V(\varphi) = \lambda\varphi^4$, particle production is very efficient for any type of coupling between the inflaton field and the scalar field χ even for small values of coupling constants. The reheating temperature in this case is $[\lambda \ln(1/\lambda)]^{-1}$ times larger than the corresponding estimates based on the first-order Born approximation [see the expressions (115) and (117)].

In the case of a quadratic inflaton potential the situation is more complicated. The particle production process depends crucially on the type of coupling between the inflaton and the scalar field χ and on the magnitudes of the coupling constants σ and h . The theory predicts not only the possible reheating temperature but also the effective energy density of the inflaton field “freezing out” below which the inflaton field energy density decrease is dominated by the cosmological redshift rather than by particle production. The inflaton, so to say, decouples from the matter it creates. In the case of the quadratic inflaton potential we obtained the lowest [Eqs. (127) and (132)] and highest [Eqs. (136) and (137)] possible boundaries of the effective energy density of the inflaton field when it freezes out. With an interaction linear in the inflaton field, besides these possible freeze-out boundaries which exist for couplings in the range (128), there is also an effective particle production boundary (129). For an interaction quadratic in the inflaton field the lowest possible freeze-out boundary (132) exists in the range (133) of the values of coupling h . If the couplings σ and h are as small as to satisfy (134) then the effective particle production condition $\Gamma \gtrsim 4H$ is achieved only at the later stages after the condition (129) starts to hold.

The situation is more complex for more complicated potentials. For example, one may wish to consider the inflaton potential of shape

$$V(\varphi) = \lambda(\varphi^2 - \eta^2)^2, \quad (139)$$

with spontaneous symmetry breaking at a scale η . In the chaotic inflation scenario the inflaton field rolls from large values of $|\varphi|$ to the minimum of its potential. After inflation it starts quasiperiodic motion. For values of the amplitude $\varphi_0 \gg \eta$ the behavior of the inflaton is as for a $\lambda\varphi^4$ potential, and particle production proceeds as described in Sec. V. But as soon as the magnitude of φ_0 becomes close to η the dynamics changes and the inflaton field starts oscillating around one of the minima of its potential at $|\varphi| = \eta$. The inflaton potential in the vicinity of its minimum can be approximated as quadratic; hence particle production (freeze-out) will be described by the theory of Sec. VI.

In parallel to our own work, Kofman, Linde, and Starobinsky have also been working on these issues for several years. Their results were recently summarized in Ref. [13]. While our paper deals only with the case when the conditions (44) hold and perturbation theory in the inflaton couplings to the other fields is valid, both the

cases (44) and opposite to (44) are considered in [13]. The resonance frequency bandwidth can be shown to be relatively narrow in the case (44), and broad in the case opposite to (44), so these two cases are called in [13] “narrow” and “broad” resonance cases, respectively. In the case of broad resonance, higher resonance bands dominate the contribution to the particle production. After the inflaton field amplitude decreases sufficiently one enters into the regime of narrow resonance which can be accurately described by perturbation theory. The results announced in [13] for the case of narrow resonance in general agree with those obtained in our paper in Secs. V and VI. There are some results for the case of broad resonance which we would like to compare with our results for the narrow resonance case. In doing this we shall use the notation of the present paper which somewhat differs from that of [13].

In the case of quartic inflaton potential the estimate of [13] $N_k(t) \propto \exp(\frac{1}{5}\sqrt{\lambda}\varphi_0 t)$ for the case of coupling $h = 6\lambda$ (self-excitation of the inflaton, considered in [13]) can be compared with our estimate $N_k(t) \propto \exp(2\mu_+ t) \simeq \exp[(h/2\lambda)\sqrt{\lambda}\varphi_0 t]$ for the case of smaller coupling ($h \ll \lambda$). We note that these estimates will be numerically of the same order of magnitude if we choose our smaller coupling to be $h \simeq 2\lambda/5$. For the values of $h \simeq 0.22\lambda$ our estimate (93) for the beginning of reheating will be of the same order as the estimate of [13] $\varphi_0 \lesssim 0.005M_P \ln^{-1}(1/\lambda)$ for $h = 6\lambda$. These estimates show that the effect of other particle production for the values of coupling about $h \gtrsim 0.3\lambda$ may be larger than the effect of inflaton self-excitation. This value of h is on the border of the applicability of perturbation theory. We cannot say anything about the reheating with couplings $h \gtrsim \lambda$ as our perturbation theory breaks down in this case.

If the inflaton potential is quadratic then, as previously discussed by Kofman, Linde, and Starobinsky (see Ref. [13]), in the case of absence of inflaton coupling to fermions and without linear (in inflaton field) coupling to scalar particles, the inflaton field will eventually decouple from the rest of the matter, and the residual inflaton oscillations may provide the (cold) dark matter of the universe. Decoupling of the inflaton field occurs somewhere between the values given by the right-hand sides of (132) (lowest possible freeze-out boundary) and (137) (highest possible freeze-out boundary).

Our analysis has not taken into account several quantum effects which might be important. As stressed by Boyanovsky *et al.* [14], particle production will lead to corrections to the mass term of the inflaton. This correction enters both in the equation of motion for the inflaton and in the equation for fluctuations. For example, in a theory with inflaton potential

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{24}\varphi^4 \quad (140)$$

coupled to a second scalar field χ , then—calculated in the Hartree approximation—quantum corrections lead to the following correction to the inflaton mass (see, e.g., [14]):

$$m^2 \rightarrow m_R^2 + \left[\frac{\lambda_R}{2} [\langle \varphi^2(t) \rangle_R - \langle \varphi^2(0) \rangle_R] + \frac{1}{2} g^2 \langle \chi^2(t) \rangle_R \right], \quad (141)$$

where the subscript R indicates renormalized values. The additional mass terms are not negligible (in terms of both \hbar and coupling constant power counting) since they are of the same order as the source term driving parametric resonance.

We will postpone a careful study of these quantum effects to future work. Note, however, that the time scale on which the terms in (141) become important is the same on which the back reaction effects we already included play a role. For example, the effective frequency of inflaton oscillations changes as the inflaton amplitude decreases, and the time scale for this change is the same as the one on which the extra terms in (141) become dominant. Hence we do not expect our qualitative conclusions to change. In particular, the importance of parametric resonance for the energy transfer from the homogeneous part of the inflaton field to other modes will persist. Details like the precise time interval over which a given mode remains in a parametric resonance instability band might well change.

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APPENDIX A

In this appendix we present necessary expressions which describe the effect of parametric resonance in the lowest resonance band.

Consider the equation for the function $Y(t)$ of the following form

$$\ddot{Y} + [\omega_0^2 + \epsilon g(\omega t)]Y = 0, \quad (A1)$$

where ω_0 and ω are constant parameters, $g(x)$ is a 2π -periodic function, and ϵ is a small number. Our task is to find the approximate solution to Eq. (A1) in the first instability band of the frequencies ω_0 . To do this we will use the Bogolyubov method of averaging [15] which

implies perturbation theory of a certain type in the small parameter ϵ .

The function $g(x)$ being periodic, it can be developed in a Fourier series as follows:

$$g(x) = \sum_{n=-\infty}^{\infty} g_n e^{inx}, \quad (A2)$$

with the amplitudes g_n satisfying

$$g_n^* = g_{-n}. \quad (A3)$$

With no loss of generality (for a small parameter ϵ) we can put $g_0 = 0$ (redefining ω_0 if necessary). For the following purposes it will be convenient to introduce the phases α_n as follows:

$$g_n = |g_n| e^{i\alpha_n}. \quad (A4)$$

Let us introduce the value Δ by the following relation:

$$\omega_0^2 = \left(\frac{p}{q} \omega \right)^2 + \epsilon \Delta, \quad (A5)$$

where p/q is a rational noncontractible number (p and q are integers). Equation (A1) then becomes

$$\ddot{Y} + \left(\frac{p}{q} \omega \right)^2 Y = -\epsilon [g(\omega t) + \Delta] Y. \quad (A6)$$

The Bogolyubov method [15] consists in looking for a solution to Eq. (A6) in the form of the following expansion in powers of ϵ :

$$Y = a \cos \psi + \sum_{s=1}^{\infty} \epsilon^s u^{(s)} \left(a, \theta, \frac{\omega}{q} t \right), \quad (A7)$$

where

$$\psi = \frac{p}{q} \omega t + \theta, \quad (A8)$$

the $u^{(s)}$ are functions periodic in their second and third arguments, a is the amplitude, and θ is the phase of the solution (A7). The values of a and θ are not constant in the Bogolyubov approach. Rather, they are functions of time t .

We will be concerned explicitly only with the first-order approximation in the small parameter ϵ . For the function $u^{(1)}$ we have

$$u^{(1)} \left(a, \theta, \frac{\omega}{q} t \right) = \sum_{n=-\infty}^{\infty} u_n(a, \theta) e^{in\omega t/q}, \quad (A9)$$

with coefficients $u_n(a, \theta)$ periodic in θ .

Clearly, in the case of $\epsilon = 0$ the solution to (A6) will be just the first term in the right-hand side of (A7), with arbitrary constant amplitude a and phase θ , so in this case we have $\dot{a} = 0$, $\dot{\theta} = 0$. For $\epsilon \neq 0$ according to the Bogolyubov method we regard the values \dot{a} and $\dot{\theta}$ as functions of a and θ and we expand these functions in powers of ϵ :

$$\dot{a} = \epsilon A(a, \theta) + O(\epsilon^2) , \quad (\text{A10})$$

$$\dot{\theta} = \epsilon B(a, \theta) + O(\epsilon^2) .$$

Putting the ansatz (A7) into Eq. (A6) and using (A10), after collecting terms of first order in ϵ we will get the following equation for the function $u^{(1)}$:

$$\ddot{u}^{(1)} + \left(\frac{p}{q}\omega\right)^2 u^{(1)} = -(g + \Delta)a \cos \psi + 2\frac{p}{q}\omega A \sin \psi + 2\frac{p}{q}\omega B a \cos \psi . \quad (\text{A11})$$

Using the expansions (A2) and (A9) we can develop both sides of Eq. (A11) in harmonics $\exp(in\omega t/q)$. On the left-hand side of (A11) we will have, to the zeroth order in ϵ ,

$$\sum_{n=-\infty}^{\infty} \left[\left(\frac{p}{q}\right)^2 - \left(\frac{n}{q}\right)^2 \right] \omega^2 u_n e^{in\omega t/q} , \quad (\text{A12})$$

and we see that terms with $n = \pm p$ in this expansion are equal to zero. The corresponding terms in the development of the right-hand side of (A11) should also vanish. These conditions allow us to determine the functions $A(a, \theta)$ and $B(a, \theta)$. If we denote by s the number

$$s = \frac{2p}{q} , \quad (\text{A13})$$

then the expressions for these functions will look as follows:

$$A(a, \theta) = \frac{a|g_s|}{s\omega} \sin(2\theta - \alpha_s) , \quad (\text{A14})$$

$$B(a, \theta) = \frac{1}{s\omega} [\Delta + |g_s| \cos(2\theta - \alpha_s)] ,$$

with the phases α_s defined in (A4) for s integer. The expressions (A14) are valid for arbitrary s , in the case of s noninteger we should simply put g_s to zero, and the (unspecified) value of α_s is of no importance.

If we make the change of variables

$$x = a \cos\left(\theta - \frac{\alpha_s}{2}\right) , \quad (\text{A15})$$

$$y = a \sin\left(\theta - \frac{\alpha_s}{2}\right) ,$$

then the system (A10) in terms of the new variables x and y will look as follows:

$$\dot{x} = \frac{\epsilon}{s\omega} (|g_s| - \Delta)y + O(\epsilon^2) , \quad (\text{A16})$$

$$\dot{y} = \frac{\epsilon}{s\omega} (|g_s| + \Delta)y + O(\epsilon^2) .$$

To first order in ϵ the system (A16) is linear and its eigenvalues are

$$\mu_{\pm} = \pm \frac{\epsilon}{s\omega} \sqrt{|g_s|^2 - \Delta^2} . \quad (\text{A17})$$

From this expression one can see that μ_{\pm} can be real only if $g_s \neq 0$, that is, if s is integer. Hence unstable growth of the solution Y to Eq. (A6) can take place only for the values of Δ [Δ is defined in (A5)]

$$\Delta < |g_s| , \quad (\text{A18})$$

and the growth (decay) rate is exponential, $Y \sim \exp(\mu_{\pm} t)$, with μ_{\pm} given by (A17).

For every integer s for which g_s is nonzero the expression (A17) gives the growth rate of the solution Y in the first instability band determined by (A5) and (A18). There are also an infinite number of instability bands whose width and growth rates are higher order in ϵ (to all orders of ϵ). Relevant expressions can be obtained to any desirable order in ϵ by keeping track of the higher order terms in the expansions (A7) and (A10). For the purpose of this paper these higher order expressions are not required.

APPENDIX B

The aim of this appendix is to provide necessary details of the resonance particle production effect studied in Sec. III of this paper, in particular, to derive the formula (60). We first consider the general problem of a harmonic oscillator with time-dependent frequency, and after that the problem of resonant scalar particle production.

1. Harmonic oscillator with time-dependent frequency

Consider an oscillator with coordinate Q and conjugate momentum P , whose frequency Ω depends on time so that the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}[P^2 + \Omega^2(t)Q^2] . \quad (\text{B1})$$

For a quantum oscillator the values Q and P are Hermitian operators with standard commutation relations (we put $\hbar = 1$)

$$[Q, P] = i . \quad (\text{B2})$$

In the Schrödinger representation, these operators are time independent.

Our aim will be to go to a convenient time-dependent frame in Hilbert space, in which the Hamiltonian (B1) is diagonal at every moment of time. To do this define time-dependent operators $a(t)$ and $a^\dagger(t)$ by

$$a = \frac{e^{i \int \Omega dt}}{\sqrt{2\Omega}} (\Omega Q + iP) , \quad (\text{B3})$$

$$a^\dagger = \frac{e^{-i \int \Omega dt}}{\sqrt{2\Omega}} (\Omega Q - iP) .$$

These operators are mutually Hermitian conjugate and have the standard commutation relations of creation-

annihilation operators

$$[a, a^\dagger] = 1 . \quad (\text{B4})$$

In terms of these operators the Hamiltonian is expressed as follows:

$$\mathcal{H} = \Omega(\frac{1}{2} + a^\dagger a) . \quad (\text{B5})$$

The orthonormal frame in Hilbert space which diagonalizes the Hamiltonian \mathcal{H} is given by time-dependent states

$$|n_t\rangle = \frac{[a^\dagger(t)]^n}{\sqrt{n!}} |0_t\rangle , \quad (\text{B6})$$

constructed from the time-dependent vacuum state $|0_t\rangle$ which is annihilated by the operator $a(t)$.

The operators $a(t)$ and $a^\dagger(t)$ obey the following equations of motion:

$$\dot{a} = \frac{\dot{\Omega}}{2\Omega} e^{2i \int \Omega dt} a^\dagger , \quad (\text{B7})$$

$$\dot{a}^\dagger = \frac{\dot{\Omega}}{2\Omega} e^{-2i \int \Omega dt} a ,$$

so these operators vary with time only when Ω varies.⁴ The solution to the equations (B7) can be written in terms of constant creation and annihilation operators a_0^\dagger and a_0 as follows:

$$a(t) = \alpha(t)a_0 + \beta^*(t)a_0^\dagger , \quad (\text{B8})$$

$$a^\dagger(t) = \beta(t)a_0 + \alpha^*(t)a_0^\dagger ,$$

with the complex functions $\alpha(t)$ and $\beta(t)$ obeying the system of equations similar to (B7)

$$\dot{\alpha} = \frac{\dot{\Omega}}{2\Omega} e^{2i \int \Omega dt} \beta , \quad (\text{B9})$$

$$\dot{\beta} = \frac{\dot{\Omega}}{2\Omega} e^{-2i \int \Omega dt} \alpha .$$

The system (B8) represents what is called the Bogolyubov transformation between two pairs of creation-annihilation operators.

If the oscillator initially (at $t = 0$) is in the vacuum state then its state $|0_0\rangle$ is annihilated by the operator a_0 and the initial conditions for the functions $\alpha(t)$ and $\beta(t)$ are

$$|\alpha(0)| = 1, \quad \beta(0) = 0 . \quad (\text{B10})$$

At the moment t it will not be in the vacuum state $|0_t\rangle$, annihilated by the operator $a(t)$. Rather, there is the following relation between the states considered:

$$|0_0\rangle = \frac{1}{\sqrt{|\alpha(t)|}} \exp\left(\frac{\beta^*(t)}{2\alpha^*(t)} [a^\dagger(t)]^2\right) |0_t\rangle , \quad (\text{B11})$$

which follows from Eqs. (B8). At the moment t the average number of the oscillator excitation level (number of quanta) is

$$N(t) = \langle 0_0 | a^\dagger(t) a(t) | 0_0 \rangle = |\beta(t)|^2 . \quad (\text{B12})$$

In the Heisenberg representation, the equation of motion for the operator $Q(t)$ is

$$\ddot{Q} + \Omega^2 Q = 0 . \quad (\text{B13})$$

The solution to this equation can be expressed in terms of the operators a_0 and a_0^\dagger so that

$$Q(t) = Q^{(-)}(t)a_0 + Q^{(+)}(t)a_0^\dagger . \quad (\text{B14})$$

Comparing this expression with Eqs. (B3) and (B8) we can express the coefficient $\beta(t)$ in terms of $Q^{(-)}(t)$ at the moment of time at which the first time derivative of the frequency Ω vanishes as follows:

$$\beta = \frac{1}{\sqrt{2\Omega}} e^{-i \int \Omega dt} (\Omega Q^{(-)} - i \dot{Q}^{(-)}) . \quad (\text{B15})$$

2. Resonance particle production

Now let us turn to the issue of scalar particle creation in the external spatially homogeneous periodic field $\epsilon g(\omega t)$ whose properties were described in Appendix A [see Eqs. (A2)–(A4)]. Remember that ϵ is a small parameter of the perturbation theory. The scalar field operator χ which describes particles with mass m_χ can be decomposed into spatial Fourier modes

$$\chi = \int \frac{d^3 k}{(2\pi)^3} Q_k e^{ikx} , \quad (\text{B16})$$

so that every mode corresponds to a quantum oscillator with Hamiltonian (B1), complex coordinate $Q = Q_k$, and frequency $\Omega = \Omega_k$ given by

$$\Omega_k^2 = \omega_k^2 + \epsilon_g(\omega t) , \quad (\text{B17})$$

where $\omega_k^2 = m_\chi^2 + k^2$. Because the frequency Ω_k depends on time, the oscillator with label k will be excited and this means particle production. We can use all the above expressions obtained for the generic oscillator by simply adding a subscript k . Thus the theory of scalar particle production in the case under consideration is only a slight modification of the oscillator formalism described above. One only has to take into account correctly the presence of an infinite number of oscillators labeled by wave numbers k and interrelations between them. The expression analogous to (B11) relating the initial scalar field vacuum to the vacuum at the moment of time t will look as follows:

⁴This property is achieved by a convenient choice of the time-dependent phase factors in the definition (B3) of the operators a and a^\dagger .

$$|0_0\rangle = \prod_k \frac{1}{\sqrt{|\alpha_k(t)|}} \exp\left(\frac{\beta_k^*(t)}{2\alpha_k^*(t)} a_k^\dagger(t) a_{-k}^\dagger(t)\right) |0_t\rangle, \quad (\text{B18})$$

where $a_k^\dagger(t)$ and $a_k(t)$ are the time-dependent creation and annihilation operators which diagonalize the scalar field Hamiltonian at the moment t . From the expression (B18) it can be seen that scalar particles are created in pairs with opposite wave numbers k . The average number of particles produced at the moment t in the k th mode will be given by an expression similar to (B12) with the only modification being the addition of the subscript k . To determine the average number N_k of particles produced we can use the formulas (B12) and (B15). We only need to know the solution for $Q_k^{(-)}(t)$. This function obeys the equation

$$\ddot{Q}_k^{(-)} + [\omega_k^2 + \epsilon g(\omega t)] Q_k^{(-)} = 0, \quad (\text{B19})$$

which is just Eq. (A1) which we considered in Appendix A. Initial conditions for $Q_k^{(-)}$ stem from the initial vacuum conditions (B10) and are

$$|Q_k^{(-)}(0)| = \frac{1}{\sqrt{2\Omega_k(0)}}, \quad (\Omega_k Q_k^{(-)} - i\dot{Q}_k^{(-)})|_{t=0} = 0. \quad (\text{B20})$$

The solutions to Eq. (B19) in the resonance band were studied in Appendix A. According to (A5) we introduce the value Δ by

$$\omega_k^2 = \left(\frac{s}{2}\omega\right)^2 + \epsilon\Delta, \quad (\text{B21})$$

where we replaced p/q in advance by the half-integer $s/2$ according to the results of Appendix A. Now the solution to $Q_k^{(-)}$ will be given by a complex linear combination of solutions of type (A7), namely,

$$Q_k^{(-)} = \frac{1}{\sqrt{2\omega_k}} (a_1 \cos \psi_1 + i a_2 \cos \psi_2) + O(\epsilon), \quad (\text{B22})$$

with

$$\psi_{1,2} = \frac{s}{2}\omega t + \theta_{1,2}. \quad (\text{B23})$$

The initial conditions for $a_{1,2}$ and $\theta_{1,2}$ stem from (B20):

$$a_1 = a_2 = 1, \quad \theta_2 - \theta_1 = \frac{\pi}{2}. \quad (\text{B24})$$

If in analogy to (A15) we introduce the variables $x_{1,2}$ and $y_{1,2}$ as⁵

$$\begin{aligned} x_1 &= a_1 \cos \theta_1, & x_2 &= a_2 \cos \theta_2, \\ y_1 &= a_1 \sin \theta_1, & y_2 &= a_2 \sin \theta_2, \end{aligned} \quad (\text{B25})$$

then to first order in the small parameter ϵ the equations for $x_{1,2}$ and $y_{1,2}$ will be identical to (A16). The initial conditions for these functions follow from (B24) as

$$\begin{aligned} x_1(0) &= y_2(0), & x_2(0) &= -y_1(0), \\ |x_1|^2 + |y_1|^2 &= |x_2|^2 + |y_2|^2 = 1. \end{aligned} \quad (\text{B26})$$

To lowest order in ϵ , the expression for the value β_k is given by

$$\beta_k = \frac{1}{2}[x_1 + ix_2 + i(y_1 + iy_2)]. \quad (\text{B27})$$

It obeys the equation

$$\ddot{\beta}_k = \mu_+^2 \beta_k, \quad (\text{B28})$$

which follows from the system (A16), with μ_+ given by (A17). The initial conditions for the value β_k stem from (B26) and are

$$\beta_k(0) = 0, \quad |\dot{\beta}_k(0)| = \mu_+|_{\Delta=0}. \quad (\text{B29})$$

Solving the linear system (B28) for the function β_k with the initial conditions (B29) we obtain the expression for the mean number of particles produced:

$$N_k \equiv |\beta_k|^2 = \frac{1}{1 - \Delta^2/|g_s|^2} \sinh^2(\mu_+ t), \quad (\text{B30})$$

where as we recall, the value μ_+ is given by (A17).

The analysis developed in this appendix can also be used in the case when the value of μ_+ is not constant in time but changes adiabatically. Namely, if

$$|\dot{\mu}_+| \ll \mu_+^2, \quad (\text{B31})$$

then the mean number of particles created can be approximated by

$$N_k \simeq \sinh^2\left(\int \mu_+ dt\right), \quad (\text{B32})$$

where the integral in the argument of the hyperbolic sine is taken over the time during which resonance condition (A18) is satisfied for the mode with wave number k , and the prefactor similar to that of (B30) has been set to unity, which is reasonable because at the edge of the resonance wave band (when $\Delta \approx |g_s|$) the adiabaticity condition (B31) ceases to be valid.

⁵The constant value of the phase α_n defined in (A4) can be eliminated without loss of generality by a shift of the values of $\theta_{1,2}$.

- [1] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, Chur, Switzerland, 1990).
- [2] A. Starobinsky, *Phys. Lett.* **91B**, 99 (1980).
- [3] A. Dolgov and A. Linde, *Phys. Lett.* **116B**, 329 (1982).
- [4] L. Abbott, E. Farhi, and M. Wise, *Phys. Lett.* **117B**, 29 (1982).
- [5] A. Albrecht, P. Steinhardt, M. Turner, and F. Wilczek, *Phys. Rev. Lett.* **48**, 1437 (1982).
- [6] D. Nanopoulos, K. Olive, and M. Srednicki, *Phys. Lett.* **127B**, 30 (1983).
- [7] M. Morikawa and M. Sasaki, *Prog. Theor. Phys.* **72**, 782 (1984).
- [8] A. Hosoya and M. Sakagami, *Phys. Rev. D* **29**, 2228 (1984).
- [9] E. Kolb and M. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990).
- [10] J. Traschen and R. Brandenberger, *Phys. Rev. D* **42**, 2491 (1990).
- [11] A. Dolgov and D. Kirilova, *Sov. Nucl. Phys.* **51**, 273 (1990).
- [12] Y. Shtanov, *Ukr. Fiz. Zh.* **38**, 1425 (1993).
- [13] L. Kofman, A. Linde, and A. Starobinsky, *Phys. Rev. Lett.* **73**, 3195 (1994).
- [14] D. Boyanovsky *et al.*, *Phys. Rev. D* **51**, 4419 (1995).
- [15] N. Bogoliubov and Y. Mitropolsky, *Asymptotic Methods in the Theory of NonLinear Oscillations* (Hindustan Publishing, Delhi, 1961).