COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Strong renormalization scheme dependence in τ-lepton decay: Fact or fiction?

J. Chýla

*Institute of Physics, Academy of Sciences of Czech Republic, Prague, Czech Republic**

(Received 26 February 1993)

The question of the renormalization scheme dependence of the τ semileptonic decay rate is reexamined in response to a recent criticism. Particular attention is payed to a distinction between a consistent quantitative description of this dependence and the actual selection of a subset of "acceptable" renormalization schemes. It is pointed out that this criticism is valid only within a particular definition of the "strength" of the renormalization scheme dependence and should not discourage further attempts to use the semileptonic τ decay rate for quantitative tests of perturbative QCD.

PACS number(s): 12.38.Cy, 11.15.Bt, 13.35.Dx

In a recent article [1] Rączka has questioned the optimism of several earlier papers [2–4] concerning the possibility to use the semileptonic decay rate of the τ lepton and, namely, the quantity r_τ appearing in the measured ratio

$$R_\tau \equiv \frac{\Gamma(\tau^- \to \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \to \nu_\tau e^- \overline{\nu}_e)} = 3\,(1 + r_\tau) \qquad (1)$$

for an accurate determination of the basic QCD scale parameter $\Lambda_{\overline{\text{MS}}}$, where $\overline{\text{MS}}$ denotes the modified minimal subtraction scheme. Let me recall that this optimism is based on the following two favorable circumstances: (1) strong suppression of the potentially dangerous nonperturbative corrections [5, 6]; (2) decreasing renormalization scheme (RS) dependence at higher orders [2, 3].

The author of [1] claims, on the other hand, to have found for r_τ a strong RS dependence at both the next-to-leading (NLO) and next-to-next-to-leading (NNLO) orders. He thus comes to the conclusion that it is impossible to use this quantity for an accurate determination of $\Lambda_{\overline{\text{MS}}}$. He motivates his own analysis of the RS dependence by the observation that the previous ones [2, 3] have considered only some of the possible RS's, while "...for a full estimate of the RS dependence ambiguity one should compare predictions in all schemes which *a priori* seem to be admissible... ." The reason for writing this paper is that, although this observation is basically correct it, if taken literally, leads to complete arbitrariness of perturbative QCD predictions for *any* physical quantity. The τ decay rate certainly plays no special role in this respect. I shall, moreover, argue that the situation is not so hopeless and that one can formulate a plausible

strategy how to construct reasonably "accurate" perturbative QCD predictions for many measurable quantities, including the τ decay rate (1).

The whole discussion, ongoing for more than a decade, of the RS dependence problem does in fact boil down to the question of finding some physically motivated algorithm *restricting* this arbitrariness. There can be no true solution of this problem, merely more or less reasonable assumptions can be made on how to select one, in one way or another "optimal," RS [7–9], and how to measure the "strength" of the RS dependence, i.e., how to estimate the "theoretical" error associated with such a choice.

Before this can be attempted it is, however, necessary to have a consistent quantitative description of all the degrees of freedom associated with general renormalization group transformations. Choosing such a consistent labeling has nothing to do with the actual problem of "resolving" the RS dependence (in the above sense), but is an inevitable prerequisite for this latter step. There is no problem in this respect and there are various distinct, though in principle equivalent, ways how to do it. The lack of clear distinction between these two separate issues is, however, a frequent phenomenon and can be detected also in [1].

In the notation of [1] [with the exception of Q which is used below to denote a generic external momentum variable instead of m_τ, appropriate to (1)], I shall discuss QCD predictions for any physical quantity depending on Q and admitting perturbative expansion of the form (in massless QCD)

$$r(Q) = a(\mu)\left[1 + r_1(\mu/Q)a(\mu) + r_2(\mu/Q)a^2(\mu) + \cdots\right].$$

$$(2)$$

The renormalized couplant $a(\mu) \equiv g^2(\mu)/4\pi^2$ obeys the equation

*Postal address: Na Slovance 2, 180 40 Prague 8, Czech Republic.

<u>51</u>   5331

$$\frac{da(\mu)}{d\ln\mu} = -\beta_0 a^2 \left[1 + c_1 a(\mu) + c_2 a(\mu)^2 + \cdots\right], \qquad (3)$$

where the coefficients β_0 and c_1 are fixed by the number of quark flavors, while all the higher ones are essentially free, defining the so-called *renormalization convention* (RC), RC=$\{c_i, i \geq 2\}$. Although not written out explicitly, both the coefficients r_k and the couplant $a(\mu)$ [when (3) is considered to the kth order] do, however, depend on all $c_i, i \leq k$ as well. Finally, they also depend on the specification which of the infinite number of solutions to (3) we have in mind. Combining this last information with that on c_i defines what is called the *referential renormalization scheme* (RRS) (for detailed discussion of all these points see, for instance, [10]). Only if this RRS is fixed does the specification of the scale μ uniquely determine the RS. Instead of (2) we can write equivalently [again suppressing the dependence of r_k and $a(kQ)$ on the RRS]

$$r(Q) = a(kQ) \left[1 + r_1(k)a(kQ) + r_2(k)a^2(kQ) + \cdots\right], \qquad (4)$$

where the arbitrariness in μ has been traded for that of the the dimensionless parameter $k \in (0, \infty)$. As the essence of Rączka's claim concerns both the NLO and NNLO, I shall in the rest of this paper concentrate on the NLO case and furthermore set, purely for reasons of technical simplicity, $c_1 = 0$ in (3). All the following considerations hold, however, for the realistic values of c_1 as well.

Under these circumstances the solutions to (3) assume the particularly simple form

$$a(\mu, \text{RRS}) = \frac{1}{\beta_0 \ln(\mu/\Lambda_{\text{RRS}})}, \qquad (5)$$

where the parameter Λ_{RRS} uniquely specifies the solution to (3). In this way of labeling the RS the selection of the RRS is just a matter of bookkeeping and so it is commonly accepted to use the $\overline{\text{MS}}$ RRS for this purpose. Once this convention is made the scale μ, or the parameter k, can be used to label uniquely all the avaliable RS's. In view of the arbitrariness in the selection of the RRS, no absolute meaning can, however, be given to the scale μ or the parameter k and thus no arguments on the existence of a "natural" scale, given for instance by Q, are sufficient to fix the RS.

I have repeated these simple facts as the author of [1] starts his criticism of the "conventional" approach with a correct but obvious and well-known observation just made, i.e., that the same μ, or equivalently k, implies different r_i's and $a(\mu)$ in different RRS's. Varying k around unity, or in any other finite interval, in any single RRS is certainly not sufficient to take properly into account all the RS available at the NLO, but without this restriction on k any such RRS is equally suitable for a consistent labeling of all RS's.

It may, however, be preferable to use such a labeling of RS's which avoids the concept of RRS and uses just one variable to uniquely fix a RS. One way of doing this is to use for this purpose the value of r_1 itself. The internal

consistency of perturbation theory implies the following relation between μ, a, and r_1:

$$r_1 = \beta_0 \ln\left(\frac{\mu}{\Lambda_{\overline{\text{MS}}}}\right) - \rho = \frac{1}{a} - \rho \;\Rightarrow\; a = \frac{1}{r_1 + \rho} \qquad (6)$$

where ρ is the renormalization group invariant [7]. It is solely through this invariant that the Q dependence of $r(Q)$ enters (2). Substituting (6) into (2) and truncating it to the NLO we get

$$r^{\text{NLO}}(\rho, r_1) = \frac{2r_1 + \rho}{(r_1 + \rho)^2} \qquad (7)$$

as an explicit function of r_1 and ρ. In Fig. 1 the dependence of r^{NLO} on r_1 is displayed for several values of ρ [only those points for which (7) stays positive are plotted]. So far only the question of an exhaustive and consistent labeling of the RS's has been discussed. Now the crucial moment arrives and some point on each curve corresponding to a given ρ must be selected as our preferred RS, together with the interval of r_1's, specifying our definition of "acceptable" RS's. Looking at the curves in Fig. 1, we conclude that (i) there is one "exceptional" point on each of the curves, namely the stationary point, defining the principle of minimal sensitivity (PMS) [7] choice of the RS [in our approximation this point coincides with the effective charges approch (ECH) of [8], defined by the condition $r_1 = 0$], and (ii) although the curves are becoming less steep functions of r_1 as ρ increases, the pattern of r_1 dependence is qualitatively the same for all values of ρ. To give the word "strong" or "weak" RS dependence a quantitative content requires that we define the range of "acceptable" r_1. In [2] we have taken as our preferred RS the PMS and/or ECH one and furthermore suggested to estimate the associated "theoretical error" by the difference

$$\Delta^{\text{theory}} = r^{\text{NLO}}(\text{PMS}) - r^{\text{NLO}}(\overline{\text{MS}}), \qquad (8)$$

i.e., considered only the subset of all RS's corresponding to $r_1 \in (0, r_1(\overline{\text{MS}}))$. This is of course a somewhat arbitrary definition and we could certainly take some other interval of r_1. We, however, consider it meaningful to assume this definition of the theoretical "error" and use
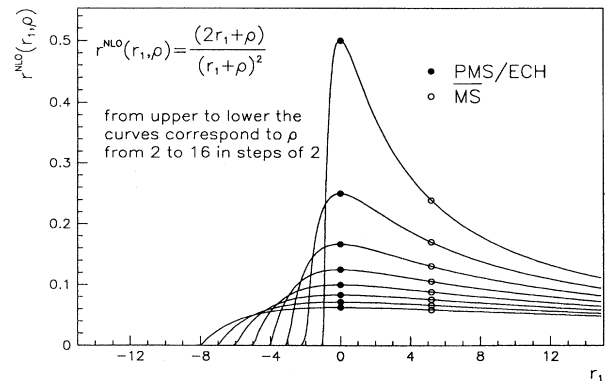

FIG. 1. $r_\tau^{\text{NLO}}(r_1, \rho)$ as a function of r_1 for several values of ρ.

it in analyses of all physical quantities for which the NLO and, if possible, also the NNLO, calculations are available. For $\Lambda_{\overline{\rm MS}}$ in the region of a few hundreds of MeV and taking into account that, for 3 quark flavors, $r_1(\mu = Q, \overline{\rm MS}) = 5.2$ the range of ρ appropriate to the τ decay rate (2) is roughly between 2 and 4. There, as is clear from two upper curves in Fig. 1, the RS dependence, i.e., the variation of the curves between $r_1 = 0$ and $r_1 = 5.2$ is quite large. What we have, however, found important is that when the same criterion is applied at the NNLO, the theoretical "error" as defined by (8) is reduced substantially. I stress the importance of comparing this kind of analysis at NLO and NNLO as only this comparison can tell us whether our choice of the RS leads to a reasonable convergence (in the pragmatical sense) as we proceed to higher orders and whether the associated "theoretical" error simultaneously decreases as it should if our procedure is sound. The results of [2, 10] are encouraging in both respects. After the τ-lepton decay rate and $e^+ e^-$ annihilation into hadrons [2] we have recently analyzed in the same way the Gross-Llewellyn-Smith sum rule [10] and intend to continue in this direction as further NNLO calculations become available.

The dependence of $r^{\rm NLO}$ on r_1 and ρ as given in (7) is quite general and holds for any physical quantity admitting perturbative expansion of the form (2). The difference between various physical quantities enters entirely through the corresponding values of ρ due to the fact that

$$\rho = \beta_0 \ln\left(\frac{Q}{\Lambda_{\overline{\rm MS}}}\right) - r_1(\mu = Q, \overline{\rm MS}) \tag{9}$$

contains both possible differences in the scale Q and in $r_1(\mu = Q, \overline{\rm MS})$.

In Fig. 2(a) the formula (7) is plotted as a function of ρ for a number of values of r_1. It can be redrawn (as in [1]) for any particular physical quantity expressible in the form (2) as a function of $\beta_0 \ln(Q/\Lambda_{\overline{\rm MS}})$ by simply shifting the origin of Fig. 2(a) by an appropriate value of r_1 and taking into account different value of external momentum Q. The content of Fig. 2(a), with curves corresponding to $r_1 \in (-3.83, 8.32)$ has been interpreted in [1] as an evidence for the strong RS dependence of the τ decay rate (2). This interval of r_1 values corresponds to the overlap between those obtained within the $\overline{\rm MS}$ RRS varying k in (4) between $1/2$ and 2 and between $1/3$ and 1 within the symmetric momentum-space subtraction RRS. No argument has, however, been put forward to justify the restriction to the above mentioned RRS and thus the selected interval of r_1 values must be considered as arbitrary as any other one. It must be born in mind that all values of r_1 are in principle equally acceptable and that even restricting the values of k in (4) to the region around unity we can always find such a RRS in which r_1 is equal to any prescribed value. Consequently, the conclusions drawn in [1] and based on this restriction of "acceptable" RS is valid only within this particular definition. Taking into account sufficiently large upper and lower bounds on r_1 would, of course, make any physical quantity "strongly" RS dependent.

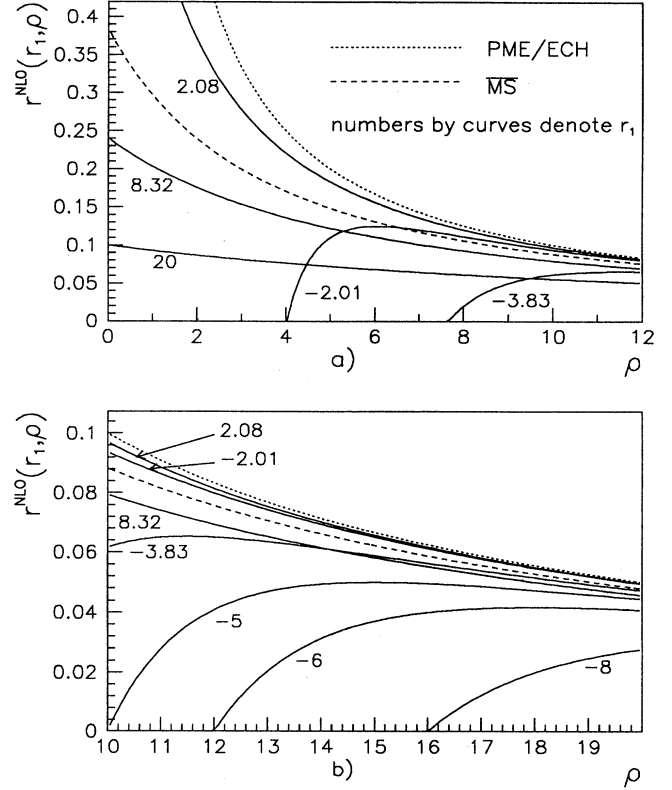To illustrate this last point let me consider the same

FIG. 2. $r_\tau^{\rm NLO}(r_1, \rho)$ as a function of ρ for several values of r_1 in the region appropriate to (a) the τ decay rate and (b) the R ration in $e^+ e^-$ annihilations.

expression (7) in the region $\rho \in (10, 20)$, which is roughly the range covered by the experiments at the DESY $e^+ e^-$ collider PETRA measuring the familiar R ratio

$$R_{e^+ e^-} \equiv \frac{\Gamma(e^+ e^- \to \text{hadrons})}{\Gamma(e^+ e^- \to \mu^+ \mu^-)} = 3(1 + r_{e^+ e^-}). \tag{10}$$

The quantity $r_{e^+ e^-}$ differs from (2), beside the range of ρ, only by the value of $r_1(\mu = Q, \overline{\rm MS}) = 1.41$ (for 5 quark flavors). In Fig. 2(b) the same curves as in Fig. 2(a) are plotted in this high ρ region [I continue to label as "$\overline{\rm MS}$" the curve corresponding to $r_1 = 5.2$ despite the fact that for (10) $r_1(\mu = Q, \overline{\rm MS}) = 1.41$ in order to maintain direct relation to Fig. 2(a)]. While the spread of the results corresponding to $r_1 \in (-3.83, 8.32)$ has decreased with respect to Fig. 2(a), extending this interval of "acceptable" r_1 just a little bit down to moderately negative values around -5 or -6, would imply "strong" RS dependence even for the R ratio (10). I do not think there is any argument why $r_1 = -3.83$ should be acceptable while $r_1 = -5$ should not. And even at energies reached at the CERN $e^+ e^-$ collider LEP (corresponding to ρ around 22) $r_1 = -11$ would be sufficient to yield $r_{e^+ e^-} = 0$.

When going beyond the NLO approximation, the measure of the strength of the RS dependence employed in [2, 10] has another advantage compared to the measure based on an *ad hoc* choice of the interval of r_1. There, new parameters, one at each further order, are needed to describe the full RS dependence. We can choose $r_i; i \geq 2$,

(or c_i; $i \geq 2$) for that purpose, but in any case if we attempt to define the "acceptable" RS by means of restrictions on these further parameters, new criteria have to be invented at each new higher order. For instance in [1] c_2 was considered in the interval $c_2 \in (-25, 25)$, with no particular reason given for these limits. In our approach, based on the selection of PMS and/or ECH and $\overline{\text{MS}}$ RS's, we do not face such problems as these criteria are reasonably (although not entirely) free of ambiguities.

Summarizing this paper, it is fair to say that the considerations presented above are just a recollection of well-known facts. But neither is there any really new idea or observation in [1]. There is no true solution to the RS dependence ambiguity, short of calculating the expansions to all orders. I have tried to stress the distinction between the task of a full and consistent description of this ambiguity and the actual problem of selecting one, or a subset, of the RS's. In this latter step some arbitrariness and subjective choice is inevitable. Bearing this in mind the conclusions of [1] have their validity only within a particularly defined measure of the strength of the RS dependence adopted by its author. They should certainly not discourage further attempts, both theoretical and experimental, to use the τ decay rate (1) as a potentially suitable place for quantitative tests of perturbative QCD.

[1] P. Rączka, Phys. Rev. D **23**, R3699 (1992).

[2] J. Chýla, A. Kataev, and S. Larin, Phys. Lett. B **267**, 269 (1991).

[3] F. Le Diberder and A. Pich, Phys. Lett. B **289**, 165 (1992).

[4] A. Pich, in *Proceedings of the XXVIIth Rancontre de Moriond*, Les Arcs, 1992, edited by Tran Than Van (Editions Frontieres, Gif-sur-Yvette, 1992).

[5] E. Braaten, Phys. Rev. Lett. **60**, 1606 (1988); **63**, 577 (1989).

[6] E. Braaten, S. Narison, and A. Pich, Nucl. Phys. **B373**, 581 (1992).

[7] P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).

[8] G. Grunberg, Phys. Rev. D **29**, 2315 (1984).

[9] S. Brodsky, G. P. Lepage, and P. Mackenzie, Phys. Rev. D **28**, 228 (1983).

[10] J. Chýla and A. Kataev, Phys. Lett. B **297**, 385 (1992).