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Determination of SU(6) Clebsch-Gordan coefficients and baryon mass and electromagnetic moment relations

Richard F. Lebed*

Department of Physics, University of California at San Diego, La Jolla, California 92093

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We develop a new method to compute and tabulate the Clebsch-Gordan coefficients of the SU(6) $\supset$ SU(3) $\times$ SU(2) product $\overline{\mathbf{56}} \otimes \mathbf{56}$, which are relevant to the nonrelativistic spin-flavor symmetry of the lightest baryons. Under the assumption that the largest representation in this product, the **2695**, gives rise to operators in a chiral expansion that produce numerically small effects, we obtain a set of relations among the masses of the baryons, as well as among their magnetic dipole and higher multipole moments. We compare the mass relations to experiment, and find numerical predictions for the Σ^0-Λ mass mixing parameter and 18 of the 27 magnetic moments in the **56**.

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I. INTRODUCTION

A generation ago, during the mid 1960s, the highly successful SU(3) model of light flavors developed by Gell-Mann and Ne'eman [1] was generalized to include the spin symmetry SU(2) in an enlarged spin-flavor symmetry group, SU(6) [2]. The increased predictive power of SU(6) over independent SU(3) $\times$ SU(2) symmetries immediately produced a number of intriguing results for the baryons, most notably the relative closeness of baryon octet and decuplet masses, the axial current coefficient ratio $F/D = 2/3$, and the famous magnetic moment ratio $\mu_p/\mu_n = -3/2$, which is experimentally true to 3%.

Yet two problems with the theory ultimately brought about its demise. The first was that mesons did not seem to fit as well as baryons into the theory; for example, why are the baryon octet and decuplet relatively close in mass, whereas the vector mesons are 2–5 times heavier than their pseudoscalar partners? Clearly SU(6) is somehow special to baryons. The other problem was much more serious, and in retrospect seems almost obvious: Mixing the compact, purely internal flavor symmetry with the noncompact Poincaré symmetry of spin angular momentum must and did ultimately lead to some nonsensical results. Such considerations gave rise to the various no-go theorems of the late 1960s, culminating in the celebrated Coleman-Mandula theorem [3], all forbidding such hybrid symmetries.

Nevertheless, there still exists the troubling matter of the μ_p/μ_n ratio and other baryonic "coincidences." Why should such good predictions exist? Although the no-go theorems tell us that SU(6) cannot be an exact symmetry of nature, there is nothing forbidding it from being a very good *approximate* symmetry. If this is the case, we may expect that a true symmetry of the universe generates predictions which are very similar to those of SU(6).

A promising candidate for such a symmetry is provided by large-N_c QCD [4]. It has recently been shown that the baryon sector of large-N_c QCD possesses a contracted spin-flavor symmetry [5–7] which is similar, but not identical, to the SU(6) spin-flavor symmetry. Results obtained from a consistent expansion in powers of $1/N_c$ allow one to explain certain results of chiral perturbation theory [which in turn relies on SU(3) symmetry] that are difficult to understand otherwise. It is a phenomenological fact that combinations of hadronic fields transforming under the largest representations of SU(3) or SU(6) tend to give rise to numerically small results, which is the origin of relations between hadron parameters. Often, but not always, this can be explained by the fact that the largest representations are accompanied by several powers of small chiral symmetry-breaking factors and are thus suppressed. In the large-N_c contracted spin-flavor symmetry, on the other hand, operators transforming under larger representations tend to be accompanied by more powers of $1/N_c$ [8]; thus we have a well-defined prescription for identifying theoretically suppressed combinations of baryonic parameters or, in other words, relations among the baryons.

It is therefore a highly relevant problem to analyze the group theory of the large-N_c contracted spin-flavor symmetry in order to find and test relations among baryon parameters, namely, masses, electromagnetic moments, and eventually decay widths and scattering amplitudes. Interesting new results have been obtained in this theory [7,9,10], but the full analysis has not yet been completed. It is also important to uncover, as is done in this work, the analogous relations within the related symmetry of SU(6), so that one can compare them to the large-N_c results [11]. A detailed comparison of the relationships between physical quantities ultimately helps us to determine how accurately each symmetry reflects reality.

In SU(6) the well-known octet and decuplet of baryons fill a single irreducible representation, the **56**; thus the operators we consider, bilinears in the baryon fields, are exactly those within the product of this representation with its conjugate. The associated Clebsch-Gordan co-

*Electronic address: rlebed@ucsd.edu

efficients for this and other SU(6) products appear in the literature [12], and were computed using the usual method of creation and destruction operators. In this paper we develop an alternate, comparatively simple and convenient method by which such group-theoretical factors may be generated. Once this is accomplished, we possess all possible information leading to relations among the baryons that depend only on SU(6) symmetry. We then need to decide only which product representations may be neglected in order to obtain the desired relations, and test their validity with experimental inputs.

This paper is organized as follows: In Sec. II, we begin with a discussion of SU(3) and its well-known Clebsch-Gordan coefficients, and how we may use them to build up the corresponding coefficients for SU(6). As a warm-up, we review the derivation of SU(3) mass relations using these coefficients in Sec. III. The purpose of Secs. II and III is pedagogical, to present the old SU(3) results in the language of pure group theory and to establish the notation we use for SU(6). We explain in Sec. IV the new method of computation of the SU(6) coefficients and their classification by additional SU(3) and isospin quantum numbers. Tables of the SU(6) Clebsch-Gordan coefficients, and the means by which relations are derived, are presented in Sec. V. The baryon relations for masses (equivalent forms of which have appeared previously in the literature) and magnetic dipole, electric quadrupole, and magnetic octupole moments (new to this work) are collected in Sec. VI, distinguished for the first time by their SU(3) and isospin content. We then use experimental values to evaluate these relations wherever possible, and estimate the size of neglected terms. Throughout this paper we stress which methods and results are old, and which are new to this work. We summarize our new results in Sec. VII.

II. SU(3) STRUCTURE OF BARYONS

We begin with a systematic classification of SU(3) representations (hereafter reps) of the octet and decuplet baryon field bilinears. This section is a new presentation of well-known results that serves to establish a simple common notation for describing group-theoretical quantities. Consider, within the effective Lagrangian, any term connecting single initial and final baryons respectively transforming under R_1- and R_2-dimensional reps:

$$\delta\mathcal{L} = \overline{R_2}\mathcal{O}R_1, \tag{1}$$

where $\mathcal{O}$ is some operator. The pattern of SU(3) breaking by this term is exhibited by the decomposition of $(\overline{R_2} \times R_1)$ into combinations transforming under all possible irreducible reps. For the octet and decuplet, these reps are

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8}_1 \oplus \mathbf{8}_2 \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}, \tag{2}$$

$$\mathbf{8} \otimes \mathbf{10} = \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{27} \oplus \mathbf{35} \tag{3}$$

(and its conjugate form $\overline{\mathbf{10}} \otimes \mathbf{8}$), and

$$\overline{\mathbf{10}} \otimes \mathbf{10} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{27} \oplus \mathbf{64}. \tag{4}$$

The projections of $\mathcal{O}$ forming the coefficients of these combinations can be labeled with the SU(3) indices of the corresponding bilinear combinations. We may then loosely speak of $\mathcal{O}$ as transforming under some rep, although in fact only the baryon field bilinears transform. This analysis is, of course, not restricted to SU(3); its verity relies only on negligible mixing from heavier states possessing the same quantum numbers.

A restriction we now place on the baryon terms in the Lagrangian is that they originate only in the strong and electromagnetic but not the weak interactions. That is, we consider only bilinears that conserve strangeness as well as electric charge or, equivalently, those with the properties $\Delta I_3 = 0$ and $\Delta Y = 0$. Note that these include "mixing" terms for any states with the same values of I_3 and Y; every octet state mixes with exactly one decuplet state, and within the octet, Σ^0-Λ mixing can occur.

It remains only to distinguish degenerate $\Delta I_3 = \Delta Y = 0$ operators within a rep. As usual, we assume the standard notation of labeling with the isospin Casimir invariant $I(I+1)$, so that x_I^R (where x is a generic coefficient name) specifies a unique "chiral coefficient" within the rep $\mathbf{R}$. It then becomes a straightforward exercise with the well-known SU(3) Clebsch-Gordan coefficients (see, e.g., Ref. [13]) to decompose bilinear terms into the forms

$$\begin{aligned} \mathbf{M}_a &= \mathcal{C}_a\mathbf{a}, \\ \mathbf{M}_b &= \mathcal{C}_b\mathbf{b}, \\ \mathbf{M}_c &= \mathcal{C}_c\mathbf{c}, \\ \mathbf{M}_{\bar{c}} &= \mathcal{C}_{\bar{c}}\bar{\mathbf{c}}, \end{aligned}$$

where

$$\mathbf{M}_a \equiv \begin{pmatrix} \bar{p}p \\ \bar{n}n \\ \overline{\Sigma^+}\Sigma^+ \\ \overline{\Sigma^0}\Sigma^0 \\ \bar{\Lambda}\Lambda \\ \overline{\Sigma^-}\Sigma^- \\ \overline{\Xi^0}\Xi^0 \\ \overline{\Xi^-}\Xi^- \\ \overline{\Sigma^0}\Lambda \\ \bar{\Lambda}\Sigma^0 \end{pmatrix}, \quad \mathbf{M}_b \equiv \begin{pmatrix} \overline{\Delta^{++}}\Delta^{++} \\ \overline{\Delta^+}\Delta^+ \\ \overline{\Delta^0}\Delta^0 \\ \overline{\Delta^-}\Delta^- \\ \overline{\Sigma^{*+}}\Sigma^{*+} \\ \overline{\Sigma^{*0}}\Sigma^{*0} \\ \overline{\Sigma^{*-}}\Sigma^{*-} \\ \overline{\Xi^{*0}}\Xi^{*0} \\ \overline{\Xi^{*-}}\Xi^{*-} \\ \overline{\Omega^-}\Omega^- \end{pmatrix}, \quad \mathbf{M}_c \equiv \begin{pmatrix} \bar{p}\Delta^+ \\ \bar{n}\Delta^0 \\ \overline{\Sigma^+}\Sigma^{*+} \\ \overline{\Sigma^0}\Sigma^{*0} \\ \bar{\Lambda}\Sigma^{*0} \\ \overline{\Sigma^-}\Sigma^{*-} \\ \overline{\Xi^0}\Xi^{*0} \\ \overline{\Xi^-}\Xi^{*-} \end{pmatrix}, \quad \mathbf{M}_{\bar{c}} \equiv \begin{pmatrix} \overline{\Delta^+}p \\ \overline{\Delta^0}n \\ \overline{\Sigma^{*+}}\Sigma^+ \\ \overline{\Sigma^{*0}}\Sigma^0 \\ \overline{\Sigma^{*0}}\Lambda \\ \overline{\Sigma^{*-}}\Sigma^- \\ \overline{\Xi^{*0}}\Xi^0 \\ \overline{\Xi^{*-}}\Xi^- \end{pmatrix},$$

$$
\mathcal{C}_a = \begin{pmatrix}
+\frac{1}{2\sqrt{2}} & +\frac{1}{2\sqrt{5}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{1}{2} & +\frac{1}{2\sqrt{3}} & +\frac{1}{2\sqrt{3}} & +\frac{1}{2\sqrt{3}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & +\frac{1}{\sqrt{10}} & 0 \\
+\frac{1}{2\sqrt{2}} & +\frac{1}{2\sqrt{5}} & +\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{10}} & 0 \\
+\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{5}} & 0 & 0 & +\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{30}} & 0 & +\frac{1}{\sqrt{6}} \\
+\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{30}} & 0 & -\sqrt{\frac{2}{3}} \\
+\frac{1}{2\sqrt{2}} & +\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2}\sqrt{\frac{3}{10}} & 0 & 0 \\
+\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{5}} & 0 & 0 & -\frac{1}{\sqrt{3}} & +\frac{1}{2\sqrt{3}} & +\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{30}} & 0 & +\frac{1}{\sqrt{6}} \\
+\frac{1}{2\sqrt{2}} & +\frac{1}{2\sqrt{5}} & +\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{2} & +\frac{1}{2\sqrt{3}} & +\frac{1}{2\sqrt{3}} & +\frac{1}{2\sqrt{3}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{10}} & 0 \\
+\frac{1}{2\sqrt{2}} & +\frac{1}{2\sqrt{5}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & +\frac{1}{\sqrt{10}} & 0 \\
0 & 0 & +\frac{1}{\sqrt{5}} & 0 & 0 & +\frac{1}{2} & -\frac{1}{2} & 0 & +\sqrt{\frac{3}{10}} & 0 \\
0 & 0 & +\frac{1}{\sqrt{5}} & 0 & 0 & -\frac{1}{2} & +\frac{1}{2} & 0 & +\sqrt{\frac{3}{10}} & 0
\end{pmatrix},
$$

$$
\mathcal{C}_b = \begin{pmatrix}
+\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{10}} & +\sqrt{\frac{3}{10}} & +\sqrt{\frac{3}{70}} & +\frac{3}{\sqrt{70}} & +\sqrt{\frac{3}{14}} & +\frac{1}{2\sqrt{35}} & +\frac{1}{2}\sqrt{\frac{3}{35}} & +\frac{1}{2\sqrt{7}} & +\frac{1}{2\sqrt{5}} \\
+\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{30}} & +\sqrt{\frac{3}{70}} & +\frac{1}{\sqrt{70}} & -\sqrt{\frac{3}{14}} & +\frac{1}{2\sqrt{35}} & +\frac{1}{2\sqrt{105}} & -\frac{1}{2\sqrt{7}} & -\frac{3}{2\sqrt{5}} \\
+\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{30}} & +\sqrt{\frac{3}{70}} & -\frac{1}{\sqrt{70}} & -\sqrt{\frac{3}{14}} & +\frac{1}{2\sqrt{35}} & -\frac{1}{2\sqrt{105}} & -\frac{1}{2\sqrt{7}} & +\frac{3}{2\sqrt{5}} \\
+\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{10}} & -\sqrt{\frac{3}{10}} & +\sqrt{\frac{3}{70}} & -\frac{3}{\sqrt{70}} & +\sqrt{\frac{3}{14}} & +\frac{1}{2\sqrt{35}} & -\frac{1}{2}\sqrt{\frac{3}{35}} & +\frac{1}{2\sqrt{7}} & -\frac{1}{2\sqrt{5}} \\
+\frac{1}{\sqrt{10}} & 0 & +\sqrt{\frac{2}{15}} & -\sqrt{\frac{5}{42}} & -\frac{3}{\sqrt{70}} & +\frac{1}{\sqrt{42}} & -\frac{2}{\sqrt{35}} & -\sqrt{\frac{5}{21}} & -\frac{1}{\sqrt{7}} & 0 \\
+\frac{1}{\sqrt{10}} & 0 & 0 & -\sqrt{\frac{5}{42}} & 0 & -\sqrt{\frac{2}{21}} & -\frac{2}{\sqrt{35}} & 0 & +\frac{2}{\sqrt{7}} & 0 \\
+\frac{1}{\sqrt{10}} & 0 & -\sqrt{\frac{2}{15}} & -\sqrt{\frac{5}{42}} & +\frac{3}{\sqrt{70}} & +\frac{1}{\sqrt{42}} & -\frac{2}{\sqrt{35}} & +\sqrt{\frac{5}{21}} & -\frac{1}{\sqrt{7}} & 0 \\
+\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{30}} & -\sqrt{\frac{3}{70}} & -\frac{2\sqrt{2}}{\sqrt{35}} & 0 & +\frac{3}{\sqrt{35}} & +\sqrt{\frac{5}{21}} & 0 & 0 \\
+\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{30}} & -\sqrt{\frac{3}{70}} & +\frac{2\sqrt{2}}{\sqrt{35}} & 0 & +\frac{3}{\sqrt{35}} & -\sqrt{\frac{5}{21}} & 0 & 0 \\
+\frac{1}{\sqrt{10}} & -\sqrt{\frac{2}{5}} & 0 & +\frac{3\sqrt{3}}{\sqrt{70}} & 0 & 0 & -\frac{2}{\sqrt{35}} & 0 & 0 & 0
\end{pmatrix},
$$

$$
\mathcal{C}_c = \begin{pmatrix}
0 & +\frac{2}{\sqrt{15}} & +\frac{1}{\sqrt{6}} & 0 & +\frac{1}{2\sqrt{10}} & +\frac{\sqrt{3}}{2\sqrt{2}} & +\frac{1}{2\sqrt{6}} & +\frac{1}{2\sqrt{2}} \\
0 & +\frac{2}{\sqrt{15}} & +\frac{1}{\sqrt{6}} & 0 & +\frac{1}{2\sqrt{10}} & -\frac{\sqrt{3}}{2\sqrt{2}} & +\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}} \\
+\frac{1}{\sqrt{5}} & +\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{6}} & +\sqrt{\frac{2}{15}} & +\frac{3}{2\sqrt{10}} & +\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}} \\
+\frac{1}{\sqrt{5}} & 0 & 0 & +\sqrt{\frac{2}{15}} & 0 & -\frac{1}{\sqrt{6}} & 0 & +\frac{1}{\sqrt{2}} \\
0 & +\frac{1}{\sqrt{5}} & 0 & 0 & -\sqrt{\frac{3}{10}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
+\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{15}} & +\frac{1}{\sqrt{6}} & +\sqrt{\frac{2}{15}} & -\frac{3}{2\sqrt{10}} & +\frac{1}{2\sqrt{6}} & +\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{2}} \\
+\frac{1}{\sqrt{5}} & +\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{10}} & -\frac{1}{\sqrt{10}} & 0 & +\frac{1}{\sqrt{6}} & 0 \\
+\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{15}} & +\frac{1}{\sqrt{6}} & -\sqrt{\frac{3}{10}} & +\frac{1}{\sqrt{10}} & 0 & -\frac{1}{\sqrt{6}} & 0
\end{pmatrix},
$$

with $\mathcal{C}_{\bar{c}} = \mathcal{C}_c$, and

$$\mathbf{a}\equiv\begin{pmatrix} a_0^1 \\ a_0^{8_1} \\ a_1^{8_1} \\ a_0^{8_2} \\ a_1^{8_2} \\ a_1^{10} \\ a_1^{\overline{10}} \\ a_0^{27} \\ a_1^{27} \\ a_2^{27} \end{pmatrix}, \quad \mathbf{b}\equiv\begin{pmatrix} b_0^1 \\ b_0^8 \\ b_1^8 \\ b_0^{27} \\ b_1^{27} \\ b_2^{27} \\ b_0^{64} \\ b_1^{64} \\ b_2^{64} \\ b_3^{64} \end{pmatrix}, \quad \mathbf{c}\equiv\begin{pmatrix} c_0^8 \\ c_1^8 \\ c_1^{10} \\ c_0^{27} \\ c_1^{27} \\ c_2^{27} \\ c_1^{35} \\ c_2^{35} \end{pmatrix}, \quad \bar{\mathbf{c}}\equiv\begin{pmatrix} \bar{c}_0^8 \\ \bar{c}_1^8 \\ \bar{c}_1^{\overline{10}} \\ \bar{c}_0^{27} \\ \bar{c}_1^{27} \\ \bar{c}_2^{27} \\ \bar{c}_1^{\overline{35}} \\ \bar{c}_2^{\overline{35}} \end{pmatrix}. \tag{5}$$

Here the $\mathbf{8}\otimes\mathbf{8}$ reps $\mathbf{8}_{1,2}$ are distinguished by the symmetry properties of their components under reflection through the origin in weight space (i.e., exchanging the component transforming with quantum numbers (I,I_3,Y) with that transforming under $(I,-I_3,-Y)$). $\mathbf{8}_{1,2}$ is symmetric (antisymmetric) under this exchange, giving, for instance, the same (opposite) contributions to the bilinears of the p and Ξ^-.

With the above normalization of the chiral coefficients a, b, c, and $\bar{c}$, the matrices $\mathcal{C}$ are orthogonal. This, of course, must be the the case, for we are merely describing the bilinears in a different basis. Because the matrices are orthogonal, we may alter the sign of any row or column and still maintain orthogonality. The phase conventions exhibited above have been chosen ultimately to match well-known quark-model results; for example, each octet term has the same singlet coefficient $a_0^1/2\sqrt{2}$. We are thus fixing the phases of the *lowest*-weight reps, the direct opposite of the usual Condon-Shortley convention.

It is easy to understand the number of chiral coefficients appearing in the octet and decuplet products. With arbitrary SU(3) breaking, one may clearly supply each bilinear with a distinct arbitrary coefficient; hence the decuplet product must have ten chiral coefficients, the decuplet-octet product eight, and the octet product ten, because the octet supports Σ^0-Λ mixing. But such mixing requires only one parameter, a mixing angle θ. In the above matrices there are two, corresponding to the bilinears $\overline{\Sigma^0}\Lambda$ and $\overline{\Lambda}\Sigma^0$. However, Hermiticity (or time-reversal invariance) of the Lagrangian reduces these to one, imposing the physical constraint $a_1^{10}=-a_1^{\overline{10}}$. Later we find a similar constraint between c_I^R and $\bar{c}_I^{\bar{R}}$.

Complete knowledge of the SU(3) group-theoretical factors already tells us a great deal about the corresponding factors for SU(6), for the quantum numbers of the latter symmetry group are assigned via the decomposition SU(6) $\supset$ SU(3) $\times$ SU(2), and the flavor and spin groups commute. Thus a chiral coefficient of any rep $\mathbf{N}$ of SU(6), distinguished by its decomposition into an R-dimensional rep of SU(3) and isospin I (henceforth denoted by $d_N^{R,I}$) must be some linear combination of all existing chiral coefficients a_I^R, b_I^R, c_I^R, and $\bar{c}_I^R$. For example, because spin and flavor commute, the bilinear combination $a_1^{8_1}$ still transforms as the $I=1$ component of an octet regardless of how we insert spins on the baryon indices. Thus, since the combinations $a_1^{8_1}$, $a_1^{8_2}$, b_1^8, c_1^8, and $\bar{c}_1^8$ span the entire subspace of $I=1$ octets formed from the baryon octet and decuplet bilinears, each $d_N^{8,1}$ must be a linear combination of these.

III. EXAMPLE: SU(3) BARYON MASS RELATIONS

As a preliminary to SU(6), let us consider how to obtain relations between baryons using only SU(3) group theory. Again, these derivations must ultimately be the same as the original ones (since both rely on group theory alone), but are designed to express the old results in a new and clearer way and pave the way for the SU(6) results. Because SU(3) multiplets take into account only flavor symmetry, we do not expect to learn anything about quantities in which the individual spin states are important (e.g., magnetic moment relations). However, we can learn about the masses. First we assume that mixing between multiplets is negligible, so that the physical baryons truly live in octet and decuplet reps of SU(3). In the usual chiral Lagrangian, SU(3) breaking is accomplished by an expansion in the quark mass (M_q) and charge (Q_q) operators; in terms of flavor indices, these are 3×3 matrices (with u, d, and s diagonal entries), and such operators X may be decomposed into octet $[X-\frac{1}{3}(\mathrm{Tr}X)\mathbf{1}]$ and singlet $[(\mathrm{Tr}X)\mathbf{1}]$ portions. At first order in SU(3) breaking only these singlet and octet operators are present; at second order, operators in the reps of Eq. (2) appear.

An important reason for the success of the chiral Lagrangian formalism is that the operators M_q and Q_q enter into the Lagrangian as operators with perturbatively small coefficients; in the case of M_q, contributions are suppressed by at least $m_s/\Lambda_\chi\approx0.2$, where m_s is the strange quark mass, and Λ_χ is the chiral symmetry-breaking scale. Terms involving only $m_{u,d}$ are suppressed by another factor of 20 or so. For Q_q, the suppression comes through powers of $e\approx0.3$, although in mass relations, charge conjugation symmetry of the strong and electromagnetic interactions permits factors of Q_q only in even numbers; there is a further suppressions of $16\pi^2$ because such mass terms come from photon loop effects in the QCD Lagrangian. Thus the true suppression is by $\alpha/4\pi\approx6\times10^{-4}$.

So now we can see explicitly why the coefficients associated with the largest reps are suppressed: Larger reps require more powers of the small symmetry-breaking reps, which in turn bring in more numerical suppressions.

Let us consider some examples, first supposing that splittings within isospin multiplets are negligible. Then all chiral coefficients of the form x_I^R with $I > 0$ must also be negligible. In this case, the only independent octet masses are N, Σ, Λ, and Ξ, whereas the only nontrivial chiral coefficients are a_0^1, $a_0^{8_1}$, $a_0^{8_2}$, and a_0^{27}. If we only work to first order in SU(3) breaking, the last of these is identically zero, and we find

$$\Delta_{\rm GMO} \equiv \tfrac{1}{2}\sqrt{\tfrac{10}{3}}a_0^{27} = \tfrac{3}{4}\Lambda + \tfrac{1}{4}\Sigma - \tfrac{1}{2}(N+\Xi) = 0, \tag{6}$$

the Gell-Mann–Okubo relation [14]. For the decuplet, the independent masses are Δ, Σ^*, Ξ^*, and Ω, whereas the nontrivial chiral coefficients are b_0^1, b_0^8, b_0^{27}, and b_0^{64}. To first order in SU(3) breaking, the vanishing of the last two coefficients gives rise to two nontrivial relations, which may be written

$$\begin{array}{rcl} 0 &=& 5(2b_0^{27}+b_0^{64}) = (\Delta-\Sigma^*) - (\Sigma^*-\Xi^*), \\ 0 &=& 10(b_0^{27}-2b_0^{64}) = (\Sigma^*-\Xi^*) - (\Omega-\Xi^*), \end{array} \tag{7}$$

Gell-Mann's famous equal-spacing rule [15].

On the other hand, if we consider only $I = 2$ operators [which we expect to be numerically well suppressed by $\alpha/4\pi$ or $(m_u - m_d)^2/\Lambda_\chi^2$], the octet provides us with the Σ equal-spacing rule [16]:

$$\Delta_\Sigma \equiv \sqrt{6}a_2^{27} = (\Sigma^+ - \Sigma^0) - (\Sigma^0 - \Sigma^-). \tag{8}$$

We caution that Σ^0 in this equation refers to the isospin $I = 1$ eigenstate rather than the mass eigenstate. In fact, we display in Sec. VI a new SU(6) relation for the mixing parameter.

Now consider second-order terms in SU(3) breaking. *A priori* we might expect to find that all of the representations within the product $\mathbf{8}\otimes\mathbf{8}$ occur, but we show that this is not the case. Because of charge conjugation symmetry of the strong interaction, the mass Lagrangian contains no terms with an odd number of Q_q factors. Thus the only second-order terms in SU(3) breaking are of the forms $(M_q \times M_q)$ and $(Q_q \times Q_q)$. Consider the product of two identical arbitrary matrices: $(X \times X)_{ij}{}^{kl}$, which contains such terms as $X_i{}^k X_j{}^l$, $X_i{}^l X_j{}^k$, and various traces of X, where i,j,k,l are flavor indices in the usual notation. It is readily seen that this product has no piece transforming under a $\mathbf{10}$, for such a tensor with the given indices has the form $A_{ijm}\epsilon^{mkl}$, and is symmetric under permutation of $\{i,j,m\}$. If we attempt to construct a product with these symmetry properties from two identical matrices, we quickly see that such a term vanishes. Similarly, the product of two identical matrices may contain no piece of a $\overline{\mathbf{10}}$.

We conclude that, to second order in SU(3) breaking, the octet chiral coefficients $a_1^{10} = a_1^{\overline{10}}$ are zero. The baryon mass relation corresponding to the vanishing of these coefficients is

$$\Delta_{\rm CG} \equiv -2\sqrt{3}a_1^{10} = (n-p) + (\Sigma^+ - \Sigma^-) - (\Xi^0 - \Xi^-) = 0, \tag{9}$$

the Coleman-Glashow relation [16]. For the decuplet, the analysis is even easier: $\mathbf{8}\otimes\mathbf{8}$ contains no $\mathbf{64}$ for *arbitrary* pairs of 3×3 matrices, and so we have four mass relations good to second order in SU(3) breaking, corresponding to the vanishing of $b_{0,1,2,3}^{64}$:

$$\Delta_1 \equiv 20b_3^{64} = \Delta^{++} - 3\Delta^+ + 3\Delta^0 - \Delta^-, \tag{10}$$

$$\Delta_2 \equiv 28b_2^{64} = \left(\Delta^{++} - \Delta^+ - \Delta^0 + \Delta^-\right) - 2\left(\Sigma^{*+} - 2\Sigma^{*0} + \Sigma^{*-}\right), \tag{11}$$

$$\Delta_3 \equiv 6(7b_1^{64} - b_3^{64}) = \left(\Delta^+ - \Delta^0\right) - \left(\Sigma^{*+} - \Sigma^{*-}\right) + \left(\Xi^{*0} - \Xi^{*-}\right), \tag{12}$$

$$\Delta_4 \equiv 35b_0^{64} = \frac{1}{4}\left(\Delta^{++} + \Delta^+ + \Delta^0 + \Delta^-\right) - \left(\Sigma^{*+} + \Sigma^{*0} + \Sigma^{*-}\right) + \frac{3}{2}\left(\Xi^{*0} + \Xi^{*-}\right) - \Omega^- \tag{13}$$

are four vanishing combinations. Notice that the first three of these are isospin breaking, and only the fourth remains in the limit that isospin is a good symmetry. The Gell-Mann–Okubo, Coleman-Glashow, and Σ equal-spacing relations and their violations were explored in chiral perturbation theory in Ref. [17], whereas similar computations for Eqs. (10)–(13) were performed in Ref. [18].

The approach of identifying relations with large, highly suppressed reps of course applies to any symmetry group, and we now proceed to apply it to SU(6). First, however, we must find the orthogonal matrix of spin-flavor baryon bilinears analogous to those in Eq. (5).

IV. DETERMINATION OF SU(6) CLEBSCH-GORDAN COEFFICIENTS

The orthogonal matrix of SU(6) group-theoretical factors can be determined most easily using tensor methods, in a manner similar to that in which we identified SU(3) mass relations in the previous section. In this case the basic reps in SU(6) breaking are no longer octets, but 6×6 traceless matrices, the $\mathbf{35}$ (adjoint) rep. The spin-1/2 octet (16 states) and spin-3/2 decuplet (40 states) of baryons neatly fill out the $\mathbf{56}$ rep, and thus the relevant products for our analysis are

$$\overline{\mathbf{56}} \otimes \mathbf{56} = \mathbf{1} \oplus \mathbf{35} \oplus \mathbf{405} \oplus \mathbf{2695} \tag{14}$$

and

$$\mathbf{35} \otimes \mathbf{35} = \mathbf{1} \oplus \mathbf{35}_1 \oplus \mathbf{35}_2 \oplus \mathbf{189} \oplus \mathbf{280} \oplus \overline{\mathbf{280}} \oplus \mathbf{405}. \tag{15}$$

In particular, since the **2695** rep does not occur in the latter product, combinations transforming under this rep give rise to relations broken only at third order.

The most straightforward approach to computing the necessary coefficients is to use the standard Wigner method of starting with the highest-weight state of the $\overline{\mathbf{56}} \otimes \mathbf{56}$ product (which is $\overline{\Delta^{-}_{-\frac{3}{2}}}\Delta^{++}_{+\frac{3}{2}}$) and applying successive SU(6) lowering operators, orthogonalizing degenerate states as necessary. Such an approach gives us not only the $\Delta I_3 = \Delta Y = 0$ bilinears, but all $56^2 = 3132$ of them. This method was employed by Cook and Murtaza [12], in a work which presents the coefficients for $\overline{\mathbf{56}} \otimes \mathbf{56}$ and other SU(6) products.

This is vastly more effort than we need to expend. To demonstrate the point, let us perform a counting of the bilinears we need: In addition to $\Delta I_3 = \Delta Y = 0$, we also impose $\Delta J_3 = 0$, where J is the total spin of the bilinear. Using again that spin and flavor commute in SU(6), we can obtain any $\Delta J_3 \neq 0$ by means of the simple SU(2) Wigner-Eckart theorem. Because the octet is spin 1/2 and the decuplet spin 3/2, octet-octet bilinears may appear only with $J = 0, 1$, octet-decuplet bilinears with $J = 1, 2$, and decuplet-decuplet bilinears with $J = 0, 1, 2, 3$, and each J multiplet possesses a unique $J_3 = 0$ state. Recalling from the previous section that the number of independent flavor bilinears (not counting Hermiticity) in the $\mathbf{8} \otimes \mathbf{8}$, $\mathbf{8} \otimes \mathbf{10}$, $\overline{\mathbf{10}} \otimes \mathbf{8}$, and $\overline{\mathbf{10}} \otimes \mathbf{10}$ products are 10, 8, 8, and 10, respectively, we find

$$10(1+1) + 8(1+1) + 8(1+1) + 10(1+1+1+1) = 92$$

independent baryon bilinears with $\Delta I_3 = \Delta Y = \Delta J_3 = 0$. The central thrust of this section, therefore, is the computation of a 92×92 orthogonal matrix.

In fact this task is simplified by the observation that the combinations of physical relevance are actually those with a well-defined J quantum number: $J = 0$ provides us with information about the baryon masses (also their "electric monopole moments" or charges, although this information is of course trivial), $J = 1$ tells us about their magnetic dipole moments, and $J = 2, 3$ about their electric quadrupole and magnetic octupole moments, respectively. This approach block-diagonalizes the 92×92 matrix according to values of J. Performing the counting above including only the single J_3 operator relevant to each value of J, we find that the $J = 0, 1, 2, 3$ blocks are, respectively, square matrices with 20, 36, 26, and 10 elements on a side. This is certainly a far cry from the full matrix of *all* bilinears, which has 56^2 entries—on each side.

There are yet further simplifications to this approach. Many of the entries will be related by means of Hermiticity of the Lagrangian. We have seen already in SU(3) how this relates the two Σ^0-Λ bilinears; the same must be true for bilinears like $\overline{p}\Delta^+$ and $\overline{\Delta^+}p$. Consequently, the chiral coefficients of octet-decuplet mixing appear only in certain characteristic combinations. We find that, of the 92 parameters at our disposal, the Hermiticity constraint reduces this number to 74.

The next task is to find the SU(3) $\times$ SU(2) content of the SU(6) multiplets. This can be accomplished by forming the products of the Young tableaux for SU(3) and SU(2) in parallel with those for SU(6), adding one block (i.e., fundamental rep index) at a time for each symmetry group. Then the content of an SU(6) rep must be such that the sum of the products of SU(3) and SU(2)-rep multiplicities adds up to the multiplicity of the SU(6) rep. As a simple example, in SU(6) the product of fundamental conjugate and fundamental reps is

$$\overline{\mathbf{6}} \otimes \mathbf{6} = \mathbf{1} \oplus \mathbf{35}, \tag{16}$$

whereas for SU(3) and SU(2) the corresponding products are

$$\overline{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}, \tag{17}$$

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}. \tag{18}$$

So writing SU(3) $\times$ SU(2) content reps as $(R,\ 2I+1)$, we have

$$\mathbf{1} = (1,1), \qquad \mathbf{35} = (1,3) + (8,1) + (8,3). \tag{19}$$

As long as we construct products one fundamental index at a time, there is never an ambiguity about how to assign content reps (at least for the $\overline{\mathbf{56}} \otimes \mathbf{56}$ product). We find the following decomposition for each value of J:

$J = 0$

SU(6) rep	SU(3) content reps
1	**1**
35	**8**
405	**1, 8, 27**
2695	**8, 10, $\overline{\mathbf{10}}$, 27, 64**

, (20)

$J = 1$

SU(6) rep	SU(3) content reps
35	**1, 8**
405	**8, 8, 10, $\overline{\mathbf{10}}$, 27**
2695	**1, 8, 8, 10, $\overline{\mathbf{10}}$, 27, 27, 27, 35, $\overline{\mathbf{35}}$, 64**

, (21)

$J=2$

SU(6) rep	SU(3) content reps
405	**1**, **8**, **27**
2695	**8**, **8**, **10**, $\overline{\mathbf{10}}$, **27**, **27**, **35**, $\overline{\mathbf{35}}$, **64**

(22)

$J=3$

SU(6) rep	SU(3) content reps
2695	**1**, **8**, **27**, **64** .

(23)

Using that the SU(3) reps **1**, **8**, **10**, $\overline{\mathbf{10}}$, **27**, **35**, $\overline{\mathbf{35}}$, and **64**, respectively, have 1, 2, 1, 1, 3, 2, 2, and 4 states with $\Delta I_3 = \Delta Y = 0$, we count 92 chiral coefficients in total, as expected, and numbers for each value of J that agree with the block-diagonalization counting for baryon bilinears given above.

The central feature that allows us to implement tensor methods is that we have explicit tensor forms for both the **35** and **56**. As previously stated, the **35** may be represented as a traceless 6×6 matrix; however, the trace adds only a harmless singlet to our analysis, and so to obtain arbitrary second-order SU(6) breaking, we require two arbitrary SU(6) matrices X and Z. The quantity we must compute is $\overline{B}BXZ$, where B is the tensor form of the **56**, and SU(6) indices are contracted in all possible ways. In fact, the very useful tensor B constructed below appears in the literature [19].

We first define the familiar SU(3) tensors. For the baryon octet,

$$O_a{}^b \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}. \tag{24}$$

The baryon decuplet in this notation, a $3\times 3\times 3$ array, may be represented as a collection of three matrices:

$$T^{abc} \equiv \begin{pmatrix} \Delta^{++} & \frac{1}{\sqrt{3}}\Delta^+ & \frac{1}{\sqrt{3}}\Sigma^{*+} \\ \frac{1}{\sqrt{3}}\Delta^+ & \frac{1}{\sqrt{3}}\Delta^0 & \frac{1}{\sqrt{6}}\Sigma^{*0} \\ \frac{1}{\sqrt{3}}\Sigma^{*+} & \frac{1}{\sqrt{6}}\Sigma^{*0} & \frac{1}{\sqrt{3}}\Xi^{*0} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}}\Delta^+ & \frac{1}{\sqrt{3}}\Delta^0 & \frac{1}{\sqrt{6}}\Sigma^{*0} \\ \frac{1}{\sqrt{3}}\Delta^0 & \Delta^- & \frac{1}{\sqrt{3}}\Sigma^{*-} \\ \frac{1}{\sqrt{6}}\Sigma^{*0} & \frac{1}{\sqrt{3}}\Sigma^{*-} & \frac{1}{\sqrt{3}}\Xi^{*-} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}}\Sigma^{*+} & \frac{1}{\sqrt{6}}\Sigma^{*0} & \frac{1}{\sqrt{3}}\Xi^{*0} \\ \frac{1}{\sqrt{6}}\Sigma^{*0} & \frac{1}{\sqrt{3}}\Sigma^{*-} & \frac{1}{\sqrt{3}}\Xi^{*-} \\ \frac{1}{\sqrt{3}}\Xi^{*0} & \frac{1}{\sqrt{3}}\Xi^{*-} & \Omega^- \end{pmatrix}. \tag{25}$$

One may assign any particular permutation of indices a,b,c to denote row, column, and submatrix in this representation, because the decuplet is completely symmetric under rearrangement of flavor indices.

Using the notation $\Uparrow$, $\uparrow$, $\downarrow$, $\Downarrow$ to denote $J_3 = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, the SU(2) spin tensors for spin 1/2 and spin 3/2 assume the forms

$$\chi^i \equiv \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \tag{26}$$

and

$$\chi^{ijk} \equiv \begin{pmatrix} \Uparrow & \frac{1}{\sqrt{3}}\uparrow \\ \frac{1}{\sqrt{3}}\uparrow & \frac{1}{\sqrt{3}}\downarrow \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}}\uparrow & \frac{1}{\sqrt{3}}\downarrow \\ \frac{1}{\sqrt{3}}\downarrow & \Downarrow \end{pmatrix}, \tag{27}$$

where the latter tensor is symmetric under exchange of indices.

Then, with the use of the Levi-Civita symbols ϵ^{ij} and ϵ^{ijk}, we construct the **56** tensor:

$$B^{aibjck} = \chi^{ijk}T^{abc} + \frac{1}{3\sqrt{2}}\left[\epsilon^{ij}\chi^k\epsilon^{abd}O_d{}^c + \epsilon^{jk}\chi^i\epsilon^{bcd}O_d{}^a + \epsilon^{ki}\chi^j\epsilon^{cad}O_d{}^b\right]. \tag{28}$$

Note that B is completely symmetric under the exchange of pairs of indices from SU(3) $\times$ SU(2), as the **56** is a symmetric rep of SU(6). The $1/3\sqrt{2}$ guarantees the singlet normalization:

$$\overline{B}_{aibjck}B^{aibjck} = \overline{p\uparrow}p\uparrow + \overline{p\downarrow}p\downarrow + \overline{\Delta^{++}\Uparrow}\Delta^{++}\Uparrow + \overline{\Delta^{++}\uparrow}\Delta^{++}\uparrow + \cdots . \tag{29}$$

Because we are interested in bilinear combinations with definite J, we also require a table of SU(2) Clebsch-Gordan coefficients; however, since we have abandoned the Condon-Shortley phase convention for the SU(3) coefficients, we must do likewise for their SU(2) analogues. Starting with Clebsch-Gordan coefficients in the Condon-Shortley convention, we choose all Clebsch-Gordan coefficients $\langle 0\,0|s+m;\ s-m\rangle$ to be the same regardless of m, and both values of $\langle 1\,0|\frac{3}{2}+m;\ \frac{1}{2}-m\rangle$ to be positive. The SU(2) relation

$$\langle j_1+m_1;\ j_2+m_2|j+m\rangle = \langle j_2-m_2;\ j_1-m_1|j-m\rangle \tag{30}$$

relates the $\frac{3}{2} \times \frac{1}{2}$ and $\frac{1}{2} \times \frac{3}{2}$ Clebsch-Gordan tables.

To obtain the SU(6) Clebsch-Gordan coefficients in the **35** rep, we simply compute the quantity $\overline{B}BXZ$ with X traceless and $Z = \mathbf{1}$. To decompose into the component SU(3) × SU(2) quantum numbers, we choose X to consist of the basis operators $\mathbf{1} \otimes \sigma_3$, $Y \otimes \mathbf{1}$, $I_3 \otimes \mathbf{1}$, $Y \otimes \sigma_3$, and $I_3 \otimes \sigma_3$. The SU(6) rep **1** is even more trivial: $X = Z = \mathbf{1}$.

One may use a similar approach for **405** and **2695** operators as well, but then one must render the products of 6×6 matrices completely traceless under any contraction, and this procedure tends to be tedious for larger reps in SU(3) × SU(2) notation. A much better approach is to find the **2695** combinations by observing that it is exactly these combinations that vanish in the quantity $\overline{B}BXZ$. We know from the SU(3) × SU(2) contents which reps appear, and we know from Sec. II that a particular SU(6) chiral coefficient $d_N^{R,I}$ is simply a linear combination of SU(3) chiral coefficients with the same quantum numbers R, I. Therefore, we form an arbitrary linear combination of the desired SU(3) chiral coefficients and seek out values of the coefficients for which this combination vanishes from $\overline{B}BXZ$; such a combination transforms under the **2695** rep. If there is more than one, we arbitrarily choose an orthogonalization to lift the degeneracy. Finally, we find the chiral coefficients $d_{405}^{R,I}$ by their orthogonality to $d_{2695}^{R,I}$, $d_{35}^{R,I}$, and $d_1^{R,I}$.

This procedure gives us all of the SU(6) Clebsch-Gordan coefficients for product states in $\overline{\mathbf{56}} \otimes \mathbf{56}$ with $\Delta I_3 = \Delta Y = \Delta J_3 = 0$. As we have pointed out, the restriction $\Delta J_3 = 0$ is of no great consequence, for we may use the Wigner-Eckart theorem to obtain coefficients with $\Delta J_3 \neq 0$. $\Delta I_3, \Delta Y \neq 0$ are not much harder; because SU(3) Clebsch-Gordan coefficients are also well known, we may use the SU(3) version of the Wigner-Eckart theorem to obtain the others. Thus *all* coefficients of this product are now known. The great advantage of this approach is that similar techniques may be applied to other product reps and other symmetry groups. The key requirement of this method is that one needs an explicit tensor representation of all fields under consideration; since these tensors are usually constructed to satisfy a specific particle content, such tensors are readily available. In our case, the SU(6) information is encapsulated in the tensor B.

V. EXHIBITION OF SU(6) CLEBSCH-GORDAN COEFFICIENTS

Here we collect the mathematical results of the procedure just described in a compact notation. Rather than exhibiting the gigantic 92×92 matrix or even the smaller diagonal blocks, we present subblocks associated with each SU(3) rep $\mathbf{R}$. Note especially that the chiral coefficients $d_N^{R,I}$, for a given R and N, are independent of the particular value of I. On the other hand, these coefficients depend implicitly upon J; when confusion could arise, we write $d_{N,J}^{R,I}$. These results, apart from the use of our chiral coefficient notation and different phase conventions already mentioned, are identical to Table V of Cook and Murtaza [12]:

$\underline{J=0}$

$$\begin{pmatrix} d_1^{1,0} \\ d_{405}^{1,0} \end{pmatrix} = \begin{pmatrix} +\sqrt{\frac{2}{7}} & +\sqrt{\frac{5}{7}} \\ +\sqrt{\frac{5}{7}} & -\sqrt{\frac{2}{7}} \end{pmatrix} \begin{pmatrix} a_0^1 \\ b_0^1 \end{pmatrix}, \qquad \begin{pmatrix} d_{405}^{27,I} \\ d_{2695}^{27,I} \end{pmatrix} = \begin{pmatrix} +\frac{1}{\sqrt{15}} & +\sqrt{\frac{14}{15}} \\ +\sqrt{\frac{14}{15}} & -\frac{1}{\sqrt{15}} \end{pmatrix} \begin{pmatrix} a_I^{27} \\ b_I^{27} \end{pmatrix},$$

$$\begin{pmatrix} d_{35}^{8,I} \\ d_{405}^{8,I} \\ d_{2695}^{8,I} \end{pmatrix} = \begin{pmatrix} 0 & +\frac{1}{\sqrt{6}} & +\sqrt{\frac{5}{6}} \\ +\sqrt{\frac{2}{5}} & +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{10}} \\ +\sqrt{\frac{3}{5}} & -\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{15}} \end{pmatrix} \begin{pmatrix} a_I^{8_1} \\ a_I^{8_2} \\ b_I^8 \end{pmatrix},$$

$$d_{2695}^{10,1} = a_1^{10}, \qquad d_{2695}^{\overline{10},1} = a_1^{\overline{10}}, \qquad d_{2695}^{64,I} = b_I^{64}; \tag{31}$$

$\underline{J=1}$

$$\begin{pmatrix} d_{35}^{1,0} \\ d_{2695}^{1,0} \end{pmatrix} = \begin{pmatrix} +\frac{\sqrt{2}}{3\sqrt{3}} & +\frac{5}{3\sqrt{3}} \\ +\frac{5}{3\sqrt{3}} & -\frac{\sqrt{2}}{3\sqrt{3}} \end{pmatrix} \begin{pmatrix} a_0^1 \\ b_0^1 \end{pmatrix},$$

$$\begin{pmatrix} d_{35}^{8,I} \\ d_{405}^{8_1,I} \\ d_{405}^{8_2,I} \\ d_{2695}^{8_1,I} \\ d_{2695}^{8_2,I} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{5}}{3\sqrt{6}} & +\frac{\sqrt{2}}{3\sqrt{3}} & +\frac{5}{3\sqrt{6}} & +\frac{\sqrt{5}}{3\sqrt{3}} & +\frac{\sqrt{5}}{3\sqrt{3}} \\ +\frac{1}{\sqrt{10}} & 0 & +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 0 & 0 & 0 & +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ +\frac{2}{\sqrt{5}} & 0 & 0 & +\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{10}} \\ +\frac{1}{3\sqrt{15}} & +\frac{5}{3\sqrt{3}} & -\frac{1}{3\sqrt{3}} & -\frac{\sqrt{2}}{3\sqrt{15}} & -\frac{\sqrt{2}}{3\sqrt{15}} \end{pmatrix} \begin{pmatrix} a_I^{8_1} \\ a_I^{8_2} \\ b_I^8 \\ c_I^8 \\ \overline{c}_I^8 \end{pmatrix},$$

$$\begin{pmatrix} d_{405}^{10,1} \\ d_{2695}^{10,1} \end{pmatrix} = \begin{pmatrix} +\frac{1}{\sqrt{5}} & +\frac{2}{\sqrt{5}} \\ +\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} a_1^{10} \\ c_1^{10} \end{pmatrix}, \qquad \begin{pmatrix} d_{405}^{\overline{10},1} \\ d_{2695}^{\overline{10},1} \end{pmatrix} = \begin{pmatrix} +\frac{1}{\sqrt{5}} & +\frac{2}{\sqrt{5}} \\ +\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} a_1^{\overline{10}} \\ c_1^{\overline{10}} \end{pmatrix},$$

$$\begin{pmatrix} d_{405}^{27,I} \\ d_{2695}^{27_1,I} \\ d_{2695}^{27_2,I} \\ d_{2695}^{27_3,I} \end{pmatrix} = \begin{pmatrix} +\frac{\sqrt{2}}{3\sqrt{5}} & +\frac{\sqrt{7}}{3} & +\frac{2}{3\sqrt{5}} & +\frac{2}{3\sqrt{5}} \\ +\frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ +\frac{\sqrt{7}}{3\sqrt{5}} & -\frac{\sqrt{2}}{3} & +\frac{\sqrt{14}}{3\sqrt{5}} & +\frac{\sqrt{14}}{3\sqrt{5}} \\ 0 & 0 & +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_I^{27} \\ b_I^{27} \\ c_I^{27} \\ \bar{c}_I^{27} \end{pmatrix},$$

$$d_{2695}^{35,I} = c_I^{35}, \qquad d_{2695}^{\overline{35},I} = \bar{c}_I^{\overline{35}}, \qquad d_{2695}^{64,I} = b_I^{64}; \tag{32}$$

$\underline{J=2}$

$$d_{405}^{1,0} = b_0^1,$$

$$\begin{pmatrix} d_{405}^{8,I} \\ d_{2695}^{8_1,I} \\ d_{2695}^{8_2,1} \end{pmatrix} = \begin{pmatrix} +\frac{2}{\sqrt{5}} & +\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{10}} \\ +\frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{5}} & -\sqrt{\frac{2}{5}} \\ 0 & +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} b_I^8 \\ c_I^8 \\ \bar{c}_I^8 \end{pmatrix},$$

$$d_{2695}^{10,1} = c_1^{10}, \qquad d_{2695}^{\overline{10},1} = \bar{c}_1^{\overline{10}},$$

$$\begin{pmatrix} d_{405}^{27,I} \\ d_{2695}^{27_1,I} \\ d_{2695}^{27_2,I} \end{pmatrix} = \begin{pmatrix} +\sqrt{\frac{7}{15}} & +\frac{2}{\sqrt{15}} & +\frac{2}{\sqrt{15}} \\ +2\sqrt{\frac{2}{15}} & -\sqrt{\frac{7}{30}} & -\sqrt{\frac{7}{30}} \\ 0 & +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} b_I^{27} \\ c_I^{27} \\ \bar{c}_I^{27} \end{pmatrix},$$

$$d_{2695}^{35,I} = c_I^{35}, \qquad d_{2695}^{\overline{35},I} = \bar{c}_I^{\overline{35}}, \qquad d_{2695}^{64,I} = b_I^{64}; \tag{33}$$

$\underline{J=3}$

$$d_{2695}^{R,I} = b_I^R \qquad \text{for } R = 1,\ 8,\ 27,\ 64. \tag{34}$$

A number of chiral coefficients contain redundant information because of symmetry under conjugation (e.g., $d_{N,J}^{10,1}$ and $d_{N,J}^{\overline{10},1}$). Others (e.g., $d_{405,1}^{8_2,I}$) necessarily vanish once we impose Hermiticity. These are the counterparts to the degrees of freedom lost from demanding only Hermitian combinations of bilinears like $(\overline{\Lambda}\Sigma^{*0} + \overline{\Sigma^{*0}}\Lambda)$, and one finds, as expected, exactly 74 physically independent chiral coefficients.

In order to obtain baryon relations, we must take into account the particular matrix elements used in defining the mass and electromagnetic moments. The matrices above are defined by bilinears in eigenstates of total J, but the various moments are defined as matrix elements connecting the states with *highest weight* in the spin-projection quantum number. The magnetic dipole moment of a particle with spin s, for example, is defined as the matrix element with angular momentum structure $\langle 1\,0|s\ -s;\ s\ +s\rangle$. In the case of transitions between particles with different spins s_1 and s_2, however, the convention is not so universal. We adopt the choice that the two particles are taken to be in the highest-weight spin states such that their combined value is still zero; that is, the spin-J multipole moment transition is defined through the matrix element

$$\langle J\,0|s_1 - \min(s_1,s_2);\ s_2 + \min(s_1,s_2)\rangle. \tag{35}$$

Note that the $J=0$ matrix elements, which give rise to masses (or electric charges), do not depend on this choice because of our previous choice of the Clebsch-Gordan convention; here the physical fact of the independence of baryon masses on individual spin states becomes most clear. The matrix elements for all multipole moments can now be obtained trivially from the SU(6) matrices by use of the SU(2) Wigner-Eckart theorem.

VI. BARYON RELATIONS

A. Estimating relation-breaking terms

As in Sec. III for the case of SU(3), we argue that the largest reps of SU(6) give rise to the most experimentally accurate relations. SU(6)-breaking operators appear in the small **35** rep; the largest rep, the **2695**, requires a product of three of these, and so the **2695** chiral coefficients contain all relations third order in SU(6) breaking. The neglect of the **2695** to obtain mass relations is quite an old approach [20]; we attempt here to justify this assumption in the modern language of effective Lagrangians. The only statement that must be verified is that all of the **35** operators possess numerically small coefficients. Certainly the quark mass and charge operators, now written in SU(3) $\times$ SU(2) notation as $M_q \otimes \mathbf{1}$ and $Q_q \otimes \mathbf{1}$, are still small, as are the corresponding operators with with spin flips, $M_q \otimes \sigma_3$ and $Q_q \otimes \sigma_3$. The only other physical operator to consider is the pure spin flip $\mathbf{1} \otimes \sigma_3$. *A priori* we see no reason this operator should have a small coefficient, but it is precisely this operator that explains the relative smallness of the breaking between the average octet and decuplet baryon masses. Thus even this operator must possess a numerically small coefficient.

In order to judge the quality of the following relations, we must be able to estimate the coefficients of these **2695** operators. Fortunately, this is a matter of simple naive dimensional analysis; we assume that any unknown dimensionless parameters are of order 1. For simplicity, let us consider the mass relations only. The numerical breaking of average octet and decuplet masses can be characterized by the number

$$\frac{m_{10} - m_8}{\frac{1}{2}(m_{10} + m_8)} \approx 0.2. \tag{36}$$

We use this to estimate the spin-flip coefficient conservatively as 0.3. Therefore, $I = 0$ operators in the **2695** contribute an amount to each baryon mass of order

$$\Lambda_\chi (0.3)^3 \approx 25 \text{ MeV}. \tag{37}$$

Isospin-breaking operators are much more heavily suppressed. Each unit of isospin breaking contributes an additional factor $(m_d - m_u)/\Lambda_\chi \approx 0.005$; alternately, for each *two* units of isospin breaking, a factor of $\alpha/4\pi$ can appear (operators with single powers of e are forbidden in masses by charge conjugation symmetry). Typical numbers are 0.5, 0.2, and 0.003 MeV for $I = 1$, 2, and 3, respectively. Note that these naive estimates apply to *individual* baryons, and large coefficients in the relations presented below must be taken into account to obtain reliable numbers. Similar arguments apply to the electromagnetic moment relations.

B. Masses

Here we exhibit the mass combinations associated with each chiral coefficient in the **2695**. There are 19 independent parameters in the $J = 0$ sector, corresponding to the octet and decuplet masses and the Σ^0-Λ mixing parameter, which we denote by β. The ten $J = 0$ chiral coefficients in the **2695**, characterized by their SU(3) decompositions, are the following:

$$\begin{array}{rl}
\underline{(\mathrm{SU}(3), I)} & \underline{\text{mass combination,}} \\
(8,0) & +(p+n) + 3(\Sigma^+ + \Sigma^0 + \Sigma^-) - 3\Lambda - 4(\Xi^0 + \Xi^-) - (\Delta^{++} + \Delta^+ + \Delta^0 + \Delta^-) + (\Xi^{*0} + \Xi^{*-}) + 2\Omega^-, \\
(8,1) & +7(p-n) + 5(\Sigma^+ - \Sigma^-) - 2(\Xi^0 - \Xi^-) - 6\sqrt{3}\beta \\
 & -(3\Delta^{++} + \Delta^+ - \Delta^0 - 3\Delta^-) - 2(\Sigma^{*+} - \Sigma^{*-}) - (\Xi^{*0} - \Xi^{*-}), \\
(27,0) & +7[3(p+n) - (\Sigma^+ + \Sigma^0 + \Sigma^-) - 9\Lambda + 3(\Xi^0 + \Xi^-)] \\
 & -2[3(\Delta^{++} + \Delta^+ + \Delta^0 + \Delta^-) - 5(\Sigma^{*+} + \Sigma^{*0} + \Sigma^{*-}) - 3(\Xi^{*0} + \Xi^{*-}) + 9\Omega^-], \\
(27,1) & +7[(p-n) - (\Xi^0 - \Xi^-) + 2\sqrt{3}\beta] - (3\Delta^{++} + \Delta^+ - \Delta^0 - 3\Delta^-) + 3(\Sigma^{*+} - \Sigma^{*-}) + 4(\Xi^{*0} - \Xi^{*-}), \\
(27,2) & +7(\Sigma^+ - 2\Sigma^0 + \Sigma^-) - 3(\Delta^{++} - \Delta^+ - \Delta^0 + \Delta^-) - (\Sigma^{*+} - 2\Sigma^{*0} + \Sigma^{*-}), \\
(10,1),(\overline{10},1) & +(p-n) - (\Sigma^+ - \Sigma^-) + (\Xi^0 - \Xi^-), \\
(64,0) & +(\Delta^{++} + \Delta^+ + \Delta^0 + \Delta^-) - 4(\Sigma^{*+} + \Sigma^{*0} + \Sigma^{*-}) + 6(\Xi^{*0} + \Xi^{*-}) - 4\Omega^-, \\
(64,1) & +(3\Delta^{++} + \Delta^+ - \Delta^0 - 3\Delta^-) - 10(\Sigma^{*+} - \Sigma^{*-}) + 10(\Xi^{*0} - \Xi^{*-}), \\
(64,2) & +(\Delta^{++} - \Delta^+ - \Delta^0 + \Delta^-) - 2(\Sigma^{*+} - 2\Sigma^{*0} + \Sigma^{*-}), \\
(64,3) & +\Delta^{++} - 3\Delta^+ + 3\Delta^0 - \Delta^-.
\end{array} \tag{38}$$

It is interesting to note that the last five of these are also SU(3) relations as well, because the SU(3) reps **10** and $\overline{\mathbf{10}}$ do not appear in the decuplet-decuplet product, and **64** does not appear in the octet-octet product. In fact, since the **64** rep appears in neither $\mathbf{8} \otimes \mathbf{10}$ nor $\overline{\mathbf{10}} \otimes \mathbf{8}$, we have the curious result that the these combinations of decuplet bilinears give not only mass but dipole, quadrupole, and octupole moment relations with the same coefficients.

We also point out that the three $I = 0$ relations, for which we may neglect mass differences within each isospin multiplet, are equivalent to the three relations derived by Dashen, Jenkins, and Manohar [7] in the large-N_c contracted spin-flavor symmetry. This is an excellent illustration of the similarity between the two symmetries. As $I = 0$ SU(6) relations, these three expressions may be obtained from the calculation of Harari and Rashid [20].

We now exhibit numerical values for these combinations. In all cases we use Particle Data Group (PDG) [21] values for the masses, with the following exceptions. First, the unknown parameter β is eliminated between the $(8,1)$ and $(27,1)$ combinations. Next, the Δ mass differences have notoriously large uncertainties; we adopt the arguments in Ref. [18] that a set consistent with chiral loop calculations is

$$\begin{aligned} \Delta^0 - \Delta^{++} &= 1.3 \pm 0.5 \text{ MeV}, \\ \Delta^+ &= 1231.5 \pm 0.3 \text{ MeV}. \end{aligned} \tag{39}$$

From the same reference, we fix the Δ^- mass, which has never been directly determined, by means of the (64, 3) relation; its corrections, including loop effects, are determined to be negligible. The results are presented in Table I. In all cases, the naive estimates of **2695** operators explain the experimental relation breakings.

The set of nine relations after the elimination of β is equivalent to the set derived by Rubinstein, Scheck, and Socolow [22], who used very similar reasoning to that above; their neglect of "three-body operators" is equivalent to the neglect of the **2695**. The difference is that the earlier authors did not distinguish the relations by SU(3) content, which is important to researchers who wish to organize renormalization effects [17,18] or other operator expansions (for example, large-N_c QCD [11]). On the one hand, the SU(3) decomposition of the **2695** in Ref. [22] is missing the (10, 1) and $(\overline{10}, 1)$ terms (one independent parameter), and on the other hand the Σ^0-Λ mixing is neglected; thus they count only nine relations.

TABLE I. Experimental values for SU(6) mass relations (MeV).

(8,0)	$+208.2 \pm 3.5$	(64,0)	$+5.9 \pm 1.7$
$\frac{3}{5}$(27,1) − $\frac{7}{5}$(8,1)	-15.4 ± 12.7	(64,1)	$+0.5 \pm 1.1$
(10,1), $(\overline{10}$,1)	$+0.3 \pm 0.6$	(64,2)	-5.2 ± 4.5
(27,0)	-278.5 ± 23.2	(64,3)	0
(27,2)	$+9.1 \pm 5.5$		

This brings us to the tenth relation, that which predicts β. We choose the unique sum of (8, 1) and (27, 1) that eliminates the troublesome Δ masses, and obtain the pretty result

$$\begin{aligned}\beta &= +\frac{1}{4\sqrt{3}}[(\Sigma^+ - \Sigma^-) + (\Xi^0 - \Xi^-) - (\Sigma^{*+} - \Sigma^{*-}) - (\Xi^{*0} - \Xi^{*-})] \\ &= -0.99 \pm 0.15 \text{ MeV}.\end{aligned} \tag{40}$$

Equivalent relations to this were first obtained in Ref. [23] by means of neglecting three-body quark operators; thus their relations for β lie within the linear span of our ten mass relations. A naive estimate of the **2695** breaking of this relation produces a further uncertainty of order 0.2–0.3 MeV. It is important to recognize that the masses above labeled Σ^0 and Λ are actually eigenvalues associated with isospin eigenstates; to obtain the mass eigenvalues, we must diagonalize a 2×2 matrix including the mixing terms. If we define the mass eigenvalues (labeled by m) via

$$\begin{pmatrix} \Sigma^0_m \\ \Lambda_m \end{pmatrix} = \begin{pmatrix} +\cos\theta & +\sin\theta \\ -\sin\theta & +\cos\theta \end{pmatrix} \begin{pmatrix} \Sigma^0 \\ \Lambda \end{pmatrix}, \tag{41}$$

then we find

$$\theta = -0.013 \pm 0.002 \text{ rad} \qquad (-0.74^\circ \pm 0.11^\circ), \tag{42}$$

where

$$\theta = \frac{1}{2}\tan^{-1}\left[\frac{\beta}{\sqrt{\left(\frac{\Sigma^0_m - \Lambda_m}{2}\right)^2 + \beta^2}}\right]. \tag{43}$$

The difference $(\Sigma^0_m - \Sigma^0) = -(\Lambda_m - \Lambda)$ turns out to be a mere 13 ± 4 keV, and thus we lose nothing by using mass eigenvalues for Σ^0 and Λ in the other mass relations.

C. Magnetic dipole moments

The $J = 1$ sector is characterized by 27 parameters, which may be thought of as the magnetic dipole moments of the octet and decuplet baryons, the eight possible transition moments between these multiplets, and the Σ^0-Λ transition moment. There are 18 independent chiral coefficients in the **2695**, given as follows:

$$\begin{array}{rl}
\underline{(\mathrm{SU}(3), I)} & \underline{\text{magnetic dipole moment combination,}} \\
(1,0) & +15[(\mu_p + \mu_n) + (\mu_{\Sigma^+} + \mu_{\Sigma^0} + \mu_{\Sigma^-}) + \mu_\Lambda + (\mu_{\Xi^0} + \mu_{\Xi^-})] \\
 & -4[(\mu_{\Delta^{++}} + \mu_{\Delta^+} + \mu_{\Delta^0} + \mu_{\Delta^-}) + (\mu_{\Sigma^{*+}} + \mu_{\Sigma^{*0}} + \mu_{\Sigma^{*-}}) + (\mu_{\Xi^{*0}} + \mu_{\Xi^{*-}}) + \mu_{\Omega^-}], \\
(8_1,0) & +[(\mu_p + \mu_n) - 2(\mu_{\Sigma^+} + \mu_{\Sigma^0} + \mu_{\Sigma^-}) + 2\mu_\Lambda + (\mu_{\Xi^0} + \mu_{\Xi^-})] \\
 & +\sqrt{2}[(\mu_{\Sigma^+\Sigma^{*+}} + \mu_{\Sigma^0\Sigma^{*0}} + \mu_{\Sigma^-\Sigma^{*-}}) + (\mu_{\Xi^0\Xi^{*0}} + \mu_{\Xi^-\Xi^{*-}})], \\
(8_1,1) & +3[(\mu_p - \mu_n) - (\mu_{\Xi^0} - \mu_{\Xi^-})] - 4\sqrt{3}\mu_{\Sigma^0\Lambda} \\
 & -\sqrt{2}[2(\mu_{p\Delta^+} + \mu_{n\Delta^0}) + (\mu_{\Sigma^+\Sigma^{*+}} - \mu_{\Sigma^-\Sigma^{*-}}) + \sqrt{3}\mu_{\Lambda\Sigma^{*0}} + (\mu_{\Xi^0\Xi^{*0}} - \mu_{\Xi^-\Xi^{*-}})], \\
(8_2,0) & +3[13(\mu_p + \mu_n) - (\mu_{\Sigma^+} + \mu_{\Sigma^0} + \mu_{\Sigma^-}) + \mu_\Lambda - 12(\mu_{\Xi^0} + \mu_{\Xi^-})] \\
 & -5[(\mu_{\Delta^{++}} + \mu_{\Delta^+} + \mu_{\Delta^0} + \mu_{\Delta^-}) - (\mu_{\Xi^{*0}} + \mu_{\Xi^{*-}}) - 2\mu_{\Omega^-}] \\
 & -6\sqrt{2}[(\mu_{\Sigma^+\Sigma^{*+}} + \mu_{\Sigma^0\Sigma^{*0}} + \mu_{\Sigma^-\Sigma^{*-}}) + (\mu_{\Xi^0\Xi^{*0}} + \mu_{\Xi^-\Xi^{*-}})], \\
(8_2,1) & +3[11(\mu_p - \mu_n) + 25(\mu_{\Sigma^+} - \mu_{\Sigma^-}) + 14(\mu_{\Xi^0} - \mu_{\Xi^-}) + 2\sqrt{3}\mu_{\Sigma^0\Lambda}] \\
 & -5[(3\mu_{\Delta^{++}} + \mu_{\Delta^+} - \mu_{\Delta^0} - 3\mu_{\Delta^-}) + 2(\mu_{\Sigma^{*+}} - \mu_{\Sigma^{*-}}) + (\mu_{\Xi^{*0}} - \mu_{\Xi^{*-}})]
\end{array}$$

$$
\begin{aligned}
& -6\sqrt{2}[2(\mu_{p\Delta^+}+\mu_{n\Delta^0})+(\mu_{\Sigma^+\Sigma^{*+}}-\mu_{\Sigma^-\Sigma^{*-}})+\sqrt{3}\mu_{\Lambda\Sigma^{*0}}+(\mu_{\Xi^0\Xi^{*0}}-\mu_{\Xi^-\Xi^{*-}})],\\
(10,1),(\overline{10},1)\quad & +2[(\mu_p-\mu_n)-(\mu_{\Sigma^+}-\mu_{\Sigma^-})+(\mu_{\Xi^0}-\mu_{\Xi^-})]\\
& -\sqrt{2}[(\mu_{p\Delta^+}+\mu_{n\Delta^0})-(\mu_{\Sigma^+\Sigma^{*+}}-\mu_{\Sigma^-\Sigma^{*-}})-(\mu_{\Xi^0\Xi^{*0}}-\mu_{\Xi^-\Xi^{*-}})],\\
(27_1,0)\quad & +[3(\mu_p+\mu_n)-(\mu_{\Sigma^+}+\mu_{\Sigma^0}+\mu_{\Sigma^-})-9\mu_\Lambda+3(\mu_{\Xi^0}+\mu_{\Xi^-})]\\
& -\sqrt{2}[2(\mu_{\Sigma^+\Sigma^{*+}}+\mu_{\Sigma^0\Sigma^{*0}}+\mu_{\Sigma^-\Sigma^{*-}})-3(\mu_{\Xi^0\Xi^{*0}}+\mu_{\Xi^-\Xi^{*-}})],\\
(27_1,1)\quad & +2[(\mu_p-\mu_n)-(\mu_{\Xi^0}-\mu_{\Xi^-})+2\sqrt{3}\mu_{\Sigma^0\Lambda}]\\
& -5\sqrt{2}[(\mu_{p\Delta^+}+\mu_{n\Delta^0})+3(\mu_{\Sigma^+\Sigma^{*+}}-\mu_{\Sigma^-\Sigma^{*-}})-2\sqrt{3}\mu_{\Lambda\Sigma^{*0}}-2(\mu_{\Xi^0\Xi^{*0}}-\mu_{\Xi^-\Xi^{*-}})],\\
(27_1,2)\quad & +4[\mu_{\Sigma^+}-2\mu_{\Sigma^0}+\mu_{\Sigma^-}]-\sqrt{2}[3(\mu_{p\Delta^+}-\mu_{n\Delta^0})+(\mu_{\Sigma^+\Sigma^{*+}}-2\mu_{\Sigma^0\Sigma^{*0}}+\mu_{\Sigma^-\Sigma^{*-}})],\\
(27_2,0)\quad & +21[3(\mu_p+\mu_n)-(\mu_{\Sigma^+}+\mu_{\Sigma^0}+\mu_{\Sigma^-})-9\mu_\Lambda+3(\mu_{\Xi^0}+\mu_{\Xi^-})]\\
& -20[3(\mu_{\Delta^{++}}+\mu_{\Delta^+}+\mu_{\Delta^0}+\mu_{\Delta^-})-5(\mu_{\Sigma^{*+}}+\mu_{\Sigma^{*0}}+\mu_{\Sigma^{*-}})-3(\mu_{\Xi^{*0}}+\mu_{\Xi^{*-}})+9\mu_{\Omega^-}]\\
& +84\sqrt{2}[2(\mu_{\Sigma^+\Sigma^{*+}}+\mu_{\Sigma^0\Sigma^{*0}}+\mu_{\Sigma^-\Sigma^{*-}})-3(\mu_{\Xi^0\Xi^{*0}}+\mu_{\Xi^-\Xi^{*-}})],\\
(27_2,1)\quad & +21[(\mu_p-\mu_n)-(\mu_{\Xi^0}-\mu_{\Xi^-})+2\sqrt{3}\mu_{\Sigma^0\Lambda}]\\
& -10[(3\mu_{\Delta^{++}}+\mu_{\Delta^+}-\mu_{\Delta^0}-3\mu_{\Delta^-})-3(\mu_{\Sigma^{*+}}-\mu_{\Sigma^{*-}})-4(\mu_{\Xi^{*0}}-\mu_{\Xi^{*-}})]\\
& +21\sqrt{2}[(\mu_{p\Delta^+}+\mu_{n\Delta^0})+3(\mu_{\Sigma^+\Sigma^{*+}}-\mu_{\Sigma^-\Sigma^{*-}})-2\sqrt{3}\mu_{\Lambda\Sigma^{*0}}-2(\mu_{\Xi^0\Xi^{*0}}-\mu_{\Xi^-\Xi^{*-}})],\\
(27_2,2)\quad & +21[\mu_{\Sigma^+}-2\mu_{\Sigma^0}+\mu_{\Sigma^-}]-10[3(\mu_{\Delta^{++}}-\mu_{\Delta^+}-\mu_{\Delta^0}+\mu_{\Delta^-})+(\mu_{\Sigma^{*+}}-2\mu_{\Sigma^{*0}}+\mu_{\Sigma^{*-}})]\\
& +21\sqrt{2}[3(\mu_{p\Delta^+}-\mu_{n\Delta^0})+(\mu_{\Sigma^+\Sigma^{*+}}-2\mu_{\Sigma^0\Sigma^{*0}}+\mu_{\Sigma^-\Sigma^{*-}})],\\
(35,1),(\overline{35},1)\quad & +(\mu_{p\Delta^+}+\mu_{n\Delta^0})-(\mu_{\Sigma^+\Sigma^{*+}}-\mu_{\Sigma^-\Sigma^{*-}})-2\sqrt{3}\mu_{\Lambda\Sigma^{*0}}+2(\mu_{\Xi^0\Xi^{*0}}-\mu_{\Xi^-\Xi^{*-}}),\\
(35,2),(\overline{35},2)\quad & +(\mu_{p\Delta^+}-\mu_{n\Delta^0})-(\mu_{\Sigma^+\Sigma^{*+}}-2\mu_{\Sigma^0\Sigma^{*0}}+\mu_{\Sigma^-\Sigma^{*-}}),\\
(64,0)\quad & +(\mu_{\Delta^{++}}+\mu_{\Delta^+}+\mu_{\Delta^0}+\mu_{\Delta^-})-4(\mu_{\Sigma^{*+}}+\mu_{\Sigma^{*0}}+\mu_{\Sigma^{*-}})+6(\mu_{\Xi^{*0}}+\mu_{\Xi^{*-}})-4\mu_{\Omega^-},\\
(64,1)\quad & +(3\mu_{\Delta^{++}}+\mu_{\Delta^+}-\mu_{\Delta^0}-3\mu_{\Delta^-})-10(\mu_{\Sigma^{*+}}-\mu_{\Sigma^{*-}})+10(\mu_{\Xi^{*0}}-\mu_{\Xi^{*-}}),\\
(64,2)\quad & +(\mu_{\Delta^{++}}-\mu_{\Delta^+}-\mu_{\Delta^0}+\mu_{\Delta^-})-2(\mu_{\Sigma^{*+}}-2\mu_{\Sigma^{*0}}+\mu_{\Sigma^{*-}}),\\
(64,3)\quad & +\mu_{\Delta^{++}}-3\mu_{\Delta^+}+3\mu_{\Delta^0}-\mu_{\Delta^-}.
\end{aligned}
\tag{44}
$$

Ideally, because the decuplet and octet-decuplet dipole moments are largely unknown (although see comments in the next section), it would be preferable to have relations written in terms of the octet only. However, the only reps distinct to a particular SU(3) product in the $J = 1$ sector are **35** and its conjugate (octet-decuplet transitions) and **64** (decuplet moments), and so such a reduction is impossible. However, once we assume the relations, there are only $27 - 18 = 9$ free moments, and exactly this many are well known; these are μ_{Ω^-} and all octet moments, including $\mu_{\Sigma^0\Lambda}$, but not μ_{Σ^0}. In terms of these, all 18 poorly known or unknown moments may be written. The predictions are presented in Table II.

Our prediction for the Δ^{++} dipole moment of $(5.42 \pm 0.49)\mu_N$ is certainly consistent with the PDG estimate $\mu_{\Delta^{++}} = 3.7$ to $7.5\mu_N$. The only other known dipole moment is $\mu_{p\Delta^+}$, which may be extracted from PDG values for photon helicity amplitudes $A_{\frac{1}{2},\frac{3}{2}}$. The relation is

$$\mu_{p\Delta^+} = -\frac{(A_{\frac{1}{2}}+\sqrt{3}A_{\frac{3}{2}})}{\sqrt{\pi\alpha k}}\frac{m_p}{1+\frac{m_p}{m_{\Delta^+}}}\sqrt{\frac{m_p}{m_{\Delta^+}}}, \tag{45}$$

where k, the photon momentum in the decay, is fixed by kinematics. This formula is obtained by comparing the amplitude for the decay in terms of $\mu_{p\Delta^+}$ (see, e.g., Ref. [22]) to the same amplitude in terms of helicity amplitudes (see, e.g., Ref. [24]). The PDG value is calculated to be $(3.53 \pm 0.09)\mu_N$, in unfavorable comparison with our prediction of $(2.52 \pm 0.23)\mu_N$. The quark model, on the other hand, predicts $2.66\mu_N$, whereas the large-N_c contracted symmetry predicts [10] the much closer $3.33\mu_N$ (both of these predictions are functions of $\mu_{p,n}$ only, and therefore have negligible uncertainties). That the SU(6) prediction is not closer to the experimental value than the quark-model prediction is surprising, because SU(6) contains the quark model, in a sense, as its lowest-order terms. We now describe this identification.

Neglecting only the **2695** means, of course, that the fit to dipole moments is made using only the **35** and **405** [the SU(6) singlet is absent for $J = 1$]. We make this restatement in order to compare to the nonrelativistic quark model (NRQM) results, which are obtained using only the **35**. To see this, note that the quark magnetic moment operator $(eQ_q/2M_q)\otimes\sigma_3$, for arbitrary values of

TABLE II. Magnetic moment predictions (in μ_N).

μ_{Σ^0}	0.86 ± 0.30	$\mu_{\Sigma^{*0}}$	0.37 ± 0.45	$\mu_{\Sigma^+\Sigma^{*+}}$	2.05 ± 0.04
$\mu_{\Delta^{++}}$	5.42 ± 0.49	$\mu_{\Sigma^{*-}}$	-2.94 ± 0.06	$\mu_{\Sigma^0\Sigma^{*0}}$	1.04 ± 0.21
μ_{Δ^+}	3.10 ± 0.46	$\mu_{\Xi^{*0}}$	0.60 ± 0.22	$\mu_{\Sigma^-\Sigma^{*-}}$	-0.26 ± 0.04
μ_{Δ^0}	0.16 ± 0.45	$\mu_{\Xi^{*-}}$	-2.46 ± 0.23	$\mu_{\Lambda\Sigma^{*0}}$	2.22 ± 0.09
μ_{Δ^-}	-3.41 ± 0.50	$\mu_{p\Delta^+}$	2.52 ± 0.23	$\mu_{\Xi^0\Xi^{*0}}$	2.07 ± 0.12
$\mu_{\Sigma^{*+}}$	3.05 ± 0.04	$\mu_{n\Delta^0}$	2.81 ± 0.23	$\mu_{\Xi^-\Xi^{*-}}$	-0.26 ± 0.12

$m_{u,d,s}$, not only fits into the **35** rep, but contains as many independent parameters (3) as the $J = 1$ part of the **35**. The NRQM results when, in addition, we set $m_u = m_d$, so that the number of independent parameters reduces to 2.

To illustrate this point, let the three initially independent parameters in the $J = 1$ part of the **35** be labeled μ_1, μ_Y, and μ_{I_3} to indicate their SU(3) content. In order to relate these parameters to quark magnetic moments, one must adopt normalizations consistent with those of the corresponding SU(3) generators:

$$\begin{aligned} \mu_1 &= +\frac{k}{\sqrt{3}}(\mu_u + \mu_d + \mu_s), \\ \mu_Y &= +\frac{k}{\sqrt{6}}(\mu_u + \mu_d - 2\mu_s), \\ \mu_{I_3} &= +\frac{k}{\sqrt{2}}(\mu_u - \mu_d), \end{aligned} \tag{46}$$

where k is a proportionality constant that is undetermined, because group theory alone does not set overall scales. The constraint $m_u = m_d$ becomes $\mu_u = -2\mu_d$, or

$$\sqrt{6}\mu_1 + \sqrt{3}\mu_Y - \mu_{I_3} = 0. \tag{47}$$

On the other hand, one may read off directly from the SU(6) Clebsch-Gordan tables:

$$\mu_{p,n} = +\frac{1}{18\sqrt{2}}(\sqrt{6}\mu_1 + \sqrt{3}\mu_Y \pm 5\mu_{I_3}), \tag{48}$$

and between these two equations one immediately obtains $\mu_p/\mu_n = -3/2$.

D. Higher multipole moments

Virtually none of the electric quadrupole or magnetic octupole moments are measured at this time; experimental values exist only for the transition quadrupole moment $Q_{p\Delta^+}$, and so a numerical analysis of the SU(6) relations would be meaningless. A discussion of the feasibility of measuring more Δ-N dipole and quadrupole transitions continues to appear in the recent literature [25]; the moments appear to be accessible in dedicated experiments, particularly those involving polarized electroproduction. The quadrupole moment of the Ω^- may soon be obtained through its spin oscillations when passing through a crystal [26]. Even the transition moments of the Σ^* and Ξ^*, for which neither parent nor daughter baryon is long lived, may be accessible through the Primakoff interactions in high-energy hyperon beams [27].

Both for mathematical completeness and for the benefit of future researchers, we display here the quadrupole moment relations. In this sector there are 18 independent parameters (10 decuplet moments and 8 octet-decuplet transitions) and 12 parameters associated with the **2695** rep. The 12 relations are given as follows:

$$\begin{array}{rl}
\underline{(\mathrm{SU}(3), I)} & \underline{\text{electric quadrupole moment combination}}, \\
(8_1,0) & +[(Q_{\Delta^{++}} + Q_{\Delta^+} + Q_{\Delta^0} + Q_{\Delta^-}) - (Q_{\Xi^{*0}} + Q_{\Xi^{*-}}) - 2Q_{\Omega^-}] \\
 & -2\sqrt{2}[(Q_{\Sigma^+\Sigma^{*+}} + Q_{\Sigma^0\Sigma^{*0}} + Q_{\Sigma^-\Sigma^{*-}}) + (Q_{\Xi^0\Xi^{*0}} + Q_{\Xi^-\Xi^{*-}})], \\
(8_1,1) & +[(3Q_{\Delta^{++}} + Q_{\Delta^+} - Q_{\Delta^0} - 3Q_{\Delta^-}) + 2(Q_{\Sigma^{*+}} - Q_{\Sigma^{*-}}) + (Q_{\Xi^{*0}} - Q_{\Xi^{*-}})] \\
 & -2\sqrt{2}[2(Q_{p\Delta^+} + Q_{n\Delta^0}) + (Q_{\Sigma^+\Sigma^{*+}} - Q_{\Sigma^-\Sigma^{*-}}) + \sqrt{3}Q_{\Lambda\Sigma^{*0}} + (Q_{\Xi^0\Xi^{*0}} - Q_{\Xi^-\Xi^{*-}})], \\
(10,1),(\overline{10},1) & +(Q_{p\Delta^+} + Q_{n\Delta^0}) - (Q_{\Sigma^+\Sigma^{*+}} - Q_{\Sigma^-\Sigma^{*-}}) - (Q_{\Xi^0\Xi^{*0}} - Q_{\Xi^-\Xi^{*-}}), \\
(27_1,0) & +4[3(Q_{\Delta^{++}} + Q_{\Delta^+} + Q_{\Delta^0} + Q_{\Delta^-}) - 5(Q_{\Sigma^{*+}} + Q_{\Sigma^{*0}} + Q_{\Sigma^{*-}}) - 3(Q_{\Xi^{*0}} + Q_{\Xi^{*-}}) + 9Q_{\Omega^-}] \\
 & +7\sqrt{2}[2(Q_{\Sigma^+\Sigma^{*+}} + Q_{\Sigma^0\Sigma^{*0}} + Q_{\Sigma^-\Sigma^{*-}}) - 3(Q_{\Xi^0\Xi^{*0}} + Q_{\Xi^-\Xi^{*-}})], \\
(27_1,1) & +8[(3Q_{\Delta^{++}} + Q_{\Delta^+} - Q_{\Delta^0} - 3Q_{\Delta^-}) - 3(Q_{\Sigma^{*+}} - Q_{\Sigma^{*-}}) - 4(Q_{\Xi^{*0}} - Q_{\Xi^{*-}})] \\
 & -7\sqrt{2}[(Q_{p\Delta^+} + Q_{n\Delta^0}) + 3(Q_{\Sigma^+\Sigma^{*+}} - Q_{\Sigma^-\Sigma^{*-}}) - 2\sqrt{3}Q_{\Lambda\Sigma^{*0}} - 2(Q_{\Xi^0\Xi^{*0}} - Q_{\Xi^-\Xi^{*-}})], \\
(27_1,2) & +8[3(Q_{\Delta^{++}} - Q_{\Delta^+} - Q_{\Delta^0} + Q_{\Delta^-}) + (Q_{\Sigma^{*+}} - 2Q_{\Sigma^{*0}} + Q_{\Sigma^{*-}})] \\
 & -7\sqrt{2}[3(Q_{p\Delta^+} - Q_{n\Delta^0}) + (Q_{\Sigma^+\Sigma^{*+}} - 2Q_{\Sigma^0\Sigma^{*0}} + Q_{\Sigma^-\Sigma^{*-}})], \\
(35,1),(\overline{35},1) & +(Q_{p\Delta^+} + Q_{n\Delta^0}) - (Q_{\Sigma^+\Sigma^{*+}} - Q_{\Sigma^-\Sigma^{*-}}) - 2\sqrt{3}Q_{\Lambda\Sigma^{*0}} + 2(Q_{\Xi^0\Xi^{*0}} - {}_{\Xi^-\Xi^{*-}}), \\
(35,2),(\overline{35},2) & +(Q_{p\Delta^+} - Q_{n\Delta^0}) - (Q_{\Sigma^+\Sigma^{*+}} - 2Q_{\Sigma^0\Sigma^{*0}} + Q_{\Sigma^-\Sigma^{*-}}), \\
(64,0) & +(Q_{\Delta^{++}} + Q_{\Delta^+} + Q_{\Delta^0} + Q_{\Delta^-}) - 4(Q_{\Sigma^{*+}} + Q_{\Sigma^{*0}} + Q_{\Sigma^{*-}}) + 6(Q_{\Xi^{*0}} + Q_{\Xi^{*-}}) - 4Q_{\Omega^-}, \\
(64,1) & +(3Q_{\Delta^{++}} + Q_{\Delta^+} - Q_{\Delta^0} - 3Q_{\Delta^-}) - 10(Q_{\Sigma^{*+}} - Q_{\Sigma^{*-}}) + 10(Q_{\Xi^{*0}} - Q_{\Xi^{*-}}), \\
(64,2) & +(Q_{\Delta^{++}} - Q_{\Delta^+} - Q_{\Delta^0} + Q_{\Delta^-}) - 2(Q_{\Sigma^{*+}} - 2Q_{\Sigma^{*0}} + Q_{\Sigma^{*-}}), \\
(64,3) & +Q_{\Delta^{++}} - 3Q_{\Delta^+} + 3Q_{\Delta^0} - Q_{\Delta^-}.
\end{array} \tag{49}$$

The situation for the octupole moments is in fact trivial. There are ten parameters and ten relations, because the $J = 3$ block of the 92×92 orthogonal matrix is identical to the pure SU(3) matrix $\mathcal{C}_b$. This in turn follows because the only combinations with $J = 3$ originate in decuplet-decuplet bilinears. The interpretation of this result is that only the **2695** rep contributes to octupole moments, and so these moments, if they are ever measured, should be numerically uniformly tiny.

VII. CONCLUSIONS

To summarize our findings, we have exhibited a new method to compute conveniently all Clebsch-Gordan coefficients associated with the product $\overline{\mathbf{56}} \otimes \mathbf{56}$ (as well as other products), and we have displayed these coefficients for the particular bilinear combinations with $\Delta I_3 = \Delta Y = \Delta J_3 = 0$. All the others, useful for baryon decay processes, can be obtained from those in this paper by means of the SU(2) or SU(3) Wigner-Eckart theorem.

From the coefficients we have compiled all baryon mass and electromagnetic moment relations resulting from ignoring the **2695** component in SU(6). Violations of the mass relations can be explained with naive estimates of the neglected operators, and we have obtained a prediction for the size of Σ^0-Λ mixing. We have shown that enough magnetic dipole moments are experimentally well known to predict the others, and have used these relations to show agreement with the experimental value for $\mu_{\Delta^{++}}$ but disagreement for $\mu_{p\Delta^+}$. The latter result may be an indication of the superiority of the large-N_c predictions in general; the verification of this statement awaits the systematic analysis of the large-N_c contracted spin-flavor symmetry.

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