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Extended loop representation of quantum gravity

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A new representation of quantum gravity is developed. This formulation is based on an extension of the group of loops. The enlarged group that we call the extended loop group behaves locally as an infinite dimensional Lie group. Quantum gravity can be realized on the state space of extended loop-dependent wave functions. The extended representation generalizes the loop representation and contains this representation as a particular case. The resulting diffeomorphism and Hamiltonian constraints take a very simple form and allow us to apply functional methods and simplify the loop calculus. In particular we show that the constraints are linear in the momenta. The nondegenerate solutions known in the loop representation are also solutions of the constraints in the new representation. An approach to the regularization problems associated with the formal calculus is performed. We show that the solutions are generalized knot invariants, smooth in the extended variables, and any framing is unnecessary.

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I. INTRODUCTION

The formulation of general relativity in terms of the Ashtekar variables has opened new perspectives in the canonical quantization program of gravity [1,2]. The new set of canonical variables introduced by Ashtekar are the triads E_i^{ax} (the projections of the tetrads onto a three-surface) and a complex SU(2) connection A_{ax}^i. The fundamental result of this approach is that the constraint equations emerging from the Hamiltonian formulation of general relativity become polynomial functions of the variables. In addition, the formalism (that uses a connection as the configuration variable) casts general relativity in a fashion that closely resembles Yang-Mills theories. This fact allows to import several useful techniques from Yang-Mills theories into general relativity. One of particular importance is the loop representation [3–5].

The loop representation provides a geometric description of the Hamiltonian formulation of the theory (gauge theories and quantum gravity) in terms of loops. The loop representation can be constructed by means of the noncanonical algebra of a complete set of gauge-invariant operators that act on a state space of loop wave functions $\psi(\gamma)$ [3,4]. Once the complete set of invariant operators is realized on the space of loops, the action of any other gauge-invariant operator (such as the Hamiltonian) can be obtained from them. Another equivalent way to obtain the loop representation is through the loop transform. The loop transform connects the states between the connection and the loop representation and choosing a factor ordering, the quantum operators acting in the connection state space can be translated to the loop wave functions. This procedure explicitly shows the role of holonomies as the basic building blocks of any loop dependent object.

The introduction of the loop representation for quantum gravity allows us to immediately code the invariance under spatial diffeomorphisms in the requirement of knot invariance. Also for the first time a large class of solutions to the Wheeler-DeWitt equation has been found in terms of nonintersecting knot invariants [4]. It is not clear however how to make these solutions correspond to nondegenerate metrics [6] (a possible solution is the idea of "weaves" [7]). Another alternative is to consider knot invariants of intersecting loops and solve the Wheeler-DeWitt equation in loop space [5]. The definition of many of these states is complicated by regularization ambiguities. Since loops are one-dimensional objects living in a three-dimensional manifold, they naturally lead to the appearance of distributional expressions. In particular the few knot invariants for which we have analytic expressions require the introduction of regularizations (framings) in the case of intersecting knots. Some invariants even require a regularization for smooth loops [8,9].

There are good reasons to consider an extension of the loop representation. From a mathematical point of view, the group of loops is not a Lie group and it is not even known how to give a manifold structure to the loop space. An extension of the loop space with the structure of an infinite dimensional Lie group has been recently proposed [10]. The usual group of loops is a subgroup of this extended group and generalized holonomies may be defined in the extended space. Moreover, since the extended representation uses fields instead of distributional objects, the regularization difficulties associated with the wave functions in the loop representation disappear. On the other side, Marolf [11] has recently studied the equivalence between the connection and the loop representation in the case of 2+1 gravity. More precisely, he has shown that some problems and ambiguities arise in the kernel of the loop transform. For instance, in the boost sector the loop transform has a very large nontrivial kernel. He also noticed that the introduction of extended loops allow us to remove many of these problems. Even though the introduction of the extended representation is not mandatory in this case (the equivalence may be recovered in the loop representation by introducing a nontrivial measure in the loop transform [12]), extended loops seem to give a very simple and natural solution to this problem.

In this paper we develop a formalism that allows us to represent quantum general relativity in the extended space. The general ideas of the extended representation have been discussed in Ref. [13]. This representation can be viewed as a generalization of the loop representation. As we will show, there are different relevant groups that can be introduced in the extended space. We choose one of them to develop a representation of quantum gravity, the group called $\mathcal{D}_0$. In this sense, the representation developed here is the most simple and general of all possibles extended representations that can be constructed for quantum gravity. This representation has the characteristic to present the constraint equations in a very simple form. Moreover, the regularization problem associated with the formal action of the constraint operators simplifies considerably, being the most relevant result the removal of the typical ambiguities associated with the framing dependence that the knot invariants have in the conventional loop representation.

We organize the paper as follows: in Sec. II we make a brief review of the definition and properties of the loop coordinates, that preceded and motivated the definition of the extended group. In Sec. III the extended loop group is introduced in a formal way. We only do here a quick review of the loop coordinates and the extended loop group in order to make the article moderately self contained. A more complete treatment of these subjects can be found in Ref. [10]. Section IV is dedicated to the construction of the extended loop representation of quantum gravity. In first place we analyze the general properties of the wave functions and the operators in the extended representation. We show that the constraints are linear in the momenta. In Sec. IV A the diffeomorphism constraint is formulated in the extended space. In Sec. IV B we consider the realization of the Hamiltonian constraint. The reduction of the extended Hamiltonian constraint to the corresponding in the loop representation is performed in Sec. IV C. This procedure allows us to clarify the meaning of some of the new ingredients that appear in the extended Hamiltonian constraint. In Sec. V we develop the formal calculus considering the action of the Hamiltonian on the second coefficient of the Alexander-Conway generalized knot polynomial. We will verify that this state is annihilated by the extended Hamiltonian constraint. The issue of the regularization is considered in Sec. VI. In Sec. VI A we show that the wave functions are smooth functionals of the extended variables (in a restricted diffeomorphism invariant domain). The regulated diffeomorphism and Hamiltonian constraints are analyzed in Sec. VI B. Some conclusions and final comments are included in Sec. VII.

II. THE LOOP COORDINATES

Because of their simple behavior under gauge transformations, holonomies have been widely used in the description of gauge theories. Holonomies can be viewed as an homomorphism going from a group structure defined in terms of equivalence classes of closed curves onto a Lie group G. Each equivalence class is called a loop and the group structure defined by them is called the group of loops. The group of loops is the basic underlying structure to all the nonlocal formulations of gauge theories in terms of holonomies [3].

As it was just mentioned, among these formulations we find the loop representation, based on a quantum representation of the Hamiltonian gauge theory in terms of loops. In the loop representation wave functions are functionals of loops and they are connected with the states in the connection representation by the loop transform

$$\psi(\gamma) = \int d_\mu[A]\psi(A)W_A(\gamma), \tag{1}$$

where

$$W_A(\gamma) = \mathrm{Tr}[H_A(\gamma)] = \mathrm{Tr}\left[P\exp\left(\oint_\gamma dy^a A_a(y)\right)\right] \tag{2}$$

is the Wilson loop functional. All the gauge invariant information present in the theory can be retrieved from the holonomy. This means that the only information we really need to know from loops is the one used in the definition of $H_A(\gamma)$. One can write the holonomy as

$$H_A(\gamma) = 1 + \sum_{n=1}^{\infty}\int dx_1^3\cdots dx_n^3 A_{a_1}(x_1)\cdots A_{a_n}(x_n)X^{a_1\cdots a_n}(x_1,\ldots,x_n;\gamma), \tag{3}$$

where the loop dependent objects X are given by

$$X^{a_1\cdots a_n}(x_1,\ldots,x_n;\gamma) = \oint_\gamma dy_n^{a_n}\cdots\oint_\gamma dy_1^{a_1}\delta(x_n-y_n)\cdots\delta(x_1-y_1)\Theta_\gamma(o,y_1,\ldots,y_n). \tag{4}$$

The Θ function orders the points along the contour starting at the origin of the loop o. These relationships define the X objects of "rank" n, that we call the multitangents of the loop γ. They behave as multivector densities under general coordinate transformations. The fundamental property is that no more information from the loop is needed in order to compute the holonomy than what is present in the multitangents fields of all rank.

It is convenient to introduce the following notation for the multitangent fields:

$$X^{a_1\cdots a_n}(x_1,\ldots,x_n;\gamma)\equiv X^{a_1x_1\cdots a_nx_n}(\gamma)\equiv X^{\mu_1\cdots\mu_n}(\gamma)\equiv X^{\boldsymbol{\mu}}(\gamma). \tag{5}$$

The boldface index $\boldsymbol{\mu}$ indicates the set of indices $(\mu_1,\ldots,\mu_n)$ and μ_i represents the pair of variables (a_i,x_i), with $a_i=1,2,3$ and $x_i\in\mathcal{R}^3$.

The X's are not independent quantities, they obey two kinds of constraints: the algebraic and differential constraints.

The algebraic constraints arise from the properties of the Θ function under the interchange of the order of the indices and have the general form

$$X^{\underline{\mu_1\cdots\mu_k}\mu_{k+1}\cdots\mu_n}\equiv\sum_{P_k}X^{P_k(\mu_1\cdots\mu_n)}=X^{\mu_1\cdots\mu_k}X^{\mu_{k+1}\cdots\mu_n}, \tag{6}$$

where the sum gives over all the permutations of the μ variables which preserve the ordering of $(\mu_1,\ldots,\mu_k)$ and the $(\mu_{k+1},\ldots,\mu_n)$ among themselves. For example, for the rank three component we have

$$X^{\underline{\mu_1}\mu_2\mu_3}\equiv X^{\mu_1\mu_2\mu_3}+X^{\mu_2\mu_1\mu_3}+X^{\mu_2\mu_3\mu_1}=X^{\mu_1}X^{\mu_2\mu_3}. \tag{7}$$

The differential constraints ensure that $H_A(\gamma)$ has the correct transformation properties under gauge transformations and can be derived directly from the definition of the multitangent fields as line integrals of distributions along closed curves. They are given by

$$\partial_{\mu_i}X^{\mu_1\cdots\mu_i\cdots\mu_n}=[\delta(x_i-x_{i-1})-\delta(x_i-x_{i+1})]X^{\mu_1\cdots\mu_{i-1}\mu_{i+1}\cdots\mu_n}, \tag{8}$$

where $\partial_{\mu_i}\equiv\partial/\partial x_i^{a_i}$. Notice that the differential constraint carries information about the origin of the loop because for $i=1$ or $i=n$, the points x_0 and x_{n+1} have to be taken as the base point o of the loop.

The first idea was to find a set of independent quantities that completely specify the loop dependent information contained in the holonomy. The solution of the constraints can be outlined in the following way:

$$\mathbf{X}\text{ fields}\leftrightarrow\ {\scriptstyle\text{solving } AC \atop \text{keeping } DC}\ \leftrightarrow\mathbf{F}\text{ fields}\leftrightarrow\text{solving } DC\leftrightarrow\mathbf{Y}\text{ fields},$$

$$X^{\boldsymbol{\mu}}(\gamma)=\{\exp[F(\gamma)]\}^{\boldsymbol{\mu}},\quad F^{\boldsymbol{\mu}}(\gamma)=\sigma^{\boldsymbol{\mu}}{}_{\boldsymbol{\nu}}Y^{\boldsymbol{\nu}}(\gamma),$$
$$F^{\boldsymbol{\mu}}(\gamma)=[\ln X(\gamma)]^{\boldsymbol{\mu}},\quad Y^{\boldsymbol{\mu}}(\gamma)=\delta_T{}^{\boldsymbol{\mu}}{}_{\boldsymbol{\nu}}F^{\boldsymbol{\nu}}(\gamma),$$

where a generalized Einstein convention of sum over repeated indices is assumed, given by

$$A_{\boldsymbol{\mu}}B^{\boldsymbol{\mu}}:=\sum_{n=0}^{\infty}\sum_{a_1=1}^{3}\cdots\sum_{a_n=1}^{3}\int d^3x_1\cdots\int d^3x_n A_{a_1x_1\cdots a_nx_n}B^{a_1x_1\cdots a_nx_n} \tag{9}$$

or, in shorthand,

$$A\cdot B\equiv A_{\boldsymbol{\mu}}B^{\boldsymbol{\mu}}:=\sum_{n=0}^{\infty}A_{\mu_1\cdots\mu_n}B^{\mu_1\cdots\mu_n} \tag{10}$$

with

$$A_{\mu_i}B^{\mu_i}:=\sum_{a_i=1}^{3}\int d^3x_i A_{a_ix_i}B^{a_ix_i}. \tag{11}$$

The F's are multivector density fields that satisfy the homogeneous algebraic constraint and the differential constraint. The Y's satisfy both homogeneous algebraic and differential constraints and define the loop coordinates associated with the loop γ. δ_T is the projector on the space of the transverse functions

$$\delta_T{}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}=\delta_{n,m}\delta_T{}^{\mu_1}{}_{\nu_1}\cdots\delta_T{}^{\mu_n}{}_{\nu_n}, \tag{12}$$

$$\delta_T{}^{ax}{}_{by}=\delta^{ax}{}_{by}-\phi^{ax}{}_{y,b}, \tag{13}$$

where ϕ is any suitable function that satisfies

$$\frac{\partial}{\partial x^a}\phi^{ax}{}_y=-\delta(x-y). \tag{14}$$

The matrix σ is nondiagonal and generates nontrivial representations of the diffeomorphism group. It can be defined by the recursive expression

$$\sigma^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}=\begin{cases}\delta_T{}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_n} & \text{if } m=n,\\ Q^{\mu_1\cdots\mu_n}_{\rho_1\cdots\rho_{n-1}}\sigma^{\rho_1\cdots\rho_{n-1}}{}_{\nu_1\cdots\nu_m} & \text{if } m<n,\\ 0 & \text{if } m>n,\end{cases} \tag{15}$$

with

$$Q^{a_1x_1\cdots a_nx_n}_{c_1y_1\cdots c_{n-1}y_{n-1}}\equiv\sum_{j=1}^{n}\delta^{a_1x_1\cdots a_{j-1}x_{j-1}}_{c_1y_1\cdots c_{j-1}y_{j-1}}(\phi^{a_jx_j}_{y_j}-\phi^{a_jx_j}_{y_{j-1}})\delta_T{}^{a_{j+1}x_{j+1}\cdots a_nx_n}_{c_jy_j\cdots c_{n-1}y_{n-1}}. \tag{16}$$

For the rank two component, the relation between the F's and the Y's fields is

$$F^{ax\,by} = Y^{ax\,by} + \phi^{ax}{}_y Y^{by} - \phi^{by}{}_x Y^{ax} - \phi^{ax}{}_z \phi^{by}{}_{z,c} Y^{cz} + \phi^{[by}{}_o Y^{ax]}. \tag{17}$$

The function ϕ fixes a prescription for the decomposition of the multitensors in transverse and longitudinal parts. The quantities σ have definite transversal properties

$$\delta_T \cdot \sigma = \delta_T, \tag{18}$$

$$\sigma \cdot \delta_T = \sigma, \tag{19}$$

and under a change of the prescription $\phi^{ax}_{1y} \to \phi^{ax}_{2y}$ we have

$$\sigma[\phi_1] = \sigma[\phi_2] \cdot \sigma[\phi_1]. \tag{20}$$

The definition of the exponential and the inverse operation that connects the multitangent fields with the algebraic free coordinates involves a particular composition law between the components of the fields. Explicitly,

$$\{\exp[F(\gamma)]\}^{\mu} := \sum_{k=0}^{\infty} \frac{1}{k!} [F \times \overset{k \text{ times}}{\cdots} \times F]^{\mu} \tag{21}$$

with

$$[F \times F]^{\mu_1 \cdots \mu_m} := \sum_{i=1}^{n-1} F^{\mu_1 \cdots \mu_i} F^{\mu_{i+1} \cdots \mu_n}. \tag{22}$$

This composition law is associative and has the important property that satisfies the differential constraint if all F's do.

The loop coordinates have several interesting applications at the level of the loop representation of gauge theories and quantum gravity. What is more important, they allow us to show that there is a local Lie group structure associated with the loop space. We will now proceed to introduce this group.

III. THE EXTENDED LOOP GROUP

Consider a set of arbitrary multitensor densities of any rank and construct with them the following vectorlike object $\mathbf{E}$:

$$\mathbf{E} = (E, E^{\mu_1}, \ldots, E^{\mu_1 \cdots \mu_n}, \ldots) \equiv (E, \vec{E}), \tag{23}$$

where E is a real number and $E^{\mu_1 \cdots \mu_n}$ (for any $n \neq 0$) is an arbitrary multivector density not restricted by any constraint. The set of all $\mathbf{E}$'s has the structure of a vector space, denoted as $\mathcal{E}$.

We can introduce a product law in $\mathcal{E}$ in the following way: given two vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ we define $\mathbf{E}_1 \times \mathbf{E}_2$ as the vector with components

$$\mathbf{E}_1 \times \mathbf{E}_2 = (E_1 E_2, E_1 \vec{E}_2 + \vec{E}_1 E_2 + \vec{E}_1 \times \vec{E}_2), \tag{24}$$

where $\vec{E}_1 \times \vec{E}_2$ is given by

$$(\vec{E}_1 \times \vec{E}_2)^{\mu_1 \cdots \mu_n} = \sum_{i=1}^{n-1} E_1^{\mu_1 \cdots \mu_i} E_2^{\mu_{i+1} \cdots \mu_n}. \tag{25}$$

We see that the $\times$ product is an extension of the composition law between the algebraic free coordinates introduced before that includes the zero rank component of the vector. In fact, it can be written as

$$(\mathbf{E}_1 \times \mathbf{E}_2)^{\mu_1 \cdots \mu_n} = \sum_{i=0}^{n} E_1^{\mu_1 \cdots \mu_i} E_2^{\mu_{i+1} \cdots \mu_n} \tag{26}$$

with the convention

$$E^{\mu_1 \cdots \mu_0} = E^{\mu_{n+1} \cdots \mu_n} = E. \tag{27}$$

The product law is associative and distributive with respect to the addition of vectors. It has a null element (the null vector) and an identity element, given by

$$\mathbf{I} = (1, 0, \ldots, 0, \ldots). \tag{28}$$

An inverse element exists for all vectors with nonvanishing zero rank component. It is given by

$$\mathbf{E}^{-1} = E^{-1}\mathbf{I} + \sum_{i=1}^{\infty} (-1)^i E^{-i-1} (\mathbf{E} - E\mathbf{I})^i \tag{29}$$

such that

$$\mathbf{E} \times \mathbf{E}^{-1} = \mathbf{E}^{-1} \times \mathbf{E} = \mathbf{I}. \tag{30}$$

The set of all vectors with nonvanishing zero rank component forms a group with the $\times$ product law.

Consider now a subset $\mathcal{X} \subset \mathcal{E}$ whose elements $\mathbf{X} \equiv (X, \vec{X})$ obey some supplementary conditions related to the differential and algebraic constraints. Introducing different types of conditions one can define different subgroups. We consider here the following three base pointed subgroups of the extended group, denoted by $\mathcal{D}_0$, $\mathcal{M}_0$, and $\mathcal{X}_0$. They are defined in the following way: $\mathcal{D}_0 \to \vec{X}$ satisfies the differential constraint and $X \neq 0$; $\mathcal{M}_0 \to \vec{X}$ satisfies the differential constraint and $[X^{-1}]^{\mu_1 \cdots \mu_n} = (-1)^n X^{\mu_n \cdots \mu_1}$ and $X = 1$; $\mathcal{X}_0 \to \vec{X}$ satisfies the differential and algebraic constraints and $X = 1$. For each of these sets one can check that they are closed under the group composition law. It is straightforward to see that the algebraic constraint also implies the condition satisfied by the elements of $\mathcal{M}_0$. These groups satisfy then the inclusion relation

$$\mathcal{X}_0 \subset \mathcal{M}_0 \subset \mathcal{D}_0. \tag{31}$$

Furthermore, the group of loops is a subgroup of the group $\mathcal{X}_0$, since any multitangent field $\mathbf{X}(\gamma)$ is contained in $\mathcal{X}_0$ and they obey the relationship

$$\mathbf{X}(\gamma_1 \circ \gamma_2) = \mathbf{X}(\gamma_1) \times \mathbf{X}(\gamma_2), \tag{32}$$

where the circle indicates the group product between loops.

An important property of the constraints is that *any* multitensor density X^{μ} that satisfies them can be used to generate a *gauge covariant* quantity

$$H_A(\mathbf{X}) = \mathbf{A} \cdot \mathbf{X} = A_{\mu} X^{\mu}, \tag{33}$$

where $\mathbf{A}$ is the following object constructed with the connections: $\mathbf{A} = (1, A_{\mu_1}, \ldots, A_{\mu_1\cdots\mu_n}, \ldots)$, with $A_{\mu_1\cdots\mu_n} = A_{\mu_1}\cdots A_{\mu_n}$ and $A_{\mu_i} \equiv A_{a_i x_i} := A_{a_i}(x_i)$. When restricted to the multitangents $\mathbf{X}(\gamma)$ associated with loops, the resulting object is the holonomy. It is this property that allows us to extend loops to a more general structure. One can in general deal with arbitrary multitensor densities X (not necessarily related to loops) and construct with them gauge invariant objects. The multitensor densities need not to be distributional functions as the multitangents associated with a loop. They could be ordinary functions on the manifold.

Matrix representations of the above groups can be generated through the generalized holonomies associated with a general connection A_{ax}. They satisfy

$$H_A(\mathbf{X}_1)H_A(\mathbf{X}_1) = H_A(\mathbf{X}_1 \times \mathbf{X}_2). \tag{34}$$

The differential constraint imposed on $\mathbf{X}$ assures $H_A(\mathbf{X})$ to be a gauge covariant quantity. The trace of the extended holonomy, that defines a generalized Wilson functional, is a gauge invariant quantity for any $\mathbf{X} \in \mathcal{D}_0$. Since one can represent any gauge invariant object using the X's, one can represent the theory *entirely* in terms of X's.

An extended representation may be introduced for any subgroup of the larger constrained group $\mathcal{D}_0$. In each case we will obtain a gauge invariant representation of a theory with particular properties derived from the constraints. For quantum gravity [where the connections are elements of the SU(2) algebra] the matrices $H_A(\mathbf{X})$ would belong to the general group GL$(2,c)$ when $\mathbf{X} \in \mathcal{D}_0$, whereas for the other subgroups the generalized holonomies would be elements of the SU(2) group (in fact, the condition satisfied by the elements of $\mathcal{M}_0$ is the weakest for this property to hold). We shall consider here the simplest case of a representation with wave functions defined on the larger group $\mathcal{D}_0$. However it is important to notice that more restricted representations could be relevant and it is still unclear which of them behaves as the dual of the connection representation.

IV. THE EXTENDED LOOP REPRESENTATION OF QUANTUM GRAVITY

Let us start by considering some general properties of the wave functions in the extended representation. These wave functions are related to the states in the connection representation through the generalized (formal) loop transform

$$\psi(\mathbf{X}) = \int d_\mu[A]\psi(A)W_A(\mathbf{X}) \tag{35}$$

with

$$W_A(\mathbf{X}) = \mathrm{Tr}[A_\mu]X^\mu. \tag{36}$$

The generalized Wilson functional satisfies a set of identities that correspond to the Mandelstam identities for the SU(2) algebra. The cyclic property of the traces implies

$$W_A(\mathbf{X}_1 \times \mathbf{X}_2) = W_A(\mathbf{X}_2 \times \mathbf{X}_1) \tag{37}$$

while taking into account the specific properties of the SU(2) algebra, one gets

$$W_A(\mathbf{X}) = W_A(\bar{\mathbf{X}}) \tag{38}$$

and

$$W_A(\mathbf{X}_1)W_A(\mathbf{X}_2) = W_A(\mathbf{X}_1 \times \mathbf{X}_2) + W_A(\mathbf{X}_1 \times \bar{\mathbf{X}}_2), \tag{39}$$

where

$$\bar{X}^{\mu_1\cdots\mu_n} \equiv (-1)^n X^{\mu_n\cdots\mu_1}. \tag{40}$$

The identities satisfied by the Wilson functional are carried over the wave functions in a direct way. We get

$$\psi(\mathbf{X}_1 \times \mathbf{X}_2) = \psi(\mathbf{X}_2 \times \mathbf{X}_1), \tag{41}$$

$$\psi(\mathbf{X}) = \psi(\bar{\mathbf{X}}), \tag{42}$$

$$\psi(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) + \psi(\mathbf{X}_1 \times \mathbf{X}_2 \times \bar{\mathbf{X}}_3)$$
$$= \psi(\mathbf{X}_2 \times \mathbf{X}_1 \times \mathbf{X}_3) + \psi(\mathbf{X}_2 \times \mathbf{X}_1 \times \bar{\mathbf{X}}_3). \tag{43}$$

Notice that due to the cyclic property of the traces any reference to the base point o is lost in ψ. Furthermore, Eqs. (38) and (42) imply that W_A and ψ are functions of the combination

$$R^{\mu_1\cdots\mu_n} = \tfrac{1}{2}[X^{\mu_1\cdots\mu_n} + (-1)^n X^{\mu_n\cdots\mu_1}], \tag{44}$$

where the $\mathbf{R}$'s satisfy the following symmetry property under the inversion of the indices:

$$R^{\mu_1\cdots\mu_n} = (-1)^n R^{\mu_n\cdots\mu_1}. \tag{45}$$

The linearity of the extended holonomy in the multivector components $X^{\mu_1\cdots\mu_n}$ induces the same property on the wave functions. This means that $\psi(\mathbf{X})$ takes the general form

$$\psi(\mathbf{X}) = D_\mu X^\mu. \tag{46}$$

The coefficients $D_{\mu_1\cdots\mu_n}$ contain all the information about $\psi(\mathbf{X})$ and satisfy a set of constraints that may be derived from the Maldestam identities. They are

$$D_{\mu_1\cdots\mu_n} = D_{(\mu_1\cdots\mu_n)_c}, \tag{47}$$

$$D_{\mu_1\cdots\mu_n} = (-1)^n D_{\mu_n\cdots\mu_1}, \tag{48}$$

$$D_{\mu_1\cdots\mu_k\mu_{k+1}\cdots\mu_n} + (-1)^k D_{\mu_k\cdots\mu_1\mu_{k+1}\cdots\mu_n} = k^{-1}D_{(\mu_1\cdots\mu_k)_c\mu_{k+1}\cdots\mu_n} + (-1)^k k^{-1} D_{(\mu_k\cdots\mu_1)_c\mu_{k+1}\cdots\mu_n} \quad \text{for all } k, \tag{49}$$

where c indicates the cyclic combination of indexes. The linearity is a remarkable property of the wave functions in the extended representation. Notice that all the wave functions known in the loop representation for quantum gravity have this property when they are written in terms of the multitangents fields. Moreover, this property will be inherited also by the operators that we can construct in the extended representation. In general, the linearity over the wave functions could be imposed by means of the "linearity constraint" $\mathcal{L}$:

$$\mathcal{L}(\mathbf{X}')\psi(\mathbf{X}) \equiv X'^{\mu\nu}\frac{\delta^2}{\delta X^\mu \delta X^\nu}\psi(\mathbf{X}) = 0, \tag{50}$$

where $\mathbf{X}'$ is any object that satisfies the differential constraints. Any observable of the theory has to commute with the linearity constraint.

In the quantum version of the canonical formulation of general relativity, the classical variables are promoted to operators that act on wave functionals of the connection in the following way:

$$\hat{A}^i_{ax}\psi(A) = A^i_{ax}\psi(A), \tag{51}$$

$$\hat{E}^{ax}_i\psi(A) = \frac{\delta}{\delta A^i_{ax}}\psi(A). \tag{52}$$

The diffeomorphism and Hamiltonian constraint in this representation are

$$\tilde{\mathcal{C}}_{ax} = \frac{\delta}{\delta A^i_{bx}}F^i_{ba}(x), \tag{53}$$

$$\tilde{\mathcal{H}}(x) = \epsilon^{ijk}\frac{\delta}{\delta A^k_{bx}}\frac{\delta}{\delta A^j_{ax}}F^i_{ba}(x). \tag{54}$$

Notice that we choose the factor ordering that puts the triads to the left. It is from this factor ordering that one can obtain the loop representation via the loop transform in a simple way. One can see that with this factor ordering the regularized diffeomorphism constraint generates infinitesimal diffeomorphism transformations on the wave functions [17].

We now analyze how to go from the connection to the extended representation by means of the loop transform. In spite that the extended loop transform is only formally defined, one can use it as an heuristic method to generate the constraints. In particular, the algebra of a set of operators in the connection representation can be correctly implemented in the linear space of extended wave functions using the formal transform. The same thing happens in the loop representation.

A complete set of gauge invariant operators (the Rovelli-Smolin operators) can be implemented in the extended space and it is possible to construct any other gauge invariant operator from them. In what follows we shall restrict to consider only the diffeomorphism and Hamiltonian operators.

A. The diffeomorphism constraint

Let us consider first the diffeomorphism constraint operator. The action of this constraint on the wave functions $\psi(\mathbf{R})$ is defined by the expression

$$\mathcal{C}_{ax}\psi(\mathbf{R}) = \int d_\mu[A]W_A(\mathbf{R})[\tilde{\mathcal{C}}_{ax}\psi(A)]. \tag{55}$$

The constraint acting on $\psi(A)$ can be applied on the generalized Wilson functional integrating (formally) by parts. So

$$\mathcal{C}_{ax}\psi(\mathbf{R}) = \int d_\mu[A]\psi(A)\left(F^i_{ab}(x)\frac{\delta}{\delta A^i_{bx}}W_A(\mathbf{R})\right). \tag{56}$$

In order to calculate the quantity in brackets in the last equation we introduce some suitable notation. Let $\delta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}$ be defined as

$$\delta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = \begin{cases} \delta^{\alpha_1}_{\beta_1}\cdots\delta^{\alpha_n}_{\beta_n} \text{ if } n(\boldsymbol{\alpha}) = n(\boldsymbol{\beta}) = n \geq 1, \\ 1 \text{ if } n(\boldsymbol{\alpha}) = n(\boldsymbol{\beta}) = 0, \\ 0 \text{ in other case,} \end{cases} \tag{57}$$

where $n(\boldsymbol{\alpha})$ is the number of indexes of the set $\boldsymbol{\alpha}$. The functional derivative of any product of A's can be written as

$$\frac{\delta}{\delta A^i_{bx}}(A_{\boldsymbol{\alpha}}) = A_{\boldsymbol{\mu}}\tau^i A_{\boldsymbol{\nu}}\delta^{\boldsymbol{\mu}bx\boldsymbol{\nu}}_{\boldsymbol{\alpha}}, \tag{58}$$

where the τ's are the generators of the SU(2) algebra. Taking the trace in the above expression we get

$$\begin{aligned}\frac{\delta}{\delta A^i_{bx}}\mathrm{Tr}[A_{\boldsymbol{\alpha}}] &= \mathrm{Tr}[\tau^i A_{\boldsymbol{\beta}}]\delta^{\boldsymbol{\beta}}_{\boldsymbol{\nu\mu}}\delta^{\boldsymbol{\mu}bx\boldsymbol{\nu}}_{\boldsymbol{\alpha}} \\ &= \mathrm{Tr}[\tau^i A_{\boldsymbol{\beta}}]\delta^{(bx\boldsymbol{\beta})_c}_{\boldsymbol{\alpha}}. \end{aligned}\tag{59}$$

The curvature tensor can be written as

$$F_{ab}(x) = A_{\boldsymbol{\nu}}\mathcal{F}_{ab}{}^{\boldsymbol{\nu}}(x), \tag{60}$$

where $\mathcal{F}_{ab}$ represents the following element of the algebra of the group:

$$\mathcal{F}_{ab}{}^{\boldsymbol{\nu}}(x) = \delta_{1,n(\boldsymbol{\nu})}\mathcal{F}_{ab}{}^{\nu_1}(x) + \delta_{2,n(\boldsymbol{\nu})}\mathcal{F}_{ab}{}^{\nu_1\nu_2}(x) \tag{61}$$

with

$$\mathcal{F}_{ab}{}^{a_1x_1}(x) = \delta^{a_1d}_{ab}\partial_d\delta(x_1 - x), \tag{62}$$

$$\mathcal{F}_{ab}{}^{a_1x_1,a_2x_2}(x) = \delta^{a_1a_2}_{ab}\delta(x_1 - x)\delta(x_2 - x). \tag{63}$$

Using (59) and (60) we obtain the following expression for the action of the diffeomorphism constraint on the generalized Wilson functional:

$$\begin{aligned}F^i_{ab}(x)\frac{\delta}{\delta A^i_{bx}}\mathrm{Tr}[A_{\boldsymbol{\alpha}}]R^{\boldsymbol{\alpha}} &= \mathrm{Tr}[F_{ab}(x)A_{\boldsymbol{\beta}}]\delta^{(ax\boldsymbol{\beta})_c}_{\boldsymbol{\alpha}}R^{\boldsymbol{\alpha}} \\ &= \mathrm{Tr}[A_{\boldsymbol{\rho}}]\delta^{\boldsymbol{\rho}}_{\boldsymbol{\nu\beta}}\mathcal{F}_{ab}(x)^{\boldsymbol{\nu}}R^{(bx\boldsymbol{\beta})_c}. \end{aligned}\tag{64}$$

The δ matrix allows us to write the group product defined in (26) as

$$(\mathbf{E}_1 \times \mathbf{E}_2)^{\boldsymbol{\rho}} = \delta^{\boldsymbol{\rho}}_{\boldsymbol{\nu\beta}}E^{\boldsymbol{\nu}}_1E^{\boldsymbol{\beta}}_2. \tag{65}$$

Notice that, in particular,

$$(\delta_\nu \times \delta_\beta)^\rho = \delta^\rho_{\nu\beta}, \tag{66}$$

where δ_α is the "vector" with components $(\delta_\alpha)^\mu = \delta^\mu_\alpha$. Introducing then (64) in (56) and using (65) we obtain

$$\mathcal{C}_{ax}\psi(\mathbf{R}) = \int d_\mu[A]\psi(A)\mathrm{Tr}[A_\rho][\mathcal{F}_{ab}(x) \times \mathbf{R}^{(bx)}]^\rho$$
$$= \psi(\mathcal{F}_{ab}(x) \times \mathbf{R}^{(bx)}), \tag{67}$$

where

$$[\mathbf{R}^{(bx)}]^\mu = R^{(bx)\mu} := R^{(bx\mu)_c}. \tag{68}$$

The diffeomorphism constraint reduces to evaluate the wave function on a new object given by the group product between an element of the algebra and a cyclic combination of elements of the group. This combination satisfies the differential constraint with respect to the $\boldsymbol{\mu}$ indexes base pointed at x.

B. The Hamiltonian constraint

A similar procedure can be followed to build up the Hamiltonian constraint in the extended representation. In this case we have to use the properties of the SU(2) algebra in order to take into account the two derivatives that appear in $\tilde{\mathcal{H}}(x)$. We have now

$$\mathcal{H}(x)\psi(\mathbf{R}) = \int d_\mu[A]\psi(A)\epsilon^{ijk}\left(F^i_{ba}(x)\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}W_A(\mathbf{R})\right). \tag{69}$$

From (59) we get the following expression for the second functional derivative:

$$\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}\mathrm{Tr}[A_\alpha] = \mathrm{Tr}\left(\tau^k\frac{\delta}{\delta A^j_{bx}}A_\beta\right)\delta^{(ax\beta)_c}_\alpha$$
$$= \mathrm{Tr}[\tau^k A_\mu \tau^j A_\nu]\delta^{\mu bx\nu}_\beta \delta^{(ax\beta)_c}_\alpha$$
$$= \mathrm{Tr}[\tau^k A_\mu \tau^j A_\nu]\delta^{(ax\mu bx\nu)_c}_\alpha. \tag{70}$$

In order to include this result in (69) we need the following well-known property of the SU(2) matrices:

$$\epsilon^{ijk}\mathrm{Tr}[\tau^k A_\mu \tau^j A_\nu] = \mathrm{Tr}[\tau^i A_\nu]\mathrm{Tr}[A_\mu] - \mathrm{Tr}[A_\nu]\mathrm{Tr}[\tau^i A_\mu]. \tag{71}$$

The product between traces of SU(2) matrices can be merged in a combination of traces in the following way:

$$\mathrm{Tr}[A_\mu]\mathrm{Tr}[A_\nu] = \mathrm{Tr}[A_\mu A_\nu] + (-1)^{n(\nu)}\mathrm{Tr}[A_\mu A_{\nu^{-1}}], \tag{72}$$

where if $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$, then $\boldsymbol{\nu}^{-1} = (\nu_n, \ldots, \nu_1)$. This fact enables us to write the above result in the form

$$\epsilon^{ijk}\mathrm{Tr}[\tau^k A_\mu \tau^i A_\nu] = (-1)^{n(\mu)}\mathrm{Tr}[\tau^i A_\nu A_{\mu^{-1}}] - (-1)^{n(\nu)}\mathrm{Tr}[\tau^i A_{\nu^{-1}} A_\mu]. \tag{73}$$

We then have

$$\epsilon^{ijk}F^i_{ba}(x)\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}\mathrm{Tr}[A_\alpha] = (-1)^{n(\mu)}\mathrm{Tr}[F_{ba}(x)A_{\nu\mu}]\{\delta^{(ax\mu^{-1}bx\nu)_c}_\alpha - (-1)^{n(\mu+\nu)}\delta^{(ax\mu bx\nu^{-1})_c}_\alpha\}$$
$$= (-1)^{n(\mu)}\mathrm{Tr}[F_{ba}(x)A_{\nu\mu}]\{\delta^{(bx\nu ax\mu^{-1})_c}_\alpha + (-1)^{n(\mu+\nu)}\delta^{(\mu ax\nu^{-1}bx)_c}_\alpha\}$$
$$= (-1)^{n(\mu)}\mathrm{Tr}[A_{\beta\nu\mu}]\mathcal{F}_{ab}{}^\beta(x)\delta^{(ax\nu bx\mu^{-1})_c}_\gamma\{\delta^\gamma_\alpha + (-1)^{n(\gamma)}\delta^{\gamma^{-1}}_\alpha\}. \tag{74}$$

So

$$\epsilon^{ijk}F^i_{ba}(x)\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}W_A(\boldsymbol{R}) = 2(-1)^{n(\mu)}\mathrm{Tr}[A_{\beta\nu\mu}]\mathcal{F}_{ab}{}^\beta(x)\delta^{(ax\nu bx\mu^{-1})_c}_\gamma R^\gamma$$
$$= 2(-1)^{n(\mu)}\mathrm{Tr}[A_\alpha]\delta^\alpha_{\beta\rho}\mathcal{F}_{ab}{}^\beta(x)[\delta^\rho_{\nu\mu}R^{(ax\nu bx\mu^{-1})_c}], \tag{75}$$

where in the first step we have used the symmetry property (45) of the R's under the inversion of the indexes. The expression between the square brackets defines a specific combination of R's that we denote

$$[\mathbf{R}^{(ax,bx)}]^\rho = R^{(ax,bx)\rho} := (\delta_\nu \times \delta_\mu)^\rho(-1)^{n(\mu)}R^{(ax\nu bx\mu^{-1})_c}. \tag{76}$$

Explicitly

$$R^{(ax,bx)\rho_1\cdots\rho_n} = \sum_{k=0}^{n}(-1)^{n-k}R^{(ax\rho_1\cdots\rho_k bx\rho_n\cdots\rho_{k+1})_c}. \tag{77}$$

An important fact is that this combination satisfies the differential constraint with respect to the ρ indexes base pointed at x. In addition, it satisfies the property

$$R^{(ax,bx)\rho^{-1}} = (-1)^{n(\rho)}R^{(bx,ax)\rho}. \tag{78}$$

Equation (75) can then be written

$$\epsilon^{ijk}F^i_{ba}(x)\frac{\delta}{\delta A^j_{bx}}\frac{\delta}{\delta A^k_{ax}}W_A(\mathbf{R})$$
$$= 2\,\mathrm{Tr}[A_\alpha](\delta_\beta \times \delta_\rho)^\alpha\mathcal{F}^\beta_{ab}(x)R^{(ax,bx)\rho}$$
$$= 2\,\mathrm{Tr}[A_\alpha](\mathcal{F}_{ab} \times \boldsymbol{R}^{(ax,bx)})^\alpha \tag{79}$$

and from it we conclude

$$\mathcal{H}(x)\psi(\mathbf{R}) = 2\psi(\mathcal{F}_{ab}(x) \times \mathbf{R}^{(ax,bx)}). \tag{80}$$

Also in this case the action of the Hamiltonian constraint reduces to evaluate the wave function on a new element. As it was just mentioned, this is a general property of the operators in the extended representation due to the linearity of the wave functions. In fact, the last expression can be written in terms of the functional derivative with respect to the $\mathbf{R}$ variables:

$$\mathcal{H}(x)\psi(\mathbf{R}) = 2[\mathcal{F}_{ab}(x) \times \mathbf{R}^{(ax,bx)}]^{\mu} \frac{\delta}{\delta R^{\mu}}\psi(\mathbf{R}). \tag{81}$$

Notice that in order for this expression to be well defined it is necessary that the term contracted with the functional derivative satisfy the differential constraint, as it happens in this case. This result explicitly shows that the Hamiltonian is linear in the "momenta" $P_{\mu} \equiv \delta/\delta \mathbf{R}^{\mu}$.

The new object where the wave function is evaluated involves a combination of multivector density fields with two indexes fixed at the point where the Hamiltonian is acting and the other indexes having a specific alternating order. We will show in the next section that this alternating order of the indexes is related to the reroutings of a loop when the above expression is particularized to loops. The appearance of a rerouting is typical of the loop representation and plays a crucial role in the quantum gravity case [14].

C. From the extended to the loop representation

As it was shown in Sec. III, the group of loops $\mathcal{L}_0$ is a subgroup of the extended group $\mathcal{D}_0$. The extended version of the constraints (67) and (80) can be particularized to $\mathcal{L}_0$ simply by substituting $\mathbf{R} \to \mathbf{R}(\gamma)$. We analyze here in detail the case of the Hamiltonian constraint.

It is a well-known fact that the Hamiltonian constraint in the loop representation has only a nontrivial action on intersecting loops [4–6]. We suppose then that at the point x the loop γ intersects itself p times; that is to say, γ has "multiplicity" p at x. We start with some suitable notation to take into account this fact.

If the loop γ has multiplicity p at x one can write it as

$$\gamma_{xx} = \gamma_{xx}^{(1)} \circ \gamma_{xx}^{(2)} \circ \cdots \circ \gamma_{xx}^{(p)}. \tag{82}$$

We denote by $[\gamma_{xx}]_i^{i+j}$ the following composition of loops base pointed at x:

$$[\gamma_{xx}]_i^{i+j} = \gamma_{xx}^{(i)} \circ \cdots \circ \gamma_{xx}^{(i+j)}. \tag{83}$$

Let us suppose that the loop named $\gamma_{xx}^{(1)}$ contains the origin o of the loops. Then

$$\gamma_o = \gamma_o^{(1)x} \circ [\gamma_{xx}]_2^p \circ \gamma_x^{(1)o}. \tag{84}$$

Here, $\gamma_o^{(1)x}$ represents the portion of $\gamma^{(1)}$ from the origin o to the point x. The loop γ_o is completely described by the multitangent fields $X^{\mu}(\gamma_o)$ of all ranks. As we know, these fields satisfy both algebraic and differential constraints. Besides, these objects have another property related to the possibility of writing a loop as the composition of open paths. In general

$$X^{\mu_1\cdots\mu_n}(\gamma_o) = \int_{\gamma_o} dz^{a_i}\delta(x_i - z)X^{\mu_1\cdots\mu_{i-1}}(\gamma_o^z)X^{\mu_{i+1}\cdots\mu_n}(\gamma_z^o). \tag{85}$$

Suppose now that the index μ_i is fixed at the point x. Then

$$X^{\mu_1\cdots\mu_i ax \mu_{i+1}\cdots\mu_n}(\gamma_o) = \sum_{m=1}^{p} X^{\mu_1\cdots\mu_i}(\gamma_o^{(1)x} \circ [\gamma_{xx}]_2^m) T_m^{ax} X^{\mu_{i+1}\cdots\mu_n}([\gamma_{xx}]_{m+1}^p \circ \gamma_x^{(1)o}), \tag{86}$$

where T_m^{ax} is the tangent at x when the loop passes the time m to this point and the following convention is assumed: $[\gamma_{xx}]_{m+1}^m \approx i_{xx}$, with i_{xx} the null path. The above expression can be easily generalized to the case of any number of indexes fixed at x.

In order to evaluate $\mathbf{R}^{(ax,bx)}(\gamma_o)$ we have to use the explicit expression of this object in terms of the multitangents fields. We have

$$\begin{aligned} R^{(ax,bx)\mu_1\cdots\mu_n} = {} & \frac{1}{2}\sum_{k=0}^{n}\sum_{l=0}^{k}(-1)^{n-k}[X^{\mu_{l+1}\cdots\mu_k bx\mu_n\cdots\mu_{k+1}ax\mu_1\cdots\mu_l} + (-1)^n X^{\mu_l\cdots\mu_1 ax\mu_{k+1}\cdots\mu_n bxc\mu_k\cdots\mu_{l+1}}] \\ & + \sum_{k=0}^{n}\sum_{l=k}^{n}(-1)^{n-k}[X^{\mu_l\cdots\mu_{k+1}ax\mu_1\cdots\mu_k bx\mu_n\cdots\mu_{l+1}} + (-1)^n X^{\mu_{l+1}\cdots\mu_n bx\mu_k\cdots\mu_1 ax\mu_{k+1}\cdots\mu_l}]. \end{aligned} \tag{87}$$

One can write the above expression in a more compact and useful form introducing the following combinations of X's:

$$X^{(ax,\vec{bx})\mu} \equiv \sum_{k=0}^{n}(-1)^{n-k}X^{(ax\mu_1\cdots\mu_k bx\mu_n\cdots\mu_{k+1})_c} \tag{88}$$

and

$$X^{(ax,\overleftarrow{bx})\mu} \equiv \sum_{k=0}^{n}(-1)^k X^{(ax\mu_k\cdots\mu_1 bx\mu_{k+1}\cdots\mu_n)_c}. \tag{89}$$

These objects satisfy definite symmetry properties under the inversion of the indexes. In term of these combinations, $\mathbf{R}^{(ax,bx)}$ simply reads

$$R^{(ax,bx)\mu} = \tfrac{1}{2}[X^{(ax,\vec{bx})\mu} + (-1)^{n(\mu)} X^{(ax,\overleftarrow{bx})\mu^{-1}}]. \tag{90}$$

We are now ready to calculate $\mathbf{R}^{(ax,bx)}(\gamma_o)$. We have

$$X^{(ax,\vec{bx})\mu}(\gamma_o) = \sum_{m=1}^{p-1}\sum_{q=m+1}^{p}[T_m^{bx}T_q^{ax}X^{\mu}([\gamma_{xx}]_1^m \circ \overline{[\gamma_{xx}]}_{m+1}^q \circ [\gamma_{xx}]_{q+1}^p) \\ +(-1)^{n(\mu)}T_m^{ax}T_q^{bx}X^{\mu^{-1}}([\gamma_{xx}]_1^m \circ \overline{[\gamma_{xx}]}_{m+1}^q \circ [\gamma_{xx}]_{q+1}^p)], \tag{91}$$

where $\overline{[\gamma_{xx}]}_{m+1}^q = \bar{\gamma}_{xx}^{(q)} \circ \cdots \circ \bar{\gamma}_{xx}^{(m+1)}$ and $\bar{\gamma}$ indicates the loop γ with opposite orientation. The inversion of the orientation of the loop (rerouting) in (91) comes from the following property of the multitangent fields:

$$X^{\mu_1\cdots\mu_n}(\bar{\gamma}) = (-1)^n X^{\mu_n\cdots\mu_1}(\gamma). \tag{92}$$

This property is a direct consequence of the algebraic constraint satisfied by the multitangent fields. The reciprocal is not true in general. Notice that (92) is the condition imposed on the elements of the group $\mathcal{M}_o$.

For the other term in (90) we find

$$(-1)^{n(\mu)}X^{(ax,\overleftarrow{bx})\mu^{-1}}(\gamma_o) = X^{(ax,\vec{bx})\mu}(\gamma_o). \tag{93}$$

Then

$$\psi[\mathcal{F}_{ab}(x) \times \mathbf{R}^{(ax,bx)}(\gamma_o)] = \int d_\mu[A]\psi(A)\mathrm{Tr}[A_{\alpha\mu}]\mathcal{F}_{ab}(x)^{\alpha}X^{(ax,\vec{bx})\mu}(\gamma_o) \\ = 2\sum_{m=1}^{p-1}\sum_{q=m+1}^{p} T_m^{[bx,}T_q^{ax]}\int d_\mu[A]\psi(A)\mathrm{Tr}[F_{ab}(x)H_A\{\boldsymbol{R}([\gamma_{xx}]_1^m \circ [\bar{\gamma}_{xx}]_{m+1}^q \circ [\gamma_{xx}]_{q+1}^p)\}]. \tag{94}$$

But

$$\mathrm{Tr}[F_{ab}(x)H_A\{\mathbf{R}(\gamma_{xx})\}] = \Delta_{ab}(x)\mathrm{Tr}[H_A\{\mathbf{R}(\gamma_{xx})\}], \tag{95}$$

where $\Delta_{ab}(x)$ is the loop derivative [15]. We conclude

$$\mathcal{H}(x)\psi(\gamma_o) = 4\sum_{m=1}^{p-1}\sum_{q=m+1}^{p} T_m^{[bx,}T_q^{ax]}\Delta_{ab}(x)\psi([\gamma_{xx}]_1^m \circ \overline{[\gamma_{xx}]}_{m+1}^q \circ [\gamma_{xx}]_{q+1}^p). \tag{96}$$

This expression corresponds to the usual Hamiltonian constraint of quantum gravity in the loop representation [5,16]. For the diffeomorphism constraint we obtain a similar result. Equation (67) reduces to the usual expression of the diffeomorphism constraint in the loop representation when one particularizes this constraint to the case of loops.

It is important to stress the relationship between the solutions of the constraints in both representations. Since loops are a particular case of multitensors, any solution found in the extended representation can be particularized to loops and would yield in the limit a solution to the usual constraints of quantum gravity in the loop representation. The converse is not necessarily true. Given a solution in the loop representation, it may not generalize to a solution in the extended representation. An example are the solutions to the Hamiltonian based on smooth nonintersecting loops, which find no analogue in the extended representation.

V. THE FORMAL CALCULUS

We see that the constraint equations in the extended representation take a very compact form and they amount, in both cases, to evaluate the wave function on a new object given by the group product between an element of the algebra and a combination of elements of the group. A point to stress is that these equations become also operatives. We can make calculations with them and the calculus turns out to be relatively simple. In this section we shall illustrate this point considering the application of the Hamiltonian constraint over a particular member of a family of solutions in the loop representation which have a generalization to the extended representation. We first analyze the nature of these solutions.

In the connection representation based on the Ashtekar variables, the exponential of the Chern-Simons form built with the Ashtekar connection is a solution of all the constraints of quantum gravity with cosmological constant [21]. This state is given by

$$\Psi_\Lambda^{\rm CS}(A) = \exp\left(-\frac{12}{\Lambda}\int \tilde{\eta}^{abc}{\rm Tr}[A_a\partial_b A_c + \tfrac{2}{3}A_aA_bA_c]\right) \qquad (97)$$

and we have

$$\mathcal{H}_\Lambda \Psi_\Lambda^{\rm CS}(A) = \left[\mathcal{H} + \frac{\Lambda}{6}\det(q)\right]\Psi_\Lambda^{\rm CS}(A) = 0, \qquad (98)$$

where Λ is the cosmological constant and q the three-metric. The loop transform of this state is related to the expectation value of the Wilson loop [17]:

$$\begin{aligned}\Psi_\Lambda[\gamma] &= \tfrac{1}{2}\langle W(\gamma)\rangle \\ &= \exp\left(a_1[\gamma]\frac{\Lambda}{6}\right)\left[1 - \left(\frac{\Lambda}{6}\right)^2 \frac{3}{2}\rho[\gamma] \right. \\ &\left. \quad - \left(\frac{\Lambda}{6}\right)^3 3\tau[\gamma] + O(\Lambda^4)\right]. \end{aligned} \qquad (99)$$

The resulting loop wave function is the Kauffman brackets knot polynomial which is a phase factor times the Jones polynomial. Evaluating the loop transform (99) using perturbative techniques of Chern-Simons theory [9], explicit expression for the Kauffman brackets coefficients can be found. In particular, the phase factor is proportional to the Gauss self-linking number: $a_1[\gamma] = -\frac{3}{4}\varphi[\gamma]$ with

$$\varphi[\gamma] = g_{ax\,by}X^{ax}(\gamma)X^{bx}(\gamma), \qquad (100)$$

being

$$\begin{aligned} g_{ax\,by} &= -\frac{1}{4\pi}\epsilon_{abc}\frac{(x-y)^c}{|x-y|^3} \\ &= -\epsilon_{abc}\partial^c\nabla^{-2}\delta(x-y) \end{aligned} \qquad (101)$$

the free propagator of Chern-Simons theory. $\rho[\gamma]$ and $\tau[\gamma]$ are the first coefficients of an expansion of the Jones polynomial in the variable $\exp(\Lambda/6)$ that can be explicitly written as linear functions of the multitangents with coefficients constructed from $g_{ax\,by}$.

The action of the Hamiltonian constraint with cosmological constant can be evaluated over the Kauffman brackets polynomial and the analysis of the resulting equations order by order in Λ leads to the conjecture that the Jones polynomial may be a state of vacuum quantum gravity [20]. This conjecture was explicitly confirmed for the first candidate $\rho[\gamma]$ in the loop representation [18,19] through a laborious (formal) computation. The expression of this knot invariant in terms of the multitangent fields is

$$\begin{aligned}\rho[\gamma] &= h_{\mu_1\mu_1\mu_3}X^{\mu_1\mu_2\mu_3}(\gamma) \\ &\quad + g_{\mu_1\mu_3}g_{\mu_2\mu_4}X^{\mu_1\mu_2\mu_3\mu_4}, \end{aligned} \qquad (102)$$

where

$$h_{\mu_1\mu_2\mu_3} = \epsilon^{\alpha_1\alpha_2\alpha_3}g_{\mu_1\alpha_1}g_{\mu_2\alpha_2}g_{\mu_3\alpha_3} \qquad (103)$$

with

$$\epsilon^{\alpha_1\alpha_2\alpha_3} = \epsilon^{c_1c_2c_3}\int d^3t\,\delta(z_1-t)\delta(z_2-t)\delta(z_3-t). \qquad (104)$$

The generalization of this knot invariant to extended loops is straightforward:

$$\rho[\gamma] = \rho[\mathbf{X}(\gamma)] \rightarrow \rho(\mathbf{X}), \qquad (105)$$

where $\mathbf{X}$ is now an element of the extended group $\mathcal{D}_0$. We now analyze the application of the Hamiltonian constraint over this state in the extended representation. By (80) we have

$$\begin{aligned}\mathcal{H}(x)\rho(\mathbf{R}) &= 2h_{\mu_1\mu_2\mu_3}[\mathcal{F}_{ab}{}^{\mu_1}(x)R^{(ax,bx)\mu_2\mu_3} + \mathcal{F}_{ab}{}^{\mu_1\mu_2}(x)R^{(ax,bx)\mu_3}] \\ &\quad + 2g_{\mu_1\mu_3}g_{\mu_2\mu_4}[\mathcal{F}_{ab}{}^{\mu_1}(x)R^{(ax,bx)\mu_2\mu_3\mu_4} + \mathcal{F}_{ab}{}^{\mu_1\mu_2}(x)R^{(ax,bx)\mu_3\mu_4}]. \end{aligned} \qquad (106)$$

We can compute the action of $\mathcal{F}_{ab}$ over the propagators. The following results are obtained:

$$\mathcal{F}_{ab}{}^{\mu_1}(x)g_{\mu_1\mu_3} = -\epsilon_{aba_3}\delta(x-x_3) - \partial_{a_3}g_{axbx_3}, \qquad (107)$$

$$\mathcal{F}_{ab}{}^{\mu_1\mu_2}(x)g_{\mu_1\mu_3}g_{\mu_2\mu_4} = g_{\mu_3[ax}g_{bx]\mu_4}, \qquad (108)$$

$$\mathcal{F}_{ab}{}^{\mu_1}(x)h_{\mu_1\mu_2\mu_3} = -g_{\mu_2[ax}g_{bx]\mu_3} + (g_{ax\,bx_2} - g_{ax\,bx_3})g_{\mu_2\mu_3} + \tfrac{1}{2}g_{ax\,bz}\epsilon^{def}[g_{\mu_3dz}\partial_{a_2}g_{ex_2fz} - g_{\mu_2dz}\partial_{a_3}g_{ex_3fz}], \qquad (109)$$

$$\mathcal{F}_{ab}{}^{\mu_1\mu_2}(x)h_{\mu_1\mu_2\mu_3} = 2h_{ax\,bx\,\mu_3}. \qquad (110)$$

In the last term of Eq. (109) an integral in z is assumed. The derivatives that appear in the above expressions can be applied over the **R**'s integrating by parts, and using the differential constraint we generate from them terms of lower rank. For example from (107) we have

$$g_{\mu_2\mu_4}\partial_{a_3}g_{ax\,bx_3}R^{(ax,bx)\mu_2\mu_3\mu_4} = g_{\mu_2\mu_4}(g_{ax\,bx_2} - g_{ax\,bx_4})R^{(ax,bx)\mu_2\mu_4}. \tag{111}$$

Performing these calculations, the following partial results are obtained for each of the four expressions quoted above:

(1) $-\epsilon_{abc}g_{\mu_1\mu_2}R^{(ax,bx)\mu_1 cx\,\mu_2} - (g_{ax\,bx_1} - g_{ax\,bx_2})g_{\mu_1\mu_2}R^{(ax,bx)\mu_1\mu_2}$,

(2) $g_{\mu_1[ax}g_{bx]\mu_2}R^{(ax,bx)\mu_1\mu_2}$,

(3) $-g_{\mu_1[ax}g_{bx]\mu_2}R^{(ax,bx)\mu_1\mu_2} + (g_{ax\,bx_1} - g_{ax\,bx_2})g_{\mu_1\mu_2}R^{(ax,bx)\mu_1\mu_2} - \epsilon^{def}g_{ax\,bz}g_{\mu_1 dz}g_{ex\,fz}R^{(ax,bx)\mu_1}$,

(4) $2h_{ax\,bx\,\mu_1}R^{(ax,bx)\mu_1}$.

Some contributions cancel each other and we finally obtain

$$\mathcal{H}(x)\rho(\mathbf{R}) = -2\epsilon_{abc}g_{\mu_1\mu_2}R^{(ax,bx)\mu_1 cx\,\mu_2} + 2[2h_{ax\,bx\,\mu_1} - \epsilon^{def}g_{ax\,bz}g_{\mu_1 dz}g_{ex\,fz}]R^{(ax,bx)\mu_1}. \tag{112}$$

In the above expression, the quantity in square brackets vanishes identically because the second term is nothing else that other form to express the first one. One can easily check this fact. Developing $R^{(ax,bx)\mu_1 cx\,\mu_2}$ we get

$$R^{(ax,bx)\mu_1 cx\,\mu_2} = -2R^{(ax\,bx\mu_1 cx\,\mu_2)_c} + R^{(cx\,ax\,\mu_1 bx\,\mu_2)_c} + R^{(bx\,cx\,\mu_1 ax\,\mu_2)_c} \tag{113}$$

and the contribution of the rank five term vanishes due to symmetry considerations. We conclude

$$\mathcal{H}(x)\rho(\mathbf{R}) = 0. \tag{114}$$

VI. THE ISSUE OF THE REGULARIZATION

The extended representation provides a new scenario to analyze the regularization problem in quantum gravity. In the loop representation regularization ambiguities appear both at the level of quantum operators *and* of the quantum states associated with the knot invariants. Whereas the first problem is common to all the representations that one can construct for quantum gravity (and lies in the fact that the constraints involve the product of operators evaluated at the same point), the second is typical of the loop representation. In the case of quantum gravity the loop wavefunctions are knot invariants and their analytic expressions require the introduction of a regularization (framing) [23]. This difficulty not only arises for the gravitational case, even in the simple case of a free Maxwell field [24], it is known that the quantum states in the loop representation (associated to real connections [25]) are ill defined and a regularization is needed.

In the extended representation the second difficulty can be solved. We are going to show that with an adequate restriction of the domain of dependence, the extended wave functions are smooth functionals of the variables. In what concerns the regularization of the constraints we shall limitate the analysis to the case of wave functions with a totally specified analytical dependence. More precisely, we shall study the action of the regularized Hamiltonian constraint over the wave functions that are formally annihilated by the constraint. A more general discussion of the regularization and renormalization of the constraints as well as the consistency of their algebra will be given elsewhere [26].

A. The smoothness of the extended wave functions

In the loop representation the coefficients of the expansion (99) of the expectation value of the Wilson loop in terms of the cosmological constant are knot invariants. It is easy to see that their generalization to the extended representation are also diffeomorphism invariants. For that, consider the extended loop transform of the exponential of the Chern-Simons form

$$\Psi_\Lambda(\mathbf{X}) = \int d_\mu[A]e^{S_\Lambda(A)}\mathrm{Tr}[\mathbf{A}\cdot\mathbf{X}] = \sum_{n=0}^{\infty}[\mathbf{g}^{(n)}\cdot\mathbf{X}]\Lambda^n, \tag{115}$$

where the dot indicates the contraction of indexes. As the measure of integration and the Chern-Simons action are diffeomorphism invariants we have

$$\Psi_\Lambda(\mathbf{X}) = \int d_\mu[A_D]e^{S_\Lambda(A_D)}\mathrm{Tr}[\mathbf{A}_D\cdot\Lambda_D\cdot\mathbf{X}], \tag{116}$$

where $\Lambda_D\cdot\mathbf{X}$ and $\mathbf{A}_D = \mathbf{A}_o\cdot\Lambda_{D^{-1}}$ are the transformed quantities under the diffeomorphism $x'^a = D^a(x)$. $\mathbf{A}_D$ is in another gauge. Because of the fact that the measure of integration, the Chern-Simons action and the trace of the extended holonomy are invariant under gauge transformations connected with the identity, we conclude $\Psi_\Lambda(\mathbf{X}) = \Psi_\Lambda(\Lambda_D \cdot \mathbf{X})$. From (115) we get, for any n,

$$\mathbf{g}^{(n)} \cdot \mathbf{X} = \mathbf{g}^{(n)} \cdot \Lambda_D \cdot \mathbf{X}. \tag{117}$$

The coefficients are then invariant under diffeomorphism transformations. Note that this result does not imply that $\rho(\mathbf{X})$ or $\tau(\mathbf{X})$ are diffeomorphism invariants. From (99) the state (117) contains to any order in the cosmological constant contributions of different knot invariants. The invariance of these coefficients has to be checked explicitly.

Let us consider now the regularity properties of the extended wave functions. Generically the multitensors X^μ are distributional, as it is directly inferred from (8). As we have shown in Sec. II, any multitensor that satisfies the differential constraint can be written in the form $\mathbf{X} = \sigma[\phi] \cdot \mathbf{Y}$, where the $\mathbf{Y}$ are transverse fields. From (17) we have, for the rank two component of the multitensor,

$$X^{ax\,by} = Y^{ax\,by} + \phi^{ax}{}_y Y^{by} - \phi^{by}{}_x Y^{ax} - \phi^{ax}{}_z \phi^{by}{}_{z,c} Y^{cz} + \phi^{[by}{}_o Y^{ax]}. \tag{118}$$

As the $\mathbf{Y}$'s satisfy the homogeneous differential constraint, they can be assumed to be smooth functions. In this case, all the divergent behavior of the $\mathbf{X}$ is concentrated in the function ϕ. The σ's control then the divergent character of the group elements.

Let us define the following set of elements of the extended space: $\mathbf{X} \in \{\mathbf{X}\}_s$ if, and only if, there exists a prescription function ϕ such that $\delta_T[\phi] \cdot \mathbf{X} = \mathbf{Y}$ is a smooth function. We shall demonstrate that the wave functions defined on this domain are smooth in the extended variables and that this property is invariant under diffeomorphism transformations.

Given a diffeomorphism transformation Λ_D, it can be shown that $\delta_{DT} \equiv \Lambda_{D^{-1}} \cdot \delta_T \cdot \Lambda_D$ is a transverse projector in the prescription

$$\phi_D{}^{ax}{}_y = J(x) \frac{\partial x^a}{\partial D^b(x)} \phi^{bD(x)}{}_{D(y)}, \tag{119}$$

where $J(x)$ is the Jacobian of the coordinate transformation and ϕ the function that fixes the prescription of the projector δ_T [10]. In this prescription $\mathbf{X} = \sigma \cdot \mathbf{Y} = \Lambda_{D^{-1}} \cdot \sigma_{D^{-1}} \cdot \Lambda_D \cdot \mathbf{Y}$. For any diffeomorphism transformation Λ_D, the transverse part of $\Lambda_D \cdot \mathbf{X}$ is a smooth function with the prescription $\phi_{D^{-1}}$. In effect

$$\delta_{D^{-1}T} \cdot (\Lambda_D \cdot \mathbf{X}) = \delta_{D^{-1}T} \cdot \sigma_{D^{-1}} \cdot \Lambda_D \cdot \mathbf{Y} = \Lambda_D \cdot \mathbf{Y}. \tag{120}$$

The set $\{\mathbf{X}\}_s$ is then invariant under diffeomorphism transformations. For any $\mathbf{X} \in \{\mathbf{X}\}_s$ we have

$$\psi(\mathbf{X}) = \mathbf{g} \cdot \mathbf{X} = \mathbf{g} \cdot \sigma[\phi] \cdot \mathbf{Y} \equiv \mathbf{g}_\phi \cdot \mathbf{Y}. \tag{121}$$

All the distributional character of the wave function is concentrated in the "metric" $\mathbf{g}_\phi$. They are well-defined functionals, smooth in the extended variables. Moreover, the diffeomorphism transformed of the wave function would be defined on the same domain and is also smooth.

B. The regularization of the constraints

We shall restrict the analysis of the regularization to the case of the point splitting method, based on a delocalization of the point where the operators are evaluated. The point splitting version of the constraints are

$$\mathcal{C}^\epsilon_{ax}\psi(\mathbf{R}) = \int d^3w \int d^3v \, f_\epsilon(w,x) f_\epsilon(v,x) \psi(\mathcal{F}_{ab}(w) \times \mathbf{R}^{(bv)}), \tag{122}$$

$$\mathcal{H}^\epsilon(x)\psi(\mathbf{R}) = 2 \int d^3w \int d^3u \int d^3v \, f_\epsilon(w,x) f_\epsilon(u,x) f_\epsilon(v,x) \psi(\mathcal{F}_{ab}(w) \times \mathbf{R}^{(au,bv)}), \tag{123}$$

where f_ϵ is any appropriate symmetric smearing of the delta function. Notice that this point splitting regularization is not uniquely determined by the formal factor ordered expression. Several sources of ambiguities arise, the first one is related with the background metric used in the smearing functions. It is also possible, but not necessary, to preserve the gauge invariance in the regularization process. Finally additional factor ordering problems may arise due to the distributional character of the fundamental fields. A more complete and lengthier discussion will be given elsewhere [26]. Here we shall proceed as follows: we shall introduce a naive point splitting and study the action of the regularized and renormalized operators on the formal solutions. We shall prove that there is a factor ordering that ensures the consistency between the known results in the connection and the loop representation.

In the previous section we have shown that the invariance under diffeomorphism of the coefficients of the expansion of the generalized transform (115) is ensured by construction. Also that, with an appropriate definition of the domain of dependence, the wave functions can be endowed with convenient regularity properties (in particular, the smoothness dependence on the extended variables can be ensured in a diffeomorphism invariant way). We check now the good behavior of the regularized diffeomorphism constraint for the particular case of the Gauss invariant $\varphi(\mathbf{R})$. From (122) we obtain

$$\mathcal{C}^\epsilon_{ax}\varphi(\mathbf{R}) = \int d^3w \int d^3v \, f_\epsilon(w,x) f_\epsilon(v,x) g_{\mu_1\mu_2} \times \mathcal{F}_{ab}{}^{\mu_1}(w) R^{(bv)\mu_2}. \tag{124}$$

This result is valid for any prescription. Because of practical computational reasons we shall restrict the domain of the wave functions to those prescriptions connected by diffeomorphisms to the "transverse" prescription, given by

$$\phi_o{}^{ax}{}_y = \frac{1}{4\pi}\frac{\partial}{\partial x_a}\frac{1}{|x-y|}. \tag{125}$$

In the transverse prescription the free Chern-Simons propagator $g_{ax\,by}$ takes the form (101). Then using (107) we get

$$\mathcal{C}^{\epsilon}_{ax}\varphi(\mathbf{R}) = -\epsilon_{abc}\int d^3w \int d^3v\, f_\epsilon(w,x) f_\epsilon(v,x) R^{(bv)cw}, \tag{126}$$

where

$$R^{(bv)cw} = Y^{bv\,cw} + Y^{cw\,bv} \tag{127}$$

is a smooth function symmetric under the interchange of the indices b and c (using the fact that the integration points are undistinguishable). The last expression is well defined and we conclude that

$$\mathcal{C}^{\epsilon}_{ax}\varphi(\mathbf{R}) = 0. \tag{128}$$

Notice that no divergences occur in (126) and we do not need to take the limit when ϵ goes to zero. The diffeomorphism constraint is perfectly well defined and no renormalization is needed. A similar result holds for the other known invariants.

Let us analyze now the action of the regularized Hamiltonian constraint on the Alexander-Conway coefficient $\rho(\mathbf{R})$. We get, in this case,

$$\begin{aligned}\mathcal{H}^{\epsilon}(x)\rho(\mathbf{R}) = &\int d^3w \int d^3u \int d^3v\, f_\epsilon(w,x) f_\epsilon(u,x) f_\epsilon(v,x)\\ &\times\{-\epsilon_{abc} g_{\mu_1\mu_2} R^{(au,bv)\mu_1 cw\,\mu_2} + [2h_{aw\,bw\,\mu_1} - \epsilon^{def} g_{aw\,bz} g_{\mu_1 dz} g_{eu\,fz}] R^{(au,bv)\mu_1}\\ &+(g_{aw\,bu} - g_{aw\,bv}) g_{\mu_1\mu_2} R^{(au\,\mu_1 bv\,\mu_2)_c}\}.\end{aligned} \tag{129}$$

The last term in the above expression (what we call the "anomalous term") appears as a consequence of the modification of the differential constraint of $\mathbf{R}^{(au,bv)}$. We have now

$$\begin{aligned}\partial_{\mu_i} R^{(au,bv)\mu_1\cdots\mu_i\cdots\mu_n} = &[\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1})] R^{(au,bv)\mu_1\cdots\not{\mu_i}\cdots\mu_n}\\ &+[\delta(x_i - u) - \delta(x_i - v)](-1)^{n-i} R^{(au\,\mu_1\cdots\mu_{i-1} bv\,\mu_n\cdots\mu_{i+1})_c}\end{aligned} \tag{130}$$

instead of the normal differential constraint. In the above expression, $x_0 = u$ and $x_{n+1} = v$. The calculation of the regulated terms involves the consideration of the divergences that comes from the group elements (through the matrix sigma) and from the metric terms. The first observation is that both types of contributions are of the same order. Let us consider, for example, the divergent contributions associated with the rank five group elements contained in $R^{(au,bv)\mu_1 cw\,\mu_2}$. One can see that all these terms are equivalent in their divergent character to

$$\phi_o{}^{au}{}_v Y^{(bv\,\mu_1 cw\,\mu_2)_c} \tag{131}$$

with $Y^{(bv\,\mu_1 cw\,\mu_2)_c}$ a regular function in the limit $\epsilon \to 0$. Notice that this expression gives the leading divergence of the rank five term in (129). But

$$\epsilon_{bca}\phi_o{}^{au}{}_v Y^{(bv\,\mu_1 cw\,\mu_2)_c} = g_{bu\,cv} Y^{(bv\,\mu_1 cw\,\mu_2)_c} \tag{132}$$

is exactly the same contribution that comes from the anomalous term. Notice that in this case $g_{\mu_1\mu_2} R^{(bv\,\mu_1 cw\,\mu_2)c}$ is a regular function in the limit $\epsilon \to 0$ due to the point splitting does not affect two consecutive indexes.

The result (113) ensures that the contribution of the rank five term in the Hamiltonian vanishes due to the same symmetry properties used in the formal calculus. One can see that the second term in (129) also vanishes when one removes the regulators. For the anomalous term we get

$$\begin{aligned}2\int d^3w \int d^3u \int d^3v\, f_\epsilon(w,x) f_\epsilon(u,x) f_\epsilon(v,x) g_{aw\,bv} g_{\mu_1\mu_2} R^{(au\,\mu_1 bv\,\mu_2)_c}\\ = \frac{2}{\sqrt{2\pi}\epsilon}\epsilon_{abc} g_{\mu_1\mu_2}\partial^{cy} R^{(ax\,\mu_1 by\,\mu_2)_c}\Big|_{y=x} + O(\epsilon),\end{aligned} \tag{133}$$

where we have used a Gaussian regulator $f_\epsilon(\vec{z}) = (\sqrt{\pi}\epsilon)^{-3}\exp(-z^2\epsilon^{-2})$. Then

$$\begin{aligned}\mathcal{H}^N(x)\rho(\mathbf{R}) &:= \lim_{\epsilon\to 0}\epsilon\mathcal{H}^{\epsilon}(x)\rho(\mathbf{R})\\ &= \left(\frac{2}{\pi}\right)^{1/2}\epsilon_{abc} g_{\mu_1\mu_2}\partial^{cy} R^{(ax\,\mu_1 by\,\mu_2)_c}\Big|_{y=x}.\end{aligned} \tag{134}$$

We conclude that the renormalized Hamiltonian constraint does not annihilate the generalized diffeomorphism invariant corresponding to the second coefficient of the Alexander-Conway knot polynomial in the (naive) point splitting regularization procedure.

A compatibility argument arises from the beginning. We know that the exponential of the Chern-Simons form is an *exact* quantum state of the Hamiltonian constraint with cosmological constant in the connection representation. What about the regulated equations order by order in Λ that correspond to the same state in the extended representation? As it was just mentioned, the formal calculus works in the expected way. But the point slitting regularization makes these equations to fail. However, the consistency can be restored by introducing a counterterm in the Hamiltonian.

A counterterm is a regularized term which has no effect over a regular Wilson functional (that is to say, over an extended Wilson functional constructed with nondistributional connections). Consider, for example, the following expression, symmetric under the interchange of the internal indices:

$$(A^i_{aw}A^j_{bu} - A^i_{aw}A^i_{bv})\frac{\delta}{\delta A^{(j}_{bv}\delta A^{i)}_{au}}W_A(\mathbf{R})$$
$$= \{\mathrm{Tr}[A_{(aw|\mu|bu)\nu}] - \mathrm{Tr}[A_{(aw|\mu|bv)\nu}]\}R^{(au\,\mu\,bv\nu)_c}. \quad (135)$$

It is clear that this term vanishes in the limit $\epsilon \to 0$ if the connections are regular functions, but it may have a nontrivial contribution if the connections are distributions. The corresponding regularized expression in the extended space is

$$C^\epsilon = R^{(au\,\mu bv\,\nu)_c}\left(\frac{\delta}{\delta R^{(aw|\mu|bu)\nu}} - \frac{\delta}{\delta R^{(aw|\mu|bv)\nu}}\right). \quad (136)$$

This term generates anomalous type contributions. For example,

$$C^\epsilon(g_{\mu_1\mu_2}R^{\mu_1\mu_2}) = 2(g_{aw\,bu} - g_{aw\,bv})R^{(au\,bu)_c}. \quad (137)$$

The addition of the counterterm C^ϵ in $\mathcal{H}$ ensures that the Kauffman brackets polynomial is annihilated (up to the second order in Λ) by the renormalized Hamiltonian constraint with cosmological constant. This is an important result. In the extended representation one can prove that the Kauffman brackets polynomial is a quantum state of gravity with cosmological constant including regularization. Notice that the introduction of the counterterm C^ϵ is equivalent to a particular choice of ordering of the connections in the Hamiltonian constraint.

VII. CONCLUSIONS

An extended loop representation for quantum gravity was constructed. The formal calculus associated with the constraints have been developed and the regularization problems discussed.

We have shown that the extended loops provide an appropriate arena for the definition of the quantum states and that the constraint equations present practical calculational advantages. These facts can be considered as an improvement of the loop representation in the description of quantum gravity. However in this approach one of the most attractive ingredients of the loop representation has been lost: the fact that the diffeomorphism constraint was easily solvable in terms of knots. A further study of the characterization of diffeomorphism invariant classes of extended loops is in order. In this sense, the fact that the knot polynomial solutions have direct analogues in the new representation can be viewed, in our opinion, as a suggestion that the topological and geometric insights of the loop representation be inherited in some sense by their extension to the generalized space.

One may wonder about the equivalence between all the possible representations that can be formulated in the extended space. As we just mentioned, representations associated with the groups $\mathcal{M}_0$ and $\mathcal{X}_0$ can be developed in a way similar to the $\mathcal{D}_0$ case. Moreover, it is still unclear which among all possible representations behave as the dual of the connection representation. These representations obey different types of first class constraints. In the case of $\mathcal{D}_0$ these constraints are the linearity constraint and the diffeomorphism and Hamiltonian constraints. The presence of other constraints besides the usual gravity constraints allow to eliminate the superfluous degrees of freedom of the theory. One can see that the elimination is complete in $\mathcal{X}_0$, where all the gauge ambiguity disappears and, in consequence, all the representations can be considered as essentially equivalent.

An important property of the extended representation of quantum general relativity (and in general of any gauge theory) is that it provides, in a natural way, a framework to develop a classical counterpart of the quantum theory. One can view the role of the multitensors as configuration variables of a canonical theory. The conjugate momenta are represented by functional derivatives. This suggests that there exists an underlying classical Hamiltonian theory that under canonical quantization yields directly the extended loop representation. This was unclear with loops, where the loop representation could only be introduced through a noncanonical quantization. For the Maxwell case this theory was studied [27] and found to be equivalent to the usual Maxwell theory. For the non-Abelian case it is yet to be studied.

In what concerns the regularization procedure some comments are in order. In the first place it is important to point out that a gauge invariant regularization procedure may be introduced. However this regularization is totally equivalent to the naive regularization we have considered here. Second, it is important to remark that the generalized knot invariants are well defined in the space of $\mathbf{R}$'s that are related with continuous $\mathbf{Y}$'s for some prescription ϕ and therefore in a diffeomorphism invariant background independent domain. However we need to impose, for practical reasons, a further, background dependent, restriction in the domain of the wave functions: the transverse prescription. This prescription allows to perform the explicit analysis of the regularization and renormalization of the constraints. Therefore the regularization procedure is clearly background dependent; the smearing functions, the Chern-Simons propagators and the extended space are in the transverse prescription.

A more geometric definition of the relevant extended space, for instance, by restricting the domain of the wave functions to some geometric smearing of loops (bands or tubes), could allow to keep for one side well-defined knot invariants and on the other side to analyze the action of the regularized constraints without any reference to a background metric.

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