Chiral sum rules for $\mathcal{L}^{WZ}_{(6)}$ parameters and its application to π^0, η, η' decays

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The chiral expansion of the low-energy processes $\pi^0 \to \gamma\gamma$ and $\eta \to \gamma\gamma$ is reconsidered with particular emphasis on the question of the evaluation of the two low-energy parameters from $\mathcal{L}^{WZ}_{(6)}$ which are involved at chiral order six. It is shown how extensive use of sum rules and saturation with resonances as well as constraints from asymptotic QCD effectively determine their values. Predictions for the widths are presented for both standard and nonstandard values of the quark mass ratio $m_s/\hat{m}$. A precise relation is established between the usual phenomenological $\eta - \eta'$ mixing parameters and those of the chiral expansion. The large size of the chiral correction to the η decay can be understood on the basis of a simple counting rule: $O(1/N_c) \sim O(m_q)$. It is shown how this counting rule eventually allows one to include the η' into the effective Lagrangian in a consistent and systematic way.

I. INTRODUCTION

The η electromagnetic decay is a neat low-energy process, free of any strong final-state interaction, and so one would expect that the chiral expansion should converge as fast as it seems to do in the case of the masses. Yet, at leading order, the width is predicted to be too small by a factor of 3 compared to experiment. The problem of the $O(p^6)$ corrections to $\eta \to 2\gamma$ (and $\pi^0 \to 2\gamma$) has been addressed several times in the literature [1, 2] (who computed the chiral loop contribution) [3, 4]. Strictly speaking, chiral perturbation theory (χPT) is unable to make any prediction in this case because there are precisely two low-energy (LE) parameters, which are finite and called t_1 and t'_1 below, which cannot be determined at present from other low-energy data (in principle the amplitudes for $\eta \to \pi^0\pi^0\gamma\gamma$ or $\gamma\gamma \to 3\pi^0$ could be used as they contain just the same parameters). However, shortly after the χPT started to be developed in a systematic way [5, 6] along with its list of LE parameters, a great deal of progress was made in the art of relating, in a rather precise way, the size of these parameters to the properties of the low-lying resonances [7, 8]. In particular, as Ecker *et al.* [8] have emphasized, whenever one can write down a rapidly converging dispersion relation it is natural to expect that saturation from the first low-lying resonances should provide an adequate approximation.

In this work, we intend to exploit in this manner specific convergence properties of dispersion relations, which can be shown to hold in QCD in the chiral limit, in order to estimate the two LE parameters which occur in the electromagnetic decay of the π^0 and the η. These sum rules are similar to the ones that were established for L_{10} [9] and for L_7 [10] on the basis of two-point functions. The parameters t_1 and t'_1 are related to three-point functions instead. We will find that t'_1 is proportional to the square root of $-L_7$ and to the amplitude $A(\eta' \to \gamma\gamma)$. This amplitude must be taken from experiment and this information feeds back into the LE parameter, as is usual for the properties of the light resonances [7]. Similarly, the parameter t_1 may be shown to encode information on the $\pi(1300)$ resonance electromagnetic decay [3]. Fortunately (because hardly anything is experimentally known on this decay channel), we show that this contribution is numerically dominated by a contribution from the asymptotic behavior of the three-point function which turns out to be canonical. A similar behavior was exploited previously in the case of three-point functions involving one scalar current and was claimed to provide an explanation, based on chiral symmetry, for the somewhat unexpected electromagnetic properties of the scalar mesons [11].

In addition to the purely phenomenological application, once it is ensured that the LE parameters are estimated to a reasonable level of accuracy, one can address the question of the rate of convergence of the expansion in powers of the quark masses. The conventional determination of the light quark running masses m_u and m_d [12] was recently challenged [13], based on an analysis of the violation of the Goldberger-Treiman relation. This analysis suggests that the value of the quark mass ratio $2m_s/(m_u + m_d)$ is 2–3 times smaller than in the standard χPT, i.e., $r \simeq r_2 = 25.9$. If true, this necessitates rearranging the quark mass terms in the chiral perturbation series in a different way [14–16] (the so-called generalized χPT) which one would expect to converge more rapidly. In practice, distinguishing between the χPT and its generalized variant is, curiously, not so easy. It is likely that only very precise measurements of π-π scattering lengths could settle the issue of which one is correct. It turns out that the amplitude $A(\eta \to \gamma\gamma)$ is very sensitive to the value of r when r is in the vicinity of the value r_2. Nevertheless, we will show that it is also possible to reproduce the experimental results under the assumption of a much smaller value of r.

The plan of the paper is as follows. The next section contains the derivation of the sum rules for the parameters t_1 and t'_1; the main results are contained in the formulas (18) and (23). Application to the η decay amplitude is then discussed in Sec. III. In particular,

we establish a connection between the chiral expansion description and the phenomenological representation in terms of an η-η' mixing angle which is widely employed (e.g., [17]). The chiral corrections to the π^0 decay are also worked out here. This analysis suggests that the value of F_{π^0} quoted in the literature is incorrect. The description of the η decay amplitude in the generalized χPT is presented at the end of Sec. III. The η-η' system is of interest also in connection with the large N_c expansion since, in the large N_c limit, the η' is a ninth Goldstone boson. This suggests still another expansion scheme where $1/N_c$ is considered as an expansion parameter together with the quark masses and the momenta. A natural choice is to count one power of $1/N_c$ on the same footing as one power of a quark mass. This point of view is discussed in Sec. IV (in particular, one observes the appearance of two different mixing angles but only one of them is relevant to the η decay) and the results are compared to those of the standard chiral expansion.

II. LOW-ENERGY PARAMETERS FROM A CHIRAL SUM-RULE METHOD

The current-algebra prediction for the amplitudes describing pseudoscalar meson decays in two photons is contained into the following term of the canonical Wess-Zumino (WZ) Lagrangian [18, 19]:

$$\mathcal{L}^{\text{WZ}}_{(4)} = \frac{N_c}{32\pi^2 F_0} \epsilon^{\mu\nu\alpha\beta} \langle \phi\, v_{\mu\nu} v_{\alpha\beta} \rangle \,, \tag{1}$$

where the totally antisymmetric tensor is normalized such that $\epsilon^{0123} = 1$ and $\phi = \sum \phi_i \lambda_i$, $i = 1, 8$. The subscript in Eq. (1) is a reminder of the chiral order. Part of the higher chiral corrections are contained in chiral Lagrangian terms which are also proportional to the ϵ tensor. At order 6 there are three independent terms which are relevant for our purposes (using relatively standard notation; see, e.g., Ref. [7])

$$\mathcal{L}^{\text{WZ}}_{(6)} = i\epsilon^{\mu\nu\alpha\beta} \Big\{ t_1 \langle \chi^{(-)} f^{(+)}_{\mu\nu} f^{(+)}_{\alpha\beta} \rangle + t'_1 \langle \chi^{(-)} \rangle \langle f^{(+)}_{\mu\nu} f^{(+)}_{\alpha\beta} \rangle$$
$$- it_2 \langle d_\lambda f^{(+)}_{\lambda\mu} \{ f^{(+)}_{\alpha\beta}, u_\nu \} \rangle + \cdots \Big\} \,. \tag{2}$$

As discussed in [20], all the other potentially relevant terms that one can write down can be reduced to the above three by the use of the so-called Shouten identity and the equation of motion. Among the constants appearing in (2) only t_2 has been estimated previously [21–23]. The parameter t_2 is phenomenologically interesting in that it controls the corrections to decays like $\pi^0 \to \gamma\gamma^*$ which were discussed in [22]. This contribution vanishes when both photons are on shell. In this case, the terms t_1 and t'_1 are the *only* contributions from $\mathcal{L}^{\text{WZ}}_{(6)}$ to the processes $\pi^0 \to 2\gamma$ and $\eta \to 2\gamma$. These two parameters are finite, reflecting the fact that the loop contribution is finite and, as a matter of fact, vanishes [1, 2]. The parameter t'_1 is analogous to some extent to the parameter L_7 of the standard $O(p^4)$ Lagrangian [6, 10]. It picks up a contribution from the pole of the η' and, from the point of view of the large N_c counting, it

is of order $O(N_c^2)$ instead of being $O(N_c)$ at most, like the other parameters. From this point of view one expects it to play a dominant role. The second parameter t_1 is unrelated to the η' resonance.

In order to obtain an estimate for t_1, t'_1, and t_2 let us take the chiral limit and consider the vector-vector–pseudoscalar correlation function, i.e.,

$$d^{abc} \epsilon_{\mu\nu\alpha\beta}\, p^\alpha q^\beta\, \Pi_{VVP}(p^2, q^2, r^2)$$

$$= \int d^4x\, d^4y\, e^{ipx+iqy} \langle 0 | T \Big[j^a_\mu(x) j^b_\nu(y) j^c_P(0) \Big] | 0 \rangle \,, \tag{3}$$

with $r = -(p+q)$, $a, b, c = 1 - 8$, and where

$$j^a_\mu(x) = \bar{\psi}(x) \frac{\lambda^a}{2} \gamma_\mu \psi(x), \quad j^c_P(x) = i\bar{\psi}(x) \frac{\lambda^c}{2} \gamma_5 \psi(x) \tag{4}$$

are the vector and the pseudoscalar currents, respectively. In the case of the singlet pseudoscalar current, i.e., when the index c is set to $c = 0$ in (3), we define a function Π^0_{VVP} from exactly the same formula. Π^0_{VVP} differs from Π_{VVP} because of the presence of the U(1) axial anomaly.

First, let us perform the low-energy expansion of Π_{VVP}. At order $O(p^6)$ one has to include a tree-level contribution from $\mathcal{L}^{\text{WZ}}_{(4)}$, a one-loop contribution involving one vertex from $\mathcal{L}^{\text{WZ}}_{(4)}$, and, finally, a tree contribution involving the parameters t_1, t'_1, and t_2 from $\mathcal{L}^{\text{WZ}}_{(6)}$ [Eq. (2)]. In the minimal subtraction scheme the scale dependence which arises from the loop contribution is canceled by that of the parameter t_2 [4, 22]; the first two parameters are scale independent. The expansion is

$$\Pi_{VVP}(p^2, q^2, r^2) = 2B_0 \left\{ \frac{N_c}{16\pi^2 r^2} + 16t_1 + 4t_2 \frac{p^2 + q^2}{r^2} \right\}$$
$$+ (\text{chiral loop}) \,. \tag{5}$$

The first term in (5) is the pole contribution from the canonical Wess-Zumino Lagrangian. In the pseudoscalar singlet channel, now, we no longer have a Goldstone boson pole contribution so that the chiral expansion starts with a constant term:

$$\Pi^0_{VVP}(p^2, q^2, r^2) = 2B_0 \{ 16t_1 + 48t'_1 \} + O(p^2, q^2, r^2) \,. \tag{6}$$

For moderate values of the momenta $p, q \sim 1$ GeV it is natural to assume that Π_{VVP} and Π^0_{VVP} are dominated by the low-lying vector and pseudoscalar resonance poles. A very useful constraint arises in the chiral limit from the behavior in the asymptotic regime $p, q \to \infty$. Indeed, a simple calculation in QCD shows that one has[1]

$$\lim_{p,q \to \infty} \Pi_{VVP}(p^2, q^2, r^2) = \lim_{p,q \to \infty} \Pi^0_{VVP}(p^2, q^2, r^2)$$

$$= -B_0 F_0^2 \frac{p^2 + q^2 + r^2}{2p^2 q^2 r^2}$$

$$\times [1 + O(\alpha_s)] + \cdots \,. \tag{7}$$

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[1] The author is indebted to Marc Knecht for noticing a sign error in the first version of the manuscript.

In other terms, under a scaling of the momenta $p \to \lambda p$, $q \to \lambda q$, Π_{VVP} scales as $1/\lambda^4$ if we let λ go to infinity. The leading term in the asymptotic behavior has the property of being canonical; i.e., it does not contain powers of logarithms caused by anomalous dimensions. This is because the scalar condensate $\langle \bar{\psi}\psi \rangle$ has the same anomalous dimension as the scalar (or the pseudoscalar) current and the vector current carries no anomalous dimension. In deriving (7) we have performed the operator product expansion in the manner of Shijman, Vainshtein, and Zakharov (SVZ) [24], i.e., taking into account the nonperturbative feature that the quark and gluon condensates are nonvanishing. SVZ argue that doing this enlarges the domain of applicability of the asymptotic behavior down to the 1–2 GeV domain. It is therefore natural to demand that the two domains match smoothly. This is really useful only in the chiral limit because the correlation function does not blow up asymptotically and one need not resort to tricks such as the Borel transformation which are necessary in the more general situation. Resonance saturation in the low- to medium-energy region amounts to approximating the correlation function by a meromorphic function having simple poles at the

location of the resonance masses. Reducing to a common denominator, one can in principle have an arbitrary polynomial in the numerator. It is here that the asymptotic conditions come into play and limit the degree of the polynomial. In the present case, where the asymptotic behavior is canonical, the term of highest order in the polynomial gets exactly determined. Let us, for instance, include one nonet of vector resonances in the vector channel and only the pion octet in the pseudoscalar channel. Then, we obtain a very simple representation

$$\Pi_{VVP}(p^2, q^2, r^2) = \frac{-B_0 F_0^2 \left(p^2 + q^2 + r^2\right) + a}{2(p^2 - M_V^2)(q^2 - M_V^2)r^2} , \qquad (8)$$

where M_V is the vector meson mass in the chiral limit. This construction bypasses the use of effective Lagrangians for resonances and yields automatically the most general form of the amplitude containing the right poles and obeying the appropriate asymptotic constraints. In the case above one has a single arbitrary parameter a. In writing down (8) we have ignored the contribution of the $\pi(1300)$ (π') multiplet. If we include the π' pole contribution into the amplitude, we obtain a more complicated representation

$$\Pi_{VVP}(p^2, q^2, r^2) = \frac{[-B_0 F_0^2 \left(p^2 + q^2 + r^2\right) + a](r^2 - M_P^2) + b(p^2 + q^2) + cr^2}{2(p^2 - M_V^2)(q^2 - M_V^2)r^2(r^2 - M_P^2)} , \qquad (9)$$

which contains two additional parameters b and c. M_P is the π' nonet mass in the chiral limit. This representation reduces to the preceding one in the limit where the mass M_P is sent to infinity. It is useful to rewrite (9) by separating out the various poles in r^2

$$\begin{aligned} \Pi_{VVP}(p^2, q^2, r^2) = &\frac{1}{2(p^2 - M_V^2)(q^2 - M_V^2)} \\ &\times \left\{ \frac{a - (p^2 + q^2)(B_0 F_0^2 + b/M_P^2)}{r^2} \right. \\ &\left. + \frac{c + b/M_P^2(p^2 + q^2)}{r^2 - M_P^2} - B_0 F_0^2 \right\} . \end{aligned}$$
$$(10)$$

The parameters a, b, and c may be related to properties of the resonances which will feed back into the low-energy parameters which interest us upon expanding the correlation function around $p^2 = q^2 = r^2 = 0$ and comparing with the representation (5). Here, we must exercise some care with respect to the chiral loop part which is not explicitly displayed in (5). Setting $p^2 = q^2 = 0$ makes it vanish [1, 2]. This allows us to unambiguously identify the parameter a from the residue of the pion pole;

$$a = \frac{N_c}{4\pi^2} B_0 M_V^4 , \qquad (11)$$

and to find a relation between t_1 and c:

$$32 B_0 t_1 = \frac{-1}{2 M_V^4} \left(B_0 F_0^2 + \frac{c}{M_P^2} \right) . \qquad (12)$$

In the case of t_2, we must consider the terms proportional to p^2. The chiral loop generates a contribution proportional to p^2/r^2 (which cancels out the scale dependence arising from t_2) and furthermore generates nonanalytic terms which have no counterpart in the parametrization (9). We will content ourselves with a simple estimate of t_2 valid in the leading N_c approximation: In this limit, the chiral loop need not be taken into account as it is subleading in N_c. Ignoring the loop, one finds

$$4 B_0 t_2 = \frac{1}{2 M_V^4} \left(\frac{a}{M_V^2} - B_0 F_0^2 - \frac{b}{M_P^2} \right) . \qquad (13)$$

There remains to find estimates for b and c. As usual, physical amplitudes are extracted from the Green's functions by taking the residue of the appropriate resonance poles and dividing out by the meson-current coupling constants. Using that, we can express c, to begin with, as a ratio of amplitudes:

$$c = a \tan \Theta \frac{A(\pi'^0 \to \gamma\gamma)}{A(\pi^0 \to \gamma\gamma)} , \qquad (14)$$

where Θ is an angle which parametrizes the strength of the coupling of the π' to the pseudoscalar current,

$$\langle 0 | j_P^a(0) | \pi'^b \rangle = \delta^{ab} B_0 F_0 \tan \Theta , \qquad (15)$$

in the chiral limit (the rationale for introducing an angle here is explained in Ref. [11]). As will be recalled in the next section [see (62)] $\tan \Theta$ can be related via a sum rule to the LE parameter L_8. The amplitudes $A(P \to \gamma\gamma)$ where P is any pseudoscalar meson are normalized everywhere in the text such that the width:

$$\Gamma = \frac{1}{64\pi} M_P^3 A^2 \ .$$ (16)

Similarly, there is a simple relation between the parameter b and the ratio of amplitudes $A(\pi \to \rho\gamma)/A(\pi \to \gamma\gamma)$:

$$\frac{A(\pi \to \rho\gamma)}{A(\pi \to \gamma\gamma)} \left(\frac{-2eF_V}{M_V} \right) = 1 + x,$$

$$x = -\frac{M_V^2}{a} \left(B_0 F_0^2 + \frac{b}{M_P^2} \right) \ ,$$ (17)

where F_V is the coupling of the vector field to the vector current. In Eq. (17), x measures the deviation from the vector meson dominance (VMD) principle. Exact VMD means that $x = 0$. Using the experimental value[2] $\Gamma = 68 \pm 7$ keV for the decay width $\rho^+ \to \pi^+\gamma$ (together with $F_V = 150$ MeV and $M_V = 770$ MeV) we obtain $x = 0.022 \pm 0.051$ which, as has long been well known, is fairly close to exact VMD. If we let M_P go to infinity in (17), we get a deviation from VMD of the order of 20%. For a better accuracy, we have to use a finite π' mass. In this description, VMD is realized from a cancellation between the parameter b and the asymptotic term $B_0 F_0^2$ [see (17)]. Furthermore, assuming the cancellation to be exact ensures that the form factor associated with the matrix element $\langle \gamma | j_\mu^a | \pi \rangle$ satisfies an unsubtracted dispersion relation.

We can now express the parameters t_1 and t_2 in terms of experimentally accessible resonance properties:

$$t_1 = \frac{-1}{64 M_V^2} \left[\frac{F_0^2}{M_V^2} + \frac{N_c}{4\pi^2} \left(\frac{M_V}{M_P} \right)^2 \tan\Theta \frac{A(\pi' \to \gamma\gamma)}{A(\pi \to \gamma\gamma)} \right] \ ,$$

$$t_2 = \frac{N_c}{64\pi^2 M_V^2} (1 + x) \ .$$ (18)

Since the value of x is compatible with zero, the expression above for t_2 reproduces the one obtained previously [21, 22]. Concerning t_1, the contribution proportional to $A(\pi' \to 2\gamma)$ is identical to the one which was identified in Ref. [3] but the first contribution (which comes from the asymptotic term in the three-point function) was not included in that paper. We claim, on the contrary, that this contribution is likely to dominate. Experimentally, there exists an upper bound for the decay $\eta(1295) \to 2\gamma$ rate, $\Gamma < 0.3$ keV. Assuming ideal mixing in the $\pi(1300)$ multiplet together with the estimate $\tan\Theta \simeq 2$ (which is obtained from the sum rule quoted in Sec. IV [Eq. (62)] which relates it to the value of the

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LE parameter L_8) this bound implies that the first term in t_1 is larger than the second by a factor of at least 3. This means that keeping the first contribution in t_1, i.e., ignoring the π' contribution, should provide, at least, the right sign and the right order of magnitude for this parameter.

Let us now consider the flavor singlet pseudoscalar channel. We can write a representation of the three-point function analogous to (10):

$$\Pi_{VVP}^0 = \frac{1}{2(p^2 - M_V^2)(q^2 - M_V^2)}$$

$$\times \left\{ \frac{a' - (p^2 + q^2)(B_0 F_0^2 + b'/M_{P'}^2)}{r^2 - M_{\eta'}^2} \right.$$

$$\left. + \frac{c' + b'/M_{P'}^2(p^2 + q^2)}{r^2 - M_{P'}^2} - B_0 F_0^2 \right\} \ .$$ (19)

Here, $M_{P'}$ is the mass of the singlet state in the $\pi(1300)$ nonet in the chiral limit. At leading N_c order we expect nonet symmetry to hold and thus $M_{P'} = M_P$. Furthermore, in this limit, the residues of the poles in (19) should be equal to the ones in (10), i.e., $a = a'$, $b = b'$, $c = c'$. One can express t_1' as a difference between Π_{VVP} and Π_{VVP}^0 in the limit $p^2 = q^2 = r^2 = 0$ (again the loop contribution vanishes):

$$96 B_0 t_1' = \lim_{r^2 \to 0} \left[\Pi_{VVP}^0(0,0,r^2) - \Pi_{VVP}(0,0,r^2) \right.$$

$$\left. + \frac{B_0 N_c}{8\pi^2 r^2} \right] \ ,$$ (20)

which gives

$$96 B_0 t_1' = \frac{-1}{2M_V^4} \left(\frac{a'}{M_{\eta'}^2} + \frac{c'}{M_{P'}^2} - \frac{c}{M_P^2} \right) \ .$$ (21)

Here, one expects a strong cancellation between the c and the c' terms. In fact, the first term in the parentheses is of order $O(N_c^2)$ while the sum of the last two terms is of order N_c^0. It is easy to check that adding further resonances would also generate corrections which are of order N_c^0 and which are suppressed by inverse powers of the resonance masses. Therefore, it appears that retaining only the first term in (21) should constitute a rather solid approximation. Next, using (19) we can express the parameter a' in terms of the amplitude $A(\eta' \to 2\gamma)$. Introducing the coupling of the η' to the pseudoscalar singlet current,

$$\langle 0 | j_P^0(0) | \eta' \rangle = B_0 G_{\eta'} \ ,$$ (22)

we obtain the following representation for the parameter t_1':

$$e^2 t_1' = \frac{-G_{\eta'}}{256 M_{\eta'}^2} \sqrt{6} A(\eta' \to \gamma\gamma)$$ (23)

[the amplitude is again normalized as in (16)]. At this point we have all the necessary ingredients to discuss the chiral corrections to the decays $\eta \to \gamma\gamma$ and $\pi^0 \to \gamma\gamma$.

III. APPLICATION TO π^0 AND η DECAYS

Let us begin with the η. Collecting the chiral Lagrangian pieces up to order $O(p^6)$ one obtains

$$\mathcal{L}_{\phi_8 \to 2\gamma} = \frac{e^2 F_{\mu\nu} \tilde{F}^{\mu\nu} \phi_8}{16\pi^2 F_0 \sqrt{3}} \left\{ 1 + \frac{5 - 2r_2}{3} T_1 + (1 - r_2) T_1' \right\} .$$

(24)

where we have introduced for convenience the dimensionless quantities proportional to t_1 and t_1':

$$T_1 = \frac{256\pi^2}{3} M_\pi^2 t_1, \qquad T_1' = \frac{1024\pi^2}{3} M_\pi^2 t_1' .$$

(25)

In (24), r_2 is the quark mass ratio $m_s/\hat{m}$ which must be expressed at $O(p^2)$ precision, i.e.,

$$r_2 = 2M_K^2/M_\pi^2 - 1 \simeq 25.9 ,$$

(26)

and, as usual $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. In order to compute the η decay amplitude from (24) we must use that $\phi_8 = \phi_\eta$ at chiral order $O(p^2)$ while $\phi_8 = \phi_\eta F_0/F_\eta$ at order $O(p^4)$, ignoring the π^0-η mixing which effect can be shown to be negligible for the η decay.

Since the loop contributions vanish for on-shell photons [1, 2], the decay rate is computed by simply using (24) at the tree level. Using relation (23) for t_1' one can write the amplitude for η decay as

$$A(\eta \to \gamma\gamma) = \frac{\alpha}{\sqrt{3}\pi F_\pi} \left\{ \frac{F_\pi}{F_\eta} + \frac{5 - 2r_2}{3} T_1 \right\}$$
$$+ \frac{\sqrt{2}}{3}(r_2 - 1) \frac{M_\pi^2}{M_{\eta'}^2} \frac{G_{\eta'}}{F_\pi} A(\eta' \to \gamma\gamma) .$$

(27)

In this formula, we have replaced F_0 everywhere by F_π: This does not modify the first term, and in the others, this replacement amounts to a modification of order $O(p^8)$. One observes that the η' amplitude has crept in via the sum-rule formula for the LE parameter t_1'.

Using this form of the amplitude we can make contact with the traditional parametrization of the η and η' decay amplitudes in terms of an η-η' mixing angle. This will allow us to establish definite relations between the parameters appearing in this representation and those of the chiral expansion. The representation which is usually employed contains three parameters θ_0, λ_0, and λ_8 (see, e.g., [17]):

$$A(\eta \to \gamma\gamma) = \frac{\alpha}{\sqrt{3}\pi F_\pi} \left(\frac{\cos\theta_0}{\lambda_8} - \frac{\sqrt{8}\sin\theta_0}{\lambda_0} \right) ,$$
$$A(\eta' \to \gamma\gamma) = \frac{\alpha}{\sqrt{3}\pi F_\pi} \left(\frac{\sin\theta_0}{\lambda_8} + \frac{\sqrt{8}\cos\theta_0}{\lambda_0} \right) . \quad (28)$$

Let us eliminate λ_0 from the second equation and replace in the first one. We obtain a form exactly similar to (27), which allows us to express λ_8 and θ_0 in terms of LE parameters:

$$\frac{1}{\cos\theta_0 \lambda_8} = \frac{F_\pi}{F_\eta} + \frac{5 - 2r_2}{3} T_1 .$$

(29)

The usual assumption made in the literature to identify

λ_8 with the ratio F_η/F_π is seen to be justified only to the extent that the correction from t_1 is negligible. Here, we find that this correction is of the order of 10%. Concerning the mixing angle, we obtain

$$\tan\theta_0 = \frac{-\sqrt{2}}{3}(r_2 - 1) \frac{M_\pi^2}{M_{\eta'}^2} \frac{G_{\eta'}}{F_\pi} .$$

(30)

A relation between the mixing angle and the parameter L_7 was first derived in [6] based on an analysis of the pseudoscalar mass formulas. We obtain a similar, but slightly different, formula upon using the sum-rule relation between $G_{\eta'}$ and L_7 [see (32) below]. The two formulas do coincide at first order in the quark masses, as they should.

Let us now make some numerical estimates. At order $O(p^4)$, first, one has the well-known result

$$\Gamma_{\eta \to 2\gamma}^{(4)} = \frac{\alpha^2 M_\eta^3}{192\pi^3 F_\pi^2} \simeq 172 \text{ eV} .$$

(31)

(In numerical applications we use the value $F_\pi = 92.4 \pm 0.2$ MeV [26].) At order $O(p^6)$ now, we can use $F_\eta = (1.3 \pm 0.05) F_\pi$ provided by [6] and t_1 is given by (18) and the following discussion. In (27) we still need to evaluate the coupling constant $G_{\eta'}$. An estimate may be obtained from the sum rule for the LE parameter L_7 [10]:

$$L_7 = \frac{-G_{\eta'}^2}{48M_{\eta'}^2} .$$

(32)

This result is derived in the chiral limit under the only assumption that the η' pole is the dominant contribution in the dispersion relation. Using the numerical value of L_7, $L_7 = -(0.4 \pm 0.15) \times 10^{-3}$ [6] one deduces that $G_{\eta'} = 133 \pm 25$ MeV up to a sign. In the large N_c limit we expect to have $G_{\eta'} = F_\pi$ which incites us to adopt the positive sign. Using this value in the expression (27) for the amplitude, together with the experimental value of the width $\Gamma(\eta' \to \gamma\gamma) = 4.29 \pm 0.19$ keV, we obtain the η width in the form

$$\Gamma(\eta \to 2\gamma) = 172 \left[(0.77 \pm 0.03) + 0.09 + (0.72 \pm 0.15) \right]^2$$
$$= 430 \pm 98 \text{ eV} ,$$

(33)

where we have displayed the contributions in the same order as they appear in (27). One observes that the third contribution, which comes from the parameter t_1', largely dominates over the one from t_1, in agreement with the expectation from large N_c arguments. The contribution from t_1 is not completely negligible; it increases the result by about 50 eV. If we had chosen the opposite sign for $G_{\eta'}$ (implying a huge deviation from the leading N_c estimate), then the width would have been of the order of 3 eV. The central value is slightly below the experimental result (the average of the two-photon data at present give $\Gamma = 510 \pm 26$ eV) but the relatively large error bar on L_7 generates an uncertainty of nearly 100 eV on our prediction [as a matter of fact, within this uncertainty, the result is compatible with both the old Primakov result ($\Gamma = 324 \pm 46$ eV) and the photon-photon results].

The uncertainty on L_7 is tightly correlated with the uncertainty on the value of the quark mass ratio r, which is $r = 25.7 \pm 2.3$ in the standard χPT at $O(p^4)$ [6]. Roughly speaking, the upper bound on Γ corresponds to the lower bound on r and vice versa.

Using our formula (30) with $G_{\eta'}$ derived from L_7, we obtain, for the mixing angle ,

$$\theta_0 = -(18.4 \pm 3.6)^\circ , \qquad (34)$$

which is smaller than the one quoted in [10], $\theta_0 = -(22 \pm 4)^\circ$. We should note here that the value of L_7 that we used does not take into account recent work which has provided estimates for the corrections to Dashen's theorem [27]. Part of the large error bar in L_7 is generated from the uncertainty associated with these corrections.

Let us now turn to the π^0 decay. In contrast to the η case it is crucial here to take the quark mass difference $m_d - m_u$ into account. The Lagrangian including the $O(p^6)$ corrections reads

$$\mathcal{L}_{\phi_3 \to \gamma\gamma} = \frac{e^2 F_{\mu\nu} \tilde{F}^{\mu\nu} \phi_3}{16\pi^2 F_0} \left\{ 1 + \left(1 - \frac{5}{3} \frac{m_d - m_u}{m_d + m_u} \right) T_1 \right. $$
$$\left. + \frac{m_u - m_d}{m_u + m_d} T_1' \right\} . \qquad (35)$$

Clearly, one expects the chiral corrections here to be much smaller than in the η case because they do not involve the strange quark mass. The wave-function renormalization in this case can be written as follows at $O(p^4)$ [6], neglecting terms which are quadratic in ϵ:

$$\begin{pmatrix} \phi_3 \\ \phi_8 \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon_1 \\ \epsilon_2 & 1 \end{pmatrix} \begin{pmatrix} F_0/F_{\pi^0} & 0 \\ 0 & F_0/F_\eta \end{pmatrix} \begin{pmatrix} \phi_\pi \\ \phi_\eta \end{pmatrix} . \qquad (36)$$

At $O(p^2)$ one has to set $\epsilon_1 = \epsilon_2 = \epsilon$ with

$$\epsilon = \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \hat{m}} \qquad (37)$$

and, numerically, $\epsilon \simeq 1.00 \times 10^{-2}$ and $\epsilon_2 \simeq 1.11 \times 10^{-2}$. Expressing next t_1' in terms of the mixing angle θ_0 and the η' decay amplitude as before, we can write the amplitude for the pion decay in the form

$$A(\pi \to \gamma\gamma) = \frac{\alpha}{\pi F_{\pi^0}} \left\{ 1 + \frac{\epsilon_2}{\sqrt{3}} + \left(1 + \frac{5 - 4r_2}{\sqrt{3}} \epsilon \right) T_1 \right\}$$
$$- 3\epsilon \tan\theta_0 \, A(\eta' \to \gamma\gamma) . \qquad (38)$$

One notices that F_{π^0} appears here and not $F_\pi \equiv F_{\pi^+}$ which is experimentally well determined. Theoretically, there are two contributions to the difference between F_{π^0} and F_{π^+}. One arises from the quark mass difference $m_d - m_u$: A simple estimate (neglecting chiral logarithms) gives

$$F_{\pi^0} = F_{\pi^+} \left\{ 1 - \frac{2}{3} \left(\frac{F_K^2}{F_\pi^2} - 1 \right) \epsilon^2 + \frac{1}{2} \epsilon_2^2 - \epsilon_1 \epsilon_2 \right\} , \qquad (39)$$

which shows that this contribution is negligibly small, of the order of 10^{-4}. The other contribution is purely electromagnetic and, being of order $O(\alpha_{\rm QED}) \simeq 10^{-2}$, it should be the dominant one. Unfortunately, no estimate of this effect seems to be available at present.

Using the value of θ_0 as discussed above, one can express the rate including the $O(p^6)$ corrections as

$$\frac{F_{\pi^0}^2}{F_{\pi^+}^2} \Gamma = 7.73 \left[1 + (6.41 - 2.49 + 12.5)10^{-3} \right]^2$$
$$= 7.98 \pm 0.08 \text{ eV} . \qquad (40)$$

In the parentheses we have displayed the contributions from the terms proportional to ϵ_2, T_1, and $\tan\theta_0$, respectively. The value of the error includes the uncertainty on F_π as well as that from L_7. Comparison with the experimental result, $\Gamma = 7.74 \pm 0.60$ eV, suggests that the value of F_{π^0} should be somewhat larger than F_π. Despite the rather large error bar on the π^0 width, the value quoted in Ref. [25], $F_{\pi^0} = 84 \pm 3$ MeV, is incompatible with our chiral expansion of the amplitude.

Let us finally discuss the description of the amplitudes $\eta(\pi^0) \to \gamma\gamma$ in the generalized χPT approach. The $O(p^4)$ Lagrangian in the Wess-Zumino sector is obviously the same as in the conventional χPT, being independent of the quark masses. The leading corrections are given by exactly the same two terms, proportional to t_1 as t_1' as before, which are counted as $O(p^5)$ rather than $O(p^6)$. The main difference, then, is that the value of the quark mass ratio r is no longer determined from the $O(p^2)$ χPT but is left as a free parameter. As a function of this parameter, we can express the η amplitude as

$$A(\eta \to \gamma\gamma) = \frac{\alpha}{\sqrt{3}\pi F_\pi} \left\{ \frac{F_\pi}{F_\eta} + \frac{5 - 2r}{3} T_1(r) \right\}$$
$$- \tan\theta_0(r) A(\eta' \to \gamma\gamma) . \qquad (41)$$

Concerning the π^0, we note first that the mixing parameters depend on r, for instance

$$\epsilon(r) = \epsilon \frac{1}{r_2 - 1 + \Delta_{\rm GMO}/2} \left(r - 1 + \frac{2r(r_2 - r)}{r + 1} - \Delta_{\rm GMO} \right) , \qquad (42)$$

where $\Delta_{\rm GMO} = 3M_\eta^2/M_\pi^2 - 2r_2 - 1$ and ϵ has the same meaning as before [Eq. (37)]. The amplitude for π^0 decay can be written as

$$A(\pi \to \gamma\gamma) = \frac{\alpha}{\pi F_{\pi^0}} \left\{ 1 + \frac{\epsilon_2(r)}{\sqrt{3}} + \left[1 + \frac{10(1 - r)}{3\sqrt{3}} \epsilon \right. \right.$$
$$\left. \left. + \frac{5 - 2r}{3\sqrt{3}} \epsilon(r) \right] T_1(r) \right\}$$
$$- [2\epsilon + \epsilon(r)] \tan\theta_0(r) A(\eta' \to \gamma\gamma) . \qquad (43)$$

The expression for $\epsilon_2(r)$ is rather lengthy and contains several $O(p^3)$ low-energy parameters which are not precisely known; however, an approximate expression can be derived

$$\epsilon_2(r) = \epsilon(r) + \epsilon \frac{2}{3}(2r + 1) \frac{r - r_2}{r^2 - 1} \left(\frac{F_K^2}{F_\pi^2} - 1 \right) . \qquad (44)$$

The determination of the mixing angle angle $\theta_0(r)$ goes in the same way as before. Instead of Eq. (30), we obtain here

$$\tan\theta_0(r) = \frac{\sqrt{2}}{3}(r-1)\frac{2\hat{m}B_0}{M_{\eta'}^2}\frac{G_{\eta'}}{F_\pi}$$

$$= \frac{1}{\sqrt{3}}\frac{M_\pi}{M_{\eta'}}\left(-\Delta_{\text{GMO}} + \frac{2(r_2-r)(r-1)}{(r+1)}\right)^{\frac{1}{2}}.$$

$$\tag{45}$$

The second equality is obtained upon using the sum rule (32). Instead of L_7, one must use the corresponding LE parameter Z_0^P in the generalized χPT (GχPT), which appears at $O(p^2)$ and is given at this order from the deviation to the Gell-Mann–Okubo mass formula (see [16]). In order to determine $T_1(r)$ one starts, as before, from the sum rule (18) for t_1. The coupling $\tan\Theta$ is estimated by using, again, the sum rule (62) replacing L_8 by the corresponding parameter A_0, in the GχPT. One ends up with the expression

$$T_1(r) = \frac{-4\pi^2}{3}\frac{M_\pi^2 F_\pi^2}{M_V^4}(1-\lambda)$$

$$-\frac{M_\pi}{M_P}\left(\frac{\lambda M_S^2-(1-\lambda)^2 M_\pi^2}{M_P^2-M_S^2}\right)^{\frac{1}{2}}\frac{A(\pi'\to\gamma\gamma)}{A(\pi\to\gamma\gamma)},$$

$$\lambda = \frac{2(r_2-r)}{r^2-1}.\tag{46}$$

Let us now consider some numerical results in this formalism for a small value of r, $r = 10$, for example. Using the experimental bound on $\eta(1295)\to\gamma\gamma$ we find

$$T_1(r=10) = -6\,10^{-3}(0.68+X), \quad |X|<1.38,\tag{47}$$

where X is the term proportional to $A(\pi'\to\gamma\gamma)$ in the formula for $T_1(r)$. This term generates an uncertainty of the order of 30 eV in the value of the η width and of the order of 0.1 eV in the π^0 width. The remaining contributions to the η width are

$$\Gamma(\eta\to\gamma\gamma)|_{r=10} = 172\left[0.77+0.02+0.95\right]^2 = 521\,\text{eV},$$

$$\tag{48}$$

where we have used that F_η/F_π is practically the same as in the standard χPT and that the value of the mixing angle for $r = 10$, as obtained from (45), is $\theta_0 \simeq 23.8°$. This result is at the upper limit of the allowed range in the standard χPT. The various factors in (45) are, however, widely different from the corresponding ones in (30), $r \ll r_2$, $2\hat{m}B_0 \ll M_\pi^2$, but this is compensated by the third factor, $G_{\eta'} \simeq 8F_\pi$ which is much larger than in the standard χPT (and deviates more from the large N_c result). The π^0 width, finally, can be expressed as

$$\Gamma(\pi^0\to\gamma\gamma)|_{r=10} = 8.07\frac{F_{\pi^+}^2}{F_{\pi^0}^2}\,\text{eV}.\tag{49}$$

Both the results for the π^0 and the η fall in the range of values which are also allowed in the standard χPT.

IV. MIXED LARGE N_C AND CHIRAL EXPANSION

The proper way to introduce the pseudoscalar singlet field $\phi_0(x)$ into the chiral Lagrangian was first discussed

in [28]. The most general Lagrangian at order $O(p^2)$ is discussed in [6]. We will follow their method here, which consists in introducing a source $\theta(x)$ associated with the winding number density $\omega = (16\pi^2)^{-1}\text{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu}$. Under an axial U(1) transformation, $\exp[i\beta(x)]$, one lets the source $\theta(x)$ transform to $\theta(x) - 2\text{tr}\beta(x)$. Then (apart from the canonical WZ part), the effective action is invariant. In the singlet sector, three kinds of invariant building blocks are available, $D_\mu\phi_0 = \partial_\mu\phi_0 - 2\text{tr}a_\mu$, $D_\mu\theta = \partial_\mu\theta + 2\text{tr}a_\mu$, and the combination $\sqrt{6}\phi_0/F_0 + \theta$. For the purpose of discussing the spectrum and electromagnetic properties it is sufficient to consider quadratic polynomials in $\phi_0(x)$. The canonical WZ action is minimally extended using nonet symmetry.

At order two the mixing angle is predicted to be $\theta_P \simeq -10°$ [1, 6] which is too small by a factor of 2 compared to phenomenological determinations [17]. It has long been recognized that going to $O(p^4)$ seems to improve the situation [1]. In Ref. [6] it is proposed to use the $O(p^4)$ Lagrangian with the η' integrated out in conjunction with the $O(p^2)$ Lagrangian which includes the η'. Although this procedure gives the correct magnitude for θ_P, one may wish to include the η' at $O(p^4)$ as well. Furthermore, the Lagrangian proposed in [6] contains terms which are leading in N_c as well as terms which are subleading. Outside of the large N_c expansion, including the η' explicitly into the chiral Lagrangian does not make much sense since the η' mass is rather large, larger than the mass of the ρ or ω mesons, for instance. On the contrary, this looks reasonable if one thinks in terms of the large N_c expansion, since $M_{\eta'}^2$ is $O(1/N_c)$. This suggests to generalize the chiral counting in order to include the large N_c expansion: A natural choice is to count a factor which is $O(1/N_c)$ on the same footing as a quark mass factor [29]. We will call this counting rule the mixed chiral expansion. It should be clear that this expansion is different from the usual chiral expansion. For instance, at leading order, the η mass does not satisfy the Gell-Mann–Okubo relation and the ratio F_π/F_η differs from 1. However, it is also a systematic expansion which could, in principle, be carried out at arbitrary high order.

We first generalize the chiral matrix U to include a nonet of fields:

$$\bar{U} = U\exp\left[i\left(\frac{\lambda_0\phi_0(x)}{F_0}\right)\right].\tag{50}$$

At order 2 of the mixed expansion one has the two terms which are $O(p^2)\times O(N_c)$ as well as a single term which is $O(p^0)\times O((N_c)^0)$:

$$\bar{\mathcal{L}}_{(2)} = \frac{F_0^2}{4}\left\{\langle D_\mu\bar{U}^\dagger D^\mu\bar{U} + \chi^\dagger\bar{U} + \bar{U}^\dagger\chi\rangle\right.$$

$$\left.-\frac{1}{3}M_0^2\left(\sqrt{6}\frac{\phi_0}{F_0} + \theta\right)^2\right\}.\tag{51}$$

It is not difficult to extend this to the next order. We note first that chiral loops need not be considered since they are of chiral order 4 and they are suppressed by one power of N_c; so we effectively count this kind of contribution as

order 6. Furthermore, the two terms of the standard $O(p^4)$ Lagrangian

$$\mathcal{L} = L_6 \langle \chi^\dagger \bar{U} + \bar{U}^\dagger \chi \rangle^2 + \hat{L}_7 \langle \chi^\dagger \bar{U} - \bar{U}^\dagger \chi \rangle^2 \qquad (52)$$

are *both* suppressed by the Zweig rule here, and so they are also effectively of order 6 (the L's pick up contributions from the resonances which are *not* included in the chiral Lagrangian; this is why $\hat{L}_7$ is different from L_7). The only terms that we need consider are

$$\begin{aligned} \bar{\mathcal{L}}_{(4)} = {} & L_5 \langle D_\mu \bar{U}^\dagger D^\mu \bar{U} (\chi^\dagger \bar{U} + \bar{U}^\dagger \chi) \rangle \\ & + L_8 \langle \chi^\dagger \bar{U} \chi^\dagger \bar{U} + \bar{U}^\dagger \chi \bar{U}^\dagger \chi \rangle \ . \end{aligned} \qquad (53)$$

In addition to these terms, which are $O(p^4) \times O(N_c)$, we have to consider the terms which are $O(p^2) \times O((N_c)^0)$. We can form two terms of this kind:

$$\mathcal{L}'_{(4)} = k_1 D_\mu \phi_0 D^\mu \phi_0 + i k_2 \frac{F_0^2}{6} \left(\sqrt{6} \frac{\phi_0}{F_0} + \theta \right) \langle \chi^\dagger \bar{U} - \bar{U}^\dagger \chi \rangle \ . \qquad (54)$$

Furthermore, in the Wess-Zumino sector, we can form a term which is $O(p^4) \times O((N_c)^0)$ and is thus on a similar footing with the true $O(p^6)$ corrections:

$$\mathcal{L}'^{\text{WZ}}_{(6)} = k_3 \epsilon^{\mu\nu\alpha\beta} \langle f^{(+)}_{\mu\nu} f^{(+)}_{\alpha\beta} \rangle \left(\sqrt{6} \frac{\phi_0}{F_0} + \theta \right) \ . \qquad (55)$$

Before proceeding, we note that the LE parameters L_5, L_8 and the quark mass ratio $r = m_s/\hat{m}$ have to be redetermined using the logic of the mixed expansion scheme adopted here. In particular, L_5 and L_8 are scale independent at order 4 now. Outside of the η-η' sector one may simply use the formulas of [6] and drop the loop contributions. L_5, to begin with, is related to the ratio $F_K/F_\pi \simeq 1.22$:

$$L_5 = \frac{F_K^2 - F_\pi^2}{8(M_K^2 - M_\pi^2)} \simeq 2.31 \times 10^{-3} \ . \qquad (56)$$

This value is very close to the result of [6], $L_5(\mu = M_\eta) = 2.3 \times 10^{-3}$, which is not very surprising, since the kaon and the η chiral logarithms are very small for this value of the scale and the pion chiral logarithm, being proportional to the pion mass squared, is small anyway.

Next, expressing the K and π masses

$$\left(1 + \frac{8M_\pi^2}{F_\pi^2} L_5 \right) M_\pi^2 = 2\hat{m} B_0 + \frac{16 M_\pi^4}{F_\pi^2} L_8 \ ,$$

$$\left(1 + \frac{8M_K^2}{F_\pi^2} L_5 \right) M_K^2 = (r+1)\hat{m} B_0 + \frac{16 M_K^4}{F_\pi^2} L_8 \ , \qquad (57)$$

we can solve for $\hat{m} B_0$ and L_8. For the latter, we find

$$L_8 = \frac{L_5}{2} + \frac{F_\pi^2}{8 M_\pi^2} \frac{r_2 - r}{r_2^2 - 1} \ . \qquad (58)$$

In order to evaluate L_8, we need to know the value of r. If we use the number given in [6], $r = 25.7$ (with a 10% uncertainty), we find $L_8 = 1.17 \times 10^{-3}$ [again fairly close to $L_8(M_\eta) = 1.1 \times 10^{-3}$]. However, recent work on the $O(e^2 m_q)$ corrections to Dashen's theorem may imply that this value of r should be revised downwards. Indeed, Gasser and Leutwyler [6] have shown that r is related to the strong part of the K^0-K^+ mass difference by the relation

$$r + 1 = (r_2 + 1) \frac{m_d - m_u}{m_s - \hat{m}} \frac{M_K^2 - M_\pi^2}{(M_{K^0}^2 - M_{K^+}^2)_{\text{QCD}}} \ . \qquad (59)$$

In this relation, Gasser and Leutwyler [6] argue that $R = (m_s - \hat{m})/(m_d - m_u)$ can be estimated from the baryon sector or from ω-ρ mixing in a consistent way, giving [12] $R = 43.5 \pm 3.2$. The strong interaction part of the K_0-K_+ mass difference can be written as

$$\begin{aligned} (M_{K^0}^2 - M_{K^+}^2)_{\text{QCD}} = {} & M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2 \\ & + \delta(M_{\pi^+}^2 - M_{\pi^0}^2) \ . \end{aligned} \qquad (60)$$

Dashen's theorem [30] states that δ is a quantity which is $O(e^2 m_q)$. Recent estimates [27] yield δ to be positive and in the range $0.4 \leq \delta \leq 0.8$. In the apparent absence of any good reason to doubt the estimate of R, using this in relation (59) implies that the central value of r should be moved down to $r \simeq 23$. This has the consequence of increasing the value of L_8 up to $L_8 = 1.43 \times 10^{-3}$. This innocent looking modification has a strong influence on the η decay, as we will see.

It is interesting to compare these results with sum-rule estimates. For L_5 [10] gives

$$L_5 = \frac{F_0^2}{4 M_S^2} \simeq 2.3 \times 10^{-3} \qquad (61)$$

(using $M_S = M_{a_0} = 980$ MeV), which is in fairly good agreement with the above evaluation. A less publicized chiral sum rule can also be derived [11], giving L_8 in the form

$$L_8 = \frac{F_0^2}{16 M_S^2} \left[1 + \tan^2 \Theta \left(1 - \frac{M_S^2}{M_P^2} \right) \right] \ , \qquad (62)$$

where Θ is defined in (15). Using the value of L_8 found above one obtains the estimate of Θ used in Sec. II.

Let us now consider the masses of the η and the η'. At order 4 we have not only a mass matrix **M** but also a kinetic matrix **K**. Both matrices are real and symmetric and the matrix elements of the mass matrix are

$$M_{11} = \frac{1}{3} M_\pi^2 \left[2r_2 + 1 + 2r_2(r_2 + 2) z + 4(r_2 - 1)^2 y \right] \ ,$$

$$M_{12} = \frac{-\sqrt{2}}{3} (r_2 - 1) M_\pi^2 \left[1 + (r_2 + 3) z + 2(r_2 - 1) y + 2k_2 - k_1 \right] \ ,$$

$$M_{22} = M_0^2 (1 - 2k_1) + \frac{1}{3} M_\pi^2 \left\{ r_2 + 2 + [(r_2 + 1)^2 + 2] z + 2(r_2 - 1)^2 y + 2(r_2 + 2)(2k_2 - k_1) \right\} \ , \qquad (63)$$

where

$$y = 4M_\pi^2 L_8/F_\pi^2, \qquad z = 4M_\pi^2 L_5/F_\pi^2 . \tag{64}$$

The mixing with the π^0 has a negligibly small effect at this level and will be ignored. The matrix elements of the kinetic matrix now are

$$K_{11} = 1 + \frac{2}{3}(2r_2 + 1)z, \quad K_{12} = -\frac{2\sqrt{2}}{3}(r_2 - 1)z,$$

$$K_{22} = 1 + \frac{2}{3}(r_2 + 2)z . \tag{65}$$

The mass squared of the η and η' mesons is given by the eigenvalues of the generalized eigenvalue problem:

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

$$= \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} M_\eta^2 & 0 \\ 0 & M_{\eta'}^2 \end{pmatrix} . \tag{66}$$

As a result, the mixing matrix $\mathbf{V}$ is not unitary (instead of satisfying $^t\mathbf{V}\mathbf{V}=1$, it satisfies $^t\mathbf{V}\mathbf{K}\mathbf{V}=1$). This implies that one has to introduce two mixing angles θ_0 and θ_8 and not just one as is usually done (this fact was pointed out in Ref. [31]):

$$\phi_8 = \frac{1}{\lambda_8}(\phi_\eta \cos\theta_8 + \phi_{\eta'} \sin\theta_8) ,$$

$$\phi_0 = \frac{1}{\lambda_0}(-\phi_\eta \sin\theta_0 + \phi_{\eta'} \cos\theta_0) . \tag{67}$$

The three parameters k_1, k_2, and M_0 which appear in the mass matrix can effectively be absorbed into two unknowns $M_0^2(1 - 2k_1)$ and $2k_2 - k_1$, which we expect to determine from fitting the η and η' masses. It turns out that one has two different solutions, which we label as $(+)$ and $(-)$. The requirement that the large N_c expansion be meaningful allows one to eliminate one of the solutions. Indeed, numerically, the solution $(+)$ corresponds to the following values for the parameters M_0 and k_2 (setting $k_1 = 0$):

$$M_0 = 898 \text{ MeV}, \qquad k_2 = 0.12 , \tag{68}$$

while the solution $(-)$ corresponds to

$$M_0 = 1226 \text{ MeV}, \qquad k_2 = -1.43 . \tag{69}$$

Clearly, the solution $(+)$ does satisfy the criterion that the contribution of the k_2 term in the mass matrix be roughly of the same magnitude as, say, the contribution from the L_8 term. This does not hold for the other solution. The possibility of reproducing exactly the η and η' masses while having only small corrections to the $O(p^2)$ Lagrangian (51) was noted by Peris [32]. One can express the mixing angles in closed form

$$\tan\theta_0 = -\left(\frac{M_{11} - K_{11}M_\eta^2}{K_{11}M_{\eta'}^2 - M_{11}} \right)^{\frac{1}{2}} , \qquad \theta_8^\pm = \theta_0 \pm \phi,$$

$$\tan\phi = \frac{-K_{12}}{(K_{11}K_{22} - K_{12}^2)^{\frac{1}{2}}} , \tag{70}$$

as well as the extra factors

$$\lambda_8 = \left(K_{11} - \frac{K_{12}^2}{K_{22}} \right)^{\frac{1}{2}} , \quad \lambda_0 = \left(K_{22} - \frac{K_{12}^2}{K_{11}} \right)^{\frac{1}{2}} . \tag{71}$$

In practice, however, these expressions should be expanded linearly in terms of the $O(p^4)$ parameters in order to remain consistent with the $O(p^4)$ precision. Before exploiting this, let us examine the results that one would obtain at order two of the mixed expansion, i.e., starting from (51). The unique parameter in that case, M_0, may be adjusted so as to reproduce the η mass. The mixing angle and the η' mass then satisfy

$$\tan\theta_P = -\frac{(2r_2 + 1)M_\pi^2 - 3M_\eta^2}{\sqrt{2}(r_2 - 1)M_\pi^2}, \tag{72}$$

$$\overset{\circ}{M}_{\eta'}^2 = M_\pi^2 \left(\frac{2r_2 + 1}{3} + \frac{\sqrt{2}(r_2 - 1)}{3\tan\theta_P} \right) .$$

Numerically, this gives $\overset{\circ}{M}_{\eta'} = 1583$ MeV, which is indeed larger than the experimental mass as expected from the bound derived in [33], and $\theta_P = -5.6°$. Note that the result differs from [6] because their mass matrix includes the term k_2 which we have considered here to be of higher order. In the mixed expansion the mixing angle is of order 1 rather than $O(m_q)$ as in the chiral expansion, yet, numerically, it comes out to be rather small at leading order. The ratio F_π/F_η differs from one already at leading order and is given by $F_\pi/F_\eta = 1/\cos\theta_P$.

At order 4, let us concentrate on the quantities which are relevant in the discussion of the η decay amplitude, that is, $\tan\theta_0$ and F_π/F_η. The former is given in closed form by Eq. (70) and the latter can be expressed as

$$\frac{F_\pi}{F_\eta} = \frac{\cos(\theta_0 - \theta_8)}{\lambda_8 \cos\theta_0}(1 + z) . \tag{73}$$

As stated above, we must expand the expressions for $\tan\theta_0$ and F_π/F_η linearly in terms of the $O(p^4)$ parameters y, z and the mass difference $M_{\eta'}^2 - \overset{\circ}{M}_{\eta'}^2$. For this purpose, let us introduce the notation

$$\rho = \tan\theta_P, \quad \rho' = \frac{3M_{\eta'}^2 - (2r_2 + 1)M_\pi^2}{\sqrt{2}(r_2 - 1)M_\pi^2} . \tag{74}$$

One finds, for the ratio F_π/F_η

$$\frac{F_\pi}{F_\eta} = (1 + \rho^2)^{\frac{1}{2}} \left\{ 1 - \frac{2}{3}z(r_2 - 1) + \frac{r_2 - 1}{\sqrt{2}}\rho\left(2y - \frac{z}{3} \right) \right.$$

$$\left. -\frac{1}{2}\rho^2 \frac{\rho\rho' - 1}{\rho^2 + 1} \right\} . \tag{75}$$

The first two terms in the parentheses may be recognized as those of the standard chiral expansion at $O(p^4)$ in the leading N_c approximation. Here, this formula is seen to receive corrections. Now we give the expansion of the mixing angle

$$-\tan\theta_0 = \frac{1}{2}\rho(3 - \rho\rho') + \frac{r_2 - 1}{\sqrt{2}}(1 - \rho^2)\left(2y - \frac{z}{3}\right).$$

$$(76)$$

Let us now return to the question of the electromagnetic widths. In the effective Lagrangian language, the new feature compared to Sec. III is that we have a piece containing ϕ_0:

$$\mathcal{L}_{\phi_0 \to 2\gamma} = \frac{8e^2}{F_0\sqrt{6(1 + 2k_1)}}F_{\mu\nu}\tilde{F}^{\mu\nu}\phi_0\left\{\frac{1}{32\pi^2} + 4k_3\right.$$

$$\left. + \frac{4(5 + r_2)}{9}M_\pi^2 t_1\right\}.$$

$$(77)$$

It can be seen that the amplitude for $\eta' \to 2\gamma$ derived from (77) conforms to the general structure predicted in [34].

The Lagrangian for ϕ_8 is the same as before (24) with the crucial difference that the t_1' term is absent [to be more precise, it no longer contains the $a'/M_{\eta'}^2$ piece in (21) and the rest being $O((N_c)^0)$ does not count at this order]. In order to discuss the η and η' decays we expand ϕ_0 and ϕ_8 using (67). In order to be consistent with the $O(p^6)$ precision we use the mixing angles at $O(p^4)$ in the $O(p^4)$ piece of the WZ Lagrangian and the mixing angle θ_P in the piece which is already $O(p^6)$.

It is clear from (77) that we can make no definite prediction for the decay of the η'. All we can do is use the experimental information to constrain the combination of the parameters k_1 and k_3 which appear in (77). Once this is done, we can make a prediction for the η decay. We can recast the η decay amplitude in a similar form as before [Eq. (27)]:

$$A(\eta \to \gamma\gamma) = \frac{\alpha}{\sqrt{3}\pi F_\pi}\left\{\frac{F_\pi}{F_\eta} + \frac{5 - 2r_2}{3\cos\theta_P}T_1\right\}$$

$$- \tan\theta_0 A(\eta' \to \gamma\gamma).$$

$$(78)$$

Note that only one of the mixing angles shows up in this expression. The expansions of F_π/F_η and $\tan\theta_0$ have been determined above [Eqs. (75), (76)]. It is interesting to investigate the sensitivity of the result upon small variations of the quark mass ratio r. We will consider the two cases $r = 25.7$, as in [6], and $r = 23$, as suggested from the corrections to Dashen's theorem. In the first situation one obtains

$$\frac{F_\pi}{F_\eta} = 0.703, \qquad \theta_0 = -20.4°,$$

$$(79)$$

$$\Gamma(\eta \to \gamma\gamma) = 439 \text{ eV} \quad (r = 25.7).$$

This value of r is the same as the one used in the chiral expansion in Sec. III (standard case). The value of the mixing angle obtained here is larger in magnitude than the one obtained in Sec. III. One could a priori expect that the resulting η width should be larger as well but it turns out that this is not the case. Indeed, there is an extra ingredient in the width which is F_π/F_η and using the

mixed expansion rules we find a result which is smaller than the chiral expansion result $F_\pi/F_\eta = 0.77 \pm 0.03$. This perhaps suggests that the uncertainty on this parameter is somewhat underestimated. These two differences cancel out to some extent and one ends up with practically the same result as before for the η width. Now for the smaller value of r we obtain

$$\frac{F_\pi}{F_\eta} = 0.710, \qquad \theta_0 = -23.9°,$$

$$(80)$$

$$\Gamma(\eta \to \gamma\gamma) = 529 \text{ eV} \quad (r = 23.0).$$

This illustrates that the uncertainty of the order of 100 eV which was obtained in Sec. III can be traced, to a large extent, to the uncertainty (of the order of 10%) of the value of the quark mass ratio $m_s/\hat{m}$ in the standard χPT.

V. CONCLUSIONS

In this paper we have discussed the chiral expansion of the amplitudes for $\pi^0 \to \gamma\gamma$ and $\eta \to \gamma\gamma$ beyond leading order. The main new contribution in this topic is the attempt made to evaluate precisely, on the basis of sum rules, the two parameters t_1 and t_1' which appear in $\mathcal{L}_{(6)}^{WZ}$. The dominating contribution, numerically, is t_1', in good agreement with the fact that it is $O(N_c^2)$ as it picks up a pole from the η' meson. We believe that our estimate of t_1' should be on the same level of reliability as the sum rule for L_7. This sum rule is, in fact, one of the ingredients in the evaluation. The contribution from t_1 was found to amount to roughly 10% of the width. Its evaluation is certainly not as precise as for t_1' but we argued that the sign and the order of magnitude should be the correct ones. An interesting peculiarity of this parameter is that it does not reflect the property of a particular resonance but rather of the continuum, which is matched to the QCD asymptotic behavior. It would be interesting to compare these results with those which obtain in other approaches, in particular improved variants of the Nambu–Jona-Lasinio (NJL) model which seem to perform well for the standard $O(p^4)$ LE parameters [35]. On the practical side, we found that the $O(p^6)$ correction raises the value of the width to a level which is compatible with the photon-photon experimental results. The prediction has a rather large uncertainty, of the order of 25%, which is generated by the parameter L_7 and, to some extent, by F_π/F_η. We have also investigated the relevance of these two amplitudes in connection with the generalized χPT which embodies values of r much smaller than r_2. It turned out that, even though individual factors are very different, they combine to give a result which is also compatible with experiment.

One of our aims was to test the convergence of chiral perturbation theory in the anomalous sector. Since the lowest-order result is so small compared to experiment, it was by no means obvious how the $O(p^6)$ correction could manage to bring the two in agreement. This can be understood qualitatively if one assumes that the simple rule that a $1/N_c$ correction has roughly the same

magnitude as a quark mass correction. The leading term in the $O(p^6)$ correction goes as $O(N_c^2) \times O(m_q)$; according to this rule, it is natural to expect that it could be of a similar size as the leading chiral order contribution which goes as $O(N_c) \times O(1)$. Terms of still higher order, on the other hand, should be much smaller. On the basis of this counting scheme it is possible to include the η' into the effective Lagrangian in a systematic way. We have discussed the question of the η decay from this mixed expansion point of view. In a sense, this proves slightly disappointing because, even in this scheme, the essential contribution to the mixing angle comes at next to leading order. The mixing angle turns out to be slightly larger than in the chiral expansion but the predictions for the width are nearly identical. There remains to explore whether this kind of expansion provides some nontrivial constraints in the η' sector.

[1] J.F. Donoghue, B.R. Holstein, and Y-C.R. Lin, Phys. Rev. Lett. **55**, 2766 (1985).

[2] J. Bijnens, A. Bramon, and F. Cornet, Phys. Rev. Lett. **61**, 1453 (1988).

[3] Riazuddin and Fayyazuddin, Phys. Rev. D **37**, 149 (1988).

[4] J.F. Donoghue and D. Wyler, Nucl. Phys. **B316**, 289 (1989).

[5] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984).

[6] J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465 (1985).

[7] G. Ecker, J. Gasser, A. Pich, and E. De Rafael, Nucl. Phys. **B321**, 311 (1989).

[8] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, and E. De Rafael, Phys. Lett. B **223**, 425 (1989).

[9] J.F. Donoghue and B.R. Holstein, Phys. Rev. D **46**, 4076 (1992).

[10] H. Leutwyler, Nucl. Phys. **B337**, 108 (1990).

[11] B. Moussallam and J. Stern, in *Two-Photon Physics: From DAPHNE to LEP 200 and Beyond*, Proceedings of the Workshop, Paris, France, 1994, edited by F. Kapusta (World Scientific, Singapore, 1994).

[12] J. Gasser and H. Leutwyler, Phys. Rep. **87**, 77 (1982).

[13] N.H. Fuchs, H. Sazdjian, and J. Stern, Phys. Lett. B **238**, 380 (1990).

[14] N.H. Fuchs, H. Sazdjian, and J. Stern, Phys. Lett. B **269**, 183 (1991).

[15] J. Stern, H. Sazdjian, and N.H. Fuchs, Phys. Rev. D **47**, 3814 (1993).

[16] J. Stern and M. Knecht, in "Second edition of the DAΦNE handbook," edited by L. Maiani, G. Pancheri and N. Paver, Report No. hep-ph 9411253 (unpublished).

[17] F. Gilman and R. Kauffman, Phys. Rev. D **36**, 2761 (1987).

[18] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).

[19] E. Witten, Nucl. Phys. **B223**, 22 (1983).

[20] R. Akhoury and A. Alfakih, Ann. Phys. (N.Y.) **210**, 81 (1991).

[21] E. Pallante and R. Petronzio, Nucl. Phys. **B396**, 205 (1993).

[22] J. Bijnens, A. Bramon, and F. Cornet, Z. Phys. C **46**, 599 (1990).

[23] J. Bijnens, Nucl. Phys. **B367**, 709 (1991).

[24] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979).

[25] Particle Data Group, K. Hikasa *et al.*, Phys. Rev. D **45**, S1 (1992).

[26] B.R. Holstein, Phys. Lett. B **244**, 83 (1990).

[27] K. Maltman and D. Kotchan, Mod. Phys. Lett. A **5**, 2457 (1990); J.F. Donoghue, B.R. Holstein, and D. Wyler, Phys. Rev. D **47**, 2089 (1993); J. Bijnens, Phys. Lett. B **306**, 343 (1993); R. Urech, Nucl. Phys. **B433**, 234 (1995).

[28] P. DiVecchia, Phys. Lett. **85B**, 357 (1979); C. Rosenzweig, J. Shechter, and C.G. Trahern, Phys. Rev. D **21**, 388 (1980); E. Witten, Ann. Phys. (N.Y.) **128**, 363 (1980).

[29] J. Stern (private communication).

[30] R. Dashen, Phys. Rev. **183**, 1245 (1969).

[31] J. Schechter, A. Subbaraman, and H. Weigel, Phys. Rev. D **48**, 339 (1993).

[32] S. Peris, Phys. Lett. B **324**, 442 (1994).

[33] H. Georgi, Phys. Rev. D **49**, 1666 (1994).

[34] G.M. Shore and G. Veneziano, Nucl. Phys. **B381**, 3 (1992).

[35] B. Holdom, J. Terning, and K. Verbeek, Phys. Lett. B **245**, 612 (1990); J. Bijnens, Ch. Bruno, and E. de Rafael, Nucl. Phys. **B390**, 501 (1993).