Massless fields in scalar-tensor cosmologies

José P. Mimoso* and David Wands
Astronomy Centre, School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, Sussex, United Kingdom
(Received 29 April 1994)

We derive exact Friedmann-Robertson-Walker cosmological solutions in general scalar-tensor gravity theories, including Brans-Dicke gravity, for stiff matter or radiation. These correspond to the long or short wavelength modes, respectively, of massless scalar fields. If present, the long wavelength modes of such fields would be expected to dominate the energy density of the Universe at early times and thus these models provide insight into the classical behavior of these scalar-tensor cosmologies near an initial singularity, or bounce. The particularly simple exact solutions also provide a useful example of the possible evolution of the Brans-Dicke (or dilaton) field ϕ and the Brans-Dicke parameter $\omega(\phi)$ at late times in spatially curved as well as flat universes. We also discuss the corresponding solutions in the conformally related Einstein metric.

PACS number(s): 04.50.+h, 04.60.−m, 98.80.Cq, 98.80.Hw

I. INTRODUCTION

If we hope to describe gravitational interactions at energy densities approaching the Planck scale it seems likely that we will need to consider Lagrangians extended beyond the Einstein-Hilbert action of general relativity. The low-energy effective action in string theory, for instance, involves a dilaton field coupled to the Ricci curvature tensor [1]. Scalar fields coupled directly to the curvature appear in all dimensionally reduced gravity theories, and their influence on cosmological models was first seriously considered by Jordan [2]. These models have been termed scalar-tensor gravity, the best known of these being the Brans-Dicke theory [3]. Gravity Lagrangians including terms of higher order in the Ricci scalar can also be cast as scalar-tensor theories [4,5] with appropriate scalar potentials.

The belief that modified gravity theories may have played a crucial role during the early Universe has recently been rekindled by extended inflation [6]. In this scenario a scalar-tensor gravity theory allows the first order phase transition of the "old" inflationary model [7] to complete. This arises because the scalar field ϕ (henceforth the Brans-Dicke field, essentially the inverse of the Newton's gravitational "constant") damps the rate of expansion and, in the original extended inflationary scenario based on the Brans-Dicke theory, turns the exponential expansion found in general relativity into power law inflation [8]. However, Brans-Dicke theory is unable to meet the simultaneous and disparate requirements placed by the post-Newtonian solar system tests [9] and by the need to keep the sizes of the bubbles nucleated during inflation within the limits permitted by the anisotropies of the microwave background [10].

This situation may be averted through the consideration of more general scalar-tensor theories in which the parameter ω, a constant in Brans-Dicke theory, is allowed to vary as $\omega(\phi)$ [11]. But in such cases one needs to understand better the cosmological behavior of these general theories, in order to assess their implications on our models of the Universe. Direct observations mainly constrain these theories at the present day in our solar system [12], imposing a lower bound $\omega > 500$ and requiring that $\omega^{-3}(d\omega/d\phi)$ should approach zero. On a cosmological scale, the principal limits arise from the consideration of effects upon the synthesis of light elements, indicating that at the time of nucleosynthesis similar bounds hold [13].

Most of the work that can be found in the literature on solutions of scalar-tensor theories concerns the particular case of Brans-Dicke theory [14–19]. The properties of more general scalar-tensor cosmologies have been discussed recently [20,21] and exact solutions derived for the vacuum and radiation models [22] (where $p = \rho/3$) corresponding to the particular situation where the scalar field is sourceless (because the matter energy-momentum tensor is traceless). In this paper we show how to extend this method to derive exact solutions for the homogeneous and isotropic cosmological models with a perfect fluid characterized by the equation of state $p = \rho$, which does act as a source for the Brans-Dicke field. These models represent the evolution a homogeneous massless scalar field [23]. Such a scalar field may describe the evolution of effectively massless fields, including in the context of superstring cosmology the antisymmetric tensor field which appears in the low energy string effective action [1,24].

As the energy density of a barotropic perfect fluid with $p = (\gamma - 1)\rho$ evolves as $\rho \propto a^{-3\gamma}$, such "stiff matter" would be expected to dominate at early times in the Universe [25] over short wavelength modes or any other mat-

*Present address: Dep. Física, F.C.L., Ed C1, Campo Grande, 1700 Lisboa, Portugal. Electronic address: fmimoso@neuron.cc.fc.ul.pt

ter with $p < \rho$. Thus our solutions provide an important indication of the possible early evolution of scalar-tensor cosmologies.

As in [22], our solutions will be given in closed form in terms of an integration depending on $\omega(\phi)$, which can be performed exactly in many cases and numerically in all cases. The general field equations are given in Sec. II for scalar-tensor gravity and we solve these in Friedmann-Robertson-Walker metrics for vacuum, stiff fluid, and radiation models in Sec. III. The equivalent picture in the conformally transformed Einstein frame is presented in Sec. IV. Conclusions on the general behavior of solutions are presented in our final section.

II. SCALAR-TENSOR GRAVITY THEORIES

The scalar-tensor field equations [26] are derived from the action[1]

$$S = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left[\phi(R-2\lambda(\phi)) - \frac{\omega(\phi)}{\phi}g^{ab}\phi_{,a}\phi_{,b} + 16\pi\mathcal{L}_m\right], \tag{2.1}$$

where R is the usual Ricci curvature scalar of the spacetime, ϕ is the Brans-Dicke scalar field, $\omega(\phi)$ is a dimensionless coupling function, and $\mathcal{L}_m$ represents the Lagrangian for the matter fields. It is clear that the scalar field plays the role which in general relativity is played by the gravitational constant, but with ϕ now a dynamical variable.

The particular case of Brans-Dicke gravity arises when we take ω to be a constant and $\lambda = 0$ in the Lagrangian of Eq. (2.1). The $\lambda(\phi)$ potential is the natural generalization of the cosmological constant Λ. It introduces terms which violate Newtonian gravity at some length scale. In what follows we will leave $\omega(\phi)$ as a free function but consider only models in which λ is zero. This should be valid at least at sufficiently early times when we expect kinetic terms to dominate, and will also avoid introducing too many free functions into our analysis. (Note that the Lagrangian is sometimes written in terms of a scalar field φ with a canonical kinetic term so that $\phi \equiv f(\varphi)$ and $\omega(\phi) \equiv f/[2(df/d\varphi)^2]$.)

Taking the variational derivatives of the action (2.1) with respect to the two dynamical variables g_{ab} and ϕ and setting $\lambda(\phi) = 0$ yields the field equations

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi\frac{T_{ab}}{\phi} + \frac{\omega(\phi)}{\phi^2}\left(g^c_a g^d_b - \frac{1}{2}g_{ab}g^{cd}\right)\phi_{,c}\phi_{,d} + \frac{1}{\phi}\left(\nabla_a\nabla_b\phi - g_{ab}\Box\phi\right), \tag{2.2}$$

$$\Box\phi = \frac{1}{2\omega(\phi)+3}\left[8\pi T - g^{cd}\omega_{,c}\phi_{,d}\right], \tag{2.3}$$

where $T = T^a_a$ is the trace of the energy-momentum tensor of the matter defined as

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\partial}{\partial g_{ab}}\left(\sqrt{-g}\mathcal{L}_m\right). \tag{2.4}$$

It is important to notice that the usual relation $\nabla_b T^{ab} = 0$ establishing the conservation laws satisfied by the matter fields holds true. This follows from the assumption that all matter fields are minimally coupled to the metric g_{ab}, which means that the principle of equivalence is guaranteed. The role of the scalar field is then that of determining the spacetime curvature (associated with the metric) produced by the matter. Matter may be a source of the Brans-Dicke field, but the latter acts back on the matter only through the metric [28].

III. FRIEDMANN-ROBERTSON-WALKER MODELS

We consider homogeneous and isotropic universes with the metric given by the usual Friedmann-Robertson-Walker (FRW) line element

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\right]. \tag{3.1}$$

The field equations for a scalar-tensor theory, where we allow the coupling parameter ω to depend on the scalar field ϕ, but restrict the potential λ to be zero, are then

$$H^2 + H\frac{\dot\phi}{\phi} - \frac{\omega(\phi)}{6}\frac{\dot\phi^2}{\phi^2} + \frac{k}{a^2} = \frac{8\pi}{3}\frac{\rho}{\phi}, \tag{3.2}$$

$$\ddot\phi + \left[3\frac{\dot a}{a} + \frac{\dot\omega(\phi)}{2\omega(\phi)+3}\right]\dot\phi = \frac{8\pi\rho}{2\omega(\phi)+3}(4-3\gamma), \tag{3.3}$$

$$\dot H + H^2 + \frac{\omega(\phi)}{3}\frac{\dot\phi^2}{\phi^2} - H\frac{\dot\phi}{\phi} = -\frac{8\pi\rho}{3\phi}\frac{(3\gamma-2)\,\omega+3}{2\omega(\phi)+3} + \frac{1}{2}\frac{\dot\omega}{2\omega(\phi)+3}\frac{\dot\phi}{\phi}. \tag{3.4}$$

The equation of motion for ϕ, in particular, demonstrates how the specific $\omega(\phi)$ determines the deviation from both

[1]One can also include an additional boundary term dependent on the extrinsic curvature of the boundary [5], as is required in general relativity [27] to allow for the variation of $g_{ab,c}$ on the boundary.

general relativity and Brans-Dicke gravity. General relativistic behavior, with ϕ =const, is only allowed for trace-free matter $[(4-3\gamma)\rho = 0]$ or $\omega \to \infty$. The presence of a variable $\omega(\phi)$, in contrast with Brans-Dicke gravity, introduces a further damping term, in addition to the Hubble damping, on the left-hand side.

Several authors have studied cosmological solutions of the Brans-Dicke theory for a FRW universe filled with a perfect fluid [3,14–17]. Nariai [29] derived power law solutions for the flat FRW universe with a perfect fluid satisfying the barotropic equation of state $p = (\gamma - 1)\rho$, with γ a constant taking values in the interval $0 \le \gamma \le 2$.

The solutions of these equations of motion are defined by four integration constants whereas the corresponding solutions in general relativity depend on only three [18]. In addition to the values of $a(t_0)$, $\dot{a}(t_0)$, and $\phi(t_0) \propto 1/G(t_0)$,[2] we now also need $\dot{\phi}(t_0)$ [or $\rho(t_0)$ instead]. Originally, as done by Brans and Dicke, and by Nariai, this extra freedom was eliminated by requiring that $\dot{\phi}a^3$ should vanish when a approaches the initial singularity at $a = 0$. In a flat FRW model this restricts one to obtaining only the power law solutions of the Brans-Dicke theory. Here we shall keep our analysis more general.

The derivation of the general barotropic Brans-Dicke solutions for the spatially flat ($k = 0$) model was done by Gurevich, Finkelstein, and Ruban [16]. Their solutions, which cover all the space of parameters of the theory, render clear a very important feature of the behavior of the flat cosmological models, namely, that for the solutions which exhibit an initial singularity (those with $\omega > -3/2$) the scalar field dominates the expansion at early times, while the later stages are matter dominated and approach the behavior of Nariai's solutions for $\omega > 2(\gamma - 5/3)/(2-\gamma)^2$. Note therefore that the stiff fluid solutions are unique in that they do not approach Nariai's power law solutions at late times other than in the limit $\omega \to \infty$.

The same solutions for the flat model in the cases of vacuum, stiff matter ($p = \rho$), and radiation ($p = \rho/3$) were rederived later by Lorenz-Petzold [17] using a different method which enabled him to also obtain solutions for the nonflat models. We use this method in an improved form [22] to derive solutions for the general scalar-tensor theories.

Using the conformal time variable η defined by the differential relation

$$dt = a\, d\eta \,, \tag{3.5}$$

and the variables

$$X \equiv \phi\, a^2 \tag{3.6}$$

and

$$Y \equiv \int \sqrt{\frac{2\omega+3}{3}}\, \frac{d\phi}{\phi} \,, \tag{3.7}$$

we can rewrite the above field equations as

$$(X')^2 + 4\,k\,X^2 - (Y'\,X)^2 = 4MXa^{4-3\gamma} \,, \tag{3.8}$$

$$[Y'\,X]' = M(4-3\gamma)\sqrt{\frac{3}{2\omega+3}}\, a^{4-3\gamma} \,, \tag{3.9}$$

$$X'' + 4\,k\,X = 3\,M(2-\gamma)\, a^{4-3\gamma} \,, \tag{3.10}$$

where the density $\rho = 3M/8\pi a^{3\gamma}$ for a barotropic fluid with M a constant. The prime denotes differentiation with respect to η. Our variables are akin to those used by Lorenz-Petzold [17] when solving for the Brans-Dicke theory. To that extent the method we explore here is a generalization of his method of obtaining decoupled equations. Note that whenever X is negative this must correspond to a negative value for ϕ. In what follows, unless otherwise explicitly stated, we shall assume that $\omega > -3/2$, to guarantee the positiveness of the function under the square root in Eq. (3.7), although it would be straightforward to redefine $Y(\phi)$ for the case of $\omega < -3/2$.

This system considerably simplifies for the two particular cases: $\gamma = 4/3$ (radiation) and $\gamma = 2$ (stiff matter). We shall show in the next section precisely how these correspond to the short and long wavelength limits of a massless field, respectively. These two limits do correspond to unusually simple cases. The energy-momentum tensor for radiation (or vacuum) is trace-free and so the equation of motion for ϕ, Eq. (3.3), has no driving term on the right-hand side. Thus it can be written as an equation of motion for the massless field $Y(\phi)$, Eq. (3.9). While stiff matter does drive the field ϕ we will show that Eq. (3.9) can still be written as an equation of motion for a redefined massless field $Z(\phi)$. In either case the full integration is again possible, provided we specify the function $\omega(\phi)$. For other values of γ, Eq. (3.9) retains an explicit dependence on a which cannot be integrated by the method adopted here. An alternative approach for these latter cases based on another method of integration of the original field equations is presented elsewhere [30].

Anisotropic cosmologies have also been considered in the literature, again principally for Brans-Dicke gravity. We will show elsewhere how our method may be extended to derive solutions for general scalar-tensor gravity in anisotropic models [31].

A. Scalar field evolution

A minimally coupled scalar field σ whose energy-momentum tensor

$$\begin{aligned} T_{ab} &= \left(g_a^c g_b^d - \frac{1}{2} g_{ab} g^{cd}\right) \sigma_{,c}\sigma_{,d} \,, \\ &= (p+\rho)\, u_a u_b + p\, g_{ab} \,, \end{aligned} \tag{3.11}$$

corresponds to a perfect fluid [23] with density

$$\rho = p = \frac{1}{2}|g^{ab}\sigma_{,a}\sigma_{,b}| \tag{3.12}$$

[2]This last, $G(t_0)$, is of course usually considered a fundamental constant in general relativity rather than an initial condition.

and normalized velocity field

$$u_a = \frac{\sigma_{,a}}{|g^{cd}\sigma_{,c}\sigma_{,d}|^{1/2}} \,. \tag{3.13}$$

The scalar field itself obeys the wave equation

$$\Box\sigma \;=\; 0 \,, \tag{3.14}$$

which in a FRW metric reduces to

$$-\ddot{\sigma} - 3H\dot{\sigma} + \sum_{i,j=1}^{3} g^{ij}\nabla_i\nabla_j\sigma \;=\; 0 \,. \tag{3.15}$$

If we consider plane wave solutions of the form $\sigma = \sigma_q(t)\exp(i\sum q_i x^i)$ then for $(q/a)^2 \gg H^2, \dot{H}, k/a^2$ this can be rewritten as

$$(a\sigma_q)'' + q^2(a\sigma_q) \;=\; 0 \,, \tag{3.16}$$

where $q^2 = \sum q_i^2$. This corresponds to the usual flat space result for a plane wave where $(a\sigma_q) \propto \exp(iq\eta)$ and thus $T_a^b = q_a q^b \sigma_q^2 \propto a^{-4}$ with the null four-vector $q_a = (q, q_i)$. Each plane wave is anisotropic as each q_i points in a particular direction. For consistency with our choice of an isotropic metric we must consider an isotropic distribution of wave vectors $P(q)$ summed over all spatial directions. This ensures that all off-diagonal terms are zero [as symmetry now requires $T_{ib} = \int q_i q_b \sigma_q^2 P(q) d^3q = \int (-q_i) q_b \sigma_q^2 P(q) d^3q = 0$ for $i \neq b$] and the isotropic pressure $p = \rho/3$, the usual result for isotropic radiation.[3]

For long wavelength modes in an FRW universe (with comoving wave number $(q/a)^2 \ll H^2, \dot{H}$) we can neglect spatial gradients in the field, and the first integral of Eq. (3.15) yields $a^3\dot{\sigma}$ =const and thus $p = \rho = \dot{\sigma}^2/2 \propto a^{-6}$, i.e., a stiff fluid.

Clearly the dividing line between these long and short wavelength modes changes as the comoving Hubble length or curvature scale evolves.[4] In a conventional (noninflationary) cosmology the comoving Hubble length shrinks as we consider earlier and earlier times in an expanding universe so that as $a \to 0$ all modes must be "outside the horizon" and evolve as a homogeneous stiff fluid, lending support to our contention that the stiff fluid solutions will be important in determining the classical behavior of scalar-tensor cosmologies near any initial singularity. In any case, as already remarked, the energy density of a barotropic perfect fluid evolves as $\rho \propto a^{-3\gamma}$ and so the energy density of a stiff fluid will eventually dominate as $a \to 0$ over any matter with a barotropic index $\gamma < 2$.

In what follows we shall consider only the extreme short and long wavelength modes of the massless field, neglecting the intermediate regimes.

B. Vacuum solutions

Let us first consider the field equations in vacuum:

$$(X')^2 - (Y'X)^2 + 4kX^2 = 0 \,, \tag{3.17}$$

$$(Y'X)' = 0 \,, \tag{3.18}$$

$$X'' + 4kX = 0 \,. \tag{3.19}$$

Both Eqs. (3.18) and (3.19) are easily integrable, and $X(\eta)$ is independent of the particular $\omega(\phi)$ dependence. Solving Eq. (3.19) yields

$$X(\eta) = \begin{cases} A\,\eta & \text{for } k = 0 \,, \\ \frac{A}{2}\sin(2\eta) & \text{for } k = +1 \,, \\ \frac{A}{2}\sinh(2\eta) & \text{for } k = -1 \,, \end{cases} \tag{3.20}$$

with A an arbitrary integration constant (see Fig. 1). In what follows we will find it most useful, and succinct, to write this as

$$X(\eta) = \frac{\pm A\tau}{1 + k\tau^2} \,, \tag{3.21}$$

in terms of the new time variable

$$\tau(\eta) = \begin{cases} |\eta| & \text{for } k = 0 \,, \\ |\tan\eta| & \text{for } k = +1 \,, \\ |\tanh\eta| & \text{for } k = -1 \,. \end{cases} \tag{3.22}$$

It is convenient to define τ as a non-negative quantity and choose the plus or minus sign in Eq. (3.21) according to whether η is greater or less than zero, respectively. This only amounts to a different choice of the integration constant and so can be absorbed in our choice of A. In practice, because $a^2 = X/\phi$ must always be non-negative only one choice of $\pm A$ corresponds to a real solution anyway. For the allowed choice of A, τ may then either in-

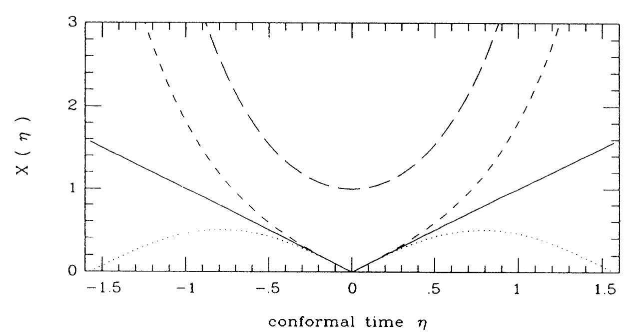

FIG. 1. The function X, defined in Eq. (3.20), plotted against conformal time η. The solid line represents $k = 0$, the dotted line $k = +1$, and the short dashed line $k = -1$ models. The long dashed line is the nonsingular function for $k = -1$ given in Eq. (3.52).

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[3] This is just a derivation in terms of a classical wave of the familiar result for an isotropic distribution of relativistic particles.

[4] Indeed this is precisely how long wavelength perturbations are produced in the inflaton field from originally short wavelength vacuum fluctuations as the comoving Hubble length shrinks during inflation [32]. This highlights the potential importance of quantum effects which we shall neglect in this purely classical treatment.

crease or decrease with conformal (and thus also with proper) time.

Note that for solutions where $\phi \to$const we have $a^2 \to X$, which is the general relativistic solution for a stiff fluid. The nonminimally coupled homogeneous Brans-Dicke field is equivalent to a nonminimally coupled stiff fluid. As $\omega \to \infty$ this coupling becomes negligible and we recover the result for a minimally coupled field.

From Eq. (3.18) we obtain

$$Y' \, X = f = \text{const} \,. \tag{3.23}$$

This latter result implies

$$Y \equiv \int \sqrt{\frac{2\omega+3}{3}} \, \frac{d\phi}{\phi} = \int \frac{f}{X} \, d\eta \,. \tag{3.24}$$

Now, notice that Eq. (3.19) has the first integral

$$X'^2 + 4\,k\,X^2 = \bar{A} \,, \tag{3.25}$$

where $\bar{A} = A^2$ for the solutions given in Eq. (3.21). Thus, substituting Eq. (3.23) into Eq. (3.17) we obtain a relation between the constants f and A such that $A = \pm f$.

Given the definition of X, and the fact that we know $X(\eta)$ from Eq. (3.20), we realize that provided we know the particular form of $\omega(\phi)$ we can obtain $\phi(\eta)$ from Eq. (3.24), and then derive the scale factor $a(\eta)$ as

$$a^2(\eta) = \frac{1}{\phi} X \,. \tag{3.26}$$

To obtain ϕ we have to invert $Y(\phi)$, given by the left-hand side of Eq. (3.24) and use the fact that we know the right-hand side:

$$Y(\eta) = \int \frac{f}{X} d\eta \; = \pm \ln \tau(\eta) \; + \; \text{const} \,. \tag{3.27}$$

We see that as $\tau \to 0$ or $\tau \to \infty$ (at early or late times) the function Y must diverge. For instance, in the case of Brans-Dicke gravity where ω is a constant, this implies that $\phi \to 0$ or $\phi \to \infty$. Only in open models does $\tau \to 1$ at late times. The evolution of Y is then frozen as the dynamics become dominated by the spatial curvature which does not couple to the Brans-Dicke field.

There is *a priori* no prescription for $\omega(\phi)$. Thus, we are led to consider some specific $\omega(\phi)$ dependences which hopefully will shed some light onto general results concerning the dependence of the solutions on the form of $\omega(\phi)$. However, even without solving these equations for a particular $\omega(\phi)$ we can come to some general conclusions about how these vacuum solutions behave. As $X \to 0$ and $(X'/X)^2 \to \infty$ the curvature becomes negligible in Eq. (3.17) and we see that

$$\left(\frac{X'}{X}\right)^2 \to (Y')^2 \,. \tag{3.28}$$

Thus using Eq. (3.7) we have

$$\dot{a} = \frac{1}{2}\left(\frac{X'}{X} - \frac{\phi'}{\phi}\right) \,, \tag{3.29}$$

$$\to \frac{1}{2}\left(1 \mp \sqrt{\frac{3}{2\omega+3}}\right)\frac{X'}{X} \,, \tag{3.30}$$

and the initial singularity (with $\dot{a} \to \pm\infty$) can only be avoided for $\omega \to 0$. As can be seen from the action, Eq. (2.1), this corresponds to the (kinetic) energy density of the Brans-Dicke field vanishing.

A necessary condition for any turning point in the evolution of the scale factor is

$$\omega(\phi) \;=\; \frac{6kX^2}{A^2 - 4kX^2} \,. \tag{3.31}$$

Thus for $k \leq 0$ a turning point can only occur when $\omega \leq 0$. This corresponds to the $(\nabla\phi)^2$ term in the action of Eq. (2.1) having the "wrong sign," in that it can contribute a negative effective energy density. Turning points can occur in closed models even if $\omega > 0$, just as they can occur in general relativity. Note that the sign of the gravitational coupling ϕ (and thus X) is irrelevant in this vacuum case.

1. Vacuum solutions in Brans-Dicke gravity

Let us consider first the case $\omega(\phi) = \omega_0$ =const, corresponding to the Brans-Dicke theory. Then

$$Y = \sqrt{\frac{2\omega_0+3}{3}} \, \ln\frac{\phi}{\phi_0} \,, \tag{3.32}$$

and thus

$$\phi = \phi_0 \tau^{\pm\beta} \,, \tag{3.33}$$

$$a^2 = \frac{A}{\phi_0}\frac{\tau^{1\mp\beta}}{1+k\tau^2} \,, \tag{3.34}$$

where we have written $\beta = \sqrt{3/(2\omega_0+3)}$. These solutions are plotted in Figs. 2 and 3.

The $k = 0$ solutions correspond to those derived by O'Hanlon and Tupper [15]. If we convert them to proper time they read $a(t) = a_0 \, t^{q_\pm}$ and $\phi = \phi_0 \, t^{(1-3q_\pm)}$, where

$$q_\pm \equiv \frac{\omega}{3\left(\omega + 1 \pm \sqrt{\frac{2\omega+3}{3}}\right)} \tag{3.35}$$

(see Fig. 4). Note that $q_\pm \to 1/3$ as $\omega \to \infty$ and we recover the general relativistic result for a stiff fluid. The $k \neq 0$ solutions were obtained by Lorenz-Petzold [17] and by Barrow [22]. As $\eta \to 0$, and thus $a \to 0$ for $\omega > 0$, they approach the $k = 0$ power law behavior.

All solutions exhibit two branches. This is a consequence of the identity $A = \pm f$ between the integration constants. Each branch corresponds to different signs of $\dot{\phi}/\phi$. In fact, the q_+ branch is associated with an increasing $|\phi|$, which means that G approaches zero in the $t \to \infty$ limit. Since this branch corresponds to a slower expansion, we shall follow Gurevich *et al.* [16] in calling it the *slow* branch. On the contrary, the q_- *fast* branch has a decreasing $|\phi|$ and G, consequently, approaches $\pm\infty$ with time.

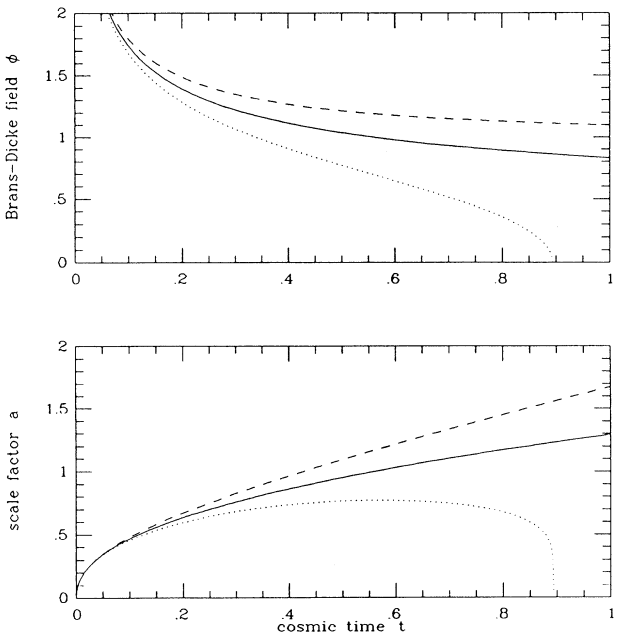

FIG. 2. Vacuum solutions for Brans-Dicke cosmologies showing Brans-Dicke field ϕ and scale factor a against proper time in the Jordan frame for the *fast* branch where $\phi = 0$ at $a = 0$. Again the solid line represents $k = 0$, the dotted line $k = +1$, and the dashed line $k = -1$ models.

Note that $\tau \to 0$ (and thus $\eta \to 0$) coincides with $a \to 0$ for both branches if and only if $\omega > 0$ (and thus $\beta < 1$), in agreement with our earlier arguments. For $\omega < 0$ the solutions do not have zero size at $\tau = 0$ but are still singular in the sense that the Ricci curvature

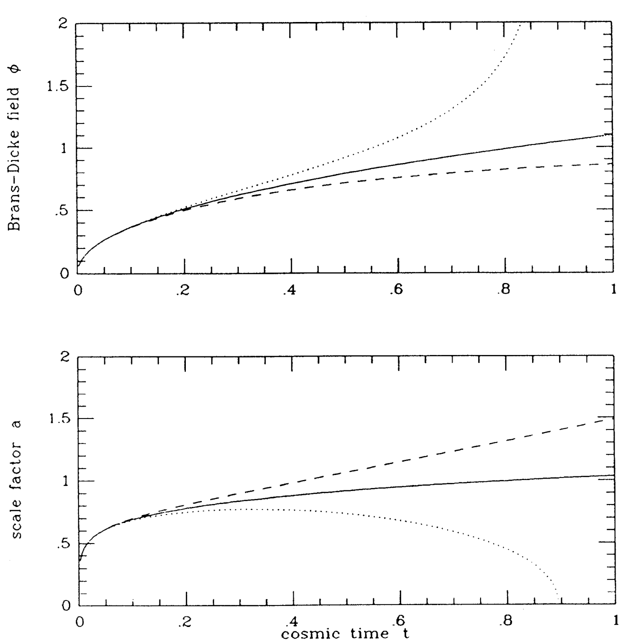

FIG. 3. Same as Fig. 2 but showing the *slow* branch.

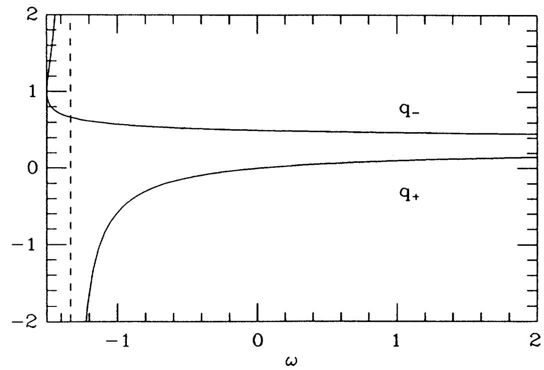

FIG. 4. Graph showing the exponents, $q_-(\omega)$ (the fast branch) and $q_+(\omega)$ (the slow branch), of power law expansion for $k = 0$ vacuum Brans-Dicke cosmologies.

scalar, for instance, diverges.

We can choose ϕ_0 to be either positive or negative and thus the sign of the gravitational "constant" is arbitrary as we would expect for solutions of the field equations in vacuum. Of course a^2 must remain positive so we require $\phi_0 A > 0$. Because of our definition of τ in Eq. (3.22) we also have two distinct solutions corresponding to whether τ decreases with time, corresponding to $\eta \leq 0$ and a collapsing universe as $\tau \to 0$ for $\omega > 0$, or increases with $\eta \geq 0$ for a universe expanding from $\tau = 0$ if $\omega > 0$.

2. *Vacuum solution with* $\omega \to \infty$

The simplest function which includes a divergent $\omega(\phi)$ at a finite value of $\phi = \phi_*$ is

$$2\omega(\phi) + 3 = (2\omega_0 + 3)\,\frac{\phi_*}{\phi_* - \phi}\,. \tag{3.36}$$

This is chosen as an example of a scalar-tensor gravity theory that mimics general relativity in the weak-field limit as $\phi \to \phi_*$ [12].

The integral in Eq. (3.7) then yields

$$\begin{aligned} Y(\phi) &= \sqrt{\frac{2\omega_0 + 3}{3}}\,\ln\left(\frac{\sqrt{\phi_*} - \sqrt{\phi_* - \phi}}{\sqrt{\phi_*} + \sqrt{\phi_* - \phi}}\right) \\ &= \pm\ln\tau + \text{const}\,, \end{aligned} \tag{3.37}$$

which in turn gives

$$\phi = \phi_*\,\frac{4\tau_*^{\beta_0}\tau^{\beta_0}}{\left(\tau_*^{\beta_0} + \tau^{\beta_0}\right)^2}\,, \tag{3.38}$$

$$a^2 = \frac{A}{\phi_*}\,\frac{\left(\tau_*^{\beta_0} + \tau^{\beta_0}\right)^2}{\tau_*^{\beta_0}\tau^{\beta_0}}\,\frac{\tau}{1 + k\tau^2}\,, \tag{3.39}$$

(see Fig. 5) where we have written $\beta_0 = \pm\sqrt{3/(2\omega_0 + 3)}$, although in fact the choice of $\pm$ is irrelevant here for $\tau_* \neq 0$. Notice again that $a \to 0$ as $\tau \to 0$ for $\omega > 0$. The function ϕ is always zero at the initial singularity $(a = 0)$ and increases towards its maximum value ϕ_*

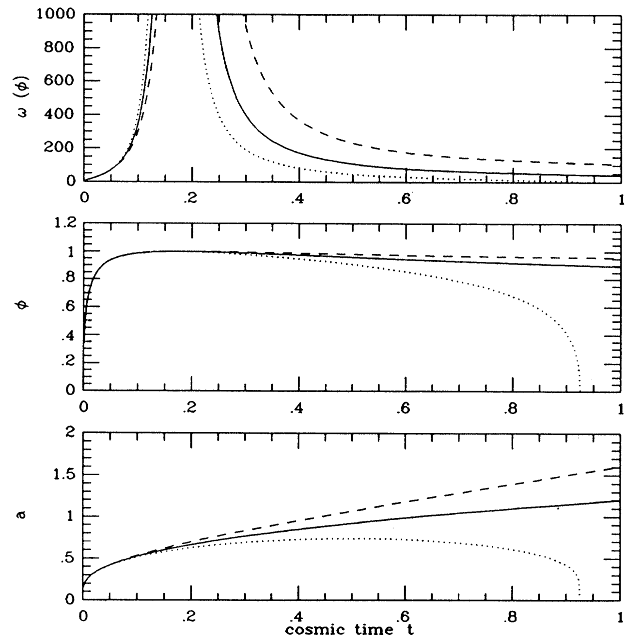

FIG. 5. Vacuum solutions for scalar-tensor gravity theory with $2\omega + 3 = 9\phi_*/(\phi_* - \phi)$ showing Brans-Dicke field and scale factor against proper time in the Jordan frame. Note that $\omega \to \infty$ is not a late time attractor. This is identical to the evolution of a Brans-Dicke model in the presence of a stiff fluid.

where $\omega \to \infty$. Because τ remains bounded ($\tau \leq 1$) in an open universe, ϕ will never reach ϕ_* if $\tau_* > 1$.

For $\tau > \tau_*$, ϕ then decreases towards zero (which it attains for $k \geq 0$ as $\tau \to \infty$). This demonstrates that, although $\dot{\phi} \to 0$ as $\phi \to \phi_*$ and $\omega \to \infty$, this is not the late time attractor solution. Instead we require that the function Y must diverge as $\tau \to \infty$ and thus $\phi \to 0$.

3. Vacuum solution with Brans-Dicke and general relativistic limits

Consider the function $\omega(\phi)$ such that

$$2\omega(\phi) + 3 = (2\omega_0 + 3)\,\frac{\phi^2}{(\phi - \phi_*)^2}\,. \tag{3.40}$$

Clearly $\omega \to$const as $\phi \to \infty$, but is divergent at $\phi = \phi_*$, giving Brans-Dicke and general relativistic limits, respectively.

Considering only $\phi > \phi_*$, initially, we have

$$Y(\phi) = \sqrt{\frac{2\omega_0 + 3}{3}}\,\ln\left(\frac{\phi}{\phi_*} - 1\right)\,, \tag{3.41}$$

and thus

$$\phi = \phi_*\left[1 + \left(\frac{\tau}{\tau_*}\right)^{\beta_0}\right]\,, \tag{3.42}$$

$$a^2 = \frac{A}{\phi_*}\,\frac{\tau_*^{\beta_0}}{\tau_*^{\beta_0} + \tau^{\beta_0}}\,\frac{\tau}{1 + k\tau^2}\,. \tag{3.43}$$

Because $Y(\phi)$ is divergent at both $\phi = \phi_*$ and $\phi \to \infty$ we again have two distinct branches according to whether $\dot{\phi}/\phi$ is increasing or decreasing. In the former case, when $\beta_0 > 0$, we find a slow branch where the initial general relativistic behavior $\phi \simeq \phi_*$ turns into the Brans-Dicke solution $\phi \propto \tau^{\beta_0}$ as $\tau \to \infty$. For $\beta_0 < 0$ we have decreasing ϕ, the fast branch, and the behavior is reversed as $\tau \to \infty$. Note that the $\tau \to \infty$ limit is only achieved for $k \geq 0$ and that the late time behavior in open models always corresponds to the general relativistic behavior with $\phi \to \phi_*(1 + \tau_*^{-\beta_0})$ as $\tau \to 1$.

For $\phi < \phi_*$ we see that $2\omega + 3$ may reach zero. This allows a far more complex behavior:

$$\phi = \phi_*\left[1 - \left(\frac{\tau}{\tau_*}\right)^{\beta_0}\right]\,, \tag{3.44}$$

$$a^2 = \frac{A}{\phi_*}\,\frac{\tau_*^{\beta_0}}{\tau_*^{\beta_0} - \tau^{\beta_0}}\,\frac{\tau}{1 + k\tau^2}\,. \tag{3.45}$$

Notice now that ϕ reaches zero when $\tau = \tau_*$. This corresponds to the divergence of the scale factor a at a finite proper time. Thus, for instance, the closed model does *not* recollapse, although its behavior is still singular as $\tau \to \tau_*$.

C. Stiff fluid solutions

We consider in this section the case where matter is described by a barotropic equation of state with $\gamma = 2$ (stiff matter), which as we have seen describes the long wavelength modes of a massless scalar field. Because the Brans-Dicke field itself is a homogeneous scalar field, we find many similarities with the vacuum solutions discussed above.

In terms of the same variables X and Y the field equations become

$$(X')^2 + 4kX^2 - (Y'X)^2 = 4M\phi\,, \tag{3.46}$$

$$[Y'X]' = -\frac{2M\phi}{X}\sqrt{\frac{3}{2\omega + 3}}\,, \tag{3.47}$$

$$X'' + 4kX = 0\,. \tag{3.48}$$

The last equation is identical to the corresponding equation for the vacuum case, and thus $X(\eta)$ is given by the same expressions [Eq. (3.21)]. This also yields the first integral

$$X'^2 + 4kX^2 = \bar{A}\,, \tag{3.49}$$

which upon insertion into the first of the field equations leads to

$$Y'X = \pm\sqrt{\bar{A} - 4M\phi}\,. \tag{3.50}$$

This requires that $\phi \leq \bar{A}/4M$. Notice also that unlike the vacuum case $\bar{A}$ could be negative, but only if ϕ is also negative and only when $k = -1$ [so that the left-hand side of Eq. (3.49) may be negative]. This gives one extra solution for $X(\eta)$ in addition to the vacuum solutions when $k = -1$ and $\bar{A} = -A^2$,

$$X(\eta) = -\frac{A}{2}\cosh 2\eta \,, \tag{3.51}$$

or, in terms of the variable τ defined in Eq. (3.22),

$$X(\tau) = -A\frac{1-k\tau^2}{1+k\tau^2} \,. \tag{3.52}$$

For $k = +1$ note that this corresponds to $X \propto \cos 2\eta$, which is equivalent simply to a different choice of the zero point of η, but in the open model we have a qualitatively different behavior when $\bar{A} < 0$. $X = a^2\phi$ remains nonzero at all times and thus we can obtain nonsingular models where a remains nonzero.

Now, from Eq. (3.50) we derive

$$Y = \int \sqrt{\bar{A} - 4M\phi}\frac{d\eta}{X} \,, \tag{3.53}$$

and thus it is useful to define

$$Z(\phi) \equiv \int \sqrt{\frac{2\omega+3}{3}} \frac{d\phi}{\phi\sqrt{\bar{A}-4M\phi}} = \pm \int \frac{d\eta}{X(\eta)} \,, \tag{3.54}$$

where the right-hand side of this equation is just $\pm \ln\tau$ as for the vacuum case. Thus, just as in the vacuum case we required the function $Y(\phi)$ to diverge as $\tau \to 0$ or $\to \infty$, in the stiff fluid case we require $Z(\phi)$ to diverge in these limits. We see that if

$$\frac{2\omega_{\rm vac}(\phi)+3}{A^2} = \frac{2\omega(\phi)+3}{\bar{A}-4M\phi} \,, \tag{3.55}$$

the vacuum solutions for $a(t)$ and $\phi(t)$ with $\omega_{\rm vac}(\phi)$ carry over to the stiff fluid solutions for $\omega(\phi)$. The reason for the equivalence becomes more apparent when we discuss the conformally transformed picture in the next section. When $\bar{A} < 0$ we see that for $2\omega + 3 > 0$ we find the vacuum equivalent $2\omega_{\rm vac} + 3 < 0$, which is why we did not find the nonsingular open models in the vacuum case.

The condition for $\dot{a} = 0$ now becomes

$$\omega \;=\; \frac{6(kX^2 - M\phi)}{\bar{A} - 4kX^2} \,, \tag{3.56}$$

confirming that $\omega > 0$ is compatible with a turning point for $k < 0$ when $\bar{A} < 0$. For $k = 0$ where we must have $\bar{A} > 0$, or as $X \to 0$, the condition becomes $\omega = -6M\phi/\bar{A}$. Thus the sign of ϕ becomes crucial. As one might expect, if the gravitational mass, M/ϕ is negative, the initial singularity can be avoided even for $\omega > 0$, while for $M/\phi > 0$ the presence of the stiff fluid requires an increasingly negative value of ω to avoid the singularity. Thus we require a negative kinetic energy density of the Brans-Dicke field to counteract the positive gravitational mass.

1. Stiff fluid solution in Brans-Dicke gravity

Proceeding as for the vacuum case, we start by considering the $\omega = \omega_0$ =const case, which enables us to compare our results with the $k = 0$ solutions existing in the literature.

For $\bar{A} = +A^2 \geq 4M\phi$ we have

$$\begin{aligned} A \times Z &= \sqrt{\frac{2\omega_0+3}{3}}\, \ln\left[\frac{A-\sqrt{A^2-4M\phi}}{A+\sqrt{A^2-4M\phi}}\right] \\ &= \pm\ln\tau + \text{const} \,. \end{aligned} \tag{3.57}$$

Notice that this is exactly the same result as found in the vacuum case with $2\omega_{\rm vac}(\phi)+3 = (2\omega_* + 3)\phi_*/(\phi_* - \phi)$ if we write $2\omega_* + 3 = (2\omega_0 + 3)$ and $\phi_* = A^2/(4M)$. This is a demonstration of the equivalence between different vacuum and stiff fluid solutions given in Eq. (3.55).

Thus, for $A > 0$,

$$\phi = \frac{A^2}{M}\frac{\tau_*^\beta\tau^\beta}{\left(\tau_*^\beta + \tau^\beta\right)^2} \,, \tag{3.58}$$

$$a^2 = \frac{M}{A}\frac{\left(\tau_*^\beta + \tau^\beta\right)^2}{\tau_*^\beta\tau^\beta}\frac{\tau}{1+k\tau^2} \,, \tag{3.59}$$

where τ_* is the constant of integration chosen to coincide with the value of τ for which ϕ reaches its maximum possible value $\phi_* = A^2/4M$. For $\tau > \tau_*$, ϕ decreases back towards zero (see Fig. 5).

If $k = -1$, τ is bounded and will never attain ϕ_* if $\tau_* > 1$. In this case ϕ remains a monotonically increasing function of τ approaching $4\phi_*/(1+\tau_*^\beta)^2$ as $\tau \to 1$ and thus t tends to infinity.

If on the other hand we consider $A < 0$, we find a solution for $\phi < 0$:

$$\phi = -\frac{A^2}{M}\frac{\tau_*^\beta\tau^\beta}{\left(\tau_*^\beta - \tau^\beta\right)^2} \,, \tag{3.60}$$

$$a^2 = -\frac{M}{A}\frac{\left(\tau_*^\beta - \tau^\beta\right)^2}{\tau_*^\beta\tau^\beta}\frac{\tau}{1+k\tau^2} \,. \tag{3.61}$$

It is possible to see that these solutions correspond to the $\omega_0 > -3/2$ solution derived by Gurevich *et al.* [16] (after the necessary translation to their time variable; Gurevich *et al.* use ξ such that $d\xi = d\eta/a^2$). Notice that $a = 0$ at both $\tau = 0$ and $\tau = \tau_*$, demonstrating that a turning point can indeed occur for $\omega > 0$ even in open or flat models in the presence of the stiff fluid if $\phi < 0$. As τ approaches τ_* from below, the solution approaches Nariai's power law solution [14], but it is clear that this is not the late time behavior suggested by Gurevich *et al.* but rather a recollapse at a finite proper time. The correct late time behavior for expanding $k = 0$ models is where they approach the vacuum solution as $\tau \to \infty$ with ϕ positive or negative.

When $\bar{A} = -A^2$, possible only for $k < 0$, we have $X(\eta)$ given by Eq. (3.52) and

$$A \times Z(\phi) = 2\sqrt{\frac{2\omega+3}{3}}\ \arctan\left(\frac{\sqrt{-4M\phi - A^2}}{A}\right)$$

$$= \pm 2\ \arctan\tau + \text{const}\ . \tag{3.62}$$

This gives

$$\phi = -\frac{A^2}{4M}\sec^2\left(c + \beta\arctan\tau\right)\ . \tag{3.63}$$

Thus $\phi \leq -A^2/4M$ as required. However $\phi \to -\infty$ whenever $\tau = \tan[(\pi/2 - c)/\beta]$, leading to $a = \sqrt{X/\phi} \to 0$. This can always occur when the arbitrary constant c is sufficiently close to $\pi/2$ regardless of the sign of k.

2. Stiff fluid solution with Brans-Dicke and general relativistic limits

If we consider again the function $2\omega(\phi) + 3 = (2\omega_0 + 3)\phi^2/(\phi - \phi_*)^2$ this time in the presence of a stiff fluid, we can integrate Eq. (3.54) for $\phi > \phi_*$ to give

$$Z(\phi) = \sqrt{\frac{2\omega_0+3}{3}}\ \frac{1}{\sqrt{A^2 - 4M\phi_*}}$$

$$\times \ln\left|\frac{\sqrt{A^2-4M\phi} - \sqrt{A^2-4M\phi_*}}{\sqrt{A^2-4M\phi} + \sqrt{A^2-4M\phi_*}}\right|\ . \tag{3.64}$$

Here ϕ must be constrained to lie within $\phi_* < \phi < A^2/4M$ and so can never reach the asymptotic Brans-Dicke limit as $\phi \to \infty$. We find

$$\phi = \frac{(\tau_*^B - \tau^B)^2\phi_* + \tau_*^B\tau^B(A^2/M)}{(\tau_*^B + \tau^B)^2}\ , \tag{3.65}$$

where we have written

$$B = \sqrt{\frac{A^2 - 4M\phi_*}{A^2}\ \frac{3}{2\omega_0+3}} < \beta_0\ . \tag{3.66}$$

Thus $\phi = \phi_*$ at $\tau = 0$, and reaches a maximum of $\phi = A^2/4M$ when $\tau = \tau_*$ (possible only for $\tau_* < 1$ in the open model). At late times for $k \geq 0$, as $\tau \to \infty$, ϕ returns to the general relativistic result, $\phi \to \phi_*$, $\omega \to 0$.

Once again for $\phi < \phi_*$ we find a considerably more complicated behavior where we may have $\phi \to 0$ for nonzero X.

D. Radiation solutions

The other case in which the nonvacuum equations of motion simplify considerably is where the energy-momentum tensor is traceless ($\gamma = 4/3$), i.e., a radiation fluid corresponding to the short wavelength modes of a massless field. As this case has been discussed elsewhere [22] we will describe the behavior only briefly for comparison with the stiff fluid case, while presenting our results in a more compact form in terms of the time coordinate $\tau(\eta)$.

The field equations in the presence of radiation with density $\rho = 3\Gamma/8\pi a^4$ where Γ is a constant, become

$$(X')^2 + 4kX^2 - (Y'X)^2 = 4\Gamma X\ , \tag{3.67}$$

$$(Y'X)' = 0\ , \tag{3.68}$$

$$X'' + 4kX = 2\Gamma\ . \tag{3.69}$$

The final equation can again be integrated directly to give the first equation where $(Y'X)^2 = A^2$ =const. Notice that unlike the stiff fluid case this constant cannot be negative. The general solution of the equation of motion for X is then

$$X = \frac{\tau(A + \Gamma\tau)}{1 + k\tau^2}\ , \tag{3.70}$$

in terms of the time coordinate $\tau(\eta)$ introduced in Eq. (3.22).

The Brans-Dicke field is not driven by matter and we have the same integral for $Y(\phi)$ as in the vacuum case, although we have a different $X(\eta)$:

$$Y(\phi) = \pm\int\frac{A d\eta}{X} = \pm\ln\left|\frac{\Gamma\tau}{A + \Gamma\tau}\right| + \text{const}\ . \tag{3.71}$$

The evolution of $\phi(\eta)$ is thus the same as for the vacuum case if we replace the function $\tau(\eta)$ by

$$s(\eta) = \left|\frac{\Gamma\tau(\eta)}{A + \Gamma\tau(\eta)}\right|\ . \tag{3.72}$$

Note that in spatially flat or closed models as $\tau \to \infty$ we find $s \to 1$, i.e., Y approaches a fixed value. Thus the evolution of Y is similar to the open models in vacuum where $\tau \to 1$ at late times. This time the field becomes frozen in as $s \to 1$ due to the radiation, which like spatial curvature does not couple to the Brans-Dicke field, dominating the dynamics. In open models as $\tau \to 1$ we have $s \to \Gamma/(A+\Gamma)$. On the other hand at early times all solutions approach the vacuum solutions as $s \simeq (\Gamma/A)\tau$ amounts simply to a rescaling of the conformal time or, equivalently, the scale factor, thus the Brans-Dicke field dominates the dynamics for $\tau \ll A/\Gamma$.

It is now straightforward to write down the radiation solutions for the particular choices of $\omega(\phi)$ given in the vacuum and stiff fluid cases.

The value of the ω at any turning point is now given by

$$\omega = \frac{6(kX^2 - \Gamma X)}{A^2 - 4(kX^2 - \Gamma X)}\ , \tag{3.73}$$

$$= -\left(\frac{6\Gamma X}{A^2 + 4\Gamma X}\right) \quad \text{for } k = 0 \text{ or } X \to 0\ . \tag{3.74}$$

The denominator must always be positive [by Eq. (3.68)] and thus we find again that to obtain a turning point with $\omega > 0$ requires either $k > 0$, which corresponds to the usual recollapse in closed models, or X (and thus ϕ) negative.

1. Radiation solution in Brans-Dicke gravity

We have

$$\phi = \phi_* \, s^{\pm\beta} \,, \tag{3.75}$$

$$a^2 = \frac{1}{\phi_*} \, \frac{s^{\mp\beta}\tau(A+\Gamma\tau)}{1+k\tau^2} \,. \tag{3.76}$$

As is well known, this Brans-Dicke solution approaches the general relativistic solution with constant ϕ at late times during the radiation-dominated era.

2. Radiation solution with Brans-Dicke and general relativistic limits

When $2\omega(\phi)+3 = (2\omega_0+3)\phi^2/(\phi-\phi_*)^2$ we find

$$\phi = \phi_* \left[1 + \left(\frac{s}{s_*}\right)^{\beta_0}\right] \,, \tag{3.77}$$

$$a^2 = \frac{1}{\phi_*} \, \frac{s_*^{\beta_0}}{s_*^{\beta_0}+s^{\beta_0}} \, \frac{\tau(A+\Gamma\tau)}{1+k\tau^2} \,. \tag{3.78}$$

Once again ϕ approaches a constant at late times; however, unlike the stiff fluid case considered earlier, this constant value may not be close to ϕ_* so this need not coincide with $\omega \to \infty$.

IV. CONFORMALLY TRANSFORMED FRAME

It has long been realized that a theory with varying gravitational coupling such as scalar-tensor gravity must be equivalent to one in which the gravitational coupling is constant but masses and lengths vary [33]. Mathematically this equivalence can be shown by using a conformally rescaled metric

$$\tilde{g}_{ab} = \left(\frac{\phi}{\phi_0}\right) g_{ab} \,. \tag{4.1}$$

ϕ_0 is just an arbitrary constant introduced to keep the conformal factor dimensionless. Written in terms of this new metric and its scalar curvature $\tilde{R}$ the scalar-tensor action given in Eq. (2.1) becomes

$$S = \frac{1}{16\pi}\int_M d^4x\sqrt{-\tilde{g}}\left\{\phi_0\tilde{R} - 16\pi\left[-\frac{1}{2}\tilde{g}^{ab}\psi_{,a}\psi_{,b} + \left(\frac{\phi_0}{\phi}\right)^2 \mathcal{L}_{\rm matter}\right]\right\} \,, \tag{4.2}$$

where we introduce a new scalar field $\psi(\phi)$ defined by

$$d\psi \equiv \sqrt{\phi_0 \, \frac{2\omega+3}{16\pi}} \, \frac{d\phi}{\phi} \,. \tag{4.3}$$

The gravitational Lagrangian is reduced simply to the Einstein-Hilbert Lagrangian of general relativity, albeit at the expense of altering the matter Lagrangian. Thus we shall refer to this as the Einstein frame.

The arbitrary dimensional constant ϕ_0 plays the role of Newton's constant, $G \equiv \phi_0^{-1}$. In order to avoid changing the signature, the conformal factor relating the metrics must be positive. So for $\phi < 0$ we must pick $\phi_0 < 0$, giving a negative gravitational constant in the Einstein frame. Not surprisingly then, the usual singularity theorems need not apply even in the Einstein frame for $\phi < 0$. Similarly, in the definition of ψ we require $\phi_0(2\omega+3) > 0$. If this were not the case we could instead define a scalar field

$$d\bar{\psi} \equiv \sqrt{-\phi_0 \, \frac{2\omega+3}{16\pi}} \, \frac{d\phi}{\phi} \,, \tag{4.4}$$

but this would have a negative kinetic energy density, again invalidating the usual singularity theorems by breaking the dominant energy condition. However, for $\phi > 0$ and $\omega > -3/2$ the FRW models must contain singularities in the conformal Einstein frame where $\tilde{a} \to 0$.

A. Vacuum solutions

The field equations are then, at least in vacuum ($\mathcal{L}_{\rm matter} = 0$), just the usual Einstein field equations of general relativity plus a massless scalar field, ψ. In particular, in a FRW universe (which remains homogeneous and isotropic under the homogeneous transformation) we have

$$\tilde{H}^2 = \frac{8\pi}{3}\frac{\hat{\rho}}{\phi_0} - \frac{k}{\tilde{a}^2} \,, \tag{4.5}$$

$$\frac{d^2\psi}{d\tilde{t}^2} + 3\tilde{H}\frac{d\psi}{d\tilde{t}} = 0 \,, \tag{4.6}$$

$$\frac{d\tilde{H}}{d\tilde{t}} + \tilde{H}^2 = -\frac{4\pi}{3}\frac{\hat{\rho}+3\hat{p}}{\phi_0} \,, \tag{4.7}$$

where the scale factor in the conformal frame $\tilde{a} = (\phi/\phi_0)^{1/2}a$, $d\tilde{t} = (\phi/\phi_0)^{1/2}dt$, and $\tilde{H} = (d\tilde{a}/d\tilde{t})/\tilde{a}$. (Note that $\tilde{t}$ is the time in the conformal frame and not to be confused with the conformally invariant time η used earlier.) The massless scalar field behaves, as it must, as a stiff fluid with density $\hat{\rho} = \hat{p} = (d\psi/d\tilde{t})^2/2$.

Notice now that the variables X and Y introduced in the preceding section correspond to the square of the conformal scale factor and the scalar field ψ, respectively:

$$X \equiv \frac{a^2}{\phi} \equiv \frac{\tilde{a}^2}{\phi_0} \,, \tag{4.8}$$

$$Y \equiv \int \sqrt{\frac{2\omega+3}{3}} \, \frac{d\phi}{\phi} \equiv \sqrt{\frac{16\pi}{3\phi_0}} \, \psi \,. \tag{4.9}$$

The equations of motion for the conformal scale factor written in terms of X and for ψ written in terms of Y and derivatives with respect to the conformal time η are then precisely Eqs. (3.17)–(3.19) solved in Sec. III B.

We can solve explicitly for X and Y as functions of η because the stiff fluid continuity equation can be integrated directly (as for any perfect barotropic fluid) to give $\hat{\rho} \propto \tilde{a}^{-6}$. These results are independent of the form of $\omega(\phi)$. A particular choice of $\omega(\phi)$ determines how ϕ is related to the stiff fluid field ψ. To obtain $\phi(\eta)$ we must be able to perform the integral in Eq. (4.9), and thus we also obtain the scale factor in the original frame, $a \equiv (\phi_0/\phi)^{1/2}\tilde{a}$.

B. Nonvacuum solutions

If we include the matter lagrangian for a perfect fluid in the original scalar-tensor frame, then there is a nontrivial interaction between this matter and the scalar field ψ in the Einstein frame:

$$\tilde{\mathcal{L}}_{\text{matter}} = \left(\frac{\phi_0}{\phi(\psi)}\right)^2 \mathcal{L}_{\text{matter}} . \tag{4.10}$$

Thus the matter energy-momentum tensor, defined in the Einstein metric,

$$\tilde{T}^{ab} \equiv \frac{2}{\sqrt{-\tilde{g}}}\frac{\partial}{\partial \tilde{g}_{ab}}\left(\sqrt{-\tilde{g}}\tilde{\mathcal{L}}_{\text{matter}}\right) , \tag{4.11}$$

is no longer independently conserved,

$$\tilde{\nabla}^a \tilde{T}_{ab} = -\frac{1}{2\sqrt{\phi_0}}\sqrt{\frac{16\pi}{2\omega+3}}\tilde{T}^a_a \psi_{,b} , \tag{4.12}$$

unless it is traceless, i.e., vacuum or radiation. The conformally transformed density $\tilde{\rho} = (\phi_0/\phi)^2\rho$ and pressure $\tilde{p} = (\phi_0/\phi)^2 p$ of the fluid become dependent on ϕ and thus ψ, so while the fluid retains the same barotropic equation of state it is no longer a perfect fluid in general. Note however that the overall energy-momentum tensor of the matter plus the ψ field must be conserved as guaranteed in general relativity by the Ricci identity.

We have the usual general relativistic equations of motion in a FRW model

$$\tilde{H}^2 = \frac{8\pi}{3}\frac{\hat{\rho}+\tilde{\rho}}{\phi_0} - \frac{k}{\tilde{a}^2} , \tag{4.13}$$

$$\frac{d\tilde{H}}{d\tilde{t}} + \tilde{H}^2 = -\frac{4\pi}{3}\frac{\hat{\rho}+\tilde{\rho}+3(\hat{p}+\tilde{p})}{\phi_0} , \tag{4.14}$$

and the interaction leads to a transfer of energy between the original fluid and the stiff (ψ) fluid:

$$\frac{d\tilde{\rho}}{d\tilde{t}} = -3\tilde{H}(\tilde{\rho}+\tilde{p}) + \frac{1}{2\sqrt{\phi_0}}\sqrt{\frac{16\pi}{2\omega+3}}(3\tilde{p}-\tilde{\rho})\frac{d\psi}{d\tilde{t}} , \tag{4.15}$$

$$\frac{d\hat{\rho}}{d\tilde{t}} = -6\tilde{H}\hat{\rho} - \frac{1}{2\sqrt{\phi_0}}\sqrt{\frac{16\pi}{2\omega+3}}(3\tilde{p}-\tilde{\rho})\frac{d\psi}{d\tilde{t}} . \tag{4.16}$$

Again we find two cases in which the problem simplifies. Firstly for radiation ($\tilde{\rho} = 3\tilde{p}$) there is no interaction and both continuity equations can be directly integrated and the conformal picture contains two noninteracting fluids:

$$\frac{8\pi}{3\phi_0}\hat{\rho} = \frac{A^2}{4\tilde{a}^6} , \tag{4.17}$$

$$\frac{8\pi}{3\phi_0}\tilde{\rho}_{\text{rad}} = \frac{\Gamma}{\tilde{a}^4} . \tag{4.18}$$

This is precisely the case considered recently by Barrow [22] (although without explicitly invoking the conformal frame) and discussed in Sec. III D.

The second case in which we can find exact solutions is where the original fluid is itself a stiff fluid (or massless scalar field) in which case although there is an interaction between the two fluids, their combined dynamical effect is that of a single perfect stiff fluid,[5] or massless scalar field χ, say

$$\frac{8\pi}{3\phi_0}\tilde{\rho}_\chi = \frac{8\pi}{3\phi_0}(\tilde{\rho}+\hat{\rho}) = \frac{\bar{A}}{4\tilde{a}^6} . \tag{4.19}$$

This is why we find exactly the same equation of motion for the scale factor in the conformal frame, $\tilde{a}^2 \propto X$, in the stiff fluid case as in the vacuum case. Notice now that in the conformal frame we must have $\bar{A} = +A^2 > 0$ for a positive energy density. The nonsingular solutions found when $k < 0$ and $\bar{A} < 0$ with a stiff fluid in the Jordan frame correspond to solutions with negative energy density in the Einstein frame.

The continuity equation for the original fluid can always be integrated to give

$$\frac{8\pi}{3\phi_0}\tilde{\rho} = \frac{M}{\tilde{a}^{3\gamma}}\left(\frac{\phi_0}{\phi}\right)^{(4-3\gamma)/2} , \tag{4.20}$$

and so in the stiff fluid case we have

$$\frac{8\pi}{3\phi_0}\hat{\rho} = \frac{4\pi}{3\phi_0}\left(\frac{d\psi}{d\tilde{t}}\right)^2 = \frac{A^2 - 4M\phi}{4\tilde{a}^6} . \tag{4.21}$$

We have $\tilde{a}^2 \propto X$ as a function of η and we must now perform the integral in Eq. (3.54) to obtain $\phi(\eta)$. The change in the relation between ϕ and the total stiff fluid density in the Einstein frame compared with the vacuum case is equivalent to a different choice of $\omega(\phi)$ (which relates ϕ to ψ), as demonstrated in Eq. (3.55). The vacuum case can of course be seen as a special case amongst the stiff fluid solutions, where $M = 0$, and thus $\omega(\phi) = \omega_{\text{vac}}(\phi)$.

We can also obtain exact solutions for radiation *and* stiff fluid in the original Jordan frame as the radiation remains decoupled in the Einstein frame and the interaction is solely between the two stiff fluids in that frame. Thus the equation of motion for the scale factor in the conformal frame is exactly the same as in the radiation only case, Eq. (3.69), while the equation for ϕ is the same as in the stiff fluid case, Eq. (3.54).

Stiff fluid plus radiation in Brans-Dicke gravity. To solve for the evolution of Brans-Dicke models (where ω_0 =const) in the presence of both radiation and a stiff

[5]The combined energy-momentum tensor of two interacting fluids is equivalent to that of a single perfect fluid provided their velocity fields are parallel. This must be true if both fluids are homogeneous as is the case here. Futhermore as they are both stiff fluids, $p = \rho$, in this case, their total pressure must be equal to their total density.

fluid, the conformal frame is particularly useful, since the evolution of the conformal scale factor (or $X \equiv \tilde{a}^2/\phi_0$) is exactly the same as for radiation only [Eq. (3.69)]. The evolution of ϕ then follows directly from Eq. (4.21) as

$$\begin{aligned}\hat{\rho} &= \frac{3\phi_0}{32\pi X}\left(\frac{2\omega+3}{3}\right)\left(\frac{1}{\phi}\frac{d\phi}{d\eta}\right)^2 ,\\ &= \frac{3\phi_0}{32\pi}\frac{A^2-4M\phi}{X^3} ,\end{aligned} \tag{4.22}$$

so that, from the definition of ψ in Eq. (4.3),

$$\sqrt{\frac{2\omega_0+3}{3}}\ \ln\left[\frac{A-\sqrt{A^2-4M\phi}}{A+\sqrt{A^2-4M\phi}}\right] = \pm\ln\left|\frac{\Gamma\tau}{A+\Gamma\tau}\right| + \text{const} . \tag{4.23}$$

Rewriting this to give ϕ and thus $a=\sqrt{X/\phi}$ yields

$$\phi = \frac{A^2}{M}\frac{s_*^\beta s^\beta}{(s_*^\beta+s^\beta)^2} , \tag{4.24}$$

$$a^2 = \frac{M}{A^2}\frac{(s_*^\beta+s^\beta)^2}{s_*^\beta s^\beta}\frac{\tau(A+\Gamma\tau)}{1+k\tau^2} , \tag{4.25}$$

where

$$s(\eta) = \left|\frac{\Gamma\tau}{A+\Gamma\tau}\right| , \tag{4.26}$$

s_* is a constant of integration, and $\beta=\sqrt{3/(2\omega_0+3)}$. Thus the behavior is very similar to that seen for stiff fluid only, except that the variable s takes the place of τ.

Unlike τ, $s \to$const at late times for $k=0$ [where $s\to 1$] as well as for $k<0$ [where $s\to\Gamma/(A+\Gamma)$]. Thus the Brans-Dicke field becomes frozen in at late times in the flat FRW universe, dominated by the friction due to Hubble expansion driven by radiation, just as it is in the open FRW model, where the expansion becomes driven by the curvature. Only in the closed universe does the dynamical effect of the stiff fluid remain important. Note, however, that the radiation delays the recollapse which occurs at $\eta > \pi/2$. This means that $\tau \equiv \tan(\eta)$ becomes negative, but the solution is still well behaved as $s>0$ and both the conformal Einstein and Jordan (for $\omega>0$) scale factors recollapse, X,$a\to 0$, when $\eta = \pi+\arctan(-A/\Gamma)$, as $s\to\infty$.

Notice once again that the presence of a stiff fluid in the Jordan frame just leads to solutions which would be obtained in the absence of the stiff fluid but with the modified $\omega_{\rm vac}(\phi)$ given in Eq. (3.55).

V. CONCLUSIONS

We have shown how to extend the procedure recently proposed by Barrow [22] to obtain the solutions for general $\omega(\phi)$ scalar-tensor gravity theories with a stiff fluid in addition to vacuum or radiation solutions in a FRW metric. These two nonvacuum cases correspond to the extreme long and short wavelength modes, respectively, of a minimally coupled massless scalar field. We show that these solutions can be obtained due to the particularly simple evolution of the corresponding scale factor in the conformally related Einstein frame which is independent of the form of $\omega(\phi)$. This is no longer true when considering other matter such as dust or a cosmological constant. Then the form of $\omega(\phi)$ affects the dynamics in the Einstein frame as well as the original Jordan frame so we cannot obtain the scalar-tensor cosmology simply from known general relativistic solutions. An alternative approach to deal with this situation is presented in [30].

In the presence of a stiff fluid the physical scale factor evolves like a vacuum scalar-tensor cosmology with a modified $\omega(\phi)\to\omega_{\rm vac}(\phi)$, as defined in Eq. (3.55). (This is because introducing a new scalar field modifies the relation between ϕ and the total energy density in the homogeneous scalar fluid in the Einstein frame.) For example the Brans-Dicke model (ω =const) in the presence of a stiff fluid evolves like a vacuum model with $\omega_{\rm vac}(\phi) \propto \phi_*/(\phi_*-\phi)$. This significantly modifies the evolution of the Brans-Dicke field leading to an upper bound on $\phi\le\phi_*$.

We find that even for functions $\omega(\phi)$ that diverge at a finite value of ϕ, this need not be a stable late time attractor for $k=0$ models, in contrast to Damour and Nordtvedt's rule [20] that $\omega\to\infty$ is a cosmological attractor. Instead (due to the absence of the damping effect of matter with $p<\rho$, required by the result of Damour and Nordtvedt) we find that the late (or early) time attractor in vacuum, as $a\to\infty$ (or $a\to 0$), is associated with the divergence of the function $Y(\phi) \propto \int\sqrt{2\omega+3}d\phi/\phi$. In the presence of a stiff fluid the function $Z(\phi) \propto \int(\sqrt{(2\omega+3)/(A^2-4M\phi)}d\phi/\phi$ must diverge as $a\to 0$ or $\to\infty$.

The stiff fluid solutions are expected to be of primary importance as the scale factor $a\to 0$. When spatial curvature is negligible ($k=0$), the condition necessary for a turning point, $\dot{a}=0$, in the stiff fluid cosmology is simply $\omega=-6M\phi/\bar{A}$, where $\bar{A}$ and M are positive constants of integration. In vacuum this reduces to $\omega=0$ and the sign of ϕ is irrelevant. In the conformally related Einstein frame we have seen that the evolution is simply that for a stiff fluid irrespective of the form of $\omega(\phi)$ and thus singularities are always present here provided $\omega > -3/2$. Only for $\omega<-3/2$ does the energy density of the stiff fluid in the Einstein frame become negative and so nonsingular behavior becomes possible.

ACKNOWLEDGMENTS

D.W. was supported by the PPARC. The authors are grateful to John Barrow and Ed Copeland for useful discussions, and acknowledge use of the Starlink computer system at Sussex.

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