Foliation by constant-mean-curvature hypersurfaces of the Schwarzschild spacetime

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In this Brief Report a procedure for the complete foliation of the Schwarzschild spacetime by spacelike hypersurfaces of constant York time is proposed.

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For a foliation of spacetime by hypersurfaces of constant mean extrinsic curvature, York defined a time parameter [1] proportional to the mean extrinsic curvature. It has been argued [2] that such hypersurfaces have special significance in a cosmological context. While it may not be possible to obtain a maximal slicing [3] ($K = 0$ hypersurface foliation), it may be possible [4] to obtain a constant mean extrinsic curvature K slicing of spacetime and hence define a York time there. Brill *et al.* [5] have provided a thorough discussion of K slicing a Schwarzschild spacetime but were unable to provide a complete foliation of it. They used only non-negative values of K. However, they conjectured that if the full range of values of K was used, a complete foliation of the Schwarzschild spacetime could be obtained. Eardly and Smarr have shown that there is a K slicing of the Schwarzschild spacetime, but have not shown explicitly how to construct it. Here we describe a procedure which we believe should explicitly produce such a slicing. We describe this procedure and provide numerical evidence that it works.

Spacelike hypersurfaces for K slicing the Schwarzschild spacetime in Kruskal-Szekeres coordinates are generated by [5]

$$\frac{dv}{du} = \frac{Av + Eu}{Au + Ev}, \tag{1}$$

where

$$E = H - K\frac{r^3}{3},\ A^2 = E^2 + r^3(r - 2m), \tag{2}$$

the Hamiltonian H being a given constant. The mean intrinsic curvature is [6]

$$R = 6\left(\frac{H^2}{r^6} - \frac{K^2}{9}\right), \tag{3}$$

which remains finite as r goes to infinity and infinite only at the singularity $r = 0$. On the other hand, the mean extrinsic curvature K varies from surface to surface. Following Brill *et al.*, all slices were chosen smooth and flat at the throat of the Einstein-Rosen bridge $u = 0$. For this purpose we must set $A = 0$. To fix H for given K, Brill *et al.* took the K surfaces such that the potential

$$V(H,K,r) = 1 = -\frac{r^3(r - 2m)}{E^2} \tag{4}$$

is minimum at $u = 0$, i.e., $dV/dr = 0$. This was done for the following reason. All the K surfaces rise up in the Kruskal diagram as u increases up to some finite distance. They must not be allowed to rise too steeply up or they will hit the singularity and thus limit the height to which they can rise.

A problem arises in foliating the Penrose diagram using the noncompact Kruskal-Szekeres coordinates. In the compactified coordinates [7]

$$\psi = \arctan(v + u) + \arctan(v - u),$$
$$\xi = \arctan(v + u) - \arctan(v - u), \tag{5}$$

the equation

$$\frac{d\psi}{d\xi} = \frac{Av(u^2 - v^2 - 1) + Eu(v^2 - u^2 - 1)}{Au(v^2 - u^2 - 1) + Ev(u^2 - v^2 - 1)} \tag{6}$$

at $u = 0 = \xi$ gives the same condition as $\left.\frac{d\psi}{d\xi}\right|_{u=0=\xi} = 0$, namely, that $A = 0$. However, the requirement that V be minimized at $u = 0$ does not guarantee that the hypersurfaces rise the least. In fact, it does not even guarantee that their rate of increase at $u = 0$ be the least. It can be verified that $\left.\frac{d^2v}{du^2}\right|_{u=0=A} = 0$ gives a different requirement from

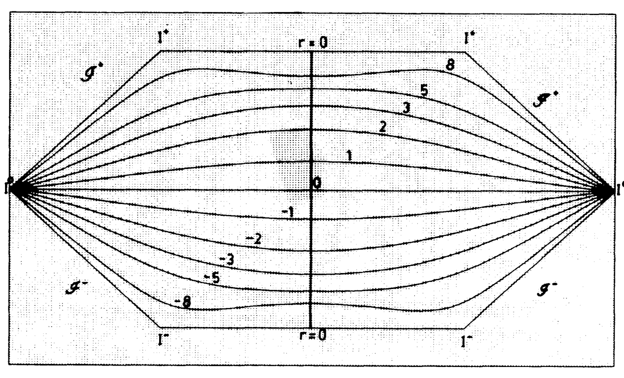

FIG. 1. Foliation of the Schwarzschild spacetime by York slicing in the Penrose diagram is shown. Only a few typical spacelike hypersurfaces are shown corresponding to $K = -0.2, -0.09, -0.05, -0.03, -0.01, 0.0, 0.01, 0.03, 0.05, 0.09, 0.2$.

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TABLE I. Twenty-one York slices for different values of the mean extrinsic curvature are described by the corresponding values for the initial value of r, r_i, the constant H, the initial value of ψ in the Penrose diagram, ψ_*, and the maximum (minimum for $K < 0$) value that ψ attains on the given hypersurface. Notice that ψ_* is the maximum value for eleven of them. The last column gives $\delta\psi = \psi^* - \psi_*$. Notice that δ_ψ rises to a maximum and then starts decreasing again so that we can see that in the limit as $K \longrightarrow \infty$, the hypersurface tends to $\{\psi = \pi/2\}U\mathcal{I}^+$ and as $K \longrightarrow -\infty$ it tends to $\{\psi = \pi/2\}U\mathcal{I}^-$.

Number	K	$r_i/2m$	H	ψ_*	ψ^*	δ_ψ
0	0.0	1.0	0.0	0.0	0.0	0.0
±1	±0.01	0.99	∓0.09527	±0.3252	±0.3252	0.0
±2	±0.03	0.951	∓0.19669	±0.6842	±0.6842	0.0
±3	±0.05	0.89	∓0.26672	±0.9552	±0.9552	0.0
±4	±0.07	0.85	∓0.28918	±1.0696	±1.0696	0.0
±5	±0.09	0.815	∓0.30022	±1.1478	±1.1478	0.0
±6	±0.11	0.785	∓0.30476	±1.2033	±1.2038	0.0001
±7	±0.15	0.751	∓0.30358	±1.2565	±1.2852	0.0287
±8	±0.2	0.7278	∓0.29824	±1.2879	±1.3654	0.0775
±9	±0.5	0.67724	∓0.26486	±1.3459	±1.3940	0.0481
±10	±1.0	0.6054792	∓0.22194	±1.4092	±1.4305	0.0213

$$\left.\frac{d^2\psi}{d\xi^2}\right|_{u=0=A} = \frac{4mr(v^2+1)(2r-3m-KE)-E^2(v^2-1)}{2vE^2} \qquad (7)$$

being zero. Further, this does *not* guarantee that the hypersurface rises least in the Penrose diagram.

The procedure adopted to solve Eq. (6) with Eq. (5) is the following. We choose a particular value of K and require that $A = 0$ at $\xi = 0$ in Eq. (2). This provides a relationship between H and the initial value, r_i, of r:

$$H = K\frac{r_i^3}{3} \pm \sqrt{r_i^3(r_i - 2m)}. \qquad (8)$$

Thus, for some K_1, if we choose some $r_i (0 < r_i < 2m)$, we get two possible choices of H, of which we choose the sign opposite to that of K. (The expression with the same sign does *not* yield a foliation.) We thus have some initial value of ψ, call it ψ_*. In general the maximum value of ψ on such a hypersurface, call it ψ^*, can be greater than ψ_*. We first try to find such an r_i that $\psi^* - \psi_*$ be least. In general such hypersurfaces do not reach I^0 in the Penrose diagram. We now fine-tune r_i to extend the point of intersection of the hypersurfaces with the line $\psi = 0$ or $\mathcal{I}^+$ so that it attains the largest possible value of ξ, say ξ^*. This value will never quite be π, but can approach it arbitrarily closely for sufficiently small step sizes in our integration. The appropriate r_i lies between those values which give hypersurfaces that intersect $\mathcal{I}^+$ and $\psi = 0$. The hypersurfaces are symmetric functions of ξ. Further, for $K_1 \longrightarrow -K_1$ we get the solution by taking $H_1 \longrightarrow -H_1$. Thus there is a reflection symmetry in both the ψ and ξ axes. For $K < 0$, ψ^* will be the minimal value instead of the maximal value. The foliation has been carried out up to $|K| = 1$ and $r_i = 1.210\,958\,4m$. The results of this foliation up to $|K| = 0.2$ are shown in Fig. 1 and the corresponding values of K, r_i, H, ψ_*, and ψ^* are given in Table I up to $|K| = 1$. The step size in all the calculations is taken to be 10^{-3} except for $K = \pm0.5$ and ±1.0, where it is 10^{-5} and 10^{-6}, respectively. In the latter cases, due to loss of sensitivity, we were unable to find hypersurfaces intersecting $\mathcal{I}^+$, though it was clear that still smaller sizes would allow us to complete the procedure. Computing time becomes prohibitive in these cases.

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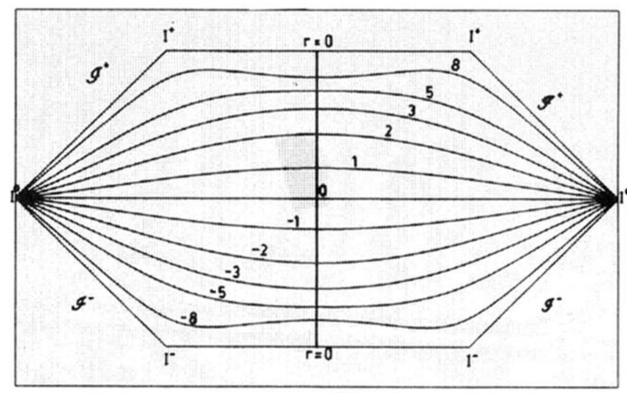

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