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Pressure of hot $g^2\phi^4$ theory at order g^5

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The order g^5 contribution to the pressure of massless $g^2\phi^4$ theory at nonzero temperature is obtained explicitly. Lower order contributions are reconsidered and two issues leading to the optimal choice of the rearranged Lagrangian for such calculations are clarified.

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I. INTRODUCTION

Rapid progress has been made recently in computations of the free energy density, at nonzero temperature (T), of massless $g^2\phi^4$ theory [1], quantum electrodynamics (QED) [2] and quantum chromodynamics (QCD) [3], to three-loop order (fourth order in coupling). For QED, the fifth (e^5) order contribution has also been obtained [4] by dressing the photon lines of the three-loop diagrams.

Compared to QED, a fifth order calculation in QCD will be more involved because gluonic self-interactions imply that many more lines in any three-loop diagram can be soft (i.e., at zero Matsubara frequency in the imaginary time formalism), and so must be dressed in order to obtain the full g^5 contribution. In this respect a fifth order calculation in QCD will resemble the same order calculation in $g^2\phi^4$ theory. In this paper we will compute the order g^5T^4 contribution to the pressure of massless $g^2\phi^4$ theory and show that by an optimal choice of rearranged Lagrangian it is possible to obtain analytical results with relative ease.

The theory we are concerned with is defined by the Euclidean Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 + \frac{g^2\mu^{2\epsilon}}{4!}\phi^4 , \tag{1}$$

where μ is the mass scale of dimensional regularization ($d = 4 - 2\epsilon$) . We use the imaginary time formalism in which the energies take discrete values, $2\pi nT$, $n \in \mathcal{Z}$. For perturbative calculations beyond leading order, it is necessary to take into account in a systematic manner the non-negligible collective effects [5]. For the theory defined by Eq. (1) this means that one uses instead the shifted Lagrangian [6]

$$\mathcal{L}_0' = \left(\mathcal{L}_0 + \frac{1}{2}m^2\phi^2\right) - \frac{1}{2}m^2\phi^2 \tag{2}$$

with $m^2 = g^2T^2/24$ the thermal mass generated at one-loop. The term within parenthesis in Eq. (2) defines a dressed propagator while the last term is a new two-point vertex which prevents overcounting. With Eq. (2) one can proceed to calculate any Green's function in the theory, order by order, in a consistent way. However for the calculation of Green's functions with *static* (zero energy) external legs, Eq. (2) is not very economical since it involves some extraneous resummation.

Recall (see, for example, [2,3]) that for a static Green's function its physical definition is already given in imaginary time, without the need to analytically continue to real time. Thus the power counting of infrared (IR) divergences may be safely done using the Euclidean propagators with discrete energies. Then, only the propagators at zero Matsubara frequency do not have an IR cutoff of order T, and it is only for these zero modes that the thermal mass $\sim gT$ is a relevant infrared cutoff. Thus instead of Eq. (2) one can use

$$\mathcal{L} = \left(\mathcal{L}_0 + \frac{1}{2}m^2\phi^2\delta_{p_0,0}\right) - \frac{1}{2}m^2\phi^2\delta_{p_0,0} \tag{3}$$

for *static* calculations, with p_0 the energy in Fourier space of $\phi(x)$. The Lagrangian in Eq. (3) is precisely what is suggested by the Braaten-Pisarski [5] resummation scheme which involves dressing "soft" lines with "hard-thermal loops." For the calculation of *static* quantities in *imaginary time*, the only soft-line is the zero-mode propagator and the only hard-thermal loop is the static one-loop thermal mass.

We remark that a resummation as in Eq. (3), which involves dressing only the zero-modes, was used very effectively by Arnold and co-workers [7,3] for their free energy calculations in gauge theories, but they used the more general expression (2) (as in Ref. [1]) for their scalar free-energy calculation. The advantage of using minimal resummation (3) also for the scalar case can be seen by the following example. The propagator from Eq. (3) is

$$\Delta(K) \equiv \frac{1-\delta_{k_0,0}}{K^2} + \frac{\delta_{k_0,0}}{k^2+m^2} . \tag{4}$$

Then the one-loop integral

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$$\int [dK]\Delta(K) = \int [dK]\frac{1-\delta_{k_0,0}}{K^2} + T\int (dk)\frac{1}{k^2+m^2} \tag{5}$$

$$= \frac{1}{2}\pi^{\frac{d-5}{2}}T^{d-2}\zeta(3-d)\,\Gamma\left(\frac{3-d}{2}\right) + \frac{T}{4\pi}\left(\frac{m^2}{4\pi}\right)^{\frac{d-3}{2}}\Gamma\left(\frac{3-d}{2}\right) , \tag{6}$$

where we have used the notation

$$\int [dK] \equiv T\sum_{k_0\,(\text{even})}\int (dk) \equiv T\sum_{k_0\,(\text{even})}\int \frac{d^{d-1}k}{(2\pi)^{d-1}} , \tag{7}$$

with $K_\mu = (k_0, \vec{k})$. Notice that the integrals in Eq. (5) were computable in closed form and the result is a term of order g^0 and another of order g. Since the bosonic one-loop integral (5) and others related to it occur frequently in Figs. 1(a)–1(g) , one easily sees that all those diagrams may be evaluated easily. By contrast the propagator of Eq. (2) is $1/(K^2+m^2)$ and its one-loop integral is only available as a high temperature expansion [3], making the evaluation of the diagrams of theory (2) more involved.

Another point that needs clarifying is the choice of m^2 in Eq. (3). In four dimensions, $m^2 = \frac{g^2T^2}{24}$. However since we are using dimensional regularization, one might wonder if the value of m^2 in d-dimensions should be used. For example, in [1] the value $\overline{m^2} = \frac{g^2T^{2-2\epsilon}}{24}\mu^{2\epsilon}$ was used (it being the hard thermal loop in $4-2\epsilon$ dimensions), while in [3] the full one-loop value in d-dimensions was used. Will these different choices affect the final result? If one is interested in renormalized values as $\epsilon \to 0$, the answer is no. Here is the proof: Let the resummation in Eq. (3) be done using some $m^2(\epsilon)$ with $m^2(0) = \frac{g^2T^2}{24}$. Now keep the ϵ dependence of $m^2(\epsilon)$ implicit, even when it hits $1/\epsilon$ terms. Since the full Lagrangian in Eq. (3) is massless, no ultraviolet mass renormalization is needed in perturbative calculations using dimensional regularization. Therefore any $\frac{m^2(\epsilon)}{\epsilon}$ terms generated when calculating a renormalized quantity (i.e., after including coupling-constant renormalization) must mutually cancel. Finally only terms of the form $m^2(\epsilon)\epsilon^n$ $(n \geq 0)$ will appear and then, as $\epsilon \to 0$, only $m^2(0)$ survives. End of proof.

In explicit calculations one finds that if the ϵ-dependence of $m^2(\epsilon)$ is expanded out then extra finite (ϵ^0) terms are generated from some diagrams because of the ϵ-dependence of the mass in the propagator, but these will cancel, order by order in g, with similar terms generated from ϵ-dependence of the thermal counterterm. The fact that one can use $m^2(0)$ rather than some $m^2(\epsilon)$ clearly simplifies calculations by reducing cancellations. On the other hand one can exploit the proof in the last paragraph to provide a cross-check on calculations. That is, one can use $\hat{m}^2 = \frac{g^2T^2}{24}(1+\epsilon A)$, where A is an *arbitrary* regular function of ϵ(which may also depend on g and T/μ), in Eq. (3) and verify that one's final renormalized result is independent of A as $\epsilon \to 0$. Such a check had been performed [8] for the pole of the propagator of $g^2\phi^4$ theory to two-loop [6], and we have also done it for the calculations in this paper. However in order to emphasize the simplicity of using $m^2(0)$, we will present here the results for this case only.

In the next section we reconsider the calculations of the pressure to order g^4 using Eq. (3) and recover the results of [1,3]. Then in Sec. III the g^5 terms are obtained and the results are summarized in Sec. IV. We conclude with some comments in Sec. V.

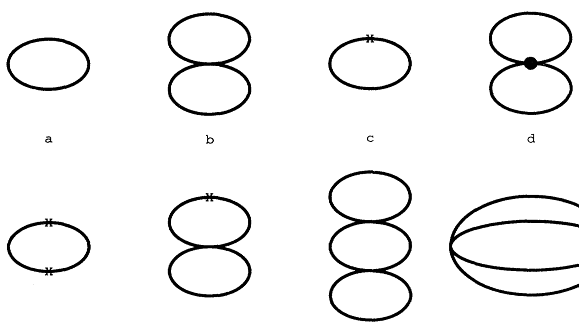

FIG. 1. Diagrams which contribute to fifth order. The propagators are given by Eq. (4), the cross represents the thermal counterterm of Eq. (3) while the blob in diagram (d) is the ultraviolet counterterm.

II. LOWER ORDERS

The diagrams which contribute to the pressure to order g^5 are shown in Fig. (1). The g^4 terms from these diagrams were extracted in Refs. [1,3]. Here we reconsider those g^4 terms using the minimal Lagrangian, Eq. (3), with $m^2 = \frac{g^2T^2}{24}$. The reader comparing the results here with those of [1,3] should note this difference; order by order (rather than diagram by diagram) our results here agree with those of [1,3]. Diagram 1(a) contributes

$$\begin{aligned}
\mathcal{P}_a &= -\frac{1}{2}\int [dP]\ln\Delta^{-1}(P) \\
&= -\frac{1}{2}\int [dP]\ln\{P^2(1-\delta_{p_0,0}) + (P^2+m^2)\delta_{p_0,0}\} \\
&= -\frac{1}{2}\int [dP]\ln P^2 - \frac{1}{2}\int [dP]\ln\left(1+\frac{m^2\delta_{p_0,0}}{P^2}\right) \\
&= -\frac{1}{2}\int [dP]\ln P^2 - \frac{T}{2}\int (dp)\ln\left(1+\frac{m^2}{p^2}\right) \qquad (8)\\
&= T^d\pi^{-d/2}\zeta(d)\Gamma(d/2) - \frac{T}{2}\Gamma\left(\frac{d-1}{2}\right)\left(\frac{m^2}{4\pi}\right)^{(d-1)/2} \\
&= \frac{\pi^2T^4}{90} + \frac{\pi^2T^4}{9\sqrt{6}}\left(\frac{g}{4\pi}\right)^3 + O(\epsilon). \qquad (9)
\end{aligned}$$

In Eqs. (8) and (9), the first term represents the ideal gas contribution while second term is the plasmon contribution [in dimensional regularization, (DR), see [2]] obtained by dressing the zero mode of the one-loop diagram.

The contributions of diagrams 1(b) through 1(g) are

$$\mu^{2\epsilon}(\mathcal{P}_b+\mathcal{P}_c) = -\frac{g^2}{8}\mu^{4\epsilon}\left[\int [dP]\Delta(P)\right]^2 + \frac{m^2}{2}\mu^{2\epsilon}\int [dP]\Delta(P)\delta_{p_0,0}$$
$$= -\frac{g^2T^4}{2^7\,3^2} - \frac{g^4T^4}{2^{10}\,3\,\pi^2} + O(\epsilon), \tag{10}$$

$$\mu^{2\epsilon}\mathcal{P}_d = -\frac{\mu^{2\epsilon}}{8}\left(\frac{3g^4}{32\pi^2\epsilon}\right)\left[\int [dP]\Delta(P)\right]^2$$
$$= \frac{1}{\epsilon}\frac{g^4T^4}{2^{12}\,\pi^2}\left(-\frac{1}{3}+\frac{g}{\pi\sqrt{6}}-\frac{g^2}{2^3\pi^2}\right) - \frac{g^4T^4}{3\,\pi^2\,2^{12}}\left(4\ln\frac{\mu}{T}+4-2\gamma-2\ln 4\pi+4\frac{\zeta'(-1)}{\zeta(-1)}\right)$$
$$+\frac{g^5T^4}{2^{12}\,\pi^3\,\sqrt{6}}\left(4\ln\frac{\mu}{T}+4-2\gamma+2\frac{\zeta'(-1)}{\zeta(-1)}+\ln\frac{6}{g^2}\right)+O(g^6), \tag{11}$$

$$\mu^{2\epsilon}(\mathcal{P}_e+\mathcal{P}_f+\mathcal{P}_g) = \frac{\mu^{2\epsilon}}{4}\int [dP]\delta_{p_0,0}\Delta^2(P)\left[m^2-\frac{g^2\mu^{2\epsilon}}{2}\int [dK]\Delta(K)\right]^2$$
$$+\frac{\mu^{2\epsilon}}{4}\int [dP](1-\delta_{p_0,0})\Delta^2(P)\left[\frac{g^2\mu^{2\epsilon}}{2}\int [dK]\Delta(K)\right]^2$$
$$= \frac{1}{\epsilon}\frac{g^4T^4}{3\,2^{12}\,\pi^2}\left(\frac{1}{3}-\frac{g}{\pi\sqrt{6}}+\frac{g^2}{2^3\pi^2}\right)+\frac{g^4T^4}{3^2\,\pi^2\,2^{12}}\left(6\ln\frac{\mu}{T}+4-\gamma-3\ln 4\pi+4\frac{\zeta'(-1)}{\zeta(-1)}\right)$$
$$-\frac{g^5T^4}{3\,2^{12}\,\pi^3\,\sqrt{6}}\left(6\ln\frac{\mu}{T}+1-\gamma+\ln\frac{3}{2\pi g^2}+2\frac{\zeta'(-1)}{\zeta(-1)}\right)+O(g^6). \tag{12}$$

Note that in $\mathcal{P}_d$ only the *one*-loop ultraviolet (UV) coupling constant renormalization counterterm is used. (We are also using minimal subtraction.) The g^4/ϵ piece is required to cancel similar pieces from diagrams 1(g) and 1(h). The g^5/ϵ and g^6/ϵ divergences in Eq. (11) are due to the mixing of IR resummation effects with the UV renormalization and will cancel against similar terms generated from diagrams 1(g) and 1(h).

The order g^4 contribution from diagram 1(h) is obtained by setting $m=0$ in the propagators since the integrals are IR finite. The result $\mathcal{P}_{h4}$ has been evaluated analytically by Arnold and Zhai and we simply quote their value (which agrees with earlier semianalytical evaluations in [1,2]):

$$\mu^{2\epsilon}\mathcal{P}_{h4} = \frac{g^4T^4}{3^3\,2^{12}\,\pi^2}\left(\frac{6}{\epsilon}+18\ln\frac{\mu^2}{4\pi T^2}-12\frac{\zeta'(-3)}{\zeta(-3)}+48\frac{\zeta'(-1)}{\zeta(-1)}-18\gamma+\frac{182}{5}\right). \tag{13}$$

Thus the sum of diagrams up to order g^4 is

$$\mathcal{P}_4 = \frac{\pi^2T^4}{9}\left\{\frac{1}{10}-\frac{1}{8}\left(\frac{g}{4\pi}\right)^2+\frac{1}{\sqrt{6}}\left(\frac{g}{4\pi}\right)^3\right.$$
$$\left.-\left(\frac{g}{4\pi}\right)^4\left[-\frac{3}{16}\ln\frac{\mu^2}{4\pi T^2}+\frac{1}{4}\frac{\zeta'(-3)}{\zeta(-3)}-\frac{1}{2}\frac{\zeta'(-1)}{\zeta(-1)}+\frac{\gamma}{16}+\frac{59}{120}\right]\right\}+O\left(\frac{g^5}{\epsilon},g^5,\frac{g^6}{\epsilon}\right). \tag{14}$$

The terms to fourth order agree with [1,3].

III. FIFTH ORDER

We now pick up the subleading pieces from diagram 1(h). For this we first rewrite $\Delta(K)$, given in Eq. (4), as (cf. Ref. [4])

$$\Delta(P) = \frac{1}{P^2}-\frac{m^2\,\delta_{p_0,0}}{p^2(p^2+m^2)}$$
$$\equiv \Delta_0(P)+\Delta^*(P)\ . \tag{15}$$

Then

$$\mu^{2\epsilon}\mathcal{P}_h = \frac{g^4\mu^{6\epsilon}}{48}\int [dK\;dQ\;dP]\Delta(K)\Delta(Q)\Delta(P)\Delta(K+Q+P)$$
$$\equiv \frac{1}{48}\{I_0+4I_1+6I_2+4I_3+I_4\}\ , \tag{16}$$

where

$$I_0 = g^4\mu^{6\epsilon}\int[dK\,dQ\,dP]\Delta_0(K)\Delta_0(Q)\Delta_0(P)\Delta_0(K+Q+P)\ ,$$
$$I_1 = g^4\mu^{6\epsilon}\int[dK\,dQ\,dP]\Delta^*(K)\Delta_0(Q)\Delta_0(P)\Delta_0(K+Q+P)\ ,$$
$$I_2 = g^4\mu^{6\epsilon}\int[dK\,dQ\,dP]\Delta^*(K)\Delta^*(Q)\Delta_0(P)\Delta_0(K+Q+P)\ ,$$
$$I_3 = g^4\mu^{6\epsilon}\int[dK\,dQ\,dP]\Delta^*(K)\Delta^*(Q)\Delta^*(P)\Delta_0(K+Q+P)\ ,$$
$$I_4 = g^4\mu^{6\epsilon}\int[dK\,dQ\,dP]\Delta^*(K)\Delta^*(Q)\Delta^*(P)\Delta^*(K+Q+P)\ .$$

The integral I_0 contributes to $\mathcal{P}_{h4}$ and was considered in the last section. We will now extract the order g^5, g^5/ϵ, and g^6/ϵ pieces from I_1 through I_4. Though our final objective is to calculate the pressure only to fifth order, we have to ensure that all subleading divergences such as g^6/ϵ [see Eq. (11)] cancel (there are no divergences beyond g^6/ϵ from these diagrams).

Consider

$$I_1 = -g^4\mu^{6\epsilon}m^2T\int\frac{(dk)}{k^2(k^2+m^2)}\int[dQ\,dP]\frac{\delta_{k_0,0}}{Q^2P^2(K+P+Q)^2}\ . \tag{17}$$

Scaling $\vec{k}\to m\vec{k}$ gives

$$I_1 = -g^4\mu^{6\epsilon}m^{1-2\epsilon}T\int\frac{(dk)}{k^2(k^2+1)}\int[dQ\,dP]\frac{1}{Q^2P^2\left[(q_0+p_0)^2+(\vec{q}+\vec{p}+m\vec{k})^2\right]}\ . \tag{18}$$

Since the external coefficient is $O(g^5)$, we need only the order g piece from the (Q,P) integrals. Write the (Q,P) integrals as (the following discussion parallels that of $I_{\rm Sun}$ in [3])

$$T^2\int\frac{(dq\,dp)}{q^2p^2(\vec{q}+\vec{p}+m\vec{k})^2}+\int[dQ\,dP]\frac{1-\delta_{q_0,0}\,\delta_{p_0,0}}{Q^2P^2\left[(q_0+p_0)^2+(\vec{q}+\vec{p}+m\vec{k})^2\right]}\ . \tag{19}$$

As the second term above is IR safe, one can expand the denominator

$$\frac{1}{(q_0+p_0)^2+(\vec{q}+\vec{p}+m\vec{k})^2} = \frac{1}{(Q+P)^2}\left[1+\frac{m^2k^2+2m\vec{k}\cdot(\vec{q}+\vec{p})}{(Q+P)^2}\right]^{-1}$$
$$= \frac{1}{(Q+P)^2}\left[1-\frac{2m\vec{k}\cdot(\vec{q}+\vec{p})}{(Q+P)^2}+O(m^2)\right]\ , \tag{20}$$

so that the second integral in Eq. (19) is

$$\int[dQ\,dP]\frac{1-\delta_{q_0,0}\,\delta_{p_0,0}}{Q^2P^2(Q+P)^2}-2m\int[dQ\,dP]\frac{1-\delta_{q_0,0}\,\delta_{p_0,0}}{Q^2P^2(Q+P)^4}\vec{k}\cdot(\vec{q}+\vec{p})+O(m^2)\,. \tag{21}$$

The first term in Eq. (21) vanishes in DR [7,2,3] while the second term vanishes when the final $\vec{k}$ integrations are performed in Eq. (18). Hence

$$I_1 = -g^4\mu^{6\epsilon}m^{1-2\epsilon}T^3\int\frac{(dk)}{k^2(k^2+1)}\int\frac{(dp\,dq)}{p^2q^2(\vec{q}+\vec{p}+m\vec{k})^2}+O(g^7)\ . \tag{22}$$

The (p,q) integrals are logarithmically sensitive to m in the infrared. They also have a logarithmic UV singularity in $d-1=3$ dimensions and so must be evaluated in $d=4-2\epsilon$ dimensions. After a standard evaluation of the (q,p) integrals one can perform the final $\vec{k}$ integrations easily by keeping only the terms to $O(\epsilon^0)$. We obtain

$$I_1 = -\frac{g^5T^4}{\sqrt{24}(32\pi^2)(8\pi)}\left(\frac{\mu}{m}\right)^{6\epsilon}\left[\frac{1}{\epsilon}+(8-3\gamma+4\ln 2+3\ln\pi)+O(\epsilon)\right]+O(g^7)\ . \tag{23}$$

Next consider I_2:

$$I_2 = g^4 m^4 \mu^{6\epsilon} T^2 \int \frac{(dk\,dq)}{k^2 q^2 (k^2+m^2)(q^2+m^2)} \int \frac{[dP]\delta_{k_0,0}\,\delta_{q_0,0}}{P^2 (K+Q+P)^2}$$
$$= g^4 m^{2(1-2\epsilon)} \mu^{6\epsilon} T^2 \int \frac{(dk\,dq)}{k^2 q^2 (k^2+1)(q^2+1)} \int \frac{[dP]}{P^2} \frac{1}{p_0^2 + (\vec{p}+m\vec{k}+m\vec{q})^2} .$$

The P integral is

$$T \int (dp) \frac{1}{p^2 (\vec{p}+m\vec{k}+m\vec{q})^2} + \int \frac{[dP]}{P^2} \frac{1-\delta_{p_0,0}}{p_0^2 + (\vec{p}+m\vec{k}+m\vec{q})^2}$$
$$= m^{d-5} T \int (dp) \frac{1}{p^2 (p+k+q)^2} + \int [dP] \frac{1-\delta_{p_0,0}}{P^4} + O(m) .$$

Thus

$$I_2 = \mu^{6\epsilon} g^4 m^{2(1-2\epsilon)} T^2 \int (dk\,dq) \frac{1}{k^2 q^2 (k^2+1)(q^2+1)} \int [dP] \frac{1-\delta_{p_0,0}}{P^4}$$
$$+ \mu^{6\epsilon} g^4 m^{1-6\epsilon} T^3 \int \frac{(dk\,dp\,dq)}{k^2 p^2 q^2 (k^2+1)(p^2+1)(\vec{k}+\vec{p}+\vec{q})^2} + O(g^7) . \tag{24}$$

The first line in (24) is of order $O(g^6)$. Since we require at most the $O(g^6/\epsilon)$ piece, we can set $d=4$ everywhere there except in the P integral, which gives the pole

$$\int [dP] \frac{1-\delta_{p_0,0}}{P^4} = \frac{\pi^2}{(2\pi)^4} \frac{T^{-2\epsilon}}{\epsilon} + O(\epsilon^0) , \tag{25}$$

and so the first line of Eq. (24) is

$$\frac{g^4 m^2 T^2}{2^8\, \pi^4\, \epsilon} + O(g^6 \epsilon^0) . \tag{26}$$

The integral in the second line of Eq. (24) is finite as $\epsilon \to 0$. Therefore we need to evaluate

$$\frac{1}{(2\pi)^9} \int \frac{d^3k\, d^3p\, d^3q}{k^2 (k^2+1) p^2 (p^2+1) q^2 (\vec{k}+\vec{p}+\vec{q})^2} . \tag{27}$$

We decouple the $(\vec{k},\vec{p},\vec{q})$ integrals by writing

$$\frac{1}{(\vec{k}+\vec{p}+\vec{q})^2} = \int d^3w \frac{\delta^3(\vec{k}+\vec{p}+\vec{q}+\vec{w})}{w^2}$$
$$= \int \frac{d^3w}{w^2} \int \frac{d^3r}{(2\pi)^3} e^{i\vec{r}\cdot(\vec{k}+\vec{p}+\vec{q}+\vec{w})} . \tag{28}$$

Inserting Eq. (28) into Eq. (27), the $(\vec{w},\vec{k},\vec{p},\vec{q})$ integrals become trivial, giving

$$\text{Eq. (27)} = \int d^3r \left(\frac{1-e^{-r}}{4\pi r}\right)^2 \left(\frac{1}{4\pi r}\right)^2 \tag{29}$$
$$= \frac{1}{(4\pi)^3} \int_0^\infty \frac{dr}{r^2} (1-e^{-r})^2$$
$$= \frac{(-1)^2}{(4\pi)^3} \mathcal{J}_2(\alpha \to 0)$$
$$= \frac{1}{(4\pi)^3} [2\ln 2] . \tag{30}$$

We have defined a function

$$\mathcal{J}_n(\alpha) \equiv \int_0^\infty \frac{dr}{r^2} (e^{-r}-1)^n e^{-\alpha r}$$
$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+\alpha-k) \ln(n+\alpha-k) , \tag{31}$$

which will appear repeatedly. The use of Eq. (28) in (27) is equivalent to evaluating the momentum integrals (27) in coordinate space (29) (see [3]). In (29) one recognizes $\frac{1}{4\pi r}$ as the coordinate space Coulomb propagator and $\frac{e^{-r}}{4\pi r}$ as the coordinate space screened Coulomb propagator. Adding Eqs. (26) and (30) gives

$$I_2 = \frac{g^4 m T^3}{(4\pi)^3} 2\ln 2 + \frac{g^4 m^2 T^2}{2^8\, \pi^4\, \epsilon} + O(g^6) . \tag{32}$$

Finally, I_3 and I_4 are both finite and their evaluation is analogous to the steps leading from Eq. (27) to (30) and utilizes Eq. (31). We find

$$I_3 = -g^4 m T^3 \mathcal{J}_3(\alpha \to 0)$$
$$= \frac{g^4 m T^3}{(4\pi)^3} [3\ln 3 - 6\ln 2] + O(\epsilon g^5) , \tag{33}$$

$$I_4 = g^4 m T^3 \mathcal{J}_4(\alpha \to 0)$$
$$= \frac{g^4 m T^3}{(4\pi)^3} [20\ln 2 - 12\ln 3] + O(\epsilon g^5). \tag{34}$$

Combining Eqs. (13), (16), (23), and (32)–(34) we get

$$\mu^{2\epsilon}\mathcal{P}_h = \frac{g^4T^4}{3^3\, 2^{12}\, \pi^2}\left(\frac{6}{\epsilon} + 18\ln\frac{\mu^2}{4\pi T^2} - 12\frac{\zeta'(-3)}{\zeta(-3)} + 48\frac{\zeta'(-1)}{\zeta(-1)} - 18\gamma + \frac{182}{5}\right) - \frac{g^5T^4}{3\sqrt{6}\, 2^{11}\pi^3}\left(\frac{1}{\epsilon} + 6\ln\frac{\mu}{m} + 8 - 3\gamma - 4\ln 2 + 3\ln\pi\right) + \frac{g^6T^4}{3\, 2^{14}\, \pi^4}\frac{1}{\epsilon} + O(g^6)\,. \tag{35}$$

The full g^5 contribution is then obtained by adding this to the value of diagrams 1(a)–1(g), some of which also contain g^5 pieces, given in Eqs. (9)–(12). The result is displayed in the following section.

IV. SUMMARY OF RESULTS

The sum of diagrams gives

$$\mathcal{P} = \frac{\pi^2T^4}{9}\left\{\frac{1}{10} - \frac{1}{8}\left(\frac{g}{4\pi}\right)^2 + \frac{1}{\sqrt{6}}\left(\frac{g}{4\pi}\right)^3 - \left(\frac{g}{4\pi}\right)^4\left[-\frac{3}{16}\ln\frac{\mu^2}{4\pi T^2} + \frac{1}{4}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{1}{2}\frac{\zeta'(-1)}{\zeta(-1)} + \frac{\gamma}{16} + \frac{59}{120}\right] + \left(\frac{g}{4\pi}\right)^5\sqrt{\frac{3}{2}}\left[-\frac{3}{4}\ln\frac{\mu^2}{4\pi T^2} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\gamma-5}{4} + \ln\frac{g^2}{24\pi^2}\right]\right\} + O(g^6)\,. \tag{36}$$

As required, all divergences, including the spurious g^6/ϵ terms, have canceled. The renormalization scale μ appears explicitly in the $\ln\left(\frac{\mu}{T}\right)$ terms and also implicitly in the coupling constant, $g = g(\mu)$. One can eliminate the $\ln\left(\frac{\mu}{T}\right)$ terms by re-expressing [1] the pressure in terms of the renormalization-group-invariant coupling $g(T)$ given by

$$g^2(T) = g^2(\mu)\left[1 + \frac{3g^2(\mu)}{(4\pi)^2}\ln\frac{T}{\mu}\right] + O(g^6)\,. \tag{37}$$

Then

$$\mathcal{P} = \frac{\pi^2T^4}{9}\left\{\frac{1}{10} - \frac{1}{8}\left(\frac{g(T)}{4\pi}\right)^2 + \frac{1}{\sqrt{6}}\left(\frac{g(T)}{4\pi}\right)^3 - \left(\frac{g(T)}{4\pi}\right)^4\left[\frac{3}{16}\ln 4\pi + \frac{1}{4}\frac{\zeta'(-3)}{\zeta(-3)} - \frac{1}{2}\frac{\zeta'(-1)}{\zeta(-1)} + \frac{\gamma}{16} + \frac{59}{120}\right] + \left(\frac{g(T)}{4\pi}\right)^5\sqrt{\frac{3}{2}}\left[\frac{3}{4}\ln 4\pi + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{\gamma-5}{4} + \ln\frac{g^2(T)}{24\pi^2}\right]\right\} + O(g^6)\,. \tag{38}$$

A simpler but perhaps less instructive way to obtain Eq. (38) is to choose $\mu = T$ in Eq. (36).

V. CONCLUSION

Using the minimally rearranged Lagrangian (3) together with an ϵ independent thermal mass $m^2 = \frac{g^2T^2}{24}$, we have verified previous [1,3] fourth-order results for the pressure of massless $g^2\phi^4$ theory and then extended the calculations to fifth order. Our final result is given by Eqs. (36) and (38). At the fifth order a coupling constant logarithm appears for the first time in the pressure of $g^2\phi^4$ theory. For QCD such coupling constant logarithms appear already at fourth order, but they do not occur in the pressure of QED (at zero chemical potential) because there are no self-interactions of photons (the only soft fields in imaginary time) and the conclusion follows by power counting [4].

It is natural to contemplate next a fifth-order calculation in QCD. Based on our experience with $g^2\phi^4$ theory and QED we expect such a calculation to be technically simpler than the corresponding fourth order (three-loop) calculation: Nontrivial three-loop diagrams which are IR finite in the bare theory and computationally difficult (e.g., I_0 in $g^2\phi^4$ theory), contribute at subleading order (g^5) when at least one of bare propagators is replaced by a zero-mode dressed propagator [cf. I_1 in Eq. (17)] so that the sum-integral over the dressed momentum line collapses to a three-dimensional UV finite integral, leaving only two overlapping frequency sums at most. Since frequency sums are the main complication in these calculations, this reduction saves effort. In practice further simplification has been observed: for the fifth order QED calculation, summing over gauge-invariant sets of diagrams results in the cancellation [4] of the terms with two overlapping frequency sums; for the scalar calculation in this paper the terms with two-overlapping frequency sums were found to contribute at higher order (g^6) (see the evaluation of I_1). That is, both the QED and scalar fifth-order calculations turned out to be easier than expected. This bonus might prevail for QCD.

As noted by Linde [9] many years ago, the perturbative evaluation of the pressure in QCD breaks down at order g^6 because of the absence of magnetic screening at lowest order. Braaten [10] has recently proposed a solution whereby one can obtain the coefficient of the g^6 contribution as a functional integral in a dimensionally reduced effective theory obtained by integrating out the hard fields in the original QCD path integral. Braaten has also described how his effective Lagrangian may be used to obtain the lower order g^5 term and it would be interesting to compare the result of that approach with one using a shifted Lagrangian in the manner of Eq. (3).

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