Kinks and bound states in the Gross-Neveu model

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(Received 29 August 1994)

We investigate static space-dependent $\sigma(x) = \langle\bar{\psi}\psi\rangle$ saddle point configurations in the two-dimensional Gross-Neveu model in the large N limit. We solve the saddle point condition for $\sigma(x)$ explicitly by employing supersymmetric quantum mechanics and using simple properties of the diagonal resolvent of one-dimensional Schrödinger operators rather than inverse scattering techniques. The resulting solutions in the sector of unbroken supersymmetry are the Callan-Coleman-Gross-Zee kink configurations. We thus provide a direct and clean construction of these kinks. In the sector of broken supersymmetry we derive the DHN saddle point configurations. Our method of finding such nontrivial static configurations may be applied also in other two-dimensional field theories.

PACS number(s): 11.10.Lm, 11.10.Kk, 11.10.St, 11.15.Pg

I. INTRODUCTION

The Gross-Neveu model [1] is a well-known two-dimensional field theory of N massless Dirac fermions ψ_a $(a = 1, \ldots, N)$ with U(N) invariant self-interactions, whose action is given by

$$S = \int d^2x \left[\sum_{a=1}^{N} \bar{\psi}_a i\partial\!\!\!/ \psi_a + \frac{g^2}{2}\left(\sum_{a=1}^{N}\bar{\psi}_a\psi_a\right)^2\right]. \quad (1)$$

We are interested in the large N limit of (1) in which $N \to \infty$ while Ng^2 is held fixed. Decomposing each Dirac spinor into two Majorana spinors one observes that S is invariant under O($2N$) flavor symmetry containing the U(N) mentioned above as a subgroup [2]. The field theory defined by (1) is a renormalizable field theory exhibiting asymptotic freedom, dynamical symmetry breaking, and dimensional transmutation. Its spectrum was calculated semiclassically (in the large N limit) in [2]. It contains the fermions in (1) which become massive, as well as a rich collection of bound states thereof [the so-called "Dashen-Hasslacher-Neveu (DHN) states" [2]]. The spectrum of (1) contains also kink configurations [4,5]. We refer to these as the Callan-Coleman-Gross-Zee (CCGZ) kinks in the sequel. These kinks are expected to be part of the spectrum of the Gross-Neveu model since dynamical breaking of the discrete chiral symmetry in the Gross-Neveu model suggests that there should be extremal field configurations that interpolate between the two minima of the *effective* potential associated with (1) in much the same way that such configurations arise in classical field theories whose potential term has two or more equivalent minima.

The Gross-Neveu model has also a system of infinitely many (nonlocal) conservation laws which forbid particle production in scattering processes in this model and enables the exact calculation of S-matrix elements in the various sectors of the model [6]. Results of such calculations are in agreement with the "large N" calculation of the spectrum.

In this paper we discuss static space-dependent $\sigma(x) = \langle\bar{\psi}\psi\rangle$ configurations that are solutions of the saddle point equation governing the effective action corresponding to (1) as $N \to \infty$. Such $\sigma(x)$ configurations correspond to nontrivial excitations of the vacuum [7,8] and are therefore important in determining the entire spectrum of the field theory in question [2] and its finite temperature behavior as well [9]. Such configurations are important also in discussing the behavior of the $1/N$ expansion of (1) at large orders [10–12].

Our discussion makes use of supersymmetric quantum mechanics and simple properties of the diagonal resolvent of one-dimensional Schrödinger operators. Using these two basic tools we are able to solve the saddle point condition for static $\sigma(x)$ configurations explicitly. The supersymmetry alluded to above relates the upper and lower components of spinors, implying that the square of the Dirac operator may be decomposed into two isospectral Schrödinger operators. It is closely related to the soliton degeneracy discussed in [13], where the soliton is considered as a degenerate doublet having fermion numbers $\pm\frac{1}{2}$. Our explicit solution of the saddle point condition in the sector of unbroken supersymmetry consists of the CCGZ kink configurations and it therefore provides a clean and direct construction of these kinks. In the sector of broken supersymmetry, our explicit solution reproduces the so-called DHN saddle point configurations [2].

In a recent paper [14], we have applied a similar method to the anharmonic O(N) oscillator and the two-dimensional O(N) vector model in the limit $N \to \infty$. In the latter case we have found that the effective action sustains in the large temperature limit extremal bilinear condensates of the O(N) vector field that are analogous to the CCGZ kinks in the Gross-Neveu model.

The paper is organized as follows. In Sec. II we analyze

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0556-2821/95/51(8)/4503(9)/$06.00

the saddle point equation for static $\sigma(x)$ configurations employing supersymmetric quantum mechanics. Using simple manipulations, we show that the latter equation may be expressed in terms of the Hamiltonian of only one of the supersymmetric sectors. In Sec. III A we resolve the saddle point equation into frequencies and demand that the extremum condition be satisfied by each Fourier component separately. This strong condition turns out to leave us always in the sector of unbroken supersymmetry. This corresponds physically to $\sigma(x)$ configurations that interpolate between the two vacua of the Gross-Neveu model at the two ends of the world, which are the CCGZ kinks that we find a explicit solutions of the saddle point equation. In Sec. III B we solve the saddle point equation without separating it into frequencies, under the assumption that there is only one bound state. This leads to the DHN $\sigma(x)$ configurations which belong to the sector of broken supersymmetry. We conclude our discussion in Sec. IV.

II. THE SADDLE POINT EQUATION FOR STATIC SOLUTIONS

Following [1] we rewrite (1) as

$$S=\int d^2x\,\left[i\bar{\psi}\partial\!\!\!/\psi-g\sigma\bar{\psi}\psi-\tfrac{1}{2}\sigma^2\right], \tag{2}$$

where $\sigma(x)$ is an auxiliary field [15].

Thus, the partition function associated with (2) is

$$\mathcal{Z}=\int\mathcal{D}\sigma\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\exp\left\{i\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dx\left[\bar{\psi}i\partial\!\!\!/\psi-g\sigma\bar{\psi}\psi-\tfrac{1}{2}\sigma^2\right]\right\}. \tag{3}$$

Gaussian integration over the Grassmannian variables is straightforward, leading to $\mathcal{Z}=\int\mathcal{D}\sigma\,\exp\{iS_{\rm eff}[\sigma]\}$ where the bare effective action is [16]

$$S_{\rm eff}[\sigma]=-\tfrac{1}{2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dx\,\sigma^2-i\frac{N}{2}{\rm Tr}\,\ln[-(i\partial\!\!\!/-g\sigma)(i\partial\!\!\!/+g\sigma)]. \tag{4}$$

The ground state of the Gross-Neveu model (1) is described by the simplest extremum of $S_{\rm eff}$ [1] in which $\sigma=\sigma_0$ is a constant that is fixed by the (bare) gap equation

$$-g\sigma_0+iNg^2{\rm tr}\int\frac{d^2k}{(2\pi)^2}\frac{1}{k\!\!\!/-g\sigma_0}=0. \tag{5}$$

Therefore, the dynamically generated mass of small fluctuations of the Dirac fields around this vacuum is

$$m=g\sigma_0=\mu\exp\left(1-\frac{\pi}{Ng_R^2(\mu)}\right), \tag{6}$$

where μ is an arbitrary renormalization scale, and the renormalized coupling $g_R(\mu)$ is related to the cutoff-dependent bare coupling via $\Lambda\exp[-\pi/Ng^2(\Lambda)]=\mu\exp[1-\pi/Ng_R^2(\mu)]$, where Λ is an ultraviolet cutoff. Since m is the physical mass of the fermions it must be a renormalization group invariant, and this fixes the scale dependence of the renormalized coupling g_R, namely, Eq. (6). From now on we will drop the subscript R from the renormalized coupling and simply denote it by Ng^2.

As was explained in the Introduction, we are interested in more complicated extrema of $S_{\rm eff}$, namely, static space-dependent solutions of the extremum condition on $S_{\rm eff}$. This condition reads generally

$$\frac{\delta S_{\rm eff}}{\delta\sigma(x,t)}=-\sigma(x,t)-i\frac{N}{2}{\rm tr}\{[2g^2\sigma(x,t)+ig\gamma^\mu\partial_\mu]\langle x,t|[\Box+g^2\sigma^2-ig\gamma^\mu\partial_\mu\sigma]^{-1}|x,t\rangle\}=0, \tag{7}$$

where "tr" is a trace over Dirac indices.

Specializing to static $\sigma(x)$ configurations, and using the Majorana representation $\gamma^1=i\sigma_3$, $\gamma^0=\sigma_2$ for γ matrices, (7) becomes

$$\begin{aligned}\frac{2i}{Ng}\sigma(x)&={\rm tr}\left[\begin{pmatrix}2g\sigma-\partial_x&0\\0&2g\sigma+\partial_x\end{pmatrix}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\begin{pmatrix}R_+(x,\omega^2)&0\\0&R_-(x,\omega^2)\end{pmatrix}\right]\\&=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}[(2g\sigma-\partial_x)R_+(x,\omega^2)+(2g\sigma+\partial_x)R_-(x,\omega^2)],\end{aligned} \tag{8}$$

where

$$R_\pm(x,\omega^2)=\left\langle x\left|\frac{1}{h_\pm-\omega^2}\right|x\right\rangle \tag{9}$$

are the diagonal resolvents of the one-dimensional Schrödinger operators

$$h_\pm=-\partial_x^2+g^2\sigma^2\pm g\sigma'(x) \tag{10}$$

evaluated at spectral parameter ω^2.

Note that $h_\pm$ are positive-semidefinite isospectral (up to zero-mode) Hamiltonians, since (10) may be brought into the form [19,20]

$$h_+=Q^\dagger Q,\quad h_-=QQ^\dagger,$$

where

$$Q=-\frac{d}{dx}+g\sigma,\quad Q^\dagger=\frac{d}{dx}+g\sigma. \tag{11}$$

These operators may be composed into a supersymmetric Hamiltonian $H = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}$ describing one bosonic and one fermionic degrees of freedom [19], which we identify with the upper and lower components of the spinors. Supersymmetry implies here that an interchange of the bosonic and fermionic sectors of H leaves dynamics unchanged, as can be seen from the fact that Eqs. (8)–(11) are invariant under the simultaneous interchanges

$$\sigma \to -\sigma, \quad h_\pm \to h_\mp, \qquad R_\pm \to R_\mp. \tag{12}$$

The isospectrality of h_+ and h_- alluded to above means that, to each eigenvector ψ_n of h_- with a *positive* eigenvalue E_n, there is a corresponding eigenvector ϕ_n of h_+ with the same eigenvalue and norm, and vice versa. The precise form of this pairing relation is

$$\phi_n = \frac{1}{\sqrt{E_n}} Q^\dagger \psi_n, \qquad \psi_n = \frac{1}{\sqrt{E_n}} Q \phi_n, \quad E_n > 0. \tag{13}$$

It is clear that the pairing in (13) fails when $E_n = 0$. Thus, in general one cannot relate the eigenvectors with the zero eigenvalue (i.e., the normalizable zero modes) of one Hamiltonian in (10) to these of the other. Should such a normalizable zero mode appear in the spectrum of one of the positive-semidefinite operators in (10), it must be the ground state of that Hamiltonian. In this case the lowest eigenvalue of the supersymmetric Hamiltonian H is zero, which is the case of unbroken supersymmetry. If such a normalizable zero mode does not appear in the spectrum, all eigenvalues of H, and, in particular, its ground-state energy, are positive, and supersymmetry is broken. Since the ground state of a Schrödinger operator is nondegenerated, $h_\pm$ can each have no more than one such a normalizable zero mode. Moreover, it is clear from (10) that in our one-dimensional case, only *one* operator in (10) may have a normalizable zero mode, since it must be annihilated either by Q or by $Q^\dagger$. In cases of unbroken supersymmetry, we will take such a normalizable zero mode to be an eigenstate of h_-, namley, the *real* function

$$\Psi_0(x) = \mathcal{N} \exp\left(-g \int_0^x \sigma(y)dy\right), \tag{14}$$

which is the normalizable solution of the differential equation

$$Q^\dagger \Psi_0 = 0 \tag{15}$$

where $\mathcal{N}$ is a normalization coefficient.

Note that a necessary condition for the normalizability of Ψ_0 is that $\sigma(x)$ have the opposite behavior at $\pm\infty$. Thus, physically, cases of unbroken supersymmetry lead to $\sigma(x)$ configurations that interpolate between the two vacua of (4) at the two ends of the world, while cases of broken supersymmetry yield $\sigma(x)$ configurations that leave and return to the same vacuum state.

Assigning the zero mode to h_- poses no loss of generality, since the other possible case is related to this one via (12). In what follows we will denote the right-hand side of (14) by Ψ_0 also in cases of broken supersymmetry where it is non-normalizable. This should not cause any confusion, since the ground state will be denoted by ψ_0, which will be equal to Ψ_0 when supersymmetry is unbroken.

By definition, the diagonal resolvents in (9) are given by eigenfunction expansions

$$R_-(x) = \sum_{n=0}^{\infty} \frac{|\psi_n(x)|^2}{E_n - \omega^2}, \qquad R_+(x) = \sum_{n>0}^{\infty} \frac{|\phi_n(x)|^2}{E_n - \omega^2} \tag{16}$$

where the sums extend over all eigenstates, including the continua of scattering states where they are understood as integrals.

Using (13), R_+ may be expressed in terms of the ψ_n's as

$$R_+(x,\omega^2) = \sum_{n>0}^{\infty} \frac{1}{E_n \Psi_0^2(x)} \frac{|W_n(x)|^2}{E_n - \omega^2}, \tag{17}$$

where $W_n(x) = \Psi_0(x)\psi_n'(x) - \Psi_0'(x)\psi_n(x)$ is the Wronskian of Ψ_0 and ψ_n. An elementary consequence of the Schrödinger equation is that $W_n'(x) = -E_n\Psi_0(x)\psi_n(x)$. Using this relation, (16) and (17) imply the important relation

$$(2g\sigma - \partial_x)R_+ + (2g\sigma + \partial_x)R_- = 2\left(2g\sigma + \frac{d}{dx}\right)\left\langle x \left| \mathcal{P} \frac{1}{h_- - \omega^2} \right| x \right\rangle. \tag{18}$$

Here $\mathcal{P}$ is the projector

$$\mathcal{P} = \mathbb{1} - \lambda|\psi_0\rangle\langle\psi_0| \tag{19}$$

that projects out the ground state of h_- when supersymmetry is unbroken ($\lambda = 1$), and is just the unit operator otherwise ($\lambda = 0$). We can also use (5) to make a frequency resolution of unity as

$$\frac{i}{Ng^2} = \int_{-\Lambda}^{\Lambda} \frac{d\omega}{\pi} \left\langle x \left| \frac{1}{-\partial_x^2 + m^2 - \omega^2 - i\epsilon} \right| x \right\rangle. \tag{20}$$

Moreover, from (18), (20), and the elementary relation

$$\left\langle x \left| \frac{1}{-\partial_x^2 + m^2 - \omega^2 - i\epsilon} \right| x \right\rangle = \frac{1}{2\sqrt{m^2 - \omega^2 - i\epsilon}}, \tag{21}$$

the frequency resolution of (8) becomes

$$\frac{d}{dx}\left\langle x \left| \mathcal{P} \frac{1}{h_- - \omega^2} \right| x \right\rangle = 2g\sigma(x)\left[\frac{1}{2\sqrt{m^2 - \omega^2}} - \left\langle x \left| \mathcal{P} \frac{1}{h_- - \omega^2} \right| x \right\rangle\right].$$

Substituting (19) into the last equation, all dependence on λ cancels out and the extremum condition (8) is shaped into its final form

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left\{\frac{dR_-(x,\omega^2)}{dx}-2g\sigma(x)\left[\frac{1}{2\sqrt{m^2-\omega^2}}-R_-(x,\omega^2)\right]\right\}, \quad (22)$$

which has to be satisfied regardless of whether or not supersymmetry is broken.

III. STATIC SOLUTIONS TO THE EXTREMUM CONDITION

A. Case of unbroken supersymmetry

The simplest way to look for solutions of the static saddle point equation (22) is to demand that it be satisfied by each frequency mode separately, namely, restricting R_- by the differential condition

$$\frac{dR_-(x,\omega^2)}{dx}=2g\sigma(x)\left[\frac{1}{2\sqrt{m^2-\omega^2}}-R_-(x,\omega^2)\right]. \quad (23)$$

Now, $R_-(x,\omega^2)$, being the diagonal resolvent of h_- at spectral parameter ω^2, is subjected to the so-called "Gel'fand-Dikii" equation [17,18]

$$-2R_-(x,\omega^2)R_-''(x,\omega^2)+(R_-'(x,\omega^2))^2 +4R_-^2(x,\omega^2)[g^2\sigma^2-g\sigma'-\omega^2]=1. \quad (24)$$

Therefore, both Eqs. (23) and (24) must hold and they form a system of coupled nonlinear differential equations in the unknowns $\sigma(x)$ and $R_-(x,\omega^2)$. Substituting R_-' and R_-'' from (23) into (24), we obtain a quadratic equation for R_- whose solutions are

$$R_-(x,\omega^2) =\frac{-g\sigma'\pm\sqrt{(g\sigma')^2+4\omega^2g^2\sigma^2-4\omega^2(m^2-\omega^2)}}{4\omega^2\sqrt{m^2-\omega^2}}. \quad (25)$$

To see what the two signs of the square root correspond to, we observe that the solution with the negative sign in front of the square root has a simple pole as a function of ω^2 at $\omega^2=0$ with a negative residue, while the other solution is regular and positive at $\omega^2=0$. Therefore, from (16) it is clear that the negative sign root corresponds to the case of unbroken supersymmetry, where the simple pole signals the existence of a normalizable zero mode in the spectrum of h_-, while the positive root solution corresponds, for similar reasons, to cases in which h_- lacks such a zero mode. We will see below that the latter solution corresponds also to the case of unbroken supersymmetry, where the zero mode is in the spectrum of h_+.

We concentrate now on the branch-cut singularity in (25) at $\omega^2=m^2$, which signals the threshold of the continuous part of the spectrum of h_-. Clearly, h_- can have no continuous spectrum other than that starting at $\omega^2=m^2$, thus leading to the conclusion that R_- in (25) can have no other branch points in the ω^2 plane. Therefore, the expression under the square root in (25) must be a perfect square as a polynamial in ω^2: namely,

$$\pm g\sigma'(x)=g^2\sigma^2(x)-m^2. \quad (26)$$

From (26) we find straightforwardly the solutions

$$g\sigma(x)=\pm m\tanh[m(x-x_0)], \quad (27)$$

which are exactly the CCGZ kinks and antikinks. Here the parameter x_0 is an integration constant that implies translational invariance of (27) since it is the arbitrary location of the kinks. Clearly, both cases in (27), and therefore both cases in (25), lead to $h_\pm$ operators that doe not break supersymmetry, since (14) and (27) imply that

$$\Psi_0=\left(\frac{m}{2}\right)^{1/2}\exp\left(-g\int_0^x\sigma(y)dy\right) =\left(\frac{m}{2}\right)^{1/2}\text{sech}[m(x-x_0)] \quad (28)$$

is the normalizable zero mode of h_- for the kink configuration, and of h_+ when $\sigma(x)$ is the antikink. Because of the zero binding energy, fermions trapped in this potential do not react back on the $\sigma(x)$ field [2].

Note that in deriving (27) above we have not set any *a priori* restrictions on R_-, thus, it is a very interesting question whether (27) are the *only* possible static extremal $\sigma(x)$ configurations in the sector of unbroken supersymmetry or not.

As it stands, our frequency decomposition of the extremum condition (23) seems to lead always to extremal $\sigma(x)$ configurations that do not break supersymmetry. We are thus unable to find in this manner the extremal $\sigma(x)$ configurations found in [2] in which $\sigma(x)$ has the shape of kink-antikink pair that are very close to each other. Such a configuration evidently breaks supersymmetry [19]. They will be discussed in the next subsection.

When $\sigma(x)$ is a kink, (25) becomes

$$R_-(x,\omega^2)=-\frac{m^2\text{sech}^2[m(x-x_0)]}{2\omega^2\sqrt{m^2-\omega^2}}+\frac{1}{2\sqrt{m^2-\omega^2}} \quad (29)$$

while for antikinks it is just the expression on the right-hand side of (21).

These statements on R_- are consistent with the explicit form of the Hamiltonians $h_\pm$. Using (10) and (27) we find, in the kink case,

$$h_+=-\partial_x^2+g^2\sigma^2+g\sigma'=-\partial_x^2+m^2$$

and

$$h_- =-\partial_x^2+g^2\sigma^2-g\sigma' =-\partial_x^2+m^2-2m^2\text{sech}^2[m(x-x_0)], \quad (30)$$

while in the antikink case $h_\pm$ interchange their roles.

Thus, in the latter case, h_- becomes the Schrödinger operator of a freely moving particles in a *constant* potential m^2 which is the reason why R_- is given by the simple expression in (21) in the antikink case. Indeed, that expression is the x independent solution of the Gel'fand-Dikii equation (24) corresponding to the constant potential $g^2\sigma^2 - g\sigma' = m^2$ that is positive at $\omega^2 = 0$.

In the kink case, we have obtained the explicit form (29) for R_- by substituting (27) into (25). As an independent check, we may deduce this expression for the diagonal resolvent for the potential $g^2\sigma^2 - g\sigma' = m^2 - 2m^2\text{sech}[m(x-x_0)]$ in other ways. The simplest one is to apply an ansatz of the form $R_- = \alpha\,\text{sech}^2(\beta x) + \gamma$ to the Gel'fand-Dikii equation. Another way to derive it is to use the general formula $R(x) = G(x,y)|_{x=y} = \psi_1(x_>)\psi_2(x_<)/W(\psi_1,\psi_2)|_{x=y}$, where the ψ's are the so-called Jost functions of the problem. In this case they are hypergeometric functions multiplied by sech factors. The Wronskian in this expression is a ratio of Γ functions dependent on ω^2 and m^2 [8].

We can deduce from (25) the CCGZ solution (27) to the extremum condition (23) and (24) by yet another method which does not rely upon the branch-cut structure of R_- but rather on its pole structure. Considering the case where the zero mode is in the spectrum of h_- we make a Laurent expansion of the appropriate expression for R_- in (25) around the simple pole at $\omega^2 = 0$. The leading term in this expansion is

$$R_- \sim -\frac{1}{\omega^2}\left[\frac{g\sigma'}{2m} + \mathrm{O}(\omega^2)\right]. \tag{31}$$

Comparing (31) to (14) and (16) we find

$$\Psi_0^2(x) = \mathcal{N}^2 \exp\left(-2g\int_0^x \sigma(y)dy\right) = \frac{g\sigma'}{2m}, \tag{32}$$

which yields

$$\mathcal{N}^2 e^{-2\phi} = \frac{\phi''}{2m}, \quad \phi(x) = g\int_0^x \sigma(y)dy. \tag{33}$$

Solving (33) we find

$$\phi(x) = \ln\cosh[m(x-x_0)] + c, \tag{34}$$

where c and x_0 are integration constants and we have imposed the normalization condition $\int_{-\infty}^{\infty}\Psi_0^2 dx = 1$ to fix $\mathcal{N} = \sqrt{m/2c}$. Clearly, then, (34) yields the CCGZ kinks upon differentiation.

We are now in a position to verify briefly that the CCGZ kink configuration leads to an h_- operator that has indeed a single normalizable zero mode as the explicit form (29) for R_- suggests. Our discussion follows [20]. In the kink sector (30) implies that the eigenstates of h_+ are simply these of freely moving particles $\phi_k(x) = e^{ikx}$, with a *continuum* of strictly positive eigenvalues $E_+ = k^2 + m^2 \geq m^2$. These states are isospectral to the eigenstates of h_-:

$$\psi_x(x) = \frac{1}{\sqrt{k^2+m^2}}Q\phi_k(x) = \left\{\frac{m\tanh[m(x-x_0)] - ik}{\sqrt{k^2+m^2}}\right\}e^{ikx}, \tag{35}$$

which are therefore the *scattering* states of h_-. The S matrix associated with h_- is thus

$$S(k) = \frac{ik-m}{ik+m} = \exp\left[i\left(\pi - 2\arctan\frac{k}{m}\right)\right]. \tag{36}$$

Since h_+ in (30) has only scattering states, h_- can have no bound states other than its zero mode (28) which must therefore be its ground state. This *single* normalizable state of h_- corresponds to the single pole of $S(k)$ in (36) at $k = im$. Note further that there are no reflected waves in any of the scattering eigenstates (35) of the Schrödinger operators in (30). This is also the case for the supersymmetry breaking $\sigma(x)$ configurations in [2] as well as in other exactly soluble models in two space-time dimensions [2].

The fact that $h_\pm$ evaluated at the extremal point must be reflectionless can be deduced even without solving the extremum condition (23) explicitly, provided one makes *a priori* an assumption that h_- has only a single bound state (namely, its ground state) regardless of whether supersymmetry is broken or not.

To this end we consider (23), in which R_- may be replaced by $R_P = \langle x|\mathcal{P}[1/(h_- - \omega^2)]|x\rangle$ as mentioned in the discussion preceding (23). Since h_- is assumed to have a single bound state, R_P contains only scattering states of h_-, whose corresponding continuous eigenvalues $E = k^2 + m^2$ start at $E = m^2$. We may therefore write the spectral resolution of R_P as

$$R_P = \int_{m^2}^{\infty} dE \frac{|\psi_E(x)|^2}{E - \omega^2} = \int_{-\infty}^{\infty} dk \frac{\rho_k(x)}{k^2+m^2-\omega^2}, \tag{37}$$

where $\rho_k(x) = 2\pi|k||\psi_k(x)|^2$. In terms of ρ_k the extremum condition (23) becomes

$$\frac{d}{dx}\frac{\rho_k(x)}{\Psi_0^2(x)} = \frac{d}{dx}\frac{1}{\Psi_0^2(x)} \tag{38}$$

whose general solution is

$$\rho_k(x) = 1 + c_k\Psi_0^2(x), \tag{39}$$

where c_k is an integration constant. Since by definition $\rho_k(x)$ cannot blow up at infinity, we must set $c_k = 0$ in the case of broken supersymmetry. Whether supersymmetry is broken or not $\rho_k(x)$ obviously obtains the asymptotic value of 1 as $x \to \pm\infty$. Therefore, the scattering states $\psi_k(x)$ of h_- are given by

$$\psi_k(x) = \sqrt{1 + c_k\Psi_0^2(x)}e^{i\alpha_k(x)}, \tag{40}$$

where α_k is a real phase. Substituting these functions into the eigenvalue equation for h_- and considering its asymptotic behavior as $x \to \pm\infty$ we see, using the boundary condition $g\sigma(\pm\infty) = \pm m$, that the phase becomes that of a free particle, which is obvious, but unimodularity of the phase factor implies further that there be only a right-moving or only a left-moving wave in $\psi_k(x)$. Therefore, h_- must be reflectionless.

The physical significance of the CCGZ kinks is as follows. As was mentioned in the Introduction, the dynamical properties of (1) are consequences of the fact that the ("large N") effective potential $V_{\rm eff}(\sigma)$ extracted from (4) has two symmetric equivalent minima at $\langle\sigma\rangle_{\rm vac} = \pm\sigma_0 \neq 0$. This causes a dynamical breakdown of the discrete (Z_2) chiral symmetry of (2) under the transformation $\psi \to \gamma_5\psi$, $\sigma \to -\sigma$, where the fermions fluctuating near the $\langle\sigma\rangle_{\rm vac} = \pm\sigma_0$ ground state acquire dynamical mass $m = \pm g\sigma_0$ [1]. In a similar manner to the appearance of kinks in classical field theories with potentials exhibiting spontaneous symmetry breaking, one should expect similar configurations to appear in field theories whose *effective* potential implies dynamical symmetry breaking. The CCGZ kinks (antikinks) are precisely such static space-dependent $\sigma(x)$ configurations that interpolate between the two minima of $V_{\rm eff}(\sigma)$, and our calculations provide explicit proof that they are indeed extremal points of (4). The various states appearing in the background of the CCGZ kink may be deduced by calculating the "partition function" of the Dirac field fluctuations in that specific $\sigma(x)$ configuration for a finite time lapse T (i.e., the trace over the time evolution operator e^{-iHT}). This has been done explicitly in [2]. The result is

$$\mathrm{Tr}e^{-iHT} = \sum_{n=0}^{2N} \frac{(2N)!}{(n)!(2N-n)!} \exp\left[-\frac{i}{2}\int_0^T dt \int_{-\infty}^{\infty} dx(\sigma_{\rm kink}^2 - \sigma_0^2)\right] \times \exp\left\{iNT\left(\sum_i \omega_i[\sigma_{\rm kink}] - \sum_i \omega_i[\sigma_0]\right) - in\omega_b[\sigma_{\rm kink}]T\right\}. \tag{41}$$

In this equation n is the total number of fermions and antifermions that are trapped in the single bound state of the kink, $\omega_i[\sigma] = \sqrt{E_i}$ is the energy of the ith state in the background of σ [21], and ω_b is the energy of the bound state, which is zero for the CCGZ kinks. Using this and also the fact that in this background $h_\pm$ in (30) are isospectral, we see that all terms in the second exponent in (41) are canceled, leaving only the first exponent which is the mass of the kink: namely [22],

$$M_{\rm kink} = \tfrac{1}{2}\int_{-\infty}^{\infty} [\sigma_0^2 - \sigma_{\rm kink}^2] = \frac{mN}{Ng^2}. \tag{42}$$

Therefore, we see that all states contributing to (41) are degenerate in energy (all having the kink mass as energy), which is a direct result of the fact that $\omega_b = 0$. Clearly there are 2^{2N} states in all that form a huge reducible supermultiplet of O($2N$). Its decomposition into irreducible components is clear from the combinatorial prefactors in (41) that simply tell us that the various bound states in the kink fall into antisymmetric tensor representations of O($2N$) (where the integer n is the rank of the tensor) [2].

We close this subsection by checking explicitly that the CCGZ kink configurations obtained above indeed extremize the effective action in (4). Substituting the kink configuration in (27) and the explicit expressions (29) and (21) for R_- and R_+ into the extremum condition (8) we find that the pole at $\omega^2 = 0$ disappears from the right-hand side of (8) in accordance with (18) leaving in the sum over frequencies only contributions from the scattering states. Thus, Eq. (8) becomes

$$\left[1 + iNg^2\int_{-\Lambda}^{\Lambda} \frac{d\omega}{2\pi}\frac{1}{\sqrt{m^2-\omega^2-i\epsilon}}\right] g\sigma(x) = 0, \tag{43}$$

implying that the term in the square brackets on the left-hand side must vanish. But vanishing of the latter is precisely the Minkowski space gap equation of the Gross-Neveu model (5) for the dynamical mass m and it must therefore hold, confirming that the kinks in (27) are indeed solutions of the extremum condition (8).

B. Case of broken supersymmetry

Enforcing the saddle point condition at each frequency component in (23) led us directly to the sector of unbroken supersymmetry without any further assumptions on $R_-(x,\omega^2)$. Therefore, in order to find static extremal $\sigma(x)$ configurations that lead to supersymmetry breaking, we must solve (22) as a whole. At a first sight this seems to be unmanageable [23], since we apparently cannot use the Gel'fand-Dikii equation (24) for R_- which was so crucial for our treatment in the preceding section. However, assuming (as in [2]) that the required $\sigma(x)$ configuration yields an h_- operator with a *single* bound state at *positive* energy $E_b = \omega_b^2 < m^2$, in addition to the obvious continuum of unbound fermions of mass m, our experience gained in the preceding section leads us to the most general form of $R_-(x,\omega^2)$ consistent with (24) and the preassumed form of the spectrum of h_-. This generic form of R_- contains enough information in order to solve (22).

In order to construct this generic form of R_-, we note the following points.

(1) R_- has scale dimension -1 in mass units as can be seen from its definition (9) or from the explicit expressions (25) and (29).

(2) Clearly, $g\sigma(x)$ must reach asymptotically either one of the vacua, namely, $|g\sigma(\pm\infty)| = m$, $g\sigma'(\pm\infty) = 0$ (actually, since we will end up indeed in the sector of broken supersymmetry, the two asymptotic values of $g\sigma$ will turn out to be equal). Thus, the h_- operator resulting from such a $\sigma(x)$ configuration must have the asymptotic form $-\partial_x^2 + m^2$ as $|x| \to \infty$. Correspondingly, its resolvent must have the asymptotic form (21).

(3) The assumption that h_- has a single bound state at positive energy $0 < \omega_b^2 < m^2$ implies that the analytic structure of $R_-(x,\omega^2)$ in the complex ω plane must in-

clude in addition to the branch cuts along the real rays $(-\infty,-m)$ and (m,∞) only two simple poles on the real axis at $\omega=\pm\omega_b$.

(4) Any ansatz for R_- must obey the Gel'fand-Dikii equation (24).

Points number (1) and (3) lead immediately to the general form

$$R_-(x,\omega^2)=\frac{Ag\sigma'+B(g\sigma)^2+C(\omega_b^2-\omega^2)+Dm^2}{(\omega_b^2-\omega^2)\sqrt{m^2-\omega^2}}, \tag{44}$$

where A, B, C, and D are dimensionless numbers yet to be determined. Point number (2) implies then that $C=\frac{1}{2}$ and $B=-D$.

In order to comply with point number (4), we substitute (44) into the Gel'fand-Dikii equation (24). Multiplying the resultant expression through by $(\omega_b^2-\omega^2)^2(m^2-\omega^2)$ we obtain an even quartic polynomial in ω [24] which must vanish. This yields three nonlinear differential equations for $g\sigma(x)$ stemming from nullification of the coefficients of ω^4, ω^2, and ω^0 in that polynomial. The condition which results from setting the coefficient of ω^4 to zero reads

$$(4A-1)(g\sigma)'+(1-4D)[(g\sigma)^2-m^2]=0, \tag{45}$$

which can be used to eliminate A and D from (44). Doing so (44) becomes

$$R_-(x,\omega^2)=\frac{g\sigma'+m^2-(g\sigma)^2}{4(\omega_b^2-\omega^2)\sqrt{m^2-\omega^2}}+\frac{1}{2\sqrt{m^2-\omega^2}}. \tag{46}$$

This expression is evidently very similar to expressions (25) and (29), the only difference being that the double pole at $\omega=0$ in those equations is resolved here into the two simple poles at $\omega=\pm\omega_b$.

We must further subject (46) to the two remaining differential equations mentioned above. It turns out that the equation resulting from setting the coefficient of ω^2 to zero is simply the derivative (with respect to x) of the equation associated with the coefficient of ω^0 [25]. The latter is the differential equation for $g\sigma(x)$ we are looking for, which must be solved subjected to the boundary conditions mentioned in point number (2) above. Equivalently, and this is what we do below, we can read off the expression for the *normalized* bound-state wave function $\psi_b(x)$ in terms of $g\sigma(x)$ from (46). The Schrödinger equation for $\psi_b(x)$ then provides the required condition for $g\sigma(x)$.

As can be seen from (16) in a similar manner to our discussion following (31) in the preceding subsection, we identify the residue of the simple pole of $R(x,E\equiv\omega^2)$ in (46) at $E=\omega_b^2$ as $-\psi_b^2(x)$. Thus,

$$\psi_b(x)=\left(\frac{g\sigma'-(g\sigma)^2+m^2}{4\sqrt{m^2-\omega_b^2}}\right)^{1/2}=\left(\frac{m^2-V}{4\sqrt{m^2-\omega_b^2}}\right)^{1/2}, \tag{47}$$

where

$$V(x)=g^2\sigma^2-g\sigma' \tag{48}$$

is the potential of h_-. Imposing the Schrödinger equation on (47) we have

$$[-\partial_x^2+V(x)]\sqrt{m^2-V(x)}=\omega_b^2\sqrt{m^2-V(x)}. \tag{49}$$

The solution to this equation, compatible with boundary conditions at infinity, is

$$V(x)=m^2-2\kappa^2\text{sech}^2[\kappa(x-x_0)], \tag{50}$$

where $\kappa^2\equiv m^2-\omega_b^2$ and x_0 is an integration constant. The corresponding bound state and resolvent are therefore

$$\psi_b(x)=\sqrt{\frac{\kappa}{2}}\text{sech}[\kappa(x-x_0)],$$

$$R_-(x,\omega^2)=\frac{\kappa^2\text{sech}^2[\kappa(x-x_0)]}{2(\omega_b^2-\omega^2)\sqrt{m^2-\omega^2}}+\frac{1}{2\sqrt{m^2-\omega^2}}, \tag{51}$$

which solve (24) and (49), as can be checked explicitly. It is not surprising at all that (51) reproduces (29) as $\omega_b\to 0$.

Finally, in order to find $g\sigma(x)$ we substitute (50) into (48). This is equivalent to solving

$$\{-\partial_x^2+m^2-2\kappa^2\text{sech}^2[\kappa(x-x_0)]\}\Psi_0(x)=0, \tag{52}$$

where $\Psi_0(x)$ is defined in terms of $g\sigma(x)$ in (14). For $\kappa^2\neq m^2$ the Schrödinger operator in (52) has no normalizable zero mode and supersymmetry is broken. Nevertheless, one can find non-normalizable solutions of the differential equation (52), which can be transformed into a hypergeometric equation [8], and extract $g\sigma(x)$ in this way. The resulting $g\sigma(x)$ configurations are those found in [2]. The specific $g\sigma(x)$ given in [2] corresponds to setting $\kappa x_0=\frac{1}{4}\ln(m+\kappa/m-\kappa)$ in (50): namely,

$$g\sigma(x)=m+\kappa\left\{\tanh\left[\kappa x-\tfrac{1}{4}\ln\frac{m+\kappa}{m-\kappa}\right]-\tanh\left[\kappa x+\tfrac{1}{4}\ln\frac{m+\kappa}{m-\kappa}\right]\right\}. \tag{53}$$

Note at this stage that we have yet to impose the saddle point condition (22). This will quantize ω_b and constrain it to the discrete set of values found by DNH [2] as we now show.

Substituting (46) into the generic static saddle point condition (22) we obtain

$$\left[\left(2g\sigma+\frac{d}{dx}\right)(g\sigma'+m^2-g^2\sigma^2)\right]\times\int_{C,n}\frac{d\omega}{2\pi}\frac{1}{(\omega_b^2-\omega^2)\sqrt{m^2-\omega^2}}=0, \tag{54}$$

where we now integrate over ω along a contour $\mathcal{C}$ in the complex ω plane, and the subindex n counts the number of fermions trapped in the single bound state of h_- produced by $g\sigma$. The contour $\mathcal{C}$ is precisely the one used to define the Feynman propagator of a free Dirac particle of mass m, and in addition, it urns right *below both* simple poles of R_- at $\omega = \pm\omega_b$ [26].

The resulting static saddle point condition (54) is solved by requiring either that the term in the square brackets vanishes, or that the integral over frequencies vanishes. The differential equation resulting from the first possibility is

$$g\sigma'' = 2g\sigma(g^2\sigma^2 - m^2), \tag{55}$$

which reproduces the kink configuration (27) discussed in the preceding section with $\omega_b = 0$. We thus focus on the other possibility, namely, that the integral in (54) vanishes.

The integral in (54) is UV finite. We can therefore deform it by folding its $\omega > \omega_b$ wing right on top of the remaining part of the real ω axis to the left of ω_b. This can be further deformed into two circles wrapped around the simple poles at $\pm\omega_b$ and a "hairpin " configuration wrapped around the left-hand cut of R_-, picking up its discontinuity across the cut [26]. In other words, this integral picks up contributions from the completely filled Dirac sea (including the pole at $\omega = -\omega_b$), where each energy state (of the Dirac operator $i\partial\!\!\!/ - g\sigma$ rather than h_-) is occupied by N fermion flavors, and the single bound state at $\omega = +\omega_b$, which is occupied by $n < N$ fermions.

Because we have pulled out a factor of N in deriving (22) we have to weigh the contribution of the simple pole of R_- at $\omega = \omega_b$ by n/N in (54) while the other contributions are weighed simply by 1. Performing the integration, we find that the contribution from the cut is $-i(1-2\theta/\pi)/m^2 \sin 2\theta(1-2\theta/\pi)$, while the poles at $\omega = \pm\omega_b$ yield, respectively, $\mp i/m^2 \sin\omega\theta$, where, following [2], we have defined

$$\omega_b = m\,\sin\theta$$

and

$$\kappa = \sqrt{m^2 - \omega_b^2} = m\,\cos\theta. \tag{56}$$

Gathering all contributions to the integral (properly weighed), (54) yields

$$\int_{\mathcal{C},n} \frac{d\omega}{2\pi}\frac{1}{(\omega_b^2-\omega^2)\sqrt{m^2-\omega^2}} = \frac{i}{m^2\sin 2\theta}\left[\frac{2\theta}{\pi} - \frac{n}{N}\right] = 0, \tag{57}$$

namely,

$$\theta_n = \frac{\pi n}{2N}, \tag{58}$$

which is precisely the result of [2]. The simple poles in R_- occur therefore at

$$\omega_b = m\,\sin\left(\frac{\pi n}{2N}\right). \tag{59}$$

One can now repeat the same analysis as in the preceding subsection, of the partition function associated with fluctuations of fermions interacting with these static $\sigma(x)$ configurations. One finds again $\mathrm{O}(2N)$ supermultiplets, where the nth supermultiplet has mass $M_n = (2N_m/\pi)\sin(\pi n/2N)$ $(n < N)$, and contains all $\mathrm{O}(2N)$ completely antisymmetric tensors of rank $n_0 \le n$, where n_0 has the same parity of n [2].

IV. CONCLUSION

In this paper we have solved the extremum condition on the effective action of the two-dimensional Gross-Neveu model for *static* $\sigma(x)$ configurations in the large N limit. Our method of calculation was direct, making use of elementary properties of one-dimensional Schrödinger operators. The natural scale that appears in the Gross-Neveu model is that of its dynamically generated mass m. Because the latter is the natural scale of the CCGZ kink configurations as well, our derivation of these kink configurations was relatively simple and straightforward. It therefore may be considered as clean and constructive proof that the CCGZ kinks are indeed static extrema of the effective action. We have also rederived the DHN extremal $\sigma(x)$ configurations in a very simple manner. To this end, however, we had to introduce their scale into the saddle point condition by hand, since it does not appear explicitly in the effective action.

Our method may be applied also to a host of other two-dimensional field theories, and in particular, to field theories that *do not* involve reflectionless static configurations, where inverse scattering methods are useless.

Note added. After submitting this paper for publication I realized that Avan and de Vega [27] have already used the diagonal resolvent of a one-dimensional Schrödinger operator to discuss solitons in "large N" vector models. However, their discussion was limited to single-particle quantum mechanics [(0+1)-dimensional quantum field theory].

ACKNOWLEDGMENTS

I would like to thank the Theory Group at Fermilab, where part of this work has been completed, for their kind hospitality. I would also like to thank Joe Lykken for inviting me to Fermilab and for valuable discussions on topics related to this work. I am also indebted to the Theory Group at the University of Texas at Austin for partial support of this visit. J. F. was supported by the Rothchild Foundation and in part by the Robert A. Welch Foundation and NSF Grant No. PHY 9009850.

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[16] Here we have used the fact that $D_1 = i\partial\!\!\!/ - g\sigma$ and $D_2 = -(i\partial\!\!\!/ + g\sigma)$ are isospectral (up to zero modes) since $\gamma_5 D_1 = D_2\gamma_5$.

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[22] It is customary to choose the renormalization point at $\mu = m$ such that $Ng^2 = \pi$.

[23] Indeed, the original analysis in [2] was carried out using sophisticated inverse scattering techniques rather than a direct calculation of the resolvent of the static Dirac operator. See also the comments made by these authors in [3] (p. 4136, first paragraph starting below Fig. 2) prior to their analysis in [2].

[24] Note that we multiply through by a polynomial of degree 6 in ω. However, the coefficient of ω^6 in the resulting polynomial vanishes identically.

[25] This result is a special case of the recurrence relations discussed in [17].

[26] That is, $\mathcal{C}$ runs right below the left branch cut of R_- in the complex ω plane, continuing right below the real ω axis up to the rightmost pole at $\omega = \omega_b$ and then crossing the real ω axis and continuing to run right above the right branch cut of R_-. This is precisely the contour used in [3]. See Figs. 1–3 (and especially the last two figures) in that reference.

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