Renormalization group and universality

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It is argued that universality is severely limited for models with multiple fixed points. As a demonstration the renormalization group equations are presented for the potential and the wave function renormalization constants in the O(N) scalar field theory. Our equations are superior compared with the usual approach which retains only the contributions that are nonvanishing in the ultraviolet regime. We find an indication for the existence of relevant operators at the infrared fixed point, contrary to common expectations. This result makes the sufficiency of using only renormalizable coupling constants in parametrizing the long distance phenomena questionable.

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I. INTRODUCTION

The modification of fundamental laws of physics with the change of observational length scale is the subject of the renormalization group (RG) [1]. Through the RG flow equation one may probe the dependence of the effective coupling constants on the characteristic length. The otherwise complicated flow pattern becomes rather simple in the vicinity of the fixed points where the linearized RG flow along with scaling provide a recipe for classifying the coupling constants via their dependence on the characteristic scale. The irrelevant coupling constants are those which decrease as the scale is moved toward the infrared (IR) direction. The physical content of the theory is insensitive to the actual choice of these coupling constants in the IR end of the region where the linearization of the RG equation is applicable. Within this regime where the usual concept of universality is recovered, the physics is parametrized by the others only, namely, the relevant and marginal coupling constants.

In the realistic models we find several scaling regimes when the renormalized trajectory passes by different fixed points. There are fixed points for the theory of everything, the grand-unified theory (GUT), standard model, QCD and QED, to mention some of them. Although the true renormalized trajectory approaches all of them for certain values of the cutoff, it reaches the first one only. In fact, in the scaling regime of, say the fixed point of QCD, some of the interactions of the standard model generate nonrenormalizable vertices in terms of the quark and gluon fields [2]. These vertices deflect the renormalized trajectory from the fixed point as we move up in energy. [Hence it is physically not too crucial whether or not the ultraviolet (UV) fixed point really exists. All we shall assume here is the scaling up to a certain energy scale.] The traditional goal of local field theory is to give an account of these vertices in terms of elementary particle exchanges in a manner which is renormalizable at higher energy. Yet renormalization, i.e., the removal of the UV cutoff is necessary only for the theory of everything. In fact, when the scaling is investigated at the other fixed points then the higher-energy reactions always make these fixed points unstable in the UV direction.

The scenario sketched above leads to a serious limitation of the use of the concept of universality. It is true that there are "islands" of autonomous scaling regimes where physics can be parametrized by relevant or marginal operators only, but these operators usually vary with scaling regimes and the matching of relevant coupling constants is rather nontrivial. Even though we can establish the importance of certain coupling constants in a given energy range, the physics at a different scale will be governed by different set of coupling constants [3].

There is one last fixed point as we move towards longer distance scale, the infrared (IR) fixed point. Macroscopic physics is characterized by the scaling at this IR fixed point. Can this fixed point have relevant operators? The answer is negative for theories with a mass gap. To see this it is sufficient to recall that the dependence of the coupling constants on the cutoff is to take into account the effects of the modes which are eliminated as the cutoff energy is lowered. At energy well below the mass gap the fluctuations are suppressed and the evolution of the coupling constants slows down. Thus the IR limit of the theory is stable.

The situation is more interesting for theories without mass gap. Realistic theories with spontaneous symmetry breaking belong to this class. For such theories, IR divergences can pile up and generate relevant coupling constant in the IR regime. It is the main result of this paper that this indeed happens in simple four-dimensional scalar models. In this case the long-distance physics of the model is not universal; i.e., it cannot be parametrized completely by the relevant and marginal coupling con-

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stants of the UV fixed point.

In arriving at this result one needs an improved version of the RG equations. The usual method of renormalizing the theory is to follow the mixing and the evolution of selectively few coupling constants. One traditionally chooses those coupling constants which are relevant or marginal in the vicinity of the UV fixed point. As we lower the cutoff to the natural mass scale m_R of the theory, the scaling properties change fundamentally. In the IR side of the natural mass scale where $k < m_R$, one deduces the scaling laws corresponding to the IR fixed point. However, the UV and the IR scaling regimes are separated by a crossover at m_R, and there is no reason whatsoever to expect the same set of relevant or marginal coupling operators for both UV and IR fixed points. Furthermore we do not have a simple power counting argument to find out the relevant operators of the IR fixed point. Thus in order to establish the scaling operators of the IR fixed point, we have to trace down the evolution and the mixing of many more operators which might well be irrelevant in the UV scaling regime. This can be achieved with a RG flow equation which is capable of handling the mixing between infinitely many coupling constants. Such an improved RG equation has been obtained in [4] and was subsequently applied for the UV scaling regime in [5] in the leading-order approximation of the derivative expansion for the renormalized action. We present in this paper the RG equation [4] applied in the next order of the derivative expansion. This allows us to verify our claim about the existence of the relevant operators at the IR fixed point in the first two orders of the derivative expansion.

The organization of the paper is the following. In Sec. II, we give a brief derivation of the RG equation [4] in the leading order of the derivative expansion for $U_k(\Phi)$ in the one-component scalar model and show the emergence of the IR singularities in certain β functions. Section III contains the technical details for deriving $U_k(\Phi)$ for the O(N) $\lambda\phi^4$ theory with two distinct wave-function renormalization constants $\tilde{\mathcal{Z}}_{k,\ell}$ and $\tilde{\mathcal{Z}}_{k,t}$ for the longitudinal and the transverse components, respectively. In Sec. IV, we derive a set of three coupled nonlinear RG flow equations for $U_k(\Phi)$, $\tilde{\mathcal{Z}}_{k,\ell}$ and $\tilde{\mathcal{Z}}_{k,t}$. Asymptotic scaling in both UV and IR limits are discussed in Sec. V. Section VI contains our conclusions. Two appendices are included supplementing the detail of deriving the RG equations for the paper.

II. ONE-COMPONENT SCALAR FIELD THEORY

Our starting point for deriving the RG equation for a system characterized by the field $\phi(x)$ is to introduce the coarse-grained "block variable:"

$$\phi_k(x) = \int_y \rho_k(x-y)\phi(y) \ , \tag{2.1}$$

where

$$\int_x = \int d^dx = \Omega \ , \tag{2.2}$$

via a smearing function $\rho_k(x)$, with k^{-1} being the characteristic linear dimension of the region over which the field averaging is performed. In this paper, we shall choose $\rho_k(x)$ to be

$$\rho_k(x) = \int_{p<k} e^{ipx} \ , \tag{2.3}$$

or $\rho_k(p) = \Theta(k-p)$, i.e., a sharp cutoff [6]. Although $\rho_k(x)$ acts as an upper cutoff, we shall use Λ as the k-independent UV cutoff for the theory.

Given a set of blocked variables $\phi_k(x)$, the blocked action $\tilde{S}_k$ can be deduced from

$$e^{-\tilde{S}_k(\Phi)} = \int D[\phi]\prod_x \delta(\phi_k(x)-\Phi(x))e^{-S(\phi)} \ , \tag{2.4}$$

where the field average Φ of a given block is chosen to coincide with the slowly varying background. By performing the functional integration in loop expansion subject to the δ-function constraints one finds

$$\begin{aligned}\tilde{S}_k(\Phi) &= S(\Phi) + \frac{1}{2}\mathrm{Tr}'\ln\frac{\partial^2 S}{\partial\phi(p)\partial\phi(-p)} \\ &\quad -\frac{1}{2}\int_p' \frac{\partial S}{\partial\phi(p)}\left(\frac{\partial^2 S}{\partial\phi(p)\partial\phi(-p)}\right)^{-1}\frac{\partial S}{\partial\phi(-p)} \\ &= S(\Phi) + \frac{1}{2}\mathrm{Tr}'\ln K - \frac{1}{2}\int_p' FK^{-1}F \ ,\end{aligned} \tag{2.5}$$

$$\int_p' = \int_k^\Lambda \frac{d^dp}{(2\pi)^d} \ , \tag{2.6}$$

where F and K are the first and the second functional derivative of the bare Lagrangian, respectively, and Tr' denotes the trace sum over internal space as well as the restricted momentum space with $k < p < \Lambda$. How blocking transformation modifies the propagator $\Delta(x-y) = K^{-1}(x-y)$ can be seen explicitly be considering a free scalar theory:

$$\int_p' F(p)\Delta(p)F(-p) \to \int_p F(p)\tilde{\Delta}(p)F(-p) \ , \tag{2.7}$$

where

$$\tilde{\Delta}(x-y) = \int_p \frac{e^{ipx}}{p^2+\mu^2}\Theta(p-k) \tag{2.8}$$

is the "blocked" propagator with an effective IR cutoff scale k. In the limit $k \to 0$, one recovers the original $\Delta(x-y)$.

Equation (2.6) is far too complicated so the derivative expansion [7] is used at this point. The form

$$\tilde{S}_k(\Phi) = \sum_{n=0}^{\infty}\int d^dx L_k^{(n)}[\Phi(x)] \tag{2.9}$$

is assumed where $L_k^{(n)}[\Phi(x)]$ is a homogeneous polynomial of order $2n$ in the space-time derivatives. We shall truncate the expansion at $n = 1$, retaining only the wave-function renormalization function $Z_k(\Phi)$ and the blocked potential $U_k(\Phi)$. Such truncation, being justifiable in the IR limit for high enough space-time dimension d, yields simpler differential equations when substituted into (2.6). In principle, however, equations that generate the scale dependence of $Z_k(\Phi)$ and the higher-order derivative terms must also be calculated in the framework of the derivative expansion in order to have a closed system.

It is worth mentioning that (2.6) gives the one-loop effective potential for $k = 0$ [6]. The modes with nonvanishing wave number are eliminated independently in the one-loop approximation. However, the result can be greatly improved by the successive elimination of the degrees of freedom. In the improved scheme, the contribution of a particular mode which has been integrated out is kept for the elimination of the next mode, thereby taking into account the interactions between the modes. By decreasing the cutoff infinitesimally from $k \to k - \Delta k$. we generate from (2.6) the evolution of the potential $L_k^0(\Phi) = U_k(\Phi)$, which for $\phi(x) = \phi$ and $d = 4$, becomes

$$U_{k-\Delta k}(\Phi) = U_k(\Phi) + \Delta_k \frac{k^3}{16\pi^2} \ln\left[\frac{Z_k(\Phi)k^2 + \partial_\Phi^2 U_k(\Phi)}{Z_k(0)k^2 + \partial_\Phi^2 U_k(0)}\right], \tag{2.10}$$

or equivalently, a differential equation of the form

$$k\partial_k U_k(\Phi) = -\frac{k^4}{16\pi^2} \ln\left[\frac{Z_k(\Phi)k^2 + \partial_\Phi^2 U_k(\Phi)}{Z_k(0)k^2 + \partial_\Phi^2 Uk(0)}\right] \tag{2.11}$$

in the limit $\Delta k \to 0$. This equation describes the renormalization of the potential with arbitrary dependence on the field Φ. The solution for $U_{k=0}(\Phi)$ differs from the usual one-loop effective potential mentioned before insofar that the effects of the operators which are irrelevant at the UV fixed point are retained during the elimination of the degrees of freedom. This difference is negligible for a weakly coupled theory as long as no new relevant operators are generated outside the UV scaling regime.

Equation (2.11) can be derived by resumming the one-loop contributions to evolution of the potential. In fact, it can be written as

$$\begin{aligned} k\partial_k U_k(\Phi) &= -\frac{k^4}{16\pi^2}\left\{\ln\left[\frac{Z_k(\Phi)k^2 + m_k^2}{Z_k(0)k^2 + m_k^2}\right] + \ln\{1 + [Z_k(\Phi)k^2 + m_k^2]^{-1}\partial_\Phi^2 V_k(\Phi)\}\right\} \\ &= -\frac{k^4}{16\pi^2}\left\{\ln\left[\frac{Z_k(\Phi)k^2 + m_k^2}{Z_k(0)k^2 + m_k^2}\right] + \frac{\partial_\Phi^2 V_k(\Phi)}{Z_k(\Phi)k^2 + m^2} - \frac{1}{2}\left(\frac{\partial_\Phi^2 V_k(\Phi)}{Z_k(\Phi)k^2 + m_k^2}\right)^2 + \cdots\right\}, \end{aligned} \tag{2.12}$$

where $\partial_\Phi^2 V_k(\Phi) = \partial_\Phi^2 U_k(\Phi) - m_k^2$ and $m_k^2 = \partial_\Phi^2 U_k(0)$. The last line contains the sum of the one-loop graphs with increasing number of $\partial_\Phi^2 V_k(\Phi)$ insertions. The external legs of these graphs which are attached to Φ in $\partial_\Phi^2 V_k(\Phi)$ are carrying zero momentum and the modes with momentum k are propagating along the loop. This is just the set of graphs one has to sum up in eliminating the modes with momentum k.

There are certainly higher loop corrections to (2.12). However, the terms of the order m in loops contains m integrations over a d-dimensional shell in the momentum space. When only few modes are eliminated, $\Delta k \approx 0$, the integration over each shell yields, on the dimensional ground, a new small parameter:

$$\zeta = \frac{\Delta k}{k}, \tag{2.13}$$

which helps suppress the higher loop contributions to the RG evolution equation in (2.11). This is the basis of the "exactness" of for a RG equation which is formally obtained in the one-loop approximation.

As the IR limit is approached with $k \to 0$, ζ becomes ill-defined. To examine the behavior of ζ in this regime, we introduce an IR cutoff by considering the system in a box of size L. With the number of degrees of freedom being $N^d = (L\Lambda/2\pi)^d$, the momentum integral measure takes on the form

$$\frac{1}{N^d}\sum_{k_\mu} \to \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d}, \tag{2.14}$$

in term of the dimensionless momentum, p_μ. In the "ultimate RG transformation," where only a single mode is eliminated each time, the small parameter would be $1/N^d$, the inverse of the number of degrees of freedom remained. Therefore, the IR limit can be reached in two different manners: (i) The limit $L \to \infty$ is taken first before $k \to 0$; and (ii) $k \to 2\pi/L$ is first taken for finite systems followed by $L \to \infty$. We immediately notice that ζ can be kept small only for case (i) and becomes of order 1 for case (ii). In another words, the limits $k \to 0$ and $L \to \infty$ do not necessarily commute. In fact, the gap of the two-dimensional σ model, whose existence is established with reasonable accuracy using procedure (ii) is absent if (i) is employed instead [8].

It seems that if the two limits $k \to 0$ and $L \to \infty$ are not commuting, procedure (ii) would be more reasonable to describe the dynamics of local interactions. In that case the RG equation, (2.11), can only represent a partial resummation of the perturbation expansion. Such a loss of the effectivity is due to the presence of the length scales, L and k^{-1}, in a system without an IR mass gap. Since in the subsequent treatments we implement the one-loop RG equation for the IR regime, the conclusions drawn from such computation are strictly relevant only for case (i). It remains to be seen if they can be carried over to the procedure (ii).

Adopting (i) as our approach, we find indication of the emergence of relevant operators at the IR fixed point for massless theories due to the following handwaving argument: For the sake of simplicity we keep the wave-function renormalization constant, $Z_k(\Phi)$, to be unity in (2.11) and introduce the β functions for the coupling constants for Φ^n as

$$\beta_n(k) = \partial_\Phi^n k\partial_k U_k(\Phi) , \tag{2.15}$$

which can be obtained by substituting (2.11) into (2.15). These β functions describe the evolution of the coupling strengths of small fluctuations around the constant background $\Phi(x) = \Phi$. Let us now consider a model where

$$\partial_\Phi^2 U_{k=0}(0) = 0 . \tag{2.16}$$

This is what one commonly calls a "massless" theory since $U_{k=0}(\Phi)$ is just the effective potential. However, this name is misleading when spontaneous symmetry breaking occurs with $\langle\phi(x)\rangle \neq 0$ since $\partial_\Phi^2 U_{k=0}(\langle\phi(x)\rangle)$ is now nonvanishing. The leading IR contribution for (2.15) comes from the highest power of $k^2 + \partial_\Phi^2 U_k(0)$ in the denominator:

$$\beta_n(k) = (-1)^n \frac{k^4}{16\pi^2}\left(\frac{\partial_\Phi^3 U_k(0)}{k^2 + \partial_\Phi^2 U_k(0)}\right)^n [1 + O(k^2)] , \tag{2.17}$$

which shows that for $\partial_\Phi^3 U_k(0) > 0$ the coupling constants for the odd powers of the field blow up in the IR limit. Although we have not found the scaling operators there ought to be relevant ones which drive the IR divergences.

Such conclusion could have been reached by considering the graphs which contribute to the evolution equation:

$$\beta_n(k) = -\frac{k^4}{16\pi^2}\partial_\Phi^n\left\{\frac{\partial_\Phi^2 V_k(\Phi)}{Z_k(\Phi)k^2 + m^2} - \frac{1}{2}\left(\frac{\partial_\Phi^2 V_k(\Phi)}{Z_k(\Phi)k^2 + m_k^2}\right)^2 + \cdots\right\} . \tag{2.18}$$

In the UV scaling regime where $k^2 \gg \partial_\Phi^2 U_k(\Phi)$, the dominant contribution comes from the graphs with least number of propagators. The evolution of the vertex with n legs is described by joining two legs of the $(n+2)$th order vertex having momentum k. One reproduces the usual β functions, e.g.,

$$\beta_2(k) = -\frac{\partial_\Phi^4 U_k(\Phi)}{16\pi^2}\frac{k^4}{k^2 + m_k^2} , \tag{2.19}$$

for the mass squared, and in the usual one-loop approximation to the ϕ^4 model where there is no sixth-order vertex at the cutoff,

$$\beta_4(k) = \frac{3[\partial_\Phi^4 U_k(\Phi)]^2}{16\pi^2}\frac{k^4}{(k^2 + m_k^2)^2} , \tag{2.20}$$

in the next-to-leading-order approximation in (2.18).

As we enter the IR regime with $k^2 \ll \partial_\Phi^2 U_k(\Phi)$, the scaling laws quickly change. For a theory with mass gap the contributions are $O(k^4/m^4)$ and the evolution slows down indicating the absence of relevant operators. But for massless theories, $\lim_{k\to 0} m_k^2 = 0$, the dominant contributions are received from graphs with the maximal number of propagators between the vertices. In the absence of other dimensional parameter, we find $\partial_\Phi^2 V_k(0) \sim k^2$ and the IR contribution to the evolution of the nth order vertex is dominated by the one-loop graph with n insertion of the vertex $\partial_\Phi^3 U_k(\Phi)$ as shown in (2.17).

Unfortunately this result is not interesting. The theory develops a nonvanishing vacuum expectation value for the field either due to the masslessness or the presence of the odd powers of the field in the potential. Thus, the coupling constants computed at $\Phi = 0$ are not characterizing the strength of the interactions of small fluctuations in the vacuum. The true vacuum with $\langle\Phi(x)\rangle \neq 0$ shields the IR divergences. However, if the theory possesses a continuous symmetry which is broken spontaneously then Goldstone's theorem guarantees the presence of the massless modes in the vacuum. The more careful repetition of this simple argument for the N-component scalar field theory is the subject of this work.

III. DERIVATIVE EXPANSION FOR THE $O(N)$ MODEL

We consider a generalized bare $O(N)$ scalar field Lagrangian of the form

$$\mathcal{L}(\phi) = \tfrac{1}{2}\tilde{Z}_a(\partial_\mu\phi^a)^2 + V(\phi) , \tag{3.1}$$

with $a = 1,\ldots,N$. The theory is chosen to be in the symmetry broken phase and the direction $a = 1$ is chosen to be in the expectation value of the field. The extra subscript in $\tilde{Z}$ is used differentiate between two wave-function renormalization constants, one for the longitudinal component and the other for the transverse ones:

$$\tilde{Z}_a = \begin{cases} \tilde{Z}_\ell, & a = 1, \\ \tilde{Z}_t, & a = 2,\ldots,N. \end{cases} \tag{3.2}$$

Alternatively, one may write (3.1) as

$$\begin{aligned}\mathcal{L}(\phi) &= \tfrac{1}{2}\tilde{Z}_\ell(\partial_\mu\phi^1)^2 + \tfrac{1}{2}\tilde{Z}_t(\partial_\mu\phi^i)^2 + V(\phi) \\ &= -\tfrac{1}{2}Z_a\phi^a\partial^2\phi^a + V(\phi)\end{aligned} \tag{3.3}$$

via the relation

$$\begin{aligned}\tilde{Z}_\ell &= \frac{d}{d\phi^1}(Z_\ell\phi^1) , \\ \tilde{Z}_t &= \frac{d}{d\phi^i}(Z_t\phi^i), \quad i = 2,\ldots,N .\end{aligned} \tag{3.4}$$

We shall split $\phi(x)$ into the slowly varying background $\chi(x)$, and the fast-fluctuating modes $\xi(x)$ such that

$$\phi^a(p) = \begin{cases} \chi^a(p), & 0 \le p \le k, \\ \xi^a(p), & k < p < \Lambda. \end{cases} \tag{3.5}$$

Noting that $\phi_k(p) = \rho_k(p)\phi(p) = \chi(p)$, one then integrates out the fast-fluctuating modes $\xi(x)$ by using the loop expansion to obtain the blocked action $\tilde{S}_k(\Phi)$ as a function of the blocked field average $\Phi = (\Phi^2)^{1/2}$:

$$\begin{aligned}\tilde{S}_k(\Phi) &= -\ln\int D[\chi]D[\xi]\prod_x \delta(\phi_k(x)-\Phi(x))\exp\{-S(\chi+\xi)\}\\
&= -\ln\int D[\chi]\prod_p \delta(\chi(p)-\Phi(p))\int D[\xi]\exp\left\{-S(\chi)-\int_p \xi^a(p)F^a(-p)-\frac{1}{2}\int_p \xi^a(p)K^{ab}(p,-p)\xi^b(-p)+\cdots\right\}\\
&= -\ln\int D[\chi]\prod_p \delta(\chi(p)-\Phi(p))\exp\left\{-S(\chi)-\frac{1}{2}\mathrm{Tr}'\ln K+\frac{1}{2}\int_p' FK^{-1}F+\cdots\right\}\\
&= S(\Phi)+\frac{1}{2}\mathrm{Tr}'\ln K(\Phi)-\frac{1}{2}\int_p' FK^{-1}F\Big|_\Phi\\
&= S(\Phi)+\delta\tilde{S}_k^1(\Phi)+\delta\tilde{S}_k^2(\Phi)\ ,\end{aligned} \tag{3.6}$$

where

$$F^a(\Phi)=\frac{\partial S}{\partial\phi^a(x)}\Big|_\Phi = -\tfrac{1}{2}Z_c^{(a)}(\Phi)\Phi^c\partial^2\Phi^c-\tfrac{1}{2}Z_a(\Phi)(\partial^2\Phi^a+\Phi^a\partial^2)+V^{(a)}(\Phi)\ , \tag{3.7}$$

$$\begin{aligned}K^{ab}(\Phi) &= \frac{\partial^2 S}{\partial\phi^a(x)\partial\phi^b(y)}\Big|_\Phi\\
&= \{-\tfrac{1}{2}Z_c^{(ab)}(\Phi)\Phi^c\partial^2\Phi^c - Z_a(\Phi)\delta^{ab}\partial^2+V^{(ab)}(\Phi)\\
&\quad -\tfrac{1}{2}Z_b^{(a)}(\Phi)(\partial^2\Phi^b+\Phi^b\partial^2)-\tfrac{1}{2}Z_a^{(b)}(\Phi)(\partial^2\Phi^a+\Phi^a\partial^2)\}\delta^4(x-y)\ ,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}Z^{(a_1a_2\cdots a_n)}(\Phi) &= \frac{\partial^n Z}{\partial\phi^{a_1}\partial\phi^{a_2}\cdots\partial\phi^{a_n}}\Big|_\Phi\ ,\\
V^{(a_1a_2\cdots a_n)}(\Phi) &= \frac{\partial^n V}{\partial\phi^{a_1}\partial\phi^{a_2}\cdots\partial\phi^{a_n}}\Big|_\Phi\ .\end{aligned} \tag{3.9}$$

Note that Tr' denotes the trace sum over the internal symmetry space as well as the restricted space-time.

As noted in the Introduction the blocked action can be expanded as

$$\tilde{S}_k(\Phi)=\int_x\left(-\frac{\mathcal{Z}_{a,k}(\Phi)}{2}\Phi^a\partial^2\Phi^a+U_k(\Phi)+O(\partial^4)\right)\ . \tag{3.10}$$

For the computation of $\mathcal{Z}_k(\Phi)$ and $U_k(\Phi)$ it is best to choose a nonconstant, slowly varying blocked field which is written as

$$\Phi^a(x)=\Phi_0^a+\tilde{\phi}^a(x)\ , \tag{3.11}$$

with $\Phi_0^a=\Phi_0\delta^{a,1}$. Simple comparison of (3.10) and (3.6) gives

$$U_k(\Phi_0)=V(\Phi_0)+\frac{1}{2\Omega}\mathrm{Tr}'\ln K^{ab}(\Phi_0) \tag{3.12}$$

and

$$\tilde{\mathcal{Z}}_{a,k}(\Phi_0)=\tilde{Z}_a(\Phi_0)\ . \tag{3.13}$$

From now on, quantities with no written arguments are understood to be evaluated at Φ_0.

In order to obtain higher-order correction for the wave-function renormalization constant, we incorporate the effect of $\tilde{\phi}$ up to quadratic order by writing

$$\begin{aligned}K^{ab} &= (K_0^{ab}+\delta K_0^{ab}+\delta K_1^{ab}+\delta K_2^{ab})\delta^4(x-y)+O(\tilde{\phi}^3,\partial^3)\ ,\\
F^a &= F_0^a+\delta F_1^a+\delta F_2^a+O(\tilde{\phi}^3,\partial^3)\ ,\end{aligned} \tag{3.14}$$

where

$$
\begin{aligned}
K_0^{ab} &= -(Z_a\delta^{ab} + Z_a^{(a)}\Phi_0\delta^{a,1}\delta^{b,1})\partial^2 + V^{(ab)} ,\\
\delta K_0^{ab} &= \tfrac{1}{2}\Phi_0(2Z_a^{(a)}\delta^{a,1}\delta^{b,1} - Z_b^{(a)}\delta^{b,1} - Z_a^{(b)}\delta^{a,1})\partial^2 ,\\
\delta K_1^{ab} &= -Z_a^{(c)}\tilde\phi^c\delta^{ab}\partial^2 - \tfrac{1}{2}(Z_b^{(a)}\partial^2\tilde\phi^b + Z_a^{(b)}\partial^2\tilde\phi^a) - \tfrac{1}{2}Z_\ell^{(ab)}\Phi_0\partial^2\tilde\phi^1\\
&\quad -\tfrac{1}{2}\{Z_b^{(a)}\tilde\phi^b + Z_a^{(b)}\tilde\phi^a + \Phi_0(\delta^{b,1}Z_b^{(ac)} + \delta^{a,1}Z_a^{(bc)})\tilde\phi^c\}\partial^2 + V^{(abc)}\tilde\phi^c ,\\
\delta K_2^{ab} &= \tfrac{1}{2}[Z_b^{(ac)}\tilde\phi^c(\partial^2\tilde\phi^b + \tilde\phi^b\partial^2) + Z_a^{(bc)}\tilde\phi^c(\partial^2\tilde\phi^a + \tilde\phi^a\partial^2) + Z_c^{(ab)}\tilde\phi^c\partial^2\tilde\phi^c + Z_a^{(cd)}\tilde\phi^c\tilde\phi^d\delta^{ab}\partial^2] + \tfrac{1}{2}V^{(abcd)}\tilde\phi^c\tilde\phi^d ,
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
F_0^a &= -\tfrac{1}{2}Z_a\Phi_0\delta^{a,1}\partial^2 + V^{(a)} ,\\
\delta F_1^a &= \tfrac{1}{2}[Z_b^{(a)}\Phi_0\delta^{b,1}\partial^2\tilde\phi^b + Z_a(\partial^2\tilde\phi^a + \tilde\phi^a\partial^2) + Z_a^{(b)}\Phi_0\delta^{a,1}\tilde\phi^b\partial^2] + V^{(ab)}\tilde\phi^b ,\\
\delta F_2^a &= -\tfrac{1}{2}[Z_b^{(a)}\tilde\phi^b\partial^2\tilde\phi^b + Z_c^{(ab)}\Phi_0\delta^{c,1}\tilde\phi^b\partial^2\tilde\phi^c + Z_a^{(b)}\tilde\phi^b(\partial^2\tilde\phi^a + \tilde\phi^a\partial^2) + Z_a^{(bc)}\Phi_0\delta^{a,1}\tilde\phi^b\tilde\phi^c\partial^2] + \tfrac{1}{2}V^{(abc)}\tilde\phi^b\tilde\phi^c .
\end{aligned}
\tag{3.16}
$$

Note that summation over repeated indices ($a,b,c,d = 1,\ldots,N$; $i,j,k, = 2,\ldots,N$) is implied, and the numerical subscripts correspond to the order of $\tilde\phi$. The effective action can now be written as

$$
\tilde S_k(\Phi) = S(\Phi_0 + \tilde\phi) + \delta\tilde S_k^1(\Phi_0 + \tilde\phi) + \delta\tilde S_k^2(\Phi_0 + \tilde\phi) , \tag{3.17}
$$

where

$$
S(\Phi_0 + \tilde\phi) = \int_x \left(-\frac{Z_a}{2}\tilde\phi^a\partial^2\tilde\phi^a + V + V^{(a)}\tilde\phi^a + \frac{1}{2}\tilde\phi^a V^{(ab)}\tilde\phi^b + \cdots \right) , \tag{3.18}
$$

$$
\begin{aligned}
\delta\tilde S_k^1 &= \tfrac{1}{2}\mathrm{Tr}'\ln(K_0 + \delta K_0 + \delta K_1 + \delta K_2)\\
&= \tfrac{1}{2}\mathrm{Tr}'\ln(K_0 + \delta K_0) - \tfrac{1}{4}\mathrm{Tr}'(K_0^{-1}\delta K_1 K_0^{-1}\delta K_1)\\
&\quad + \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n\mathrm{Tr}'[(K_0^{-1}\delta K_0)^n K_0^{-1}\delta K_1] + \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n\mathrm{Tr}'[(K_0^{-1}\delta K_0)^n K_0^{-1}\delta K_2] + \cdots ,
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
-\delta\tilde S_k^2 &= \frac{1}{2}\int_p' (F_0 + \delta F_1 + \delta F_2)(K_0 + \delta K_0 + \delta K_1 + \delta K_2)^{-1}(F_0 + \delta F_1 + \delta F_2) + \cdots\\
&= \frac{1}{2}\int_p' \Big\{ F_0K_0^{-1}(1 + K_0^{-1}\delta K_0)^{-1}F_0 + F_0K_0^{-1}(1 + K_0^{-1}\delta K_0)^{-1}\delta F_1 + \delta F_1K_0^{-1}(1 + K_0 - 1\delta K_0)^{-1}F_0\\
&\quad - F_0K_0^{-2}\delta K_1(1 + K_0^{-1}\delta K_0)^{-2}F_0 + F_0K_0^{-1}(1 + K_0^{-1}\delta K_0)^{-1}\delta F_2 + \delta F_2K_0^{-1}(1 + K_0^{-1}\delta K_0)^{-1}F_0\\
&\quad - F_0K_0^{-2}\delta K_1(1 + K_0^{-1}\delta K_0)^{-1}\delta F_1 - \delta F_1K_0^{-2}\delta K_1(1 + K_1^{-1}\delta K_0)^{-1}F_0 + F_0K_0^{-1}[(K_0^{-1}\delta K_1)^2(1 + K_0^{-1}\delta K_0)^{-3}\\
&\quad - (K_0^{-1}K_2)(1 + K_0^{-1}\delta K_0)^{-2}]F_0 + \delta F_1K_0^{-1}(1 + K_0^{-1}\delta K_0)^{-1}\delta F_1 \Big\} + \cdots .
\end{aligned}
\tag{3.20}
$$

To illustrate the above formalism, we consider the $O(N)$ scalar $\lambda\phi^4$ theory defined by

$$
V(\phi) = \frac{\mu^2}{2}\phi^2(x) + \frac{\lambda}{4!}[\phi^2(x)]^2 , \tag{3.21}
$$

where

$$
\phi^2 = \phi^a\phi^a . \tag{3.22}
$$

With

$$
V^{(ab)} = \left(\mu^2 + \frac{\lambda}{6}\Phi_0^2\right)\delta^{ab} + \frac{\lambda}{3}\Phi_0^2\delta^{a,1}\delta^{b,1} , \tag{3.23}
$$

the matrix K_0^{ab} in the momentum space representation has the eigenvalues

$$
\mathcal{K}^a = \begin{cases} \Delta_\ell^{-1} = \tilde Z_\ell p^2 + u_\ell, & a = 1,\\ \Delta_t^{-1} = \tilde Z_t p^2 + u_t, & a = 2,\ldots,N, \end{cases} \tag{3.24}
$$

$$
u_\ell = \mu^2 + \frac{\lambda}{2}\Phi_0^2 ,\qquad u_t = \mu^2 + \frac{\lambda}{6}\Phi_0^2 , \tag{3.25}
$$

which yields

$$
\mathrm{Tr}'\ln K_0^{ab} = \Omega\int_p' \{\ln\Delta_\ell^{-1} + (N-1)\ln\Delta_t^{-1}\} . \tag{3.26}
$$

Note that (3.4) implies

$$\begin{aligned} \tilde{Z}_\ell &= Z_\ell + Z_\ell^{(1)}\Phi_0 \,, \\ \tilde{Z}_\ell^{(1)} &= 2Z_\ell^{(1)} + Z_\ell^{(11)}\Phi_0 \,, \\ \tilde{Z}_\ell^{(11)} &= 3Z_\ell^{(11)} + \cdots \,, \\ \tilde{Z}_\ell^{(i)} &= Z_\ell^{(i)} + Z_\ell^{(1i)}\Phi_0 \,, \\ \tilde{Z}_\ell^{(ij)} &= Z_\ell^{(ij)} + \cdots \,, \end{aligned} \tag{3.27}$$

$$\begin{aligned} \tilde{Z}_t &= Z_t \,, \\ \tilde{Z}_t^{(1)} &= Z_t^{(1)} \,, \\ \tilde{Z}_t^{(i)} &= 2Z_t^{(i)} \,. \end{aligned} \tag{3.28}$$

In evaluating (3.19) and (3.20), we act on the x-dependent $\tilde{\phi}$ in the trace by the derivative operators contained in F and K. The trace is performed in the plane-wave basis by commuting the momentum operator $p_\mu = i\partial_\mu$ to the right end of the expressions. The commutation relations utilized for this procedure are tabulated in Appendix A. Note that whenever an operator p_μ acts on $\tilde{\phi}$ it yields $i\partial_\mu\tilde{\phi}$ which contributes in the IR because its Fourier decomposition is vanishing above the scale k. When the operator p_μ reaches the right end of the expression it gets replaced by the trace integration variable, p_μ. Although this momentum value is in the ultraviolet, the contribution represents a simple number which multiplies the $\tilde{\phi}$ dependence in the infrared. Therefore, a matrix element of the form $\mathrm{Tr}'\mathcal{F}(\tilde{\phi})p_\mu\mathcal{G}(\tilde{\phi})$ can be separated into $\mathrm{Tr}'\mathcal{F}(\tilde{\phi})i\partial_\mu[\mathcal{G}(\tilde{\phi})]$ and $\mathrm{Tr}'\mathcal{F}(\tilde{\phi})\mathcal{G}(\tilde{\phi})p_\mu$ using the commutation techniques. Iterating this algorithm, the contributions to $\mathcal{Z}$ which are of the form $\mathrm{Tr}'\mathcal{F}(\tilde{\phi})\partial^2\mathcal{G}(\tilde{\phi})$ can be isolated. Moreover, since the blocked action is real we can actually commute the derivative operators to the left end instead within the trace, as was done in [7] and adopted here in this paper.

One arrives after lengthy algebra at

$$\begin{aligned} \delta K_1^{ab} = p^2\{&[Z_a^{(c)}\delta^{ab} + \tfrac{1}{2}\Phi_0(Z_b^{(ac)}\delta^{b,1} + Z_a^{(bc)}\delta^{a,1})]\tilde{\phi}^c + \tfrac{1}{2}Z_b^{(a)}\tilde{\phi}^b + \tfrac{1}{2}Z_a^{(b)}\tilde{\phi}^a\} \\ &-[Z_a^{(c)}\delta^{ab} + \tfrac{1}{2}\Phi_0(Z_b^{(ac)}\delta^{b,1} + Z_a^{(bc)}\delta^{a,1})]\partial^2\tilde{\phi}^c - [Z_b^{(a)}\partial^2\tilde{\phi}^b + Z_a^{(b)}\partial^2\tilde{\phi}^a] \\ &-\tfrac{1}{2}Z_\ell^{(ab)}\Phi_0\partial^2\tilde{\phi}^1 - 2i[Z_a^{(c)}\delta^{ab} + \tfrac{1}{2}\Phi_0(Z_b^{(ac)}\delta^{b,1} + Z_a^{(bc)}\delta^{a,1})]p_\mu\partial_\mu\tilde{\phi}^c \\ &+V^{(abc)}\tilde{\phi}^c - ip_\mu[Z_b^{(a)}]\partial_\mu\tilde{\phi}^b + Z_a^{(b)}\partial_\mu\tilde{\phi}^a] + \cdots \,, \end{aligned} \tag{3.29}$$

$$\begin{aligned} \delta K_2^{ab} = \tfrac{1}{2}p^2\{&Z_b^{(ac)}\tilde{\phi}^b + Z_a^{(bc)}\tilde{\phi}^a + Z_a^{(cd)}\delta^{ab}\tilde{\phi}^d\}\tilde{\phi}^c + \frac{\lambda}{6}(\delta^{ab}\tilde{\phi}^c\tilde{\phi}^c + 2\tilde{\phi}^a\tilde{\phi}^b) \\ &-\tfrac{1}{2}[Z_b^{(ac)}\tilde{\phi}^c\partial^2\tilde{\phi}^b + Z_a^{(bc)}\tilde{\phi}^c\partial^2\tilde{\phi}^a + Z_c^{(ab)}\tilde{\phi}^c\partial^2\tilde{\phi}^c] + \cdots \,, \end{aligned} \tag{3.30}$$

$$\begin{aligned} \delta F_1^a = \tfrac{1}{2}p^2&[Z_a\tilde{\phi}^a + Z_\ell^{(b)}\Phi_0\tilde{\phi}^b\delta^{a,1}] + \delta^{ab}u_t\tilde{\phi}^b + \frac{\lambda}{3}\Phi_0^2\tilde{\phi}^1\delta^{a,1} - Z_a\partial^2\tilde{\phi}^a \\ &-\tfrac{1}{2}(Z_b^{(a)}\delta^{b,1} + Z_a^{(b)}\delta^{a,1})\Phi_0\partial^2\tilde{\phi}^b - ip_\mu[Z_a\partial_\mu\tilde{\phi}^a + Z_\ell^{(b)}\Phi_0\delta^{a,1}\partial_\mu\tilde{\phi}^b] + \cdots \,, \end{aligned} \tag{3.31}$$

and

$$\delta F_2^a = \tfrac{1}{2}p^2[Z_a^{(b)}\tilde{\phi}^a + Z_\ell^{(bc)}\Phi_0\tilde{\phi}^c\delta^{a,1}]\tilde{\phi}^b - \tfrac{1}{2}[Z_a^{(b)}\tilde{\phi}^a + Z_b^{(a)}\tilde{\phi}^b + Z_\ell^{(ab)}\Phi_0\tilde{\phi}^1]\partial^2\tilde{\phi}^b + \tfrac{1}{2}V^{(abc)}\tilde{\phi}^b\tilde{\phi}^c + \cdots \,. \tag{3.32}$$

As for the calculation of $\delta\tilde{S}_k^2$, since no x integration occurs in (3.20), it is only the Fourier transform of $\tilde{\phi}$. If $\tilde{\phi}(p)$ is constrained such that $p < k$, then the contribution of $\delta\tilde{S}_k^2$ to the blocked action vanishes since the p integration is performed over the range $k > p$.

In order to identify the contribution to the wave-function renormalization constants and the blocked potential we now rewrite the blocked action $\tilde{S}_k(\Phi)$ as

$$\tilde{S}_k(\Phi_0 + \tilde{\phi}) = \int_x [-\tfrac{1}{2}\mathcal{Z}_k^{ab}\tilde{\phi}^a\partial^2\tilde{\phi}^b + U_k + (U_k^{(a)}, \tilde{\phi}^a) + \tfrac{1}{2}(\tilde{\phi}^a, U_k^{(ab)}\tilde{\phi}^b) + \cdots] \,, \tag{3.33}$$

from which one obtains the $\tilde{\phi}$-independent blocked potential:

$$U_k = V + \tfrac{1}{2}\mathrm{Tr}'\ln(K_0 + \delta K_0) = V + \tfrac{1}{2}\int_p' [\ln\Delta_\ell^{-1} + (N-1)\ln\Delta_t^{-1}] \,. \tag{3.34}$$

For the wave-function renormalization constants, we first collect terms proportional to $\tilde{\phi}^1\partial^2\tilde{\phi}^1$ for commuting $\mathcal{Z}_{k,\ell}$ in the longitudinal direction. Use of (3.27) and (3.28) yields (see Appendix B)

$$\tilde{\mathcal{Z}}_{k,\ell} = \tilde{Z}_\ell + \frac{1}{2}\int_p' \{a_{11} + a'_{11} + \tilde{a}_{11} + a^*_{11} - 2b_{11}\}$$
$$= \tilde{Z}_\ell + \tfrac{1}{2}\int_p' \Bigg\{ \Delta_\ell^2[\tilde{Z}_\ell u_\ell \Delta_\ell^2 \lambda^2 \Phi_0^2 - Z_\ell^{(1)}\lambda\Phi_0(1 - u_\ell\Delta_\ell + 2u_\ell^2\Delta_\ell^2) - (Z_\ell^{(1)})^2 p^2(1 + u_\ell^2\Delta_\ell^2)]$$
$$+(N-1)u_t\Delta_t^3\left[\Delta_t\left(\tilde{Z}_t\frac{\lambda^2}{9}\Phi_0^2 - (Z_t^{(1)})^2p^2u_t\right) + \frac{\lambda}{3}\Phi_0 Z_t^{(1)}(1 - 2u_t\Delta_t)\right]$$
$$-Z_\ell^{(ii)}\Phi_0\Delta_\ell^2\left(Z_t^{(1)}p^2 + \frac{\lambda}{3}\Phi_0\right) - \tfrac{1}{4}(\tilde{Z}_\ell^{(i)})^2p^2\Delta_\ell\Delta_t(4 + u_\ell^2\Delta_\ell^2 + u_t^2\Delta_t^2)$$
$$+\frac{\theta}{4}Z_\ell^{(i)}\Phi_0\Delta_\ell\Delta_t p^2[Z_\ell^{(j)}Z_\ell^{(ji)}\Phi_0\Delta_t p^2 - 8Z_\ell^{(i1)}] + 3Z_\ell^{(11)}\theta\Delta_\ell + Z_\ell^{(ii)}\Delta_t\Bigg\}\,. \tag{3.35}$$

One may replace the Z factors above by the corresponding $\tilde{Z}$'s since the difference involves contributions having more than two orders of derivative in the Z's. Finally, to obtain $\tilde{\mathcal{Z}}_{k,\ell}(\Phi)$, we simply replace Φ_0 in $\tilde{\mathcal{Z}}_{k,\ell}(\Phi_0)$ by Φ, with implied "normal ordering" such that all p dependences be moved to the front of the Φ-dependent expressions. As noted in [7], there is no ambiguity in this procedure provided that we carefully compare the terms in the expansions of (3.10) and (3.17).

In a similar manner, we can write down the Φ-dependent wave-function renormalization constant in the transverse ij direction as

$$\tilde{\mathcal{Z}}_{k,t(ij)} = \tilde{Z}_t + \frac{1}{2}\int_p' \{a_{ij} + a'_{ij} + \tilde{a}_{ij} + a^*_{ij} - 2b_{ij}\}$$
$$= \tilde{Z}_t + \frac{1}{2}\int_p' \Bigg\{ -\tilde{Z}_\ell^{(i)}\tilde{Z}_\ell^{(j)}p^2\left[u_\ell^2\Delta_\ell^4 + \Delta_t^2\left(3 + \frac{2N+3}{2}u_t^2\Delta_t^2\right)\right]$$
$$-(\tilde{Z}_t^{(k)})^2\delta^{ij}p^2\Delta_t^2(2 + u_t^2\Delta_t^2) + \tfrac{1}{6}\tilde{Z}_\ell^{(ij)}\Phi_0\Delta_\ell\Delta_t\{-3\tilde{Z}_t^{(1)}p^2(2 + u_\ell^2\Delta_\ell^2 + u_t^2\Delta_t^2)$$
$$+\lambda\Phi[u_\ell\Delta_\ell(1 - 2u_\ell\Delta_\ell) + u_t\Delta_t(1 - 2u_t\Delta_t)]\}$$
$$+\frac{\lambda}{3}\Phi\delta^{ij}\Delta_\ell\Delta_t\left(\frac{\lambda}{3}\Phi(\tilde{Z}_\ell u_\ell\Delta_\ell^2 + \tilde{Z}_t u_t\Delta_t^2) - \frac{\tilde{Z}_t^{(1)}}{2}(4 - u_\ell\Delta_\ell - u_t\Delta_t + 2u_\ell^2\Delta_\ell^2 + 2u_t^2\Delta_t^2\right)$$
$$-\frac{p^2}{4}\Delta_\ell\Delta_t[(\tilde{Z}_t^{(1)})^2\delta^{ij}(4 + u_\ell^2\Delta_\ell^2 + u_t^2\Delta_t^2) + \tilde{Z}_\ell^{(ik)}\tilde{Z}_\ell^{(kj)}\Phi^2(u_\ell^2\Delta_\ell^2 + u_t^2\Delta_t^2)]$$
$$+\delta^{ij}\left(\theta\Delta_\ell\tilde{Z}_\ell^{(11)} + \tilde{Z}_t^{(kk)}\Delta_t + \frac{\theta}{4}\tilde{Z}_\ell^{(k)}\Phi p^2\Delta_\ell\Delta_t(\tilde{Z}_\ell^{(\ell)}\tilde{Z}_t^{(\ell k)}\Phi p^2\Delta_t - 4\tilde{Z}_t^{(k1)})\right)$$
$$+2\Delta_t\left(\tilde{Z}_t^{(ij)} + \frac{\theta}{2}\tilde{Z}_\ell^{(i)}\Phi p^2\Delta_\ell(\tfrac{1}{2}\tilde{Z}_\ell^{(k)}\tilde{Z}_t^{(kj)}\Phi p^2\Delta_t - \tilde{Z}_t^{(j1)})\right)\Bigg\}\,. \tag{3.36}$$

Consider the limiting case in which the derivative terms of the $\tilde{Z}$'s can be neglected, the above expressions can be reduced to

$$U_k = V + \frac{1}{2}\int_p' \left\{ \ln\left(1 + \frac{u_\ell}{p^2}\right) + (N-1)\ln\left(1 + \frac{u_t}{p^2}\right)\right\}, \tag{3.37}$$

$$\tilde{\mathcal{Z}}_{k,\ell} = \tilde{Z}_\ell + \frac{\lambda^2\Phi^2}{2}\int_p' \left\{\tilde{Z}_\ell u_\ell\Delta_\ell^4 + \frac{N-1}{9}\tilde{Z}_t u_t\Delta_t^4\right\}$$
$$= \tilde{Z}_\ell + \frac{\lambda^2\Phi^2}{192\pi^2}\left[\frac{(3\tilde{Z}_\ell k^2 + u_\ell)u_\ell}{\tilde{Z}_\ell(\tilde{Z}_\ell k^2 + u_\ell)^3} + \frac{N-1}{9}\frac{(3\tilde{Z}_t k^2 + u_t)u_t}{\tilde{Z}_t(\tilde{Z}_t k^2 + u_t)^3}\right], \tag{3.38}$$

and

$$\tilde{\mathcal{Z}}_{k,t} = \tilde{Z}_t\,, \tag{3.39}$$

in agreement with that obtained in [7] for the one-component case.

IV. RENORMALIZATION-GROUP FLOW EQUATIONS

Equation (3.34) gives the contribution of the modes between $k < p < \Lambda$ to the blocked action in the one-loop independent mode approximation since the systematic feedbacks from the high modes to the low ones are neglected. In order to improve upon such approximation, we first consider the case when the cutoff is changed infinitesimally from $k \to k - \Delta k$, leading to

$$k\partial_k U_k = -\frac{k^4}{16\pi^2}\left\{\ln\left(1+\frac{\lambda\Phi^2/2}{\tilde{Z}_\ell k^2+\mu^2}\right) +(N-1)\ln\left(1+\frac{\lambda\Phi^2/6}{\tilde{Z}_t k^2+\mu^2}\right)\right\}, \tag{4.1}$$

which is a linear partial differential equation. The equation is not yet suitable for the systematical repetition of the elimination of the modes since the right-hand side of (4.1) is derived by using the specific potential, (3.21). Since elimination of modes changes the specific structure of the Lagrangian, it is better to start the whole computation with a general potential. Upon replacing the Φ-dependent terms on the right-hand side of (4.1) by

$$U_k^{(11)}(\Phi) = \frac{\partial^2 U_k(\Phi)}{\partial\Phi_1^2}, \quad U_k^{(22)}(\Phi) = \frac{\partial^2 U_k(\Phi)}{\partial\Phi_2^2}, \tag{4.2}$$

one obtains a new RG equation:

$$k\partial_k U_k(\Phi) = -\frac{k^4}{16\pi^2}\left\{\ln\left(\frac{\tilde{Z}_{k,\ell}(\Phi)k^2+U_k^{(11)}(\Phi)}{\tilde{Z}_{k,\ell}(0)k^2+U_k^{(11)}(0)}\right) +(N-1)\ln\left(\frac{\tilde{Z}_{k,t}(\Phi)k^2+U_k^{(22)}(\Phi)}{\tilde{Z}_{k,t}(0)k^2+U_k^{(22)}(0)}\right)\right\}, \tag{4.3}$$

which accumulates the effects of the eliminated modes in a systematic way as we lower the cutoff. Contrary to (4.1), Eq. (4.3) is now a nonlinear partial differential flow equation. In the same manner, we can write down the corresponding RG flow equations for the $\tilde{Z}$'s:

$$\begin{aligned}k\partial_k\tilde{Z}_{k,\ell} = -\frac{k^4}{16\pi^2}\Bigg(&\Delta_{k,\ell}^2\{\tilde{Z}_{k,\ell}U_k^{(11)}\Delta_{k,\ell}^2(U_k^{(111)})^2-\tilde{Z}_{k,\ell}^{(1)}U_k^{(111)}[1-U_k^{(11)}\Delta_{k,\ell}+2(U_k^{(11)})^2\Delta_{k,\ell}^2]\\&-(\tilde{Z}_{k,\ell}^{(1)})^2k^2[1+(U_k^{(11)})^2\Delta_{k,\ell}^2]\}+(N-1)U_k^{(22)}\Delta_{k,t}^3\{\tilde{Z}_{k,t}^{(1)}U_k^{(221)}(1-2U_k^{(22)}\Delta_{k,t})\\&+\Delta_{k,t}[\tilde{Z}_{k,t}(U_k^{(221)})^2-(\tilde{Z}_{k,t}^{(1)})^2U_k^{(22)}k^2]\}-\tilde{Z}_{k,\ell}^{(ii)}\Phi\Delta_{k,t}^2(\tilde{Z}_{k,t}^{(1)}k^2+U_k^{(221)})\\&-\frac{k^2}{4}(\tilde{Z}_{k,\ell}^{(i)})^2\Delta_{k,\ell}\Delta_{k,t}[4+(U_k^{(11)})^2\Delta_{k,\ell}^2+(U_k^{(22)})^2\Delta_{k,t}^2]+\tilde{Z}_{k,\ell}^{(ii)}\Delta_{k,t}\\&+\frac{\theta_k}{4}\tilde{Z}_{k,\ell}^{(i)}\Phi\Delta_{k,\ell}\Delta_{k,t}k^2[\tilde{Z}_{k,\ell}^{(j)}\tilde{Z}_{k,\ell}^{(ji)}\Phi\Delta_{k,t}k^2-8\tilde{Z}_{k,\ell}^{(i1)}]+3\tilde{Z}_{k,\ell}^{(11)}\theta_k\Delta_{k,\ell}\Bigg),\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}k\partial_k\tilde{Z}_{k,t(ij)} = -\frac{k^4}{16\pi^2}\Bigg\{&-\tilde{Z}_{k,\ell}^{(i)}\tilde{Z}_{k,\ell}^{(j)}k^2\left[(U_k^{(11)})^2\Delta_{k,\ell}^4+\Delta_{k,t}^2\left(3+\frac{2N+3}{2}(U_k^{(22)})^2\Delta_{k,t}^2\right)\right]\\&-(\tilde{Z}_{k,t}^{(k)})^2\delta^{ij}k^2\Delta_{k,t}^2[2+(U_k^{(22)})^2\Delta_{k,t}^2]\\&+\tfrac{1}{6}\tilde{Z}_{k,\ell}^{(ij)}\Phi\Delta_{k,\ell}\Delta_{k,t}\{-3\tilde{Z}_{k,t}^{(1)}k^2[2+(U_k^{(11)})^2\Delta_{k,\ell}^2+(U_k^{(22)})^2\Delta_{k,t}^2]\\&+U_k^{(111)}[U_k^{(11)}\Delta_{k,\ell}(1-2U_k^{(11)}\Delta_{k,\ell})+U_k^{(22)}\Delta_{k,t}(1-2U_k^{(22)}\Delta_{k,t})]\}\\&+U_k^{(221)}\delta^{ij}\Delta_\ell\Delta_t\Bigg(U_k^{(221)}(\tilde{Z}_{k,\ell}U_k^{(11)}\Delta_{k,\ell}^2+\tilde{Z}_{k,t}U_k^{(22)}\Delta_{k,t}^2)\\&-\frac{\tilde{Z}_{k,t}^{(1)}}{2}[4-U_k^{(11)}\Delta_{k,\ell}-U_k^{(22)}\Delta_{k,t}+2(U_k^{(11)})^2\Delta_{k,\ell}^2+2(U_k^{(22)})^2\Delta_{k,t}^2]\Bigg)\\&-\frac{k^2}{4}\Delta_{k,\ell}\Delta_{k,t}\{(\tilde{Z}_{k,t}^{(1)})^2\delta^{ij}[4+(U_k^{(11)})^2\Delta_{k,\ell}^2+(U_k^{(22)})^2\Delta_{k,t}^2]\\&+\tilde{Z}_{k,\ell}^{(ik)}\tilde{Z}_{k,\ell}^{(kj)}\Phi^2[(U_k^{(11)})^2\Delta_{k,\ell}^2+(U_k^{(22)})^2\Delta_{k,t}^2]\}\\&+\delta^{ij}\left(\theta_k\Delta_{k,\ell}\tilde{Z}_{k,t}^{(11)}+\tilde{Z}_{k,t}^{(kk)}\Delta_{k,t}+\frac{\theta_k}{4}\tilde{Z}_{k,\ell}^{(k)}\Phi k^2\Delta_{k,\ell}\Delta_{k,t}(\tilde{Z}_{k,\ell}^{(\ell)}\tilde{Z}_{k,t}^{(\ell k)}\Phi k^2\Delta_{k,t}-4\tilde{Z}_{k,t}^{(k1)})\right)\\&+2\Delta_{k,t}\left(\tilde{Z}_{k,t}^{(i,j)}+\frac{\theta_k}{2}\tilde{Z}_{k,\ell}^{(i)}\Phi k^2\Delta_{k,\ell}(\tfrac{1}{2}\tilde{Z}_{k,\ell}^{(k)}\tilde{Z}_{k,t}^{(kj)}\Phi k^2\Delta_{k,t}-\tilde{Z}_{k,t}^{(j1)})\right)\Bigg\},\end{aligned} \tag{4.5}$$

where

$$\theta_k = [1-\tfrac{1}{4}(k^2)^2\Delta_{k,\ell}\Delta_{k,t}(\tilde{Z}_{k,\ell}^{(i)}\Phi)^2]^{-1}, \tag{4.6}$$

and

$$\Delta_{k,\ell}^{-1} = \tilde{Z}_{k,\ell}(\Phi)k^2 + U_k^{(11)}(\Phi) \ ,$$
$$\Delta_{k,t}^{-1} = \tilde{Z}_{k,t}(\Phi)k^2 + U_k^{(22)}(\Phi) \ . \tag{4.7}$$

So far the only approximation we made was to make truncation in the derivative expansion up to $O(\partial^2)$. The complicated flow equations can further be simplified if one also truncates in the amplitude of the fluctuations which is another expansion parameter of the Landau-Ginsburg method. By taking a constant background Φ_0 along the longitudinal direction and replacing Φ_0 by the general inhomogeneous Φ, we have

$$\begin{aligned} U_k^{(11)} &= 2(U_k' + 2U_k''\Phi^2) \ , \\ U_k^{(22)} &= 2U_k' \ , \\ U_k^{(111)} &= 4(3U_k'' + 2U_k'''\Phi^2)\mathbf{\Phi} \ , \\ U_k^{(221)} &= 4U_k'''\mathbf{\Phi} \ , \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \tilde{Z}_k^{(1)} &= 2\tilde{Z}_k'\mathbf{\Phi} \ , \\ \tilde{Z}_k^{(i)} &= 0 \ , \\ \tilde{Z}_k^{(11)} &= 4(\tilde{Z}_k''\Phi^2 + \tilde{Z}_k') \ , \\ \tilde{Z}_k^{(ij)} &= 2\tilde{Z}_k'\delta^{ij} \ , \end{aligned} \tag{4.9}$$

where $\mathbf{\Phi}$ points in the 1 direction and the prime denotes differentiation with respect to Φ^2. The three coupled nonlinear differential RG flow equations now become (see Appendix B)

$$k\partial_k U_k(\Phi) = -\frac{k^4}{16\pi^2}\left\{\ln\left[\frac{\tilde{Z}_{k,\ell}(\Phi)k^2 + 2(U_k'(\Phi) + 2U_k''(\Phi)\Phi^2)}{\tilde{Z}_{k,\ell}(0)k^2 + 2U_k'(0)}\right] + (N-1)\ln\left[\frac{\tilde{Z}_{k,t}(\Phi)k^2 + 2U_k'(\Phi)}{\tilde{Z}_{k,t}(0)k^2 + 2U_k'(0)}\right]\right\} , \tag{4.10}$$

$$\begin{aligned} k\partial_k\tilde{Z}_{k,\ell} &= -\frac{k^4}{16\pi^2}(a_{k,\ell} + a'_{k,\ell} + \tilde{a}_{k,\ell} + a^*_{k,\ell} - 2b_{k,\ell}) \\ &= -\frac{k^4}{16\pi^2}[4\Delta_{k,\ell}^2\Phi^2\{-(\tilde{Z}'_{k,\ell})^2k^2[1 + 4\Delta_{k,\ell}^2(U_k' + 2U_k''\Phi^2)^2] \\ &\quad +8\tilde{Z}_{k,\ell}\Delta_{k,\ell}^2(U_k' + 2U_k''\Phi^2)(3U_k'' + 2U_k'''\Phi^2)^2 \\ &\quad -2\tilde{Z}'_{k,\ell}(3U_k'' + 2U_k'''\Phi^2)[1 - 2(U_k' + 2U_k''\Phi^2)\Delta_{k,\ell} + 8(U_k' + 2U_k''\Phi^2)^2\Delta_{k,\ell}^2]\} \\ &\quad +4(N-1)\Delta_{k,t}^2\Phi^2(4U_k'\Delta_{k,t}\{\Delta_{k,t}[2\tilde{Z}_{k,t}(U_k'')^2 - (\tilde{Z}'_{k,t})^2k^2U_k'] + \tilde{Z}'_{k,t}U_k''(1 - 4U_k'\Delta_{k,t})\} \\ &\quad -\tilde{Z}'_{k,\ell}(\tilde{Z}'_{k,t}k^2 + 2U_k'')) + 12(\tilde{Z}''_{k,\ell}\Phi^2 + \tilde{Z}'_{k,\ell})\Delta_{k,\ell} + 2(N-1)\tilde{Z}'_{k,\ell}\Delta_{k,t}] \ , \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} k\partial_k\tilde{Z}_{k,t} &= -\frac{k^4}{16\pi^2}(a_{k,t} + a'_{k,t} + \tilde{a}_{k,t} + a^*_{k,t} - 2b_{k,t}) \\ &= -\frac{k^4}{16\pi^2}[2\Delta_{k,\ell}\Delta_{k,t}\Phi^2(- \tilde{Z}'_{k,\ell}k^2[\tilde{Z}'_{k,t} + 2(U_k')^2\Delta_{k,t}^2\Delta_{k,t}(\tilde{Z}'_{k,\ell} + 2\tilde{Z}'_{k,t})] \\ &\quad +16\tilde{Z}_{k,t}U_k'(U_k'')^2\Delta_{k,t}^2 - (\tilde{Z}'_{k,t})^2k^2[1 + 2(U_k')^2\Delta_{k,t}^2] + 4\tilde{Z}'_{k,\ell}\Delta_{k,t}U_k'U_k''(1 - 4U_k'\Delta_{k,t}) \\ &\quad -4\tilde{Z}'_{k,t}U_k''[1 - U_k'\Delta_{k,t} + 4(U_k')^2\Delta_{k,t}^2] + 4U_k''\{4\tilde{Z}_{k,\ell}U_k''(U_k' + 2U_k''\Phi^2)\Delta_{k,\ell}^2 \\ &\quad -\tilde{Z}'_{k,t}[1 - (U_k' + 2U_k''\Phi^2)\Delta_{k,\ell} + 4(U_k' + 2U_k''\Phi^2)^2\Delta_{k,\ell}^2]\} \\ &\quad +4\tilde{Z}'_{k,\ell}U_k''\Delta_{k,\ell}(U_k' + 2U_k''\Phi^2)[1 - 4(U_k' + 2U_k''\Phi^2)\Delta_{k,\ell}] \\ &\quad -\tilde{Z}'_{k,\ell}k^2[\tilde{Z}'_{k,t} + 2(\tilde{Z}'_{k,\ell} + 2\tilde{Z}'_{k,t})(U_k' + 2U_k''\Phi^2)^2\Delta_{k,\ell}^2]) \\ &\quad +4(\tilde{Z}''_{k,t}\Phi^2 + \tilde{Z}'_{k,t})\Delta_{k,\ell} + 2(N+1)\tilde{Z}'_{k,t}\Delta_{k,t}] \ . \end{aligned} \tag{4.12}$$

In the limiting case where $N = 1$ and the derivative couplings are neglected, we have

$$k\partial_k\tilde{Z}_{k,\ell} = -\frac{2k^4}{\pi^2}\tilde{Z}_{k,\ell}\Phi^2\Delta_{k,\ell}^4(U_k' + 2U_k''\Phi^2)(3U_k'' + 2U_k'''\Phi^2)^2 \ . \tag{4.13}$$

V. ASYMPTOTIC SCALINGS

A. Ultraviolet regime

As can be seen from (4.8), the $O(N)$ scalar model possesses a natural mass scale:

$$m_R^2 = \frac{\partial^2}{\partial\Phi_1^2} U_{k=0}(\sigma) = 4\sigma^2 U_k'' \ , \tag{5.1}$$

where $\sigma = \langle\Phi\rangle$, the vacuum expectation value. This mass scale is what separates the UV from the IR regimes. For sufficiently deeply in the UV or the IR regime, and in the linearizable vicinity of the fixed point(s), one finds asymptotic scaling. The calculation presented above reproduces the expected perturbative results in the UV scaling regime. To make contact with the usual RG equation where only the relevant and the marginal operators associated with the UV fixed point are followed, we turn to the RG coefficient functions for the longitudinal modes:

$$\beta_n(k) = k\partial_k g_n \ , \tag{5.2}$$

where

$$g_n = \partial_{\Phi_1}^n U_k(\sigma) \ . \tag{5.3}$$

In order to find the true critical exponents we have to make the coupling constants dimensionless by the help of k. To this end we introduce

$$\tilde\beta_n(k) = k\partial_k \tilde g_n \ , \tag{5.4}$$

with

$$\tilde g_n = k^{n-4}\partial_{\Phi_1}^n U_k(\sigma) \ . \tag{5.5}$$

The naive power counting for determining the sign of the critical exponents is especially transparent in this case. In fact, (5.2) now becomes

$$\tilde\beta_n(k) = k^{n-4}(k\partial_k + n - 4)g_n \ , \tag{5.6}$$

where $k\partial_k g_n$ is treated as small in perturbation expansion. Thus, g_n is classified as being relevant or irrelevant for $n < 4$ or $n > 4$, respectively.

In the case of spontaneous symmetry breaking for which $\sigma \neq 0$, the evolution equations of the quadratic and quartic coupling constants take on the forms

$$\begin{aligned}\beta_2 = \gamma_m &= k\partial_k[4\sigma^2 U_k''(\sigma)] \\ &= -\frac{k^4}{4\pi^2}\{\Delta_{k,\ell}(3U_k'' + 12\sigma^2 U_k''' + 4\sigma^2 U_k'''') - 3\sigma^2\Delta_{k,\ell}^2(3U_k'' + 2\sigma^2 U_k''')^2 \\ &\quad +(N-1)[\Delta_{k,t}(U_k'' + 2\sigma^2 U_k''') - 4\sigma^2\Delta_{k,t}^2(U_k'')^2]\} \ ,\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\beta^4 &= k\partial_k[4(3U_k'' + 12\sigma^2 U_k''' + 4\sigma^4 U_k'''')] \\ &= -\frac{3k^4}{2\pi^2}(5\Delta_{k,\ell}U_k''' - 2\Delta_{k,\ell}^2[9(U_k'')^2 + 192\sigma^2 U_k'' U_k''' + 224\sigma^4(U_k''')^2] \\ &\quad +96\Delta_{k,\ell}^3\sigma^2(U_k'' + 4\sigma^2 U_k''')(3U_k'' + 2\sigma^2 U''')^2 + 64\Delta_{k,\ell}^4\sigma^4(3U_k'' + 2\sigma^2 U_k''')^4 \\ &\quad +(N-1)\{\Delta_{k,t}U_k''' - 2\Delta_{k,t}^2[(U_k'')^2 + 12\sigma^2 U_k'' U_k''' + 4\sigma^4(U_k''')^2] \\ &\quad +32\Delta_{k,t}^3\sigma^2(U_k'')^2(U_k'' + 2\sigma^2 U_k''') - 64\Delta_{k,t}^4\sigma^4(U_k'')^4\} + \cdots) \ ,\end{aligned} \tag{5.8}$$

where the ellipsis denotes $O(U_k^{(4)})$ terms, and the inverse propagators now become

$$\begin{aligned}\Delta_{k,\ell}^{-1} &= \tilde{\mathcal{Z}}_{k,\ell}k^2 + m_R^2 \ , \\ \Delta_{k,t}^{-1} &= \tilde{\mathcal{Z}}_{k,t}k^2 \ .\end{aligned} \tag{5.9}$$

In addition, there are also flow equations for terms which are odd in σ due to spontaneous breaking of symmetry. For example, we have

$$\begin{aligned}\beta_3 &= k\partial_k[4\sigma(3U_k'' + 2\sigma^2 U_k''')] \\ &= -\frac{\sigma k^4}{2\pi^2}\{15\Delta_{k,\ell}U_k''' - 18\Delta_{k,\ell}^2(3U_k'' + 2\sigma^2 U_k''')(U_k'' + 4\sigma^2 U_k''') + 16\Delta_{k,\ell}^3\sigma^2(3U_k'' + 2\sigma^2 U_k''')^2 \\ &\quad +(N-1)\Delta_{k,t}[3U_k''' - 6\Delta_{k,t}U_k''(U_k'' + 2\sigma^2 U_k''') + 16\Delta_{k,t}^2\sigma^2(U_k'')^3] + \cdots\} \ .\end{aligned} \tag{5.10}$$

The RG coefficient functions for the wave-function renormalization constants are given as

$$\begin{aligned}\gamma_{k,\ell} &= \tilde{\mathcal{Z}}_{k,\ell}^{-1}k\partial_k\tilde{\mathcal{Z}}_{k,\ell}|_\sigma \\ &= -\frac{\tilde{\mathcal{Z}}_{k,\ell}^{-1}k^4}{16\pi^2}(4\Delta_{k,\ell}^2\sigma^2\{-(\tilde{\mathcal{Z}}_{k,\ell}')^2k^2[1 - 16(U_k'')^2\Delta_{k,\ell}^2(\sigma^2)^2] \\ &\quad -2\tilde{\mathcal{Z}}_{k,\ell}'(3U_k'' + 2U_k'''\sigma^2)[1 - 4U_k''\Delta_{k,\ell}\sigma^2 + 32(U_k'')^2\Delta_{k,\ell}^2(\sigma^2)^2] \\ &\quad +16\tilde{\mathcal{Z}}_{k,\ell}U_k''\Delta_{k,\ell}^2\sigma^2(3U_k'' + 2U_k'''\sigma^2)^2\} - 4(N-1)\tilde{\mathcal{Z}}_{k,\ell}'(\tilde{\mathcal{Z}}_{k,t}'k^2 + 2U_k'')\Delta_{k,t}^2\sigma^2 \\ &\quad +12(\tilde{\mathcal{Z}}_{k,\ell}''\sigma^2 + \tilde{\mathcal{Z}}_{k,\ell}')\Delta_{k,\ell} + 2(N-1)\tilde{\mathcal{Z}}_{k,\ell}'\Delta_{k,t}) \ ,\end{aligned} \tag{5.11}$$

and

$$\gamma_{k,t} = \tilde{Z}_{k,t}^{-1} k\partial_k \tilde{Z}_{k,t}$$
$$= -\frac{\tilde{Z}_{k,t}^{-1}k^4}{16\pi^2}[2\Delta_{k,\ell}\Delta_{k,t}\sigma^2(-\tilde{Z}'_{k,t}k^2(2\tilde{Z}'_{k,\ell}+\tilde{Z}'_{k,t})$$
$$+8U''_k\{4\tilde{Z}_{k,\ell}(U''_k)^2\Delta^2_{k,\ell}\sigma^2 - \tilde{Z}'_{k,t}[1-U''_k\Delta_{k,\ell}\sigma^2+8(U''_k)^2\Delta^2_{k,\ell}(\sigma^2)^2]\}$$
$$+8\tilde{Z}'_{k,\ell}(U''_k)^2\Delta_{k,\ell}\sigma^2[1-8U''_k\Delta_{k,\ell}\sigma^2-(\tilde{Z}'_{k,\ell}+2\tilde{Z}'_{k,t})k^2\Delta_{k,\ell}\sigma^2])$$
$$+4(\tilde{Z}''_{k,t}\sigma^2+\tilde{Z}'_{k,t})\Delta_{k,\ell}+2(N+1)\tilde{Z}'_{k,t}\Delta_{k,t}] . \tag{5.12}$$

Using $U''_k = \lambda_R/12$ by neglecting $O(U'''_k)$, i.e., dropping the irrelevant coupling constants in the UV scaling regime, and setting $m_R^2 = \lambda_R\sigma^2/3$, the above expressions become

$$\beta_2 = -\frac{\lambda_R k^4}{16\pi^2}\left\{\left(\Delta_{k,\ell}+\frac{N-1}{3}\Delta_{k,t}\right)-\lambda_R\sigma^2\left(\Delta^2_{k,\ell}+\frac{N-1}{9}\Delta^2_{k,t}\right)\right\} , \tag{5.13}$$

$$\beta_3 = \frac{\sigma\lambda_R^2 k^4}{48\pi^2}\left\{3\Delta^2_{k,\ell}(3-2\lambda_R\sigma^2\Delta_{k,\ell})+(N-1)\Delta^2_{k,t}\left(1-\frac{2\lambda_R}{9}\sigma^2\Delta_{k,t}\right)\right\} , \tag{5.14}$$

$$\beta_4 = \frac{\lambda_R^2 k^4}{16\pi^2}\left\{3\Delta^2_{k,\ell}[1-4\lambda_R\sigma^2\Delta_{k,\ell}+2\lambda_R^2\sigma^4\Delta^2_{k,\ell}]+\frac{(N-1)}{27}\Delta^2_{k,t}[1-12\lambda_R\sigma^2\Delta_{k,t}+2\lambda_R^2\sigma^4\Delta^2_{k,t}]\right\} , \tag{5.15}$$

$$\gamma_{k,\ell} = \tilde{Z}_{k,\ell}^{-1}k\partial_k\tilde{Z}_{k,\ell}$$
$$= -\frac{\tilde{Z}_{k,\ell}^{-1}k^4}{16\pi^2}\left\{4\Delta^2_{k,\ell}\sigma^2\left[-(\tilde{Z}'_{k,\ell})^2k^2\left(1+\frac{\lambda^2}{9}\Delta^2_{k,\ell}(\sigma^2)^2\right)+\frac{\lambda^3}{12}\tilde{Z}_{k,\ell}\Delta^2_{k,\ell}\sigma^2\right.\right.$$
$$\left.-\frac{\lambda}{2}\tilde{Z}'_{k,\ell}\left(1-\frac{\lambda}{3}\Delta_{k,\ell}\sigma^2+\frac{2\lambda^2}{9}\Delta^2_{k,\ell}(\sigma^2)^2\right)\right]-4(N-1)\tilde{Z}'_{k,\ell}\left(\tilde{Z}'_{k,t}k^2+\frac{\lambda}{6}\right)\Delta^2_{k,t}\sigma^2$$
$$\left.+12(\tilde{Z}''_{k,\ell}\sigma^2+\tilde{Z}'_{k,\ell})\Delta_{k,\ell}+2(N-1)\tilde{Z}'_{k,\ell}\Delta_{k,t}\right\} , \tag{5.16}$$

$$\gamma_{k,t} = \tilde{Z}_{k,t}^{-1}k\partial_k\tilde{Z}_{k,t}$$
$$= -\frac{\tilde{Z}_{k,t}^{-1}k^4}{16\pi^2}\left(2\Delta_{k,\ell}\Delta_{k,t}\sigma^2\left\{-\tilde{Z}'_{k,t}k^2(2\tilde{Z}'_{k,\ell}+\tilde{Z}'_{k,t})\right.\right.$$
$$+\frac{2\lambda}{3}\left[\frac{\lambda^2}{36}\tilde{Z}_{k,\ell}\Delta^2_{k,\ell}\sigma^2-\tilde{Z}'_{k,t}\left(1-\frac{\lambda}{12}\Delta_{k,\ell}\sigma^2+\frac{\lambda^2}{18}\Delta^2_{k,\ell}(\sigma^2)^2\right)\right]$$
$$\left.+\frac{\lambda^2}{18}\tilde{Z}'_{k,\ell}\Delta_{k,\ell}\sigma^2\left(1-\frac{2\lambda}{3}\Delta_{k,\ell}\sigma^2-(\tilde{Z}'_{k,\ell}+2\tilde{Z}'_{k,t})k^2\Delta_{k,\ell}\sigma^2\right)\right\}$$
$$\left.+4(\tilde{Z}''_{k,t}\sigma^2+\tilde{Z}'_{k,t})\Delta_{k,\ell}+2(N+1)\tilde{Z}'_{k,t}\Delta_{k,t}\right) . \tag{5.17}$$

The leading-order part of the above coefficient functions for $k^2/m_R^2 \to \infty$ and $\tilde{Z}_{k,\ell}, \tilde{Z}_{k,t} = 1 + O(\lambda_R)$,

$$\beta_2 = -\frac{N+2}{48\pi^2}\lambda_R k^2 ,$$

$$\beta_3 = -\frac{\sigma\lambda_R^2}{48\pi^2}(N+8) ,$$

$$\beta_4 = -\frac{\lambda_R^2}{16\pi^2}\left(\frac{N+8}{27}\right) ,$$

$$\gamma_{k,\ell} = -\frac{\lambda_R^3}{48\pi^2}\left(\frac{\sigma^2}{k^2}\right)^2 \to 0 ,$$

$$\gamma_{k,t} = -\frac{\lambda_R^3}{48\pi^2}\left(\frac{\sigma^2}{3k^2}\right)^2 \to 0 . \tag{5.18}$$

The last two lines are in agreement with the well-known fact that there is no need for wave-function renormalization at the one-loop order. In fact, their coefficient functions are vanishing in the ultraviolet. When symmetry breaking appears, $\tilde{\mathcal{Z}}_{k,\ell}$ and $\tilde{\mathcal{Z}}_{k,t}$ evolve differently from each other as we leave the UV regime.

If one wishes to compare the above results with the usual ϵ expansion in critical $\lambda\phi^4$ theory in $4-\epsilon$ dimension, it is sufficient to consider a symmetrical theory in which one sets $U_k'(0)=0$ and defines a dimensionless coupling constant $\tilde{\lambda}_k = 12k^{-\epsilon}U_k''(0) = 12k^{-\epsilon}\lambda_k$. From (5.8), one is led to

$$\beta_4 = k\partial_k\tilde{\lambda}_k = -\epsilon\tilde{\lambda}_k + \frac{3\tilde{\lambda}_k^2}{16\pi^2}\left[\frac{1}{\tilde{\mathcal{Z}}_{k,\ell}} + \frac{N-1}{9}\frac{1}{\tilde{\mathcal{Z}}_{k,t}}\right] - \frac{3U_k'''(0)k^2}{16\pi^2}\left[\frac{5}{\tilde{\mathcal{Z}}_{k,\ell}} + \frac{N-1}{\tilde{\mathcal{Z}}_{k,t}}\right], \quad (5.19)$$

which for $\tilde{\mathcal{Z}}_{k,\ell} = \tilde{\mathcal{Z}}_{k,t} = 1$ reproduces the leading-order results obtained by standard perturbation method.

B. Infrared regime

The scaling is more involved in the IR regime where the expansion $k^2 \ll m_R^2$ should be applied [6]. The strong nonlinearities of the RG equation prevented us from constructing the corresponding scaling operators. Instead we only argue that relevant operators must exist in the IR scaling regime.

Let us first begin with the naive argument outlined in the Introduction where $\tilde{\mathcal{Z}}$ is set to be unity. In fact, due to Goldstone's theorem which asserts $U_{k=0}^{(22)}(\sigma) = U_{k=0}'(\sigma) = 0$, we take

$$U_k^{(22)}(\sigma) = O(k^\rho) \quad (5.20)$$

with $\rho = 2$. In this limit with $U_K^{(22)}(\sigma) = ck^2$, the leading-order contribution to the β function becomes

$$\tilde{\beta}_n(k) = \frac{(-1)^n k^n}{16\pi^2}\left(\frac{U_k^{(122)}(\sigma)}{k^2 + U_k^{(22)}(\sigma)}\right)^n [1+O(k^2)] + (n-4)\tilde{g}_n \approx \frac{(-1)^n}{16\pi^2}\left(\frac{U_k^{(122)}(\sigma)}{(1+c)k}\right)^n [1+O(k^2)]\ , \quad (5.21)$$

where

$$U_k^{(122)}(\sigma) = \frac{\partial^3 U_k(\sigma)}{\partial\Phi_2^2\partial\Phi_1} = 4\sigma U_k'' = \frac{\lambda_R}{3}\sigma + \cdots\ . \quad (5.22)$$

Since $U_k^{(122)}(\sigma)$ is finite for noncritical system inside the symmetry broken phase the β functions develop power-like IR divergences when $U_k^{(122)}(\sigma) > 0$.

This simple argument relies on the leading order of the derivative expansion. Having gone through the computation of the wave-function renormalization constants we can verify that our conclusion remains valid in the next order of the derivative expansion, too. In particular, we show that the IR power singularities of the β functions persist for the RG equation (4.10)–(4.12). In the presence of a nontrivial wave-function renormalization constant the β function reads as

$$\tilde{\beta}_n(k) = \frac{(-1)^n k^n}{16\pi^2}\left(\frac{\tilde{\mathcal{Z}}_{k,t}^{(1)}(\sigma)k^2 + U_k^{(122)}(\sigma)}{\tilde{\mathcal{Z}}_{k,t}(\sigma)k^2 + U_k^{(22)}(\sigma)}\right)^n [1+O(k^2)] + (n-4)\tilde{g}_n \approx \frac{(-1)^n}{16\pi^2}\left(\frac{U_k^{(122)}(\sigma)}{k^{1-\nu}}\right)^n [1+O(k^2)]\ , \quad (5.23)$$

where the asymptotic behavior

$$\tilde{\mathcal{Z}}_{k,t} = O(k^{-\nu}) \quad (5.24)$$

$\nu \geq 0$ was assumed.

In order to show that $\nu < 1$, which would imply the persistence of IR singularity in (5.23), we verify the consistency of (4.11) and (4.12) by assuming (5.20), (5.24), and

$$\tilde{\mathcal{Z}}_{k,\ell} = O(k^{-\mu})\ . \quad (5.25)$$

A similar power-law dependence on k also applies to the $\tilde{\mathcal{Z}}_k'$'s and the $\tilde{\mathcal{Z}}_k''$'s since differentiation with respect to Φ^2 does not affect the power counting in k. While $\Delta_{k,\ell}$ remains finite as $k\to 0$ due to the presence of mass gap, the transverse propagator is scaled as

$$\Delta_{k,t} = O(k^{\nu-2})\ . \quad (5.26)$$

Substituting the above scalings in (4.11) and (4.12) for matching the leading singularities gives

$$-\mu = \min\{6-2\mu, 4-\mu, -4+3\nu+\rho, 2-\mu+\nu, -2+2\nu+\rho, \ldots\}\ , \quad (5.27)$$

and

$$-\nu = \min\{4-\mu, 4-\nu, -2\mu+3\nu+2\rho, -2+2\nu+\rho, -\mu+2\nu+\rho, 2, -2-\mu+3\nu+2\rho, \nu+\rho, 2+\rho, 2-\mu+\nu, 4-2\mu+\nu+\cdots\}\ . \quad (5.28)$$

Relying on the method of independent-mode approximation in [6], we have $\rho = 2$ which implies $\mu = 2$ and $\nu = 0$. Thus, we conclude that presence of IR singularity in the β function (5.23) remains unchanged after taking into consideration the wave-function renormalization constants. Note that the longitudinal wave-function renormalization constant $\mathcal{Z}_\ell$ is found to be quadratically divergent in the IR limit. This is in contradiction with the usual assumption [9]

$$0 < \tilde{\mathcal{Z}} < 1\ , \quad (5.29)$$

made in Minkowski space-time. Thus the invariance of $\tilde{\mathcal{Z}}$ under the Wick rotation, the one-loop RG equation, i.e., $\zeta \approx 0$ and (5.29) are inconsistent. It is not clear how to describe the vacuum but a deviation from the weakly coupled perturbative scenario is expected due to the Goldstone modes.

We emphasize again that the difference between our β functions (5.7), (5.8), and the usual ones in (5.18) stem from the terms $O(\sigma^2/k^2)$ which are neglected at the UV fixed point. These are just the pieces which make the UV and the IR scaling laws different.

In the course of investigating the IR scaling behavior, our computations are free of IR divergences. This is because for a finite value of the adjustable cutoff, k, the elimination of the degrees of freedom in (3.6) contains the integration for modes with finite momentum $k < p < \Lambda$ and the usual singularities of the massless theories do not show up. They appear only when $k \to 0$, the limit where singularity appears in the β functions. The determination of IR scaling operators is rather involved since it requires a complete resummation of the singularities which emerge in that limit.

VI. SUMMARY

Most of the applications of the RG are related to the investigation of the impact of local field operators which are introduced into the theory in the UV regime. The concept of universality which is being supported by the linearized RG equations can be used as long as the phenomena we are interested lie within the linearizability of the UV fixed point. Such phenomena can therefore be described by a simple Hamiltonian containing few relevant operators. This certainly is the case for the critical, i.e., massless ϕ^4 model near four dimensions where both the UV and the IR fixed points are Gaussian. The classification of the scaling operators is valid for all length scales in this model.

The situation is radically different for massive theories. There we have two energy scales, the UV cutoff and the mass gap. There is no reason to expect the same scaling laws at both sides of the mass gap. In fact, the renormalized trajectory runs towards large values of the mass squared as we approach the IR regime. As the vacuum expectation value of the field reaches $O(1/\epsilon)$, where $\epsilon = 4 - d$ then the three-point vertex becomes of order 1 in the symmetry broken phase and the expansion around the Gaussian fixed point is not applicable.

This is the generic situation for high-energy physics where the renormalized trajectory passes by the vicinities of different UV fixed points and we find different scaling laws as different energy ranges. The usual concept of universality is not applicable here since the relevant coupling constants of an UV fixed point parametrize only the physics of a given energy range. The trajectory is driven away from the region of the linearizability by the relevant or the irrelevant coupling constants as we move towards lower or higher energies, respectively. For the sake of definiteness we considered the $O(N)$ model in this paper which has a single finite scale and exhibits only two fixed points, an UV and an IR.

It is usually claimed that there is only one (completely trivial) relevant coupling constant at the IR fixed point, the mass. But this claim ignores the IR divergences of the massless theories which may generate relevant operators as the observational scale approaches the IR regime. Another class of models where the IR scaling might be rather nontrivial is where a symmetry is broken spontaneously. This case is interesting because it emphasizes the importance of

$$\zeta = \frac{2\pi}{Lk} = \frac{\kappa}{L} , \tag{6.1}$$

the ratio of the observational length scale, $\kappa = 2\pi k^{-1}$, and the IR cutoff, L. For L close to the characteristic mass scale of the theory there is no symmetry breaking and the evolution equation for the effective coupling constants reflects the symmetrical dynamics. For large but finite L the symmetry is still preserved but the spontaneous symmetry-breaking scheme becomes a good approximation for the dynamics. Spontaneous breakdown of symmetry can take place only asymptotically in the limit $L \to \infty$. Since κ can never exceed L for a finite system, we have $\zeta < 1$. Therefore, pattern of symmetry breaking can be uncovered in the evolution equation only for $\zeta \sim 0$. In fact, for $\zeta \sim 1$, one detects the symmetry-restoring long-range slow fluctuations, and the observables at this energy scale truly reflect the symmetrical dynamics. Thus the characteristic size of the system $\sim L$ should be much larger than the observational length scale κ in order to recover the usual picture of symmetry breaking.

There is another rather technical reason for staying in the region of small ζ. The higher loop contributions to the RG equations are suppressed by the inverse of the number of the modes in the blocked system which is $O(\zeta^d)$. Thus the studies of systems undergoing spontaneous symmetry breaking using the one-loop RG equation requires the removal of the IR cutoff by sending $L \to \infty$ before $\kappa \to \infty$, thereby making $\zeta = 0$.

The locality is lost at the IR fixed point which may lead to the breakdown of the derivative expansion and global scaling operators. All we known is that classical physics is recovered at the IR fixed point of massless theories. In this case the soft particle emission allows the spread of the energy from the microscopical to macroscopical length scales. On the other hand, when the theory possesses a mass gap, the energy cannot be distributed to arbitrary long distances and the IR physics is still controlled by coherent quantum effects, e.g., superconductivity.

We found IR singularities in the one-loop β function for the odd vertices of the $O(N)$ model when the first two orders in the derivative expansion of the renormalized Lagrangian were retained. This supports the notion of strong-coupling IR physics of the Goldstone modes. The amplification of the effective coupling strengths can be understood by recalling that the "restoring force" for the fluctuations, i.e., the eigenvalue of the small fluctuation operator is vanishing in the IR limit of a massless theory. Consequently large fluctuations are always present in the IR regime and invalidate the expansion methods.

The limitation of the concept of universality due to the existence of several fixed points can be nicely demonstrated in the $O(N)$ model. Consider the coupling constant of an odd vertex in the spontaneously broken phase. Being irrelevant in the vicinity of the UV fixed point, it decreases as we move in the IR direction. Its value should be small when we reach the crossover region between the

UV and IR scaling, at the mass gap. Universality, i.e., the insensitivity on the irrelevant initial conditions of the renormalized trajectory seems to be holding down to this energy scale. But as we continue our journey in the IR direction our coupling constant starts to grow. Although we could ignore it on the UV side of the mass gap, it plays an important role on the IR end. Furthermore, its actual value may depend strongly on the ultraviolet initial value of the renormalized trajectory. The suppression which produced the universal behavior down to the mass gap turns out to be an amplification in the IR scaling regime and the UV value of this coupling constant influences the long-distance features of the model. This possibility raises questions on the sufficiently of renormalized field theories in describing low energy phenomena.

We believe that only few nonrenormalizable operators become important in this manner. To demonstrate this point consider the standard model. Despite all complications in the IR regime the experiments performed in the vicinity of the crossover, ~ 100 GeV, can be parametrized by the help of the renormalizable coupling constants. Where then is the room for the possible violation of universality? The conjectured strong-coupling physics in the IR plays an important part in forming the vacuum but remains virtually invisible at higher energy. The only important parameters they provide are the values of the condensates. These condensates appear under the disguise of renormalizable coupling constants, say lepton masses in the usual scheme. But the relation between the mass and the condensate bridges the energy scale of the mass and zero and thus its tree-level form is highly questionable. On the one hand, the order parameters of the spontaneously broken symmetries are formed in the asymptotical IR regime. On the other hand, they parametrize the effective vertices at the crossover.

It is reasonable to expect that the only impact of the IR modes on the physics of the higher-energy processes is the generation of the symmetry-breaking condensates. Thus universality actually holds when the physics is parametrized by the help of the renormalizable coupling constants and the condensates. It is useless if we make an attempt to derive the values of the condensates starting with the UV parameters, from one fixed point only.

This scenario leaves universality unharmed for ferromagnets. In fact, the condensate of the nonlinear σ model is a unit vector and there is no possibility of changing its length. In contrast of this situation, the physics of the superconductors may show nonuniversal features. In particular, the supercurrent density might depend on nonrenormalizable coupling constants of QED which are provided by theories of higher-energy scale, such as the standard model. In turn, the Higgs condensate of the standard model is a nonuniversal function of the bare parameters of a GUT, etc.

One would object the speculations about relevant operators for the IR fixed point of a superconductor since there is no gap in the physical spectrum. But the massless excitations are present in the gauge-dependent sector where the Higgs mechanism relegates them and continues to influence the dynamics of the gauge-invariant modes. Their presence can be seen from the long-range confining forces acting between two magnetic charges. In fact, in the absence of massless modes all interactions are screened. The situation is similar to the QCD vacuum where the long-range confining modes coexist with finite range the Yukawa forces due to the massive glueball exchanges.

In closing we repeat again that our results rely on the derivative expansion. It would be of key importance to support or disclaim its validity for the four-dimensional models in the IR regime.

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APPENDIX A

We collect here the commutation relations which were used in deducing the wave-function renormalization constants. All the relations derived in this Appendix are based upon the simple rule

$$[\tilde{\phi}, p_\mu] = i\partial_\mu \tilde{\phi} \; . \tag{A1}$$

Note that the appearance of the negative sign which differs from the conventional definition is due to the fact that the space-time traces are evaluated in the plane-wave basis with all p dependences being moved to the *left* of x-dependent field operators. Repeated use of the above gives

$$[\tilde{\phi}, p^2] = -\partial^2\tilde{\phi} - 2ip_\mu\partial_\mu\tilde{\phi} \; , \tag{A2}$$

$$\begin{aligned}[\tilde{\phi}, p_\mu p^2] = &-(p_\mu\partial^2\tilde{\phi} + 2p_\nu\partial_\nu\partial_\mu\tilde{\phi}) \\ &-i(p^2\partial_\mu\tilde{\phi} + 2p_\mu p_\nu\partial_\nu\tilde{\phi}) + \cdots \; ,\end{aligned} \tag{A3}$$

$$[\tilde{\phi}, (p^2)^2] = -3p^2\partial^2\tilde{\phi} - 4ip^2p_\mu\partial_\mu\tilde{\partial} + \cdots \; , \tag{A4}$$

$$[[\tilde{\phi}, p^2], p^2] = -4p_\mu p_\nu\partial_\mu\partial_\nu\tilde{\phi} + \cdots \; , \tag{A5}$$

$$\begin{aligned}[\tilde{\phi}, \Delta] &= \tilde{Z}\Delta^2[p^2, \tilde{\phi}] + \tilde{Z}^2\Delta^3[p^2, [p^2, \tilde{\phi}]] + \cdots \\ &= \tilde{Z}u\Delta^3\partial^2\tilde{\phi} + 2i\tilde{Z}\Delta^2p_\mu\partial_\mu\tilde{\phi} + \cdots \; ,\end{aligned} \tag{A6}$$

$$[\tilde{\phi}^a\tilde{\phi}^b, p^2] = [\tilde{\phi}^a\tilde{\phi}^b, \Delta] = 0 \; , \tag{A7}$$

$$\begin{aligned}[\tilde{\phi}, p^2\Delta] &= -\tilde{Z}^{-1}u[\tilde{\phi}, \Delta] \\ &= -u^2\Delta^3\partial^2\tilde{\phi} - 2iu\Delta^2p_\mu\partial_\mu\tilde{\phi} + \cdots \; ,\end{aligned} \tag{A8}$$

$$\begin{aligned}[\tilde{\phi}, \Delta p_\mu] = &\tilde{Z}\Delta^2(u\Delta p_\mu\partial^2\tilde{\phi} + 2p_\nu\partial_\nu\partial_\mu\tilde{\phi}) \\ &+i\Delta(2Z\Delta p_\mu p_\nu\partial_\nu\tilde{\phi} - \partial_\mu\tilde{\phi}) + \cdots \; ,\end{aligned} \tag{A9}$$

$$[\tilde{\phi},(p^2)^2\Delta] = \tilde{Z}^{-1}[\tilde{\phi},p^2] + \tilde{Z}^{-2}u^2[\tilde{\phi},\Delta] = -p^2\Delta(1+u\Delta+u^2\Delta^2)\partial^2\tilde{\phi} - 2ip^2\Delta(1+u\Delta)p_\mu\partial_\mu\tilde{\phi} + \cdots \,, \quad (A10)$$

$$[\tilde{\phi},p^2\Delta p_\mu] = -u\Delta^2(u\Delta p_\mu\partial^2\tilde{\phi} + 2p_\nu\partial_\nu\partial_\mu\tilde{\phi}) - i\Delta(2u\Delta p_\mu p_\nu\partial_\nu\tilde{\phi} + p^2\partial_\mu\tilde{\phi}) + \cdots \,, \quad (A11)$$

$$[\tilde{\phi},\Delta_\ell\Delta_t] = \Delta_\ell\Delta_t\{(\tilde{Z}_t u_t\Delta_t^2 + \tilde{Z}_\ell u_\ell\Delta_\ell^2 - \tilde{Z}_\ell\tilde{Z}_t\Delta_\ell\Delta_t p^2)\partial^2\tilde{\phi} + 2i(\tilde{Z}_\ell\Delta_\ell + \tilde{Z}_t\Delta_t)p_\mu\partial_\mu\tilde{\phi}\} + \cdots \,, \quad (A12)$$

$$[\tilde{\phi},\Delta^2] = -\tilde{Z}\Delta^3(1-3u\Delta)\partial^2\tilde{\phi} + 4i\tilde{Z}\Delta^3 p_\mu\partial_\mu\tilde{\phi} + \cdots \,, \quad (A13)$$

$$[\tilde{\phi},p^2\Delta_\ell\Delta_t] = \Delta_\ell\Delta_t\{(\tilde{Z}_t p^2 u_t\Delta_t^2 + \tilde{Z}_\ell p^2 u_\ell\Delta_\ell^2 - u_\ell u_t\Delta_\ell\Delta_t)\partial^2\tilde{\phi} + 2i(1-u_\ell\Delta_\ell - u_t\Delta_t)p_\mu\partial_\mu\tilde{\phi}\} + \cdots \,, \quad (A14)$$

$$[\tilde{\phi},p^2\Delta^2] = u\Delta^3(2-3u\Delta)\partial^2\tilde{\phi} + 2i\Delta^2(1-2u\Delta)p_\mu\partial_\mu\tilde{\phi} + \cdots \,, \quad (A15)$$

After generating the derivative terms with the above commutation relations, one may untangle the x- and p-dependent terms with the following useful relations:

$$(p^2f_1 - 2ip_\mu\partial_\mu f_1 - \partial^2 f_2 + f_3)\Delta = \Delta\{p^2 f_1 - \partial^2 f_2 + f_3 + \tilde{Z}\Delta[p^2(1+u\Delta)\partial^2 f_1 + u\Delta\partial^2 f_3] + 2ip_\mu\Delta[-u\partial_\mu f_1 + \tilde{Z}\partial_\mu f_3]\} + \cdots \,, \quad (A16)$$

$$\begin{aligned}(p^2f_1 - 2ip_\mu\partial_\mu f_1 - \partial^2 f_2 + f_3)\Delta(p^2g_1 - 2ip_\nu\partial_\nu g_1 - \partial^2 g_2 + g_3) \\ = \Delta\{p^2(p^2f_1+f_3)g_1 + (p^2f_1+f_3)g_3 - (p^2f_1+f_3)\partial^2 g_2 - (p^2g_1+g_3)\partial^2 f_2 \\ +[p^2(2-u^2\Delta^2)g_1 + \tilde{Z}p^2\Delta(1+u\Delta)g_3]\partial^2 f_1 + [(1+u\Delta-u^2\Delta^2)g_1 + \tilde{Z}u\Delta^2 g_3]\partial^2 f_3 \\ +2ip_\mu\Delta(p^2g_1+g_3)(-u\partial_\mu f_1 + \tilde{Z}\partial_\mu f_3) + \cdots\} \,, \quad (A17)\end{aligned}$$

which for $g=f$ reduces to

$$\begin{aligned}(p^2f_1 - 2ip_\mu\partial_\mu f_1 - \partial^2 f_2 + f_3)\Delta(p^2f_1 - 2ip_\nu\partial_\nu f_1 - \partial^2 f_2 + f_3) \\ = \Delta\{p^2(p^2f_1+f_3)f_1 + (p^2f_1+f_3)f_3 - 2(p^2f_1+f_3)\partial^2 f_2 \\ +p^2(2-u^2\Delta^2)f_1\partial^2 f_1 + \tilde{Z}\Delta^2 u f_3\partial^2 f_3 + (2+u\Delta-2u^2\Delta^2)f_3\partial^2 f_1 \\ +2ip_\mu\Delta(p^2f_1+f_3)(-u\partial_\mu f_1 + \tilde{Z}\partial_\mu f_3) + \cdots\} \,. \quad (A18)\end{aligned}$$

Similarly, we have

$$\begin{aligned}(p^2f_1 - 2ip_\mu\partial_\mu f_1 - \partial^2 f_2 + f_3)(p^2g_1 - 2ip_\nu\partial_\nu g_1 - \partial^2 g_2 + g_3) \\ = (p^2)^2 f_1g_1 + p^2(f_1g_3 + f_3g_1) + f_3g_3 - 2ip_\mu(p^2g_1+g_3)\partial_\mu f_1 \\ +(p^2f_1+f_3)(\partial^2 g_1 - \partial^2 g_2) - (p^2g_1+g_3)\partial^2 f_2 + \cdots \,, \quad (A19)\end{aligned}$$

and

$$\begin{aligned}(f_1 - \partial^2 f_2 - 2ip_\mu\partial_\mu f_3)[g_0 + (p^2)^2 g_1 + p^2 g_2 - \partial^2 g_3 - 2ip^2 p_\nu\partial_\nu g_4 - 2ip_\nu\partial_\nu g_5] \\ = f_1g_0 + (p^2)^2 f_1g_1 + p^2 f_1g_2 - \partial^2 f_2 g_0 - p^2(3\partial^2 f_1 + p^2\partial^2 f_2 + 2p^2\partial^2 f_3) \\ -(\partial^2 f_1 + p^2\partial^2 f_2 + p^2\partial^2 f_3)g_2 - f_1\partial^2 g_3 + p^2(3f_1 + p^2 f_3)\partial^2 g_4 + (2f_1 + p^2 f_3)\partial^2 g_5 + \cdots \,. \quad (A20)\end{aligned}$$

APPENDIX B

In computing the effective blocked action $\tilde{S}_k$, one encounters the $N\times N$ matrix M of the form

$$M = 1 + K_0^{-1}\delta K_0 = \begin{pmatrix} 1 & a^T \\ b & I \end{pmatrix} \quad (B1)$$

where

$$a^T = (a_\ell Z_\ell^{(2)} \cdots a_\ell Z_\ell^{(N)}), \quad a_\ell \equiv \tfrac{1}{2}p^2\Phi_0\Delta_\ell \,, \quad (B2)$$

and

$$b = \begin{pmatrix} a^t Z_\ell^{(2)} \\ \vdots \\ a_t Z_\ell^{(N)} \end{pmatrix}, \quad a_t \equiv \tfrac{1}{2}p^2\Phi_0\Delta_t \,. \quad (B3)$$

One can easily verify that its inverse M^{-1} takes on the form

$$M^{-1} = \begin{pmatrix} \theta & -\theta a^T \\ -\theta b & I + \theta b a^T \end{pmatrix} \,, \quad (B4)$$

where

$$\theta = (1 - a^T b)^{-1} = [1 - a_\ell a_t (Z_\ell^{(i)})^2]^{-1} . \tag{B5}$$

Employing the relations

$$\mathrm{Tr}' \ln(1 + K_0^{-1}\delta K) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}'\{(K_0^{-1}\delta K)^n\} \tag{B6}$$

and

$$\sum_{n=0}^{\infty}(-1)^n \mathrm{Tr}'[(K_0^{-1}\delta K_0)^n K_0^{-1}\delta K_\alpha] = \mathrm{Tr}'[(1 + K_0^{-1}\delta K_0)^{-1} K_0^{-1}\delta K_\alpha] , \tag{B7}$$

we readily obtain

$$\begin{aligned}\delta \tilde{S}_k^1 &= \tfrac{1}{2}\mathrm{Tr}' \ln(K_0 + \delta K_0 + \delta K_1 + \delta K_2) \\ &= \tfrac{1}{2}\mathrm{Tr}' \ln(K_0 + \delta K_0) - \tfrac{1}{4}\mathrm{Tr}'(K_0^{-1}\delta K_1 K_0^{-1}\delta K_1) \\ &\quad + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \mathrm{Tr}'[(K_0^{-1}\delta K_0)^n K_0^{-1}\delta K_1] + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \mathrm{Tr}'[(K_0^{-1}\delta K_0)^n K_0^{-1}\delta K_2] + \cdots \\ &= \tfrac{1}{2}\mathrm{Tr}' \ln(K_0 + \delta K_0) - \tfrac{1}{4}\mathrm{Tr}'(K_0^{-1}\delta K_1 K_0^{-1}\delta K_1) + \tfrac{1}{2}\mathrm{Tr}'[(1 + K_0^{-1}\delta K_0)^{-1} K_0^{-1}\delta K_1] \\ &\quad + \tfrac{1}{2}\mathrm{Tr}'[(1 + K_0^{-1}\delta K_0)^{-1} K_0^{-1}\delta K_2] + \cdots .\end{aligned} \tag{B8}$$

In terms of matrix elements,

$$\begin{aligned}\mathrm{Tr}'[(1 + K_0^{-1}\delta K_0)^{-1} K_0^{-1}\delta K_\alpha] = \int_x \int_p' \Big\{ &\theta \Delta_\ell (\delta K_\alpha)^{11} + \theta \Delta_t a_\ell a_t Z_\ell^{(i)} Z_\ell^{(j)} (\delta K_\alpha)^{ji}] \\ &+ \Delta_t (\delta K_\alpha)^{ii} - \theta(a_\ell \Delta_t + a_t \Delta_\ell) Z_\ell^{(i)} (\delta K_\alpha)^{i1} \Big\}\end{aligned} \tag{B9}$$

and

$$\begin{aligned}\mathrm{Tr}'(K_0^{-1}\delta K_1 K_0^{-1}\delta K_1) = \int_x \int_p' \Big\{ &\Delta_\ell (\delta K_1)^{11} \Delta_\ell (\delta K_1)^{11} + \Delta_t (\delta K_1)^{ij} \Delta_t (\delta K_1)^{ji} \\ &+ [\Delta_\ell (\delta K_1)^{1i} \Delta_t (\delta K_1)^{il} + + \Delta_t (\delta K_1)^{il} \Delta_\ell (\delta K_1)^{1i}]\} .\end{aligned} \tag{B10}$$

The complicated commutator algebra can be simplified by noting that the matrix elements take on the forms

$$\begin{aligned}(\delta K_1)^{ab} &= p^2 \alpha_1 - 2ip_\mu \partial_\mu \alpha_1 - \partial^2 \alpha_2 + \alpha_3 , \\ (\delta K_2)^{ab} &= p^2 \beta_1 \tilde{\phi}^c - \beta_2 \partial^2 \tilde{\phi}^c + \beta_3 ,\end{aligned} \tag{B11}$$

$$\begin{aligned}(\delta K_1)^{11} &: \begin{cases} \alpha_1 = \tilde{Z}_\ell^{(1)} \tilde{\phi}^1 + \tilde{Z}_\ell^{(i)} \tilde{\phi}^i, \\ \alpha_2 = \frac{3}{2}\tilde{Z}_\ell^{(1)} \tilde{\phi}^1 + \tilde{Z}_\ell^{(i)} \tilde{\phi}^i, \\ \alpha_3 = \lambda \Phi_0 \tilde{\phi}^1, \end{cases} \\ (\delta K_1)^{1i} &: \begin{cases} \tilde{\alpha}_1 = \frac{1}{2}(\tilde{Z}_\ell^{(i)} \tilde{\phi}^1 + \tilde{Z}_t^{(1)} \tilde{\phi}^i + \tilde{Z}_\ell^{(ij)} \Phi_0 \tilde{\phi}^j), \\ \tilde{\alpha}_2 = \tilde{Z}_\ell^{(i)} \tilde{\phi}^1 + \tilde{Z}_t^{(1)} \tilde{\phi}^i + \frac{1}{2}\tilde{Z}_\ell^{(ij)} \Phi_0 \tilde{\phi}^j, \\ \tilde{\alpha}_3 = \frac{\lambda}{3}\Phi_0 \tilde{\phi}^i, \end{cases}\end{aligned} \tag{B12}$$

$$(\delta K_1)^{ij} : \begin{cases} \alpha_1' = Z_t^{(c)} \delta^{ij} \tilde{\phi}^c + \frac{1}{2}(Z_t^{(i)} \tilde{\phi}^j + Z_t^{(j)} \tilde{\phi}^i), \\ \alpha_2' = Z_t^{(c)} \delta^{ij} \tilde{\phi}^c + Z_t^{(i)} \tilde{\phi}^j + Z_t^{(j)} \tilde{\phi}^i + \frac{1}{2} Z_\ell^{(ij)} \Phi_0 \tilde{\phi}^1, \\ \alpha_3' = \frac{\lambda}{3}\Phi_0 \delta^{ij} \tilde{\phi}^1, \end{cases} \tag{B13}$$

$$(\delta K_2)^{11} : \begin{cases} \beta_1 = Z_\ell^{(1c)} \tilde{\phi}^1 + \frac{1}{2} Z_\ell^{(cd)} \phi^d, \\ \beta_2 = Z_\ell^{(1c)} \tilde{\phi}^1 + \frac{1}{2} Z_c^{(11)} \tilde{\phi}^c, \\ \beta_3 = \frac{\lambda}{6}(\tilde{\phi}^c \tilde{\phi}^c + 2\tilde{\phi}^1 \tilde{\phi}^1), \end{cases} \tag{B14}$$

$$(\delta K_2)^{1i} : \begin{cases} \tilde{\beta}_1 = \frac{1}{2}(Z_t^{(1c)}\tilde{\phi}^i + Z_\ell^{(ic)}\tilde{\phi}^1), \\ \tilde{\beta}_2 = \frac{1}{2}(Z_t^{(1c)}\tilde{\phi}^i + Z_\ell^{(ic)}\tilde{\phi}^1 + Z_c^{(1i)}\tilde{\phi}^c), \\ \tilde{\beta}_3 = \frac{\lambda}{3}\tilde{\phi}^1\tilde{\phi}^i, \end{cases} \tag{B15}$$

$$(\delta K_2)^{ij} : \begin{cases} \beta_1' = \frac{1}{2}(Z_t^{(ic)}\tilde{\phi}^j + Z_t^{(jc)}\tilde{\phi}^i + Z_t^{(cd)}\delta^{ij}\tilde{\phi}^d), \\ \beta_2' = \frac{1}{2}(Z_t^{(ic)}\tilde{\phi}^j + Z_t^{(jc)}\tilde{\phi}^i + Z_c^{(ij)}\tilde{\phi}^c), \\ \beta_3' = \frac{\lambda}{6}(\delta^{ij}\tilde{\phi}^c\tilde{\phi}^c + 2\tilde{\phi}^i\tilde{\phi}^j). \end{cases} \tag{B16}$$

Substituting (B14)–(B16) into (B9) for δK_2 yields

$$\mathrm{Tr}'[(1+K_0^{-1}\delta K_0)^{-1}K_0^{-1}\delta K_2] = \int_x \int_p' \{b_{11}\tilde{\phi}^1\partial^2\tilde{\phi}^1 + b_{ij}\tilde{\phi}^i\partial^2\tilde{\phi}^j\} , \tag{B17}$$

where

$$\begin{aligned} b_{11} &= -\tfrac{1}{2}(3Z_\ell^{(11)}\theta\Delta_\ell + Z_\ell^{(ii)}\Delta_t) - \frac{\theta}{8}Z_\ell^{(i)}\Phi_0\Delta_\ell\Delta_t p^2[Z_\ell^{(j)}Z_\ell^{(ji)}\Phi_0\Delta_t p^2 - 8Z_\ell^{(i1)}] , \\ b_{ij} &= -\frac{1}{2}\delta^{ij}\left[\theta\Delta_\ell Z_t^{(11)} + Z_t^{kk}\Delta_t + \frac{\theta}{4}Z_\ell^{(k)}\Phi_0 p^2\Delta_\ell\Delta_t(Z_\ell^{(\ell)}Z_t^{(\ell k)}\Phi_0 p^2\Delta_t - 4Z_t^{(k1)})\right] \\ &\quad -\Delta_t\left[Z_t^{(ij)} + \frac{\theta}{2}Z_\ell^{(i)}\Phi_0 p^2\Delta_\ell(\tfrac{1}{2}Z_\ell^{(k)}Z_t^{(kj)}\Phi_0 p^2\Delta_t - Z_t^{(j1)})\right] . \end{aligned} \tag{B18}$$

For the δK_1-dependent terms, after much tedious algebra with the help of the relations found in Appendix A, we have

$$\begin{aligned} \Delta_\ell(\delta K_1)^{11}\Delta_\ell(\delta K_1)^{11} &= \Delta_\ell^2\{(p^2)^2\alpha_1^2 + 2p^2\alpha_1\alpha_3 + \alpha_3^2 + 2p^2\alpha_1(\partial^2\alpha_1 - \partial^2\alpha_2) - 2\alpha_3\partial^2\alpha_2 \\ &\quad -p^2u_\ell^2\Delta_\ell^2\alpha_1\partial^2\alpha_1 + \tilde{Z}u_\ell\Delta_\ell^2\alpha_3\partial^2\alpha_3 + (2+u_\ell\Delta_\ell - 2u_\ell^2\Delta_\ell^2)\alpha_3\partial^2\alpha_1 + \cdots\} \\ &= a_{11}\tilde{\phi}^1\partial^2\tilde{\phi}^1 + a_{ij}\tilde{\phi}^i\partial^2\tilde{\phi}^j + \cdots , \end{aligned} \tag{B19}$$

$$\begin{aligned} a_{11} &= -\Delta_\ell^2\{(Z_\ell^{(1)})^2p^2(1+u_\ell^2\Delta_\ell^2) + Z_\ell^{(1)}\lambda\Phi_0(1-u_\ell\Delta_\ell + 2u_\ell^2\Delta_\ell^2) - \tilde{Z}_\ell u_\ell\Delta_\ell^2\lambda^2\Phi_0^2\} , \\ a_{ij} &= -Z_\ell^{(i)}Z_\ell^{(j)}p^2u_\ell^2\Delta_\ell^4 , \end{aligned} \tag{B20}$$

$$\Delta_t(\delta K_1)^{ij}\Delta_t(\delta K_1)^{ji} = a_{11}'\tilde{\phi}^1\partial^2\tilde{\phi}^1 + a_{ij}'\tilde{\phi}^i\partial^2\tilde{\phi}^j + \cdots , \tag{B21}$$

$$\begin{aligned} a_{11}' &= (N-1)u_t\Delta_t^3\left[\Delta_t\left(\frac{\lambda^2}{9}\Phi_0^2\tilde{Z}_t - (Z_t^{(1)})^2p^2u_t\right) + \frac{\lambda}{3}\Phi_0 Z_t^{(1)}(1-2u_t\Delta_t)\right] \\ &\quad -Z_\ell^{(ii)}\Phi_0\Delta_t^2\left(Z_t^{(1)}p^2 + \frac{\lambda}{3}\Phi_0\right) , \\ a_{ij}' &= -Z_t^{(i)}Z_t^{(j)}p^2\Delta_t^2\left(3 + \frac{2N+3}{2}u_t^2\Delta_t^2\right) - \tfrac{1}{2}(Z_t^{(k)})^2\delta^{ij}p^2\Delta_t^2(2+u_t^2\Delta_t^2) , \end{aligned} \tag{B22}$$

$$\Delta_\ell(\delta K_1)^{1i}\Delta_t(\delta K_1)^{i1} = \tilde{a}_{11}\tilde{\phi}^1\partial^2\tilde{\phi}^1 + \tilde{a}_{ij}\tilde{\phi}^i\partial^2\tilde{\phi}^j , \tag{B23}$$

$$\begin{aligned} \tilde{a}_{11} &= -\tfrac{1}{4}(\tilde{Z}_\ell^{(i)})^2p^2\Delta_\ell\Delta_t(2+u_t^2\Delta_t^2) , \\ \tilde{a}_{ij} &= \Delta_\ell\Delta_t\left\{\tfrac{1}{6}\tilde{Z}_\ell^{(ij)}\Phi_0[-3\tilde{Z}_t^{(1)}p^2(1+u_t^2\Delta_t^2) + \lambda\Phi_0 u_t\Delta_t(1-2u_t\Delta_t)]\right. \\ &\quad -\delta^{ij}\left[\tfrac{1}{4}(\tilde{Z}_t^{(1)})^2p^2(2+u_t^2\Delta_t^2) - \frac{\lambda}{3}\Phi_0\left(\frac{\lambda}{3}\Phi_0\tilde{Z}_t u_t\Delta_t^2 - \frac{\tilde{Z}_t^{(1)}}{2}(2-u_t\Delta_t + 2u_t^2\Delta_t^2)\right)\right] \\ &\quad \left. -\tfrac{1}{4}\tilde{Z}_\ell^{(ik)}\tilde{Z}_\ell^{(kj)}\Phi_0^2p^2u_t^2\Delta_t^2\right\} , \end{aligned} \tag{B24}$$

$$\Delta_t(\delta K_1)^{i1}\Delta_\ell(\delta K_1)^{1i} = a^*_{11}\tilde{\phi}^1\partial^2\tilde{\phi}^1 + a^*_{ij}\tilde{\phi}^i\partial^2\tilde{\phi}^j \ , \tag{B25}$$

$$\begin{aligned} a^*_{11} &= -\tfrac{1}{4}(\tilde{Z}^{(i)}_\ell)^2 p^2\Delta_\ell\Delta_t(2+u_\ell^2\Delta_\ell^2) \ , \\ a^*_{ij} &= \Delta_\ell\Delta_t\Bigg\{\tfrac{1}{6}\tilde{Z}^{(ij)}_\ell\Phi_0[-3\tilde{Z}^{(1)}_t p^2(1+u_\ell^2\Delta_\ell^2)+\lambda\Phi_0 u_\ell\Delta_\ell(1-2u_\ell\Delta_\ell)] \\ &\quad -\delta^{ij}\left[\tfrac{1}{4}(\tilde{Z}^{(1)}_t)^2p^2(2+u_\ell^2\Delta_\ell^2)-\frac{\lambda}{3}\Phi_0\left(\frac{\lambda}{3}\Phi_0\tilde{Z}_\ell u_\ell\Delta_\ell^2-\frac{\tilde{Z}^{(1)}_t}{2}(2-u_\ell\Delta_\ell+2u_\ell^2\Delta_l^2)\right)\right] \\ &\quad -\tfrac{1}{4}\tilde{Z}^{(ik)}_\ell\tilde{Z}^{(kj)}_\ell\Phi_0^2p^2u_\ell^2\Delta_\ell^2\Bigg\} \ . \end{aligned} \tag{B26}$$

In arriving at the above expressions, O(4) invariance has been used:

$$\int_x \tilde{\phi}\partial_\mu\partial_\nu\tilde{\phi}\int_p p_\mu p_\nu = \frac{1}{4}\int_x\tilde{\phi}\partial^2\tilde{\phi}\int_p p^2 \ . \tag{B27}$$

The coefficients obtained above become much simpler when $O(N)$ symmetry is invoked which allows us to make the following substitutions when seeking for the RG flow equations:

$$\begin{aligned} u_\ell &\to U^{(11)}_k = 2(U'_k+2U''_k\Phi^2) \ , \\ u_t &\to U^{(22)}_k = 2U'_k \ , \\ \lambda\Phi &\to U^{(111)}_k = 4(3U''_k+2U'''_k\Phi^2)\mathbf{\Phi} \ , \\ \frac{\lambda}{3}\Phi &\to U^{(221)}_k = 4U''_k\mathbf{\Phi} \ , \end{aligned} \tag{B28}$$

and

$$\begin{aligned} \tilde{Z}^{(1)} &\to \tilde{Z}^{(1)}_k = 2\tilde{Z}'_k\mathbf{\Phi} \ , \\ \tilde{Z}^{(i)} &\to \tilde{Z}^{(i)}_k = 0 \ , \\ \tilde{Z}^{(11)} &\to \tilde{Z}^{(11)}_k = 4(\tilde{Z}''_k\Phi^2+\tilde{Z}'_k) \ , \\ \tilde{Z}^{(ij)} &\to \tilde{Z}^{(ij)}_k = 2\tilde{Z}'_k\delta^{ij} \ , \end{aligned} \tag{B29}$$

where the prime notation denotes differentiation with respect to Φ^2. Thus we have

$$\begin{aligned} b_{k,\ell} &= -6(\tilde{Z}''_{k,\ell}\Phi^2+\tilde{Z}'_{k,\ell})\Delta_{k,\ell}-(N-1)\tilde{Z}'_{k,\ell}\Delta_{k,t} \ , \\ b_{k,t} &= -2(\tilde{Z}''_{k,t}\Phi^2+\tilde{Z}'_{k,t})\Delta_{k,\ell}-(N+1)\tilde{Z}'_{k,t}\Delta_{k,t} \ , \end{aligned} \tag{B30}$$

$$\begin{aligned} a_{k,\ell} = -4\Delta^2_{k,\ell}\Phi^2\Bigg\{&(\tilde{Z}'_{k,\ell})^2k^2[1+4\Delta^2_{k,\ell}(U'_k+2U''_k\Phi^2)^2]+2(3U''_k+2U'''_k\Phi^2) \\ &\times\Bigg(\tilde{Z}'_{k,\ell}[1-2(U'_k+2U''_k\Phi^2)\Delta_{k,\ell}+8(U'_k+2U''_k\Phi^2)^2\Delta^2_{k,\ell}] \\ &-4\tilde{Z}_{k,\ell}\Delta^2_{k,\ell}(U'_k+2U''_k\Phi^2)(3U''_k+2U'''_k\Phi^2)\Bigg)\Bigg\} \ , \end{aligned} \tag{B31}$$

$$a'_{k,\ell} = 4(N-1)\Delta^2_{k,t}\Phi^2(-\tilde{Z}'_{k,\ell}(\tilde{Z}'_{k,t}k^2+2U''_k)+4U'_k\Delta_{k,t}\{\Delta_{k,t}[2\tilde{Z}_{k,t}(U''_k)^2-(\tilde{Z}'_{k,t})^2k^2U'_k]+\tilde{Z}'_{k,t}U''_k(1-4U'_k\Delta_{k,t})\}) \ , \tag{B32}$$

$$\begin{aligned} \tilde{a}_{k,t} = 2\Delta_{k,\ell}\Delta_{k,t}\Phi^2\{&16\tilde{Z}_{k,t}U'_k(U''_k)^2\Delta^2_{k,t}-\tilde{Z}'_{k,\ell}k^2[\tilde{Z}'_{k,t}+2(U'_k)^2\Delta^2_{k,t}(\tilde{Z}'_{k,\ell}+2\tilde{Z}'_{k,t})] \\ &-(\tilde{Z}'_{k,t})^2k^2[1+2(U'_k)^2\Delta^2_{k,t}]+4\tilde{Z}'_{k,\ell}\Delta_{k,t}U'_kU''_k(1-4U'_k\Delta_{k,t}) \\ &-4\tilde{Z}'_{k,t}U''_k[1-U'_k\Delta_{k,t}+4(U'_k)^2\Delta^2_{k,t}]\} \ , \end{aligned} \tag{B33}$$

$$a^*_{k,t} = 2\Delta_{k,\ell}\Delta_{k,t}\Phi^2(4U''_k\{4\tilde{Z}_{k,\ell}U''_k(U'_k + 2U''_k\Phi^2)\Delta^2_{k,\ell}$$
$$-\tilde{Z}'_{k,t}[1 - (U'_k + 2U''_k\Phi^2)\Delta_{k,\ell} + 4(U'_k + 2U''_k\Phi^2)^2\Delta^2_{k,\ell}]\}$$
$$+4\tilde{Z}'_{k,\ell}U''_k\Delta_{k,\ell}(U'_k + 2U''_k\Phi^2)[1 - 4(U'_k + 2U''_k\Phi^2)\Delta_{k,\ell}]$$
$$-\tilde{Z}'_{k,\ell}k^2[\tilde{Z}'_{k,t} + 2(\tilde{Z}'_{k,\ell} + 2\tilde{Z}'_{k,t})(U'_k + 2U''_k\Phi^2)^2\Delta^2_{k,\ell}]) \ , \qquad \text{(B34)}$$

and

$$a_{k,t} = a'_{k,t} = \tilde{a}_{k,\ell} = a^*_{k,\ell} = 0 \ . \qquad \text{(B35)}$$

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