Quantum aspects of supersymmetric Maxwell Chern-Simons solitons

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We study the various quantum aspects of the $N = 2$ supersymmetric Maxwell Chern-Simons vortex systems. The fermion zero modes around the vortices will give rise to the degenerate states of vortices. We analyze the angular momentum of these zero modes and apply the result to get the supermultiplet structures of the vortex. The leading quantum correction to the mass of the vortex coming from the mode fluctuations is also calculated using various methods depending on the value of the coefficient of the Chern-Simons term κ being zero, infinite, and finite, separately. The mass correction is shown to vanish for all cases. Fermion numbers of vortices are also discussed.

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I. INTRODUCTION

Abelian gauge theory in 2+1 dimensions with the Chern-Simons (CS) term [1,2] has attracted much interest. Matter fields coupled with this term are believed to describe anyons with fractional spin and fractional statistics. Such important planar phenomena as high T_c superconductivity and fractional quantum Hall effect have added more interest to the field theory models with a CS term. The characters of allowed solitons are also affected by the presence of the CS term. As is well known, the usual (2+1)-dimensional Abelian Higgs model supports only electrically neutral vortices as topologically stable soliton solutions [3]. On the other hand, the CS term makes the vortices [4] electrically charged, which are (extended) anyons [5]. We have quite a rich vortex structure depending on whether the matter fields are relativistic [6] or nonrelativistic [7] and whether we have more than one CS field [8]. In this work we are mainly interested in the case with relativistic matter fields coupled to the gauge field with both Maxwell and CS terms in general. As a special limit of this general model we get the Abelian Higgs model and the "minimal" Chern-Simons Higgs model (i.e., without the Maxwell term in the action).

With some special choice of the scalar potential in (2+1)-dimensional gauge models Ref. [9], one can obtain interesting limiting theories in which the minimum energy static soliton solutions satisfy first-order differential equations, called the Bogomol'nyi [10] or self-duality equations. This special potential becomes a specific scalar quartic potential for the Abelian Higgs model, while in the case of the "minimal" Chern-Simons Higgs model it becomes a specific sixth-order potential form [6,11]. The appearance of self-dual structures for certain special Higgs potentials can be ascribed to the extended supersymmetry [12–14]. Requiring an $N = 2$ supersymmetry guarantees this special form of the potential. There also exists an $N = 1$ supersymmetric model which produces exactly the same bosonic part of the Lagrangian as that of the $N = 2$ model. The fermion number is, however, not preserved in this case. We will mainly consider the model with more symmetry, i.e., the model with the $N = 2$ supersymmetry.

A remarkable feature with these self-dual systems is the existence of static multivortex solutions which represent static configurations of vortices with unit flux without any interaction energy between them. This interpretation is supported by counting independent zero modes [11,15] to the boson fluctuation equations in the background field of a particular soliton solution. These bosonic zero modes are related to the collective modes of the solitons and play an important role in understanding the dynamics of the slowly moving vortices [16].

Fermions around the vortices also have zero modes. For the models under study it is found that *all* fermion zero modes around the general multivortex background are closely related to the corresponding bosonic zero modes. The $N = 2$ supersymmetry is crucial [14]. They are very important in the quantum study of the models, representing the degeneracy of the soliton states (in contradistinction to bosonic zero modes which become collective coordinates) [17]. In supersymmetric models in particular, they account for the soliton supermultiplet structure [18].

The organization of this paper is as follows. In Sec. II we review the properties of the $N = 2$ supersymmetric Maxwell Chern-Simons system. The particle contents, the properties of the vortices, the symmetries and the corresponding algebra of the Lagrangian are reviewed. In Sec. III we will quantize the fluctuating modes around the self-dual vortices. By analyzing the angular momentum fermion zero modes we describe the multiplet structures of the vortices. Spin contents of the degenerate supermultiplet of the vortex states are also calculated. Then, in Sec. IV we calculate the mass correction to the vortices. We will do this for the value of κ to be zero, infinite and finite, separately. Section V contains the summary of our work and discussions. Some technical

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details related with the spin assignment, supermultiplets, and phase shift analysis are described in the Appendixes.

II. SUPERSYMMETRIC MAXWELL CHERN-SIMONS THEORY

The Lagrangian for the Maxwell-Chern-Simons system with $N = 2$ supersymmetry is given by [14]

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F, \quad (2.1)$$

where

$$\mathcal{L}_B = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda - |D_\mu\phi|^2 - \tfrac{1}{2}(\partial_\mu N)^2 - \tfrac{1}{2}(e|\phi|^2 + \kappa N - ev^2)^2 - e^2N^2|\phi|^2 \quad (2.2)$$

and

$$\mathcal{L}_F = i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\chi}\gamma^\mu\partial_\mu\chi + \kappa\bar{\chi}\chi - i\sqrt{2}e(\bar{\psi}\chi\phi - \bar{\chi}\psi\phi^*) + eN\bar{\psi}\psi. \quad (2.3)$$

Here, $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, N a real scalar, ϕ a complex charged scalar, and ψ (χ) is a complex charged (neutral) 2-component spinor. Our metric tensor $\eta^{\mu\nu}$ has the signature $(-,+,+)$. We will choose the γ matrices as $\gamma^\mu = (\sigma_3, i\sigma_2, i\sigma_1)$.

When the coupling strength κ for the Chern-Simons term becomes zero, the above Lagrangian reduces to the $N = 2$ supersymmetric Abelian Higgs model [12]. The scalar potential in this limit allows only the symmetry broken vacuum. In another extreme limit of very large κ (with the ratio e^2/κ fixed), the neutral scalar field N (spinor field χ) can be represented in terms of the complex scalar field ϕ (spinor field ψ) as

$$N = -\frac{1}{\kappa}e(|\phi|^2 - v^2), \quad \chi = -\frac{i}{\kappa}\sqrt{2}e\phi^*\psi, \quad (2.4)$$

and the Lagrangian becomes the supersymmetric extension of the minimal self-dual Chern-Simons Higgs model given in Ref. [13]:

$$\mathcal{L}^{(2)}_{\rm CS} = \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda - |D_\mu\phi|^2 - \frac{e^4}{\kappa^2}|\phi|^2(|\phi|^2 - v^2)^2 + i\bar{\psi}\gamma^\mu D_\mu\psi - \frac{e^2}{\kappa}(3|\phi|^2 - v^2)\bar{\psi}\psi. \quad (2.5)$$

The above theory posesses various kinds of symmetries and their corresponding currents. The energy-momentum vectors related with the translational symmetry is given by

$$P^\mu = \int d^2x\Theta^{0\mu} \quad (2.6)$$

with the energy-momentum tensor given by

$$\Theta_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda + D_\mu\phi^* D_\nu\phi + D_\nu\phi^* D_\mu\phi + \partial_\mu N\partial_\nu N - i\bar{\psi}\gamma_{(\mu}D_{\nu)}\psi - i\bar{\chi}\gamma_{(\mu}\partial_{\nu)}\chi + \eta_{\mu\nu}\mathcal{L}. \quad (2.7)$$

This energy-momentum tensor is gauge invariant corresponding to the translation supplemented with an appropriate gauge transformation. The generators for the Lorentz symmetry can be also found. For example, the canonical angular momentum operator J is given by

$$J = \int d^2x[\epsilon_{ij}x^iP_j - \tfrac{1}{2}(\bar{\psi}\psi + \bar{\chi}\chi)] = J_B + J_F \quad (2.8)$$

with the contribution from the bosonic fields,

$$J_B = \int d^2x\epsilon_{ij}x^i[\partial^0 N\partial^j N + D^0\phi^* D^j\phi + D^j\phi^* D^0\phi + F^{0k}F^j{}_k], \quad (2.9)$$

and that from fermions

$$J_F = -i\int d^2x\epsilon_{ij}x^i[\bar{\psi}\gamma^0 D_j\psi + \bar{\chi}\gamma^0\partial_j\chi] - \tfrac{1}{2}\int d^2x(\bar{\psi}\psi + \bar{\chi}\chi). \quad (2.10)$$

The theory in Eq. (2.1) possesses the supersymmetry

$$\begin{aligned} &\delta_\eta A_\mu = i(\bar{\eta}\gamma_\mu\chi - \bar{\chi}\gamma_\mu\eta), \\ &\delta_\eta\phi = \sqrt{2}\bar{\eta}\psi, \qquad \delta_\eta N = i(\bar{\chi}\eta - \bar{\eta}\chi), \\ &\delta_\eta\psi = -\sqrt{2}(i\gamma^\mu\eta D_\mu\phi - \eta F), \\ &\delta_\eta\chi = \gamma^\mu\eta(\partial_\mu N - f_\mu) + i\eta G, \end{aligned} \quad (2.11)$$

where

$$F = eN\phi, \quad f_\mu = -\frac{i}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}, \quad G = e|\phi|^2 + \kappa N - ev^2. \quad (2.12)$$

Here the spinor parameter η should be taken as being *complex* Grassmannian. The corresponding supercharges are

$$Q = \sqrt{2}\int d^2x\big[(D_\mu\phi)^*\gamma^\mu\gamma^0\psi - \sqrt{2}iF^*\gamma^0\psi - i(\partial_\mu N + f_\mu)\gamma^\mu\gamma^0\chi - G\gamma^0\chi\big]. \quad (2.13)$$

The algebra of these with the supercharges are given by

$$\{Q_\alpha, \bar{Q}^\beta\} = 2\gamma^{\mu\beta}_\alpha P_\mu - 2\delta^\beta_\alpha ev^2\Phi, \quad (2.14)$$

with $\Phi = \int d^2xB$.

The particle spectrum will form representations of the above symmetry algebra. General structure of supermultiplet and the spin assignment will be described in Appendix B. In the case of $\kappa = 0$ where there is only symmetry broken vacuum, all the particles are massive with the same mass $\sqrt{2}|ev|$. We have two (massive) vector modes with spin 1 and -1, two real scalar modes of spin 0, and four spinor modes with two of spin 1/2 and the other two of spin $-1/2$. For the fermions, the sign of the mass term will depend on the spin. Two sets of the $N = 2$ supermultiplets are formed. One set is with one of spin 1, two of spin 1/2, and one of spin 0. The other is with one of spin -1, two of spin $-1/2$, and one of spin 0.

For the Chern-Simons theory, the potential allows both symmetry unbroken and symmetry broken vacua. In the symmetry broken vacuum, we have 4 degrees of freedom with the equal masses $2e^2v^2/|\kappa|$ forming one set of $N = 2$ supermultiplet. The spin contents for $\kappa > 0$ are one of spin -1, two of spin $-1/2$, and one of spin 0. With $\kappa < 0$, all the spins in the supermultiplet will change signs. In the unbroken vacuum sector, all the four modes are massive again with masses equal to $e^2v^2/|\kappa|$. These are split into two supermultiplets. One supermultiplet consists with each of spin 0 and spin 1/2, and the other with each of spin 1/2 and 1. If the sign of κ becomes negative, the signs of all the spins are also changed.

In the general case with a finite value of κ, there are also two degenerate ground states, i.e., a symmetric one where $\phi = 0, N = ev^2/\kappa$ and an asymmetric one where $|\phi| = v, N = 0$. The particle contents in the symmetry unbroken phase are the complex scalar ϕ and the Dirac fermion ψ with the mass $e^2v^2/|\kappa|$ and the neutral scalar N, the gauge field A_μ, and the Dirac fermion χ with another mass equal to $|\kappa|$. They form four $N = 2$ supermultiplets with 2 degrees of freedom. One supermultiplet consists of spins 1 and 1/2 and the other three consist of spins 1/2 and 0. In the broken phase we still have two mass scales $\frac{1}{2}[\kappa^2 + 4e^2v^2 \pm \sqrt{\kappa^2(\kappa^2 + 8e^2v^2)}]$. The fields corresponding to these mass eigenstates are obtained as some combination of the original fields. We have two $N = 2$ supermultiplets. The spin contents in the supermultiplet with the mass $m_+^2 = \frac{1}{2}[\kappa^2 + 4e^2v^2 + \sqrt{\kappa^2(\kappa^2 + 8e^2v^2)}]$ are one with spin 1, two of spin 1/2, and one of spin 0. The spins in the other supermultiplet with $m_-^2 = \frac{1}{2}[\kappa^2 + 4e^2v^2 - \sqrt{\kappa^2(\kappa^2 + 8e^2v^2)}]$ will be that of the m_+ with all the signs changed. The brief analysis of the above supermultiplet structures is done in Appendix B.

Now, let us briefly review the structure of self-dual vortices in the Maxwell-Chern-Simons theory. The bosonic Lagrangian $\mathcal{L}_B$ in Eq. (2.5) is enough for the classical solution. In this theory, there are two degenerate ground states as mentioned above. It is known that topological solitons exist in the asymmetric phase with the asymptotic behavior

$$N(\mathbf{r}) \to 0, \quad |\phi(\mathbf{r})| \to v \quad \text{as} \quad r \to \infty \tag{2.15}$$

and a quantized flux $\Phi = \pm\frac{2\pi}{e}n$ ($n=$ positive integer). Nontopological solitons also exist in the symmetric phase. We will consider only the topological vortex for simplicity. These vortices satisfy the field equations. Especially, they satisfy the Gauss law constraint

$$\partial_i F^{i0} + \kappa F_{12} + eJ^0 = 0 \tag{2.16}$$

with $J^0 = -i(\phi^* D^0\phi - D^0\phi^*\phi)$. Integrating over the whole space then tells us that a configuration with the magnetic flux Φ carries the electric charge $Q_E \equiv \int d^2x J^0 = -\frac{\kappa}{e}\Phi$. In this theory it has also been shown [9] that the energy of the configuration is bounded from below by the relation

$$E \geq ev^2|\Phi| = 2\pi v^2 n, \tag{2.17}$$

and is saturated if the configurations satisfy the "self-duality" equations

$$\begin{aligned}
&(D_1 \pm iD_2)\phi \equiv D_\pm\phi = 0,\\
&F_{12} \pm (e|\phi|^2 + \kappa N - ev^2) = 0,\\
&A^0 \mp N = 0,
\end{aligned} \tag{2.18}$$

together with the Gauss law (2.16). The upper (lower) sign corresponds to a positive (negative) value of the magnetic flux Φ. Whenever we need the explicit choice of the self-dual background field configuration, we will choose the upper sign corresponding to the vortex.

In the case of $\kappa = 0$ we may consistently set $A^0 = N = 0$ and Eq. (2.18) will become the self-duality equations for Landau-Ginzburg vortices [10]. On the other hand, in the case of $\kappa \to \infty$, we have, instead,

$$A^0 = \left(\frac{\kappa}{2e^2}\right)\frac{F_{12}}{|\phi|^2}, \tag{2.19}$$

and the self-duality equations reduce to those of Ref. [6]: viz.,

$$\begin{aligned}
&D_\pm\phi = 0,\\
&F_{12} \pm \frac{2e^3}{\kappa^2}|\phi|^2(|\phi|^2 - v^2) = 0.
\end{aligned} \tag{2.20}$$

We can take the spherical ansatz for those classical vortices with vorticity n on top of each other:

$$\phi = f(r)e^{in\theta}, \qquad eA^i = \epsilon_{ij}\frac{x_j}{r^2}(a(r) - n). \tag{2.21}$$

The functions are related as

$$a(r) = r\frac{d}{dr}\ln f(r), \tag{2.22}$$

and can be solved using the Bogomolyi equation. Now consider the angular momentum in the presence of the classical vortices given by the spherical ansatz. With the Gauss law now containing the fermion charge density in J^0, the angular momentum from the bosonic fields in (2.9) can be written as

$$\begin{aligned}
J_B = \int d^2x \epsilon_{ij} x^i \Big[&\partial^0 N \partial^j N + D^0\phi^* \partial^j\phi + \partial^j\phi^* D^0\phi\\
&+ F^{0k}F^j{}_k + A^j(-\partial_k E^k + \kappa F_{12} + e\bar{\psi}\gamma^0\psi)\Big].
\end{aligned} \tag{2.23}$$

The leading contribution to the angular momentum in (2.23) for the vortices will be from the background classical vortex configuration $J_{\rm cl}$ and from the fermion zero modes ΔJ_0. $J_B \sim J_{\rm cl} + \Delta J_0$. For vortices with the spherical ansatz, these are given by [9]

$$J_{\rm cl} = -\frac{\pi\kappa}{e^2}n^2 \tag{2.24}$$

and

$$\Delta J_0 = -\int d^2x\, a(r)\psi^\dagger\psi. \tag{2.25}$$

The fermionic field contribution to the angular momentum in Eq. (2.10) under the vortex background becomes

$$J_F = \int d^2x[\psi^\dagger(-i\partial_\theta + a(r) - \tfrac{1}{2}\sigma_3 - n)\psi + \chi^\dagger(-i\partial_\theta - \tfrac{1}{2}\sigma_3)\chi]. \tag{2.26}$$

The total contribution from the fermion modes will then become

$$J_F + \Delta J_0 = \int d^2x(\psi^\dagger, \chi^\dagger)\left[-i\partial_\theta - \frac{1}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} - \begin{pmatrix}n & 0\\ 0 & 0\end{pmatrix}\right]\begin{pmatrix}\psi\\ \chi\end{pmatrix}. \tag{2.27}$$

This will be used in the next section.

III. QUANTIZATION

Now, we want to quantize the theory in the soliton sector. The general procedure of the quantization of fields around the soliton has been well developed [19]. We decompose the fields around the classical vortex configuration as

$$\Phi = \Phi_{\rm cl} + \delta\Phi. \tag{3.1}$$

The field $\delta\Phi$ is the fluctuating modes around the classical vortex configuration $\Phi_{\rm cl}$. Plugging Eq. (3.1) into the Lagrangian we have the Lagrangian in the form of

$$\mathcal{L} = \mathcal{L}_{\rm cl} + \mathcal{L}_{(2)} + \mathcal{L}_{\rm int}.$$

Here, $\mathcal{L}_{\rm cl}$ is the same as the bosonic Lagrangian in Eq. (2.2) except that all the fields are the classical vortex configuration. This is the tree level contribution to the Lagrangian coming from the classical vortex configuration. $\mathcal{L}_{(2)} = \mathcal{L}_B^{(2)} + \mathcal{L}_F^{(2)}$ is the quadratic piece in terms of the fluctuation fields. For a general value of κ, the quadratic part of the bosonic fluctuations is given by

$$\begin{aligned}\mathcal{L}_B^{(2)} = & -\tfrac{1}{4}\delta F_{\mu\nu}\delta F^{\mu\nu} + \tfrac{1}{4}\kappa\epsilon^{\mu\nu\lambda}\delta F_{\mu\nu}\delta A_\lambda - |D_\mu\delta\phi|^2 - e^2\delta A_\mu^2|\phi|^2 \\ & -ie\delta A_\mu(\delta\phi^* D_\mu\phi + \phi^* D_\mu\delta\phi) - \tfrac{1}{2}(\partial_\mu\delta N)^2 - \frac{1}{2}\left(\phi^*\delta\phi + \phi\delta\phi^* + \frac{\kappa}{e}\delta N\right)^2 \\ & -e^2\left(|\phi|^2 + \frac{\kappa}{e}N - v^2\right)|\delta\phi|^2 - e^2(\delta N)^2|\phi|^2 - e^2N^2|\delta\phi|^2 - e^2N\delta N(\phi^*\delta\phi + \phi\delta\phi^*),\end{aligned} \tag{3.2}$$

and that of the fermionic fluctuations by

$$\mathcal{L}_F^{(2)} = i\bar\psi\gamma^\mu D_\mu\psi + i\bar\chi\gamma^\mu\partial_\mu\chi + \kappa\bar\chi\chi - i\sqrt{2}e(\bar\psi\chi\phi - \bar\chi\psi\phi^*) + eN\bar\psi\psi. \tag{3.3}$$

All the bosonic fields above are the classical background. Terms in $\mathcal{L}_{\rm int}$ are the higher order interaction terms.

The equations of motion for the fluctuating fields around the self-dual vortices can be obtained by varying the quadratic piece of the Lagrangian in the above. First, the equations of motion for the bosonic fluctuations are

$$\begin{aligned}&(\partial_\mu\partial^\mu - 2e^2|\phi|^2)\delta A^\nu - \partial^\nu\partial_\mu\delta A^\mu - \kappa\epsilon^{\mu\nu\lambda}\partial_\mu\delta A_\lambda - ie(\phi^*\overleftrightarrow{D}^\nu\delta\phi + \delta\phi^*\overleftrightarrow{D}^\nu\phi) = 0,\\ &(-\partial_t^2 + D_-D_+ - 2ieA^0\partial_0)\delta\phi - ie(D_-\phi)\delta A_+ - \kappa e\phi\delta N + 2eA^0\phi(\delta A^0 - \delta N) - ie\phi\partial_\mu\delta A^\mu - e^2\phi(\phi^*\delta\phi + \phi\delta\phi^*) = 0,\\ &(\partial_\mu\partial^\mu - 2e^2|\phi|^2 - \kappa^2)\delta N - e(\kappa + 2eN)(\phi^*\delta\phi + \phi\delta\phi^*) = 0.\end{aligned} \tag{3.4}$$

Time independent solutions of these equations are the zero modes. The bosonic zero mode fluctuations may also be obtained by considering the variation of the self-duality equations (2.18) around the given classical vortex configuration as given in Ref. [15]. Among the zero modes we eliminate those related with the gauge transformation by imposing the gauge-fixing condition. The index theorem or its variants can be used to count the number of bosonic zero modes satisfying the equations of motion and the gauge-fixing condition. The bosonic zero modes for the $\kappa = 0$ [20], those for the $\kappa = \infty$ case [11] and those for a general value of κ were studied [15]. The results are that in the background of a topological vortex configuration with vorticity n, there exist $2n$ bosonic zero modes for any value of κ. They correspond to the collective coordinates associated with the vortices. The quantization of the bosonic zero modes will give rise to the excitation of the collective coordinates, e.g., the momentum. This is consistent with the interpretation of these zero modes as being related to translation of individual vortices.

The general time dependent modes can be quantized as usual to describe the bosonic particle excitations around the vortices.

We now turn to the fermion fields. The Dirac equation for fermion fields around the vortex is

$$i\partial_0 \begin{pmatrix} \psi \\ \chi \end{pmatrix} = H_F \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{3.5}$$

where the Hamiltonian H_F is given by

$$\begin{aligned} H_F &= \begin{pmatrix} -i\gamma^0\boldsymbol{\gamma}\cdot\mathbf{D} + e(-\gamma^0 N + A^0) & \sqrt{2}ie\gamma^0\phi \\ -\sqrt{2}ie\phi^*\gamma^0 & -i\gamma^0\boldsymbol{\gamma}\cdot\nabla - \gamma^0\kappa \end{pmatrix} \\ &= -i\begin{pmatrix} 0 & D_+ & -\sqrt{2}e\phi & 0 \\ D_- & 2ieN & 0 & \sqrt{2}e\phi \\ \sqrt{2}e\phi^* & 0 & -i\kappa & \partial_+ \\ 0 & -\sqrt{2}e\phi^* & \partial_- & i\kappa \end{pmatrix}. \end{aligned} \tag{3.6}$$

The background fields in the above are for vortex configuration corresponding to the upper sign in Eq. (2.18).

Solutions of the Dirac equation of fermions around the vortex in the above equation can be decomposed as

$$\Psi = \sum a_i \Psi_i^0 + \sum_\omega b\Psi_{+\omega} + \sum_\omega d^\dagger \Psi_{-\omega}. \tag{3.7}$$

Here Ψ_i^0 are the zero modes and $\sum_\omega b\Psi_{+\omega}$ ($\sum_\omega d^\dagger\Psi_{-\omega}$) are the positive (negative) energy solutions. The fermionic zero modes are analyzed in Ref. [14] using the index theorem and also the relation between the bosonic zero modes and the fermion zero modes. There are $2n$ fermion zero modes around the vortex configuration with the winding number n as in the case of bosonic zero modes. This is deeply related with the $N = 2$ supersymmetry of the theory [14].

The quantization of the zero modes of the fermions will be relevant to the multiplet contents of the vortices. To do this, we need to know the angular momentum of the zero modes. The quantum mechanical angular momentum operator $\mathcal{J}$ can be read from the field theoretical expression in Eq. (2.27) as

$$\mathcal{J} = \begin{pmatrix} -i\epsilon_{ij}\partial_j - \frac{1}{2}\sigma_3 - n & 0 \\ 0 & -i\epsilon_{ij}\partial_j - \frac{1}{2}\sigma_3 \end{pmatrix}. \tag{3.8}$$

It is straightforward to show that the angular momentum operator $\mathcal{J}$ commutes with the Hamiltonian H_F. We decompose the modes into angular momentum eigenstates. The general mode with the angular momentum quantum number j is given by

$$\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \\ \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} = \begin{pmatrix} h_1(r)e^{i(j+\frac{1}{2}+n)\theta} \\ h_2(r)e^{i(j-\frac{1}{2}+n)\theta} \\ h_3(r)e^{i(j+\frac{1}{2})\theta} \\ h_4(r)e^{i(j-\frac{1}{2})\theta} \end{pmatrix} e^{-i\omega t}. \tag{3.9}$$

The Dirac equation for this mode becomes

$$\omega\begin{pmatrix} h_1(r) \\ h_2(r) \\ h_3(r) \\ h_4(r) \end{pmatrix} = -i\begin{pmatrix} 0 & \partial_r - \frac{a+j-1/2}{r} & -\sqrt{2}evf & 0 \\ \partial_r + \frac{a+j+1/2}{r} & 2ieN & 0 & \sqrt{2}evf \\ \sqrt{2}evf & 0 & -i\kappa & \partial_r - \frac{j-1/2}{r} \\ 0 & -\sqrt{2}evf & \partial_r + \frac{j+1/2}{r} & i\kappa \end{pmatrix}\begin{pmatrix} h_1(r) \\ h_2(r) \\ h_3(r) \\ h_4(r) \end{pmatrix}. \tag{3.10}$$

We first consider the fermion zero modes in the $\kappa \to 0$ limit. In that case we have two sets of equations:

$$\begin{pmatrix} \partial_r - \frac{a+j-1/2}{r} & -\sqrt{2}evf \\ -\sqrt{2}evf & \partial_r + \frac{j+1/2}{r} \end{pmatrix}\begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix} = 0 \tag{3.11}$$

and

$$\begin{pmatrix} \partial_r + \frac{a+j+1/2}{r} & \sqrt{2}evf \\ \sqrt{2}evf & \partial_r - \frac{j-1/2}{r} \end{pmatrix}\begin{pmatrix} h_1(r) \\ h_4(r) \end{pmatrix} = 0. \tag{3.12}$$

The second set of equations (3.12) does not allow any normalizable solution. The first set of equations (3.11) can be combined as

$$\left(\partial_r^2 + \frac{\partial_r}{r} - \frac{(j-1/2)^2}{r^2} - 2e^2v^2f^2\right)\left(\frac{h_2}{\sqrt{2}evf}\right) = 0. \tag{3.13}$$

The asymptotic behavior of $h_2(r)$ in (3.13) will be proportional to $e^{-\sqrt{2}evr}$. Near the origin, h_2 behaves as

$$h_2 \sim r^n(A_1 r^{j-\frac{1}{2}} + A_2 r^{-(j-\frac{1}{2})}). \tag{3.14}$$

We expect a single solution matching the boundary condition at infinity with two free parameters at the origin. To get the regular solution at the origin with two free parameters, j is restricted to half integral values in $-n + \frac{1}{2} \le j \le n - \frac{1}{2}$. The value $j = n + \frac{1}{2}$ is discarded since the corresponding $\psi_\downarrow$,

$$\psi_\downarrow \sim (A_1 r^{2n} + A_2)e^{2in\theta}, \tag{3.15}$$

has bad behavior at $r = 0$. For each of the above $2n$ solutions of $\psi_\downarrow$ (h_2), the function $\psi_\uparrow(h_3(r))$ is determined through Eq. (3.11). Hence we have $2n$ independent zero modes. This result agrees with that in Ref. [14] based on the index theorem.

For the general value of κ, we can get two coupled second-order differential equations of h_2 and h_4 from (3.10):

$$\left(\partial_r^2 + \frac{\partial_r}{r} - \frac{(j-1/2)^2}{r^2} - 2e^2v^2f^2\right)\left(\frac{h_2}{\sqrt{2}evf}\right) + i\kappa h_4 = 0 \quad (3.16)$$

and

$$\left(\partial_r^2 + \frac{\partial_r}{r} - \frac{(j-1/2)^2}{r^2} - \kappa^2 - 2e^2v^2f^2\right)h_4 - i(\kappa + 2eN)2e^2v^2f^2\left(\frac{h_2}{\sqrt{2}evf}\right) = 0. \quad (3.17)$$

The remaining functions h_3 and h_1 can be determined by h_2 and h_4 through the third line and first line in the Dirac equation in (3.10). The general solutions of Eqs. (3.16) and (3.17) are expected to have four free parameters, to be adjusted to the boundary conditions at the origin and infinity. Near $r \sim 0$, the regularity of the solution gives us only three free parameters for $-n + \frac{1}{2} \leq j \leq n - \frac{1}{2}$. The leading orders in the power series expansions in that range of the angular momentum are given by

$$h_2 \sim r^n(A_1 r^{j-\frac{1}{2}} + A_2 r^{-(j-\frac{1}{2})}), \quad h_4 \sim B r^{|j-\frac{1}{2}|}. \quad (3.18)$$

If the angular momentum is out of the above range, then we have at most two free parameters. In the asymptotic region, among the four parameters, two will conrrespond to the unphysical divergent solutions and only two free parameters will show up in the convergent solutions as

$$h_2 \sim C_1 e^{-m_1 r} + C_2 e^{-m_2 r},$$

$$\frac{1}{i\kappa}h_4 \sim \frac{{m_1}^2 - 2e^2v^2}{\kappa^2}C_1 e^{-m_1 r} + \frac{{m_2}^2 - 2e^2v^2}{\kappa^2}C_2 e^{-m_2 r}, \quad (3.19)$$

where ${m_1}^2$ and ${m_2}^2$ are eigenvalues of the mass matrix

$$\begin{pmatrix} 2e^2v^2 & \kappa^2 \\ \kappa^2 + 2e^2v^2 & 2e^2v^2 \end{pmatrix}. \quad (3.20)$$

Matching the solutions to have the regularity in (3.18) at the origin and the integrability in (3.19) will then leave us only one free parameter that comes from the homogeneity of the differential equations. In other words, we have $2n$ solutions, one for each j in the range of $-n + \frac{1}{2} \leq j \leq n - \frac{1}{2}$. Note that we do not expect any solution if j is not in the above range, since we have less parameters in the power series solutions near the origin. Specifically, for $n = 1$, we have two modes with $j = \pm\frac{1}{2}$.

Based on this analysis, we quantize the theory. For simplicity, we consider the case of single vortex with the winding number $n = 1$. Multivortex case can be similary done when they are widely separated. The quantization b and d in Eq. (3.7) (with their conjugates) for nonzero modes are the same as that in the vacuum sector and these modes describe the fermions around the soliton. We have two fermion zero modes with the angular momentum $\pm\frac{1}{2}$. The quantization for these modes will be

$$\{a_i, a_j^\dagger\} = \delta_{ij} \quad (i,j = 1,2). \quad (3.21)$$

The subscript 1 represents for $j = -\frac{1}{2}$ mode and 2 for $j = \frac{1}{2}$. According to the Jackiw-Rebbi interpretation [17], the soliton states will be degenerate due to the fermion zero modes and the quantum multiplet structure and the spin contents of the vortices will form a representation of the algebra relations in Eq. (3.21). We will then have four degenerate soliton states from the fermion zero mode algebra in (3.21):

$$|--\rangle, \quad |+-\rangle = a_1^\dagger|--\rangle, \quad |-+\rangle = a_2^\dagger|--\rangle,$$
$$|++\rangle = a_1^\dagger a_2^\dagger|--\rangle. \quad (3.22)$$

The algebra in Eq. (3.21) is not the same as the $N = 2$ supersymmetric (SUSY) algebra with the central charge described in Appendix B. The discrepancy comes from the fact that only the mode corresponding to $j = -\frac{1}{2}$ can be obtained by supertranslation of the vortex configuration. To see this, note that we can always get one fermion zero modes by the supersymmetry tranformation in Eq. (2.11) to the classical bosonic vortex background. We then get one fermion zero modes $\Psi_1^{(0)}$ proportional to

$$\psi_\uparrow = 2\sqrt{2}F, \quad \psi_\downarrow = \sqrt{2}iD_-\phi, \quad \chi_\uparrow = 2iG,$$
$$\chi_\downarrow = -2\partial_- N. \quad (3.23)$$

It is straightforward to check that the spin of this zero mode is $j = -\frac{1}{2}$ using the angular momentum operator in (3.8). Hence this is the zero mode corresponding to a_1 in (3.21). We now write the supercharge in (2.13) in a two-component form

$$Q = \int d^2x \begin{pmatrix} \sqrt{2}(D_0\phi^* - iF^*)\psi_\uparrow - i(\partial_0 N + f_0 - iG)\chi_\uparrow + \sqrt{2}(D_-\phi)^*\psi_\downarrow - i(\partial_+ N + f_+)\chi_\downarrow \\ \sqrt{2}(D_0\phi^* + iF^*)\psi_\downarrow - i(\partial_0 N + f_0 + iG)\chi_\downarrow - \sqrt{2}(D_+\phi)^*\psi_\uparrow + i(\partial_- N + f_-)\chi_\uparrow \end{pmatrix}. \quad (3.24)$$

Note that the down component of the supercharge with the background self-dual bosonic fields vanishes. The nonvanishing upper component $Q_\uparrow$ of the supercharge becomes

$$2\int d^2x \left(-\sqrt{2}iF^*\psi_\uparrow - G\chi_\uparrow + \frac{1}{\sqrt{2}}(D_-\phi)^*\psi_\downarrow - if_+\chi_\downarrow\right). \quad (3.25)$$

The algebra of the supercharge will then be that of $N = 2$ with the central charge in Appendix B. The nonvanishing supercharge is in the form of $Q_\uparrow = \int d^2x \Psi_1^{(0)}\Psi$. From the orthogonality of the modes, this is proportional to a_1 and so the operator a_2 corresponding to the other zero mode anticommutes with the supercharge. In other words, among the two independent supertranslations to the self-dual background configurations, one from $Q_\downarrow$ acts trivially and only the other one from $Q_\uparrow$ will give us the fermionic zero mode corresponding to $j=-\frac{1}{2}$. The other zero mode is not obtained from supersymmetry. The algebra between a_1 and $a_1^\dagger$ is the same as the $N=2$ SUSY algebra with the central term in Appendix B and will be realized as a doublet state. On the other hand, the doublet representation of the algebra from a_2 and $a_2^\dagger$ will transform as a singlet under the $N=2$ SUSY. Hence the above four degenerate solition states will form two sets of $N=2$ supermultiplets rather than one. One supermultiplet will be by $|--\rangle$ and $|+-\rangle$ with angular momenta $J_{\rm cl}$ and $J_{\rm cl}-\frac{1}{2}$, respectively. The other one is by $|-+\rangle$ and $|++\rangle$ with angular momenta $J_{\rm cl}+\frac{1}{2}$ and $J_{\rm cl}$, respectively. Here $J_{\rm cl}$ is the leading contribution to the angular momentum from the classical bosonic field configuration of self-dual vortices.

We now calculate the fermion number of the soliton by taking the expectation value of the fermion number operator for the degenerate soliton states:

$$\langle\pm\pm|\tfrac{1}{2}\int d^2x[\Psi(x)^\dagger,\Psi(x)]|\pm\pm\rangle=\pm\tfrac{1}{2}\pm\tfrac{1}{2}+\eta(H_F)\ . \tag{3.26}$$

The constant pieces differing on the vortex structures are from the fermion zero modes and $\eta(H_F)$, the so-called η invariant, given by

$$\eta(H_F)=\frac{1}{2}\left(\sum_{\omega>0}-\sum_{\omega<0}\right) \tag{3.27}$$

is from the nonzero modes.

For the Landau-Ginzburg model, i.e., $\kappa=0$ case, there exists a constant matrix $\begin{pmatrix}\gamma^0 & 0\\ 0 & -\gamma^0\end{pmatrix}$ that anticommutes with the Hamiltonian. This matrix matches the positive energy solution with the negative energy solution with the norm and the density of states preserved, making the value η to be 0. This can be also shown easily by direct evaluation of η using the method in Ref. [21], for example. Then from Eq. (3.26), the four degenerate states of the vortex (3.22) carry fermion numbers -1, 0, 0, 1, respectively.

The Hamiltonian (3.6) for the general value of κ no longer has such a structure. Niemi and Semenoff developed a method to calculate η for such a general case [21] which is based on the works of Atiyah, Patodi, and Singer [22]. Following them, let us introduce one parameter family of Hamiltonian $H(\tau)$ which interpolates the Hamiltonian $H_0\equiv H_F(\kappa=0)$ (when $\tau=-\infty$) and the Hamiltonian H_F (when $\tau=\infty$). The Dirac operator $D_\tau\equiv i\gamma^0(\partial_\tau-H_\tau)$ is defined on the extended manifold $\mathcal{M}$ of $R^1\times D^2$ where $R^1=\{\tau\}$ and D^2 is a disk of radius R in the usual x-y plane. The result is [21], in the $R\to\infty$ limit,

$$-\tfrac{1}{2}\eta(H_F)=\text{Index}(D_\tau)-\tfrac{1}{2}\eta(H_0)+\tfrac{1}{2}\eta(\text{Re}(P))\ . \tag{3.28}$$

Here, Index(D_τ) is the index of D_τ in the extended manifold $\mathcal{M}$ and P is the operator defined by projecting the operator D_τ onto the boundary of the disk D^2 for each value of τ. For our case, it becomes

$$\text{Re}P=-i\gamma^0\partial_\tau+i\begin{pmatrix}\sigma_1\cos\theta+\sigma_2\sin\theta & 0\\ 0 & \sigma_1\cos\theta+\sigma_2\sin\theta\end{pmatrix}\partial_\theta \tag{3.29}$$

which is almost a free equation. One can easily see that $\eta(\text{Re}P)=0$ and $\eta(H_0)=0$. Hence we get $\eta(H_F)=-2\text{Index}(D_\tau)$. The index of D_τ is hard to evaluate. Since the index is an integer, η is an even integer in general. But we expect its value to be zero, since η is shown to be zero in the above when $\kappa=0$. In other words, there is no contribution to the fermion number from the nonzero modes. Then the fermion number of the vortex is the same as that in the $\kappa=0$ case.

IV. QUANTUM CORRECTION TO THE MASS OF THE VORTEX

The quantization of the nonzero modes in Eq. (3.7) will correspond to the particle creation and annihilation around vortices. The leading mass correction comes from the quantum fluctuation of these modes. We will get this correction by comparing the bosonic and fermionic modes.

The quantization of the nonzero modes in Eq. (3.7) will correspond to the mass correction of the vortex and particle creation and annihilation around vortices. The mass correction comes from the sum of those mode contributions [23,24]. We will get this correction by comparing the bosonic and fermionic modes. First, let us calculate the quantum correction to the mass of the vortex. The leading quantum correction to the vortex mass comes from the quantum fluctuation modes given by

$$\begin{aligned}\Delta M&=\sum\omega_B-\sum\omega_F\\&=\int d\lambda\sqrt{\lambda}\left(\frac{dn_B(\lambda)}{d\lambda}-\frac{dn_F(\lambda)}{d\lambda}\right).\end{aligned} \tag{4.1}$$

The frequency ω_B (ω_F) are eigenvalues and $n_B(n_F)$ are the number of states upto eigenvalue λ of the bosonic (fermionic) fluctuating modes.

We first evaluate the mass correction for the usual Maxwell theory ($\kappa=0$). The equations for the fermionic modes around the self-dual vortex with the positive frequency ω_F becomes

$$i\omega_F\begin{pmatrix}U\\V\end{pmatrix}=\begin{pmatrix}0 & D_F\\ -D_F^\dagger & 0\end{pmatrix}\begin{pmatrix}U\\V\end{pmatrix}, \qquad (4.2)$$

where

$$U=\begin{pmatrix}\psi_\uparrow\\ \chi_\downarrow\end{pmatrix}, \qquad V=\begin{pmatrix}\psi_\downarrow\\ \chi_\uparrow\end{pmatrix},$$

and the Dirac-like operator D_F is defined as

$$D_F=\begin{pmatrix}D_+ & -\sqrt{2}e\phi\\ -\sqrt{2}e\phi^* & \partial_-\end{pmatrix}. \qquad (4.3)$$

Here the bosonic fields ϕ and A_i in the covariant derivatives are the classical background fields of the self-dual vortex. Apply the Dirac-like operator D_F to the equations of fermions in Eq. (4.2) to get

$$\omega_F^2\begin{pmatrix}U\\V\end{pmatrix}=\begin{pmatrix}D_FD_F^\dagger & 0\\ 0 & D_F^\dagger D_F\end{pmatrix}\begin{pmatrix}U\\V\end{pmatrix}. \qquad (4.4)$$

The contribution from the fermionic modes to the soliton mass is given by

$$\sum\omega_F=\sum\omega_U+\sum\omega_V. \qquad (4.5)$$

Note that if (U,V) is the mode corresponding to ω_F then $(U,-V)$ is the mode corresponding to $-\omega_F$. Let us represent $\frac{dn_+}{d\lambda}$ ($\frac{dn_-}{d\lambda}$) as the density of states of the operator of $D_FD_F^\dagger$ ($D_F^\dagger D_F$). Then the density of the states for the fermion modes in Eq. (4.2) is half of those of the second-order equation in Eq. (4.4),

$$\frac{dn_F(\lambda)}{d\lambda}=\frac{1}{2}\left(\frac{dn_+(\lambda)}{d\lambda}+\frac{dn_-(\lambda)}{d\lambda}\right), \qquad (4.6)$$

since only half of the solution of the above equation correspond to the positive frequency solution in Eq. (4.2).

We now turn to the bosonic fluctuations. The equations of motion in Eq. (3.4) for $\kappa=0$ become

$$\begin{aligned}(\partial_\mu\partial^\mu-2e^2|\phi|^2)\delta A_+-\partial_+\{\partial_\mu\delta A^\mu+ie(\phi^*\delta\phi-\phi\delta\phi^*)\}+2ie(D_-\phi)^*\delta\phi&=0,\\ (-\partial_t^2+D_-D_+-2e^2|\phi|^2)\delta\phi-ie\phi\{\partial_\mu\delta A^\mu+ie(\phi^*\delta\phi-\phi\delta\phi^*)\}-ieD_-\phi\delta A_+&=0,\\ (\nabla^2-2e^2|\phi|^2)\delta A^0+\frac{d}{dt}\{\nabla_i\delta A^i+ie(\phi^*\delta\phi-\phi\delta\phi^*)\}&=0,\\ (\partial_\mu\partial^\mu-2e^2|\phi|^2)\delta N&=0. \qquad (4.7)\end{aligned}$$

The gauge field is written in terms of $\delta A_+=\delta A_1+i\delta A_2$ for convenience. The classical background fields ϕ and A_i appearing in the above equations are the configuration for the self-dual vortices satisfying the self-dual equations (self-dual) with $\kappa=0$. Among the fluctuations satisfying the above equation (4.12), we have to subtract those fluctuation modes corresponding to the gauge transformation. For this purpose, we choose the physical gauge condition as follows:

$$\nabla_i\delta A^i+ie(\phi^*\delta\phi-\phi\delta\phi^*)=0. \qquad (4.8)$$

The above equation (4.7) for δA^0 with this gauge condition then requires that

$$\delta A^0=0, \qquad (4.9)$$

since the operator $\nabla^2-2e^2|\phi|^2$ is negative definite. Other equations for the bosonic fluctuations can then be written as

$$(\partial_\mu\partial^\mu-2e^2|\phi|^2)\delta A_++2ie(D_-\phi)^*\delta\phi=0, \qquad (4.10)$$

$$(-\partial_t^2+D_-D_+-2e^2|\phi|^2)\delta\phi-ieD_-\phi\delta A_+=0, \qquad (4.11)$$

$$(\partial_\mu\partial^\mu-2e^2|\phi|^2)\delta N=0. \qquad (4.12)$$

To find the equations for the fluctuation modes corresponding to the gauge transformations, we take an infinitesimal gauge transformation $\delta_G\delta A_+=\partial_+\Lambda$ and $\delta_G\delta\phi=ie\Lambda\phi$ to Eq. (4.12). We then get the equations for the gauge transformation modes satisfying

$$(\partial_\mu\partial^\mu-2e^2|\phi|^2)\Lambda=0. \qquad (4.13)$$

We have to subtract the degree of freedom satisfying this equation. Since this equation is the same as that of the real scalar field of δN in Eq. (4.12), the contribution of the fluctuation corresponding to the gauge degree of freedom Λ satisfying Eq. (4.13) and that of the neutral scalar δN in Eq. (4.12) cancels out in the contribution to the correction of the mass. Hence the only bosonic contribution is from the complex fields δA_+ and $\delta\phi$ and the vortex mass correction from bosons becomes

$$\sum\omega_B=\sum\omega_{\delta\phi}+\sum\omega_{\delta A_+}. \qquad (4.14)$$

To get the density of the states we write the equations for δA_+ and $\delta\phi$ in Eqs. (4.7) in matrix form as

$$-\partial_t^2\begin{pmatrix}\delta\phi\\ \frac{-i}{\sqrt{2}}\delta A_+\end{pmatrix}$$

$$=\begin{pmatrix}D_-D_+-2e^2|\phi|^2 & \sqrt{2}e\phi\\ \sqrt{2}e\phi^* & \partial_i{}^2-2e^2|\phi|^2\end{pmatrix}\begin{pmatrix}\delta\phi\\ \frac{-i}{\sqrt{2}}\delta A_+\end{pmatrix} \qquad (4.15)$$

$$=D_F^\dagger D_F\begin{pmatrix}\delta\phi\\ \frac{-i}{\sqrt{2}}\delta A_+\end{pmatrix}, \qquad (4.16)$$

where the Dirac operator D_F is defined in Eq. (4.3). From this we see that the density of states for the bosons are n_+. With the result for fermions in Eq. (4.6), this gives

$$\frac{dn_B(\lambda)}{d\lambda} - \frac{dn_F(\lambda)}{d\lambda} = \frac{1}{2}\left(\frac{dn_+(\lambda)}{d\lambda} - \frac{dn_-(\lambda)}{d\lambda}\right), \quad (4.17)$$

and hence the mass correction becomes

$$\Delta M = \sum \omega_B - \sum \omega_F \quad (4.18)$$
$$= \frac{1}{2}\int d\lambda \left(\frac{dn_+(\lambda)}{d\lambda} - \frac{dn_-(\lambda)}{d\lambda}\right)\sqrt{\lambda}. \quad (4.19)$$

We can perform the integration by calculating the density of states from the phase shift. This method is summarized in Appendix C and we get zero for the value of the above integration.

Evaluation of the integrand can also be done with the help of the index formula [20]:

$$I(z) = \mathrm{Tr}\left(\frac{z}{z + D_F^\dagger D_F} - \frac{z}{z + D_F D_F^\dagger}\right)$$
$$= \int_0^\infty d\lambda \frac{z}{z+\lambda}\left(\frac{dn_+(\lambda)}{d\lambda} - \frac{dn_-(\lambda)}{d\lambda}\right). \quad (4.20)$$

The index is easily calculated and the result is $2n$. For the index to be independent of z, the integrand in the above should be

$$\frac{dn_+(\lambda)}{d\lambda} - \frac{dn_-(\lambda)}{d\lambda} = 2n\delta(\lambda). \quad (4.21)$$

By plugging this result into the Eq. (4.19) we see that the mass correction to the vortex in $N = 2$ model at the one-loop level vanishes:

$$\Delta M = 0.$$

This result agrees with Ref. [25].

As a check for the independence of the choice of the gauge, let us describe this in the covariant gauge. We choose the background gauge by adding the following gauge-fixing term to the Lagrangian:

$$\mathcal{L}_{\rm GF} = -\tfrac{1}{2}\left\{\partial_\mu \delta A^\mu + ie(\phi^*\delta\phi - \phi\delta\phi^*)\right\}^2. \quad (4.22)$$

The equations for the modes in this gauge are then obtained from the quadratic pieces of the Lagrangian by adding:

$$\mathcal{L}_B^{(2)} + \mathcal{L}_F^{(2)} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm gh}. \quad (4.23)$$

The ghost Lagrangian from the gauge fixing is given by

$$\mathcal{L}_{\rm gh} = \bar{c}(\partial_\mu\partial^\mu - 2e^2|\phi|^2)c. \quad (4.24)$$

The equations for the fermionic modes are the same as before. For the equations of the bosonic fluctuations, those for δA_+, $\delta\phi$, and δN are the same as before in (4.10) and (4.11). The equations for δA^0 which are dynamical in this gauge and the ghost field c are exactly same as that of field δN in (4.12). Hence the contribution from the bosonic fields and ghosts is given by

$$\sum \omega_B = \sum \omega_{\delta\phi} + \sum \omega_{\delta A_+} + \frac{1}{2}\sum \omega_{\delta A^0} + \frac{1}{2}\sum \omega_{\delta N} - \sum \omega_c \quad (4.25)$$
$$= \sum \omega_{\delta\phi} + \sum \omega_{\delta A_+}. \quad (4.26)$$

We have used the fact that the contribution to the mass from the ghost fields and that from fields δN and δA^0 cancel out since the equation for the ghost fields are the same as those for fields δN and δA^0 in Eq. (4.12). We have shown that the only contribution comes from δA_+ and $\delta\phi$. Since equations for these fields are the same as those in the Coulomb gauge, the remaining arguments are the same as before and so we get the same result for the mass correction for the vortex.

Now consider another extreme limit of the Lagrangian with $\kappa \to \infty$. The equation of motion of the fermion modes is

$$-\partial_0 \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = D_{(\infty)} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \quad (4.27)$$

with

$$D_{(\infty)} = \begin{pmatrix} 2i\frac{e^2}{\kappa}|\phi|^2 & D_+ \\ D_- & -2i\frac{e^2}{\kappa}(2|\phi|^2 - v^2) \end{pmatrix}. \quad (4.28)$$

The equations of bosonic fluctuations can be obtained in a straighforward way from the Lagrangian (2.5):

$$\kappa\epsilon^{\mu\nu\lambda}\partial_\nu\delta A_\lambda - 2e^2|\phi|^2\delta A^\mu - ie(\phi^*\overleftrightarrow{D}^\mu\delta\phi + \delta\phi^*\overleftrightarrow{D}^\mu\phi) = 0 \quad (4.29)$$

and

$$D_0^2\delta\phi + ie(2\delta A^0 D_0 + \partial_0\delta A^0)\phi = D_i^2\delta\phi - ie(\delta A^i D_i\phi + D_i\delta A^i\phi) - \frac{e^4}{\kappa^2}(9|\phi|^4 - 8v^2|\phi|^2 + v^4)\delta\phi - \frac{e^4}{\kappa^2}\phi^2\delta\phi^*(6|\phi|^2 - 4v^2). \quad (4.30)$$

We choose the gauge for the spatial gauge fields as

$$\nabla_i\delta A^i + 2i\frac{e^3}{\kappa}(2|\phi|^2 - v^2)(\phi^*\delta\phi - \phi\delta\phi^*) = -\frac{e}{\kappa}(\phi^*\partial^0\delta\phi - \partial^0\delta\phi^*\phi). \quad (4.31)$$

The zeroth component of Eq. (4.29) and the gauge-fixing condition (4.31) can be combined into the complex equation (and its complex conjugate)

$$\partial_-\delta A_+ + 4i\frac{e^3}{\kappa^2}(2|\phi|^2 - v^2)\phi^*\delta\phi - 2i\frac{e^2}{\kappa}|\phi|^2 Q = 2\frac{e}{\kappa}\phi^*\partial_0\delta\phi, \quad (4.32)$$

where

$$\mathcal{Q} = \delta A^0 + \frac{e}{\kappa}(\phi^* \delta\phi + \delta\phi^* \phi). \tag{4.33}$$

The spatial component of Eq. (4.29) can be written as

$$\partial_0 \delta A_+ + 2i\frac{e^2}{\kappa}|\phi|^2 \delta A_+ - 2\frac{e}{\kappa}\phi^* D_+ \delta\phi + \partial_+ \mathcal{Q} = 0. \tag{4.34}$$

We will show that the gauge-fixing condition and the above equations of motion gives δA^0 so that $\mathcal{Q}$ in the above equations is zero, and δA^0 is fixed as

$$\delta A^0 = -\frac{e}{\kappa}(\phi^* \delta\phi + \delta\phi^* \phi). \tag{4.35}$$

To show this we first rewrite the equations for the scalar field (4.30) using the gauge-fixing condition and Eq. (4.32) as

$$\partial_0^2 \delta\phi = D_- D_+ \delta\phi - 4\frac{e^4}{\kappa^2}(2|\phi|^2 - v^2)^2 \delta\phi$$
$$-ie\delta A_+ (D_- \phi) \tag{4.36}$$
$$+ie\frac{1}{\phi^*}(|\phi|^2 - v^2)\partial_- \delta A_+ - ie\phi\partial_0 \mathcal{Q}. \tag{4.37}$$

Comparing this equation with the equation obtained by taking the time derivative of Eq. (4.32) we get

$$\nabla^2 \mathcal{Q} \equiv \nabla^2 \left(\delta A^0 + \frac{e}{\kappa}(\phi^* \delta\phi + \delta\phi^* \phi) \right) = 0, \tag{4.38}$$

hence $\mathcal{Q} = 0$. Equations (4.32) and (4.34) then become

$$\phi^* \partial_0 \delta\phi = \frac{\kappa}{2e}\partial_- \delta A_+ + 4i\frac{e^2}{\kappa}(2|\phi|^2 - v^2)\phi^* \delta\phi \tag{4.39}$$

and

$$\partial_0 \delta A_+ = -2i\frac{e^2}{\kappa}|\phi|^2 \delta A_+ - 2\frac{e}{\kappa}\phi^* D_+ \delta\phi. \tag{4.40}$$

And they can replace the set of Eqs. (4.30), (4.32), and (4.34). The imaginary part of Eq. (4.39) is nothing but the gauge-fixing condition in Eq. (4.31). We can also easily see that the two equations (4.39) and (4.40) with δA_0 given as in Eq. (4.35) imply the equation of motion of the scalar fluctuation in Eq. (4.30). Hence the physically relevant bosonic fluctuation modes are described by the above two equations.

To compare these equations with the fermionic modes we write Eqs. (4.39) and (4.40):

$$-\partial_0 \begin{pmatrix} -\frac{\kappa}{2e}\frac{\delta A_+}{\phi^*} \\ \delta\phi \end{pmatrix} = D_{(\infty)} \begin{pmatrix} -\frac{\kappa}{2e}\frac{\delta A_+}{\phi^*} \\ \delta\phi \end{pmatrix}. \tag{4.41}$$

Note that this equation has precisely the same form as that for the fermionic modes in Eq. (4.27). This means that the density of the modes as well as the spectrum of the bosons are equal to that of fermions. Hence the quantum correction for the mass of the vortex vanishes identically.

Finally, let us consider the general case with finite κ. The equations for the the bosonic fluctuation fields are given in Eq. (3.4). We have to fix the gauge to eliminate those fluctuations corresponding to the gauge transformation. We choose the background-type gauge condition

$$\partial_i \delta A^i + \kappa\delta\mathcal{F} + ie(\phi^* \delta\phi - \phi\delta\phi^*) = -\partial_0 \delta N, \tag{4.42}$$

where $\delta\mathcal{F}$ satisfies the equation

$$\partial_0 \delta\mathcal{F} - \epsilon_{ij}\partial_i \delta A_j - \kappa\delta N - e(\phi^* \delta\phi + \phi\delta\phi^*) = 0. \tag{4.43}$$

The equations of motion in Eq. (3.4) with the help of the gauge-fixing condition in Eq. (4.43) can be written as

$$(\partial_0^2 + D_- D_+ - 2e^2|\phi|^2)\delta\phi - 2ieA^0 \partial_0 \delta\phi - ieD_- \phi\delta A_+ - \kappa e\delta N\phi + 2eA^0 \phi(\delta A^0 - \delta N) + ie\kappa\phi\delta\mathcal{F} = 0, \tag{4.44}$$

$$(\partial_i^2 - 2e^2|\phi|^2)\delta A^0 - (\kappa^2 + \partial_0^2)\delta N - e(\kappa + 2eN)(\phi^* \delta\phi + \delta\phi^* \phi) = 0, \tag{4.45}$$

$$(\partial_\mu \partial^\mu - 2e^2|\phi|^2)\delta A_+ + \kappa\partial_+ \delta\mathcal{F} + i\kappa(\partial_+ \delta N + \partial_0 \delta A_+) + 2ie(D_- \phi)^* \delta\phi = 0, \tag{4.46}$$

$$(\partial_\mu \partial^\mu - \kappa^2 - 2e^2|\phi|^2)\delta N - e(\kappa + 2eN)(\phi^* \delta\phi + \phi\delta\phi^*) = 0. \tag{4.47}$$

We compare Eqs. (4.45) and (4.47) to get

$$(\partial_i^2 - 2e^2|\phi|^2)(\delta A^0 - \delta N) = 0. \tag{4.48}$$

Then

$$\delta A^0 = \delta N. \tag{4.49}$$

This relation reduces (4.44) and (4.46) in the form

$$(-\partial_0^2 + D_- D_+ - 2e^2|\phi|^2)\delta\phi - 2ieN\partial_0 \delta\phi$$
$$+ie\kappa\phi(\delta\mathcal{F} + i\delta N) - ieD_- \phi\delta A_+ = 0, \tag{4.50}$$

$$(-\partial_0^2 + \nabla^2 - 2e^2|\phi|^2)A_+$$
$$+i\kappa\partial_0 \delta A_+ + \kappa\partial_+(\delta\mathcal{F} + i\delta N) + 2ie(D_- \phi)^* \delta\phi = 0. \tag{4.51}$$

We have a set of Eqs. (4.50), (4.51), and (4.47) which are supplemented by the gauge condition (4.42) and (4.43) or

$$(\partial_0 + i\kappa)(\delta\mathcal{F} + i\delta N) + i\partial_- \delta A_+ - 2e\phi^* \delta\phi = 0\ . \tag{4.52}$$

We now turn to the Dirac equation given in Eq. (3.5). We can remove $\psi_\uparrow$ from this equation by using the first line of that equation:

$$-\partial_0\psi_\uparrow = D_+\psi_\downarrow - \sqrt{2}e\phi\chi_\uparrow. \tag{4.53}$$

Then we have

$$(-\partial_0^2 + D_-D_+ - 2e^2|\phi|^2)\psi_\downarrow - 2ieN\psi_\downarrow - \sqrt{2}e(D_-\phi)\chi_\uparrow + \sqrt{2}ie\kappa\phi\chi_\downarrow = 0 \tag{4.54}$$

and

$$(-\partial_0^2 + \nabla^2 - 2e^2|\phi|^2)\chi_\uparrow + i\kappa\partial_0\chi_\uparrow - \sqrt{2}e(D_-\phi)^*\psi_\downarrow + i\kappa\partial_+\chi_\downarrow = 0, \tag{4.55}$$

together with the last line of the Dirac equation:

$$\partial_0\chi_\downarrow - \sqrt{2}e\phi^*\psi_\downarrow + \partial_-\chi_\uparrow + i\kappa\chi_\downarrow = 0. \tag{4.56}$$

Comparison of bosonic equations (4.50), (4.51), and (4.52) with fermionic equations (4.54), (4.55), and (4.56) gives us the relation

$$\psi_\downarrow = \delta\phi, \quad \chi_\uparrow = \frac{i}{\sqrt{2}}\delta A_+, \quad \chi_\downarrow = \frac{1}{\sqrt{2}}(\delta\mathcal{F} + i\delta N). \tag{4.57}$$

Note that this identification was pointed in Ref. [14] for the zero mode case. Here we show that the identification holds also for nonzero mode. The number of fermion states n_F is simply related with the sum of the phase shifts of $\psi_\uparrow$, $\psi_\downarrow$, $\chi_\uparrow$, and $\chi_\downarrow$. In Appendix C we will show that the sum of phase shift of $\psi_\uparrow$ and $\chi_\downarrow$ is equal to the sum of phase shift of $\psi_\downarrow$ and $\chi_\uparrow$. Therefore

$$n_F = n_{\psi_\downarrow} + n_{\chi_\uparrow}. \tag{4.58}$$

For the bosonic case, the physical degrees of freedoms are $\delta\phi$, δN, and δA_+ where the gauge degree of freedom is subtracted. However Eq. (4.49) says that $\partial_0\delta N = i\omega\delta N$ (assuming the time dependence as $e^{-i\omega t}$) is nothing but

$$\nabla_i\delta A^i - i\kappa\delta\mathcal{F} + ie(\phi^*\delta\phi - \phi\delta\phi^*) = -\omega\delta N, \tag{4.59}$$

which is a gauge degree of freedom. So we assert that, as in the case of $\kappa = 0$, we would subtract δN instead of subtracting the gauge degree. Then bosonic degree of freedom is given by $\delta\phi$ and δA_+. Then

$$n_B = n_{\delta\phi} + n_{\delta A_+} \tag{4.60}$$

and so we get

$$n_F = n_B. \tag{4.61}$$

Therefore there is no mass correction:

$$\Delta M = 0. \tag{4.62}$$

V. SUMMARY AND DISCUSSION

We have studied various quantum aspects of the $N = 2$ supersymmetric Maxwell Chern-Simons theory. First we identified the mass spectrum, the spin contents, and the supermultiplet structures of the particles both in the broken and the unbroken sectors.

Then, we analyzed the vortex sector, the main subject of this paper. Starting from the canonical angular momentum we evaluated the leading quantum correction to the classical value of the angular momentum of the vortex coming from the fermion zero modes. For the supermultiplet structure of vortices, fermion zero modes play an important role. The algebra by the operators of the fermion zero modes around the winding number $n = 1$ vortex is larger than that of the $N = 2$ SUSY algebra with the central charge. They provide two supermultiplets with the relative spin difference half, rather than single supermultiplet. This is in contrast with the case in the monopoles in the 3+1 dimensions or kinks in the 1+1 dimensions. The fermion number of the vortex is also calculated.

Leading quantum correction to the mass of the vortices is calculated separately depending upon whether $\kappa = 0$, $\kappa = \infty$, or finite κ. The mass correction can be obtained either through the index theorem or by comparing the modes between bosons and fermions. In all cases we do not see any mass correction.

Self-duality is deeply related to the underlying supersymmetry. Here we have considered only $N = 2$ supersymmetric model. For the model with $N = 1$ supersymmetry that allows self-dual vortices is not treated. We expect mass correction in this model since we do not see any simple way of matching the bosonic and fermionic contributions.

In the models with the self-duality, the mass of the vortex will be simply related to the magnetic flux of the objects at the tree level. An interesting question is whether this so-called Bogomolnyi bound will still be saturated at the quantum level. There are some models in two and four dimensions known to satisfy the Bogomolnyi bound at the quantum level [26,18]. To show the saturation at the quantum level, Olive and Witten [26] used the argument that the size of the supermultiplets for particles and solitons cannot change abruptly by perturbation. This argument seems not to be directly applied in our case. First, we do not have any self-duality for particles since we do not have any conserved charge for the particles in the broken sector. In the vortex sector, for the single vortex with the winding number 1 to be specific, we have four degenerate vortex states from two fermion zero modes. These form two irreducible supermultiplets of size two with the Bogomolnyi bound saturated. If the bound is not saturated, we might have only one supermultiplet of size four. Whether the bound is saturated or not, the total size of the states remains the same. On the other hand, in those models in other dimensions mentioned above, the supermultiplets of the degenerate solitons form single irreducible representation of the superalgebra with the bound saturated. If the bound does not become saturated by the perturbation, we need more states, which is unlikely. This is the difference between our models and those in other dimensions. This difference is related to the fact that all the fermion zero modes in our case do not come from the supersymmetric transformation of the vortex for winding number one unlike those other models.

One way to check the saturation of the bound at the quantum level is to calculate directly the quantum corrections in Eq. (2.17). We have shown that the leading quantum correction to the mass vanish. The magnetic flux on the right-hand side of Eq. (2.17) may not get any quantum correction. We still need the quantum correction to the coupling constants in the presence of the vortex background to check the validity of the quantum Bogomolnyi bound. This is quite an interesting open problem. For some quantities for the particles in the vacuum sector, there exist some perturbative calculations [29].

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APPENDIX A: SPINS

In this appendix we will describe how to determine the spin of elementary excitations. The spin of the fermion coupled with the CS field was considered in Ref. [27]. It was found that the fermions carry spin $\pm 1/2$ and the sign of spin is determined by the sign of the mass term in the Lagrangian. The spin of vectors in the case of the unbroken Maxwell CS gauge theory was considered in Ref. [2] while that of broken CS theory considered in Ref. [28]. In the following we will describe how to determine the spin of gauge field of the Maxwell CS gauge theory in the broken phase.

We start from the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda - \frac{1}{2}\mu^2 A_\mu A^\mu \ . \quad \text{(A1)}$$

This may be considered coming from the spontaneously broken theory of the Maxwell-Chern-Simons Higgs in the unitary gauge. The canonical variables of this system are A_i and $\pi_i = F_{0i} + \frac{\kappa}{2}\epsilon_{ij}A^j$. We separate the longitudinal and transverse components using the identification

$$A^i = \epsilon_{ij}\hat{\nabla}_j\varphi - \hat{\nabla}_i\chi \ , \quad \text{(A2)}$$

$$\pi_i = \epsilon_{ij}\hat{\nabla}_j\pi_\varphi - \hat{\nabla}_i\pi_\chi \ , \quad \text{(A3)}$$

with the abbreviation $\hat{\nabla}_i = \nabla_i/\sqrt{-\nabla^2}$.

With these new degrees of freedom, φ, χ, π_φ, and π_χ, the Hamiltonian density for the system in (A1) is written as

$$\mathcal{H} = \frac{1}{2}\left(\pi_\varphi + \frac{\kappa}{2}\chi\right)^2 + \frac{1}{2}\left(\pi_\chi - \frac{\kappa}{2}\varphi\right)^2 + \frac{1}{2}(\nabla_i\varphi)^2 + \frac{1}{2}\frac{1}{\mu^2}\sqrt{-\nabla^2}\left(\pi_\chi + \frac{\kappa}{2}\varphi\right)^2 + \frac{1}{2}\mu^2(\varphi^2 + \chi^2). \quad \text{(A4)}$$

In the derivation of (A4) we solved the Gauss law for A^0 as

$$A^0 = \frac{1}{\mu^2}\left(\nabla_i\pi_i + \frac{\kappa}{2}\epsilon_{ij}\nabla_i A_j\right) = \frac{1}{\mu^2}\sqrt{-\nabla^2}\left(\pi_\chi + \frac{\kappa}{2}\varphi\right). \quad \text{(A5)}$$

By varying the Hamiltonian we can write the equation of motion as

$$\partial_t^2\begin{pmatrix}\varphi \\ \pi_\chi/\kappa\end{pmatrix} = \begin{pmatrix}\nabla^2 - \mu^2 - \frac{\kappa^2}{2} & \kappa^2 \\ \mu^2 + \frac{\kappa^2}{4} & \nabla^2 - \mu^2 - \frac{\kappa^2}{2}\end{pmatrix}\begin{pmatrix}\varphi \\ \pi_\chi/\kappa\end{pmatrix} \quad \text{(A6)}$$

with

$$\pi_\varphi + \frac{\kappa}{2}\chi = \dot{\varphi}, \qquad \left(\mu^2 + \frac{\kappa^2}{4}\right)\chi + \frac{\kappa}{2}\pi_\varphi = -\dot{\pi}_\chi \ . \quad \text{(A7)}$$

Note that the usual notions of variable and conjugate momentum for χ and π_χ has been reversed. The mass matrix on the right-hand side of (A6) has two eigenvalues:

$$m_\pm^2 = \mu^2 + \frac{\kappa^2}{2} \pm |\kappa|\sqrt{\mu^2 + \frac{\kappa^2}{4}}. \quad \text{(A8)}$$

Now, we consider the generators of the Poincaré algebra. The Hamiltonian is the integration of its density (A4). The momentum is

$$P^i = \int d^2x\left[\varphi\nabla_i\dot{\varphi} + \frac{1}{\mu^2}\left(\pi_\chi + \frac{\kappa}{2}\varphi\right)\nabla_i\left(\dot{\pi}_\chi + \frac{\kappa}{2}\dot{\varphi}\right)\right] \ . \quad \text{(A9)}$$

The angular momentum is

$$J = -\int d^2\epsilon_{ij}x^i\left[\dot{\varphi}\nabla_j\varphi + \frac{1}{\mu^2}\left(\dot{\pi}_\chi + \frac{\kappa}{2}\dot{\varphi}\right) \times\nabla_j\left(\pi_\chi + \frac{\kappa}{2}\varphi\right)\right]. \quad \text{(A10)}$$

This means that φ and π_χ behave as spin-0 fields. However the boost generators have an infrared problem which was expected from the decomposition, and removal of this infrared divergence will fix the spin, as first noted in Ref. [2]. After some manipulation we have the following expression for the boost generator:

$$B^i = \int d^2x\left[x^i\mathcal{H} + \kappa\dot{\varphi}\epsilon_{ij}\frac{\nabla_j}{-\nabla^2}\varphi - \dot{\varphi}\epsilon_{ij}\frac{\nabla_j}{-\nabla^2}\left(\pi_\chi + \frac{\kappa}{2}\varphi\right) - \left(\dot{\pi}_\chi + \frac{\kappa}{2}\dot{\varphi}\right)\epsilon_{ij}\frac{\nabla_j}{-\nabla^2}\varphi\right] \ . \quad \text{(A11)}$$

The equation of motion (A6) can be rewritten as

$$\partial_t^2 \begin{pmatrix} \varphi \\ (\pi_\chi + \frac{\kappa}{2}\varphi)/\mu \end{pmatrix} = \begin{pmatrix} \nabla^2 - \mu^2 - \kappa^2 & \kappa\mu \\ \kappa\mu & \nabla^2 - \mu^2 \end{pmatrix} \begin{pmatrix} \varphi \\ (\pi_\chi + \frac{\kappa}{2}\varphi)/\mu \end{pmatrix} . \tag{A12}$$

We can diagonalize the matrix on the right-hand side by the new fields ξ and η:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \mu\kappa/N_+ & (m_-^2 - \mu^2)/N_+ \\ (m_+^2 - \mu^2)/N_- & \mu\kappa/N_- \end{pmatrix} \times \begin{pmatrix} \varphi \\ (\pi_\chi + \frac{\kappa}{2}\varphi)/\mu \end{pmatrix} \tag{A13}$$

with two suitable normalization constants N_+ and N_-. With the new degrees of freedom ξ and η the generators of the Poincaré algebra have the form

$$H = \tfrac{1}{2}\int d^2x \Big[\dot{\xi}^2 + \dot{\eta}^2 + (\nabla_i\xi)^2 + (\nabla_i\eta)^2 + m_+^2\xi^2 + m_-^2\eta^2 \Big] , \tag{A14}$$

$$P_i = \int d^2x \left[\xi\nabla_i\dot{\xi} + \eta\nabla_i\dot{\eta} \right] , \tag{A15}$$

$$J = -\int d^2x x^i\epsilon_{ij} \left[\dot{\xi}\nabla_j\xi + \dot{\eta}\nabla_j\eta \right] , \tag{A16}$$

$$B^i = \int d^2x \left[x^i\mathcal{H} + m_+\dot{\xi}\epsilon_{ij}\frac{\nabla_j}{-\nabla^2}\xi - m_-\dot{\eta}\epsilon_{ij}\frac{\nabla_j}{-\nabla^2}\eta \right] , \tag{A17}$$

with $m_\pm = \sqrt{m_\pm^2} = \sqrt{\mu^2 + \frac{\kappa^2}{4}} \pm \frac{\kappa}{2}$. Note that in all formulas, ξ and η have the same contribution except for B_i where the signs of infrared singular terms are opposite. By direct application of the argument in Ref. [2], one can show that after removing the infrared singularity the angular momentum generator (A16) has additional terms which are propotional to $\frac{m_+}{|m_+|}$ and $-\frac{m_-}{|m_-|}$, respectively. From those terms we can determine the spin of ξ and η as $+1$ and -1.

APPENDIX B: SUPERMULTIPLETS

In this appendix we will describe the supermultiplet structures. We will consider both the particle and vortex supermultiplets. The structure will depend on whether or not we have the central charge. In the case of the vortices or the symmetry unbroken sector of the vacuum, we will have the central charges while in the particle spectrum in the symmetry broken sector we have no central charges. For simplicity we start from the SUSY alsgebra without the central charge

$$\{Q_A^\alpha, Q_B^\beta\} = 2\bar{\sigma}_\mu^{\alpha\beta}\delta_{AB}P^\mu \tag{B1}$$

with

$$\bar{\sigma}_\mu = (I, \sigma_3, \sigma_1) . \tag{B2}$$

We work in the Majorana represention and Q^α corresponds to the real spinor in SL$(2,R)$ in this appendix. The index A, B denotes the indices for the extended algebra. In the case of $N = 1$, we have only one supercharge Q_α. In the massive case (this is sufficient in our analysis), we can take the rest frame and the algebra becomes

$$\{Q^\alpha, Q^\beta\} = 2\delta^{\alpha\beta}M . \tag{B3}$$

Introducing $a = \frac{1}{2\sqrt{M}}(Q^1 - iQ^2)$, $a^\dagger = \frac{1}{2\sqrt{M}}(Q^1 + iQ^2)$, we get the algebra

$$\{a, a^\dagger\} = 1 \quad \{a, a\} = 0 \quad \{a^\dagger, a^\dagger\} = 0 . \tag{B4}$$

We can construct the Clifford vacuum $|\Omega(j)\rangle$ with spin j by setting $a|\Omega(j)\rangle = 0$. The angular momentum operator will then be realized as $J = ja^\dagger a$ in this representation. We will then have two states, i.e., $|\Omega(j)\rangle$ with spin j and $a^\dagger|\Omega(j)\rangle$ with spin $j + \frac{1}{2}$. To verify this last fact we note

$$[J, Q^\alpha] = \tfrac{1}{2}Q_\alpha \tag{B5}$$

and hence

$$[J, a] = -\tfrac{1}{2}a, \quad [J, a^\dagger] = \tfrac{1}{2}a^\dagger. \tag{B6}$$

We now turn to the case of $N = 2$ without the central charge. The indices A and B will run from 1 to 2. In this case we can rewrite the algebra in Eq. (B1) as

$$\{Q^\alpha, Q^{*\beta}\} = 2\bar{\sigma}_\mu P^\mu \tag{B7}$$

with the complex notation $Q^\alpha = \frac{1}{\sqrt{2}}(Q_1^\alpha - iQ_2^\alpha)$. By defining

$$a_1 = \frac{1}{2\sqrt{M}}(Q^1 - iQ^2), \quad a_1^\dagger = \frac{1}{2\sqrt{M}}(Q^{*1} + iQ^{2*}),$$
$$a_2 = \frac{1}{2\sqrt{M}}(Q^1 + iQ^2), \quad a_2^\dagger = \frac{1}{2\sqrt{M}}(Q^{*1} - iQ^{2*}) , \tag{B8}$$

we get the algebra

$$\{a_1, a_1^\dagger\} = \{a_2, a_2^\dagger\} = 1 , \tag{B9}$$

$$\{a_1, a_2\} = \{a_1^\dagger, a_2^\dagger\} = \{a_1, a_2^\dagger\} = \{a_2, a_1^\dagger\} = 0 , \tag{B10}$$

and

$$[J, a_1] = -\tfrac{1}{2}a_1, \quad [J, a_2] = \tfrac{1}{2}a_2. \tag{B11}$$

Based on the algebra, we have four states:

$$\{|\Omega(j)\rangle, a_1^\dagger|\Omega(j)\rangle, a_2^\dagger|\Omega(j)\rangle, a_1^\dagger a_2^\dagger|\Omega(j)\rangle\} \tag{B12}$$

with angular momentum $(j, j + \frac{1}{2}, j - \frac{1}{2}, j)$.

Now consider the supermultiplets in the presence of the central charges. The central charge changes the superalgebra

$$\{Q^\alpha, Q^{*\beta}\} = 2\bar{\sigma}_\mu^{\alpha\beta} P^\mu - (\sigma_2)^{\alpha\beta} 2C \tag{B13}$$

with $C = ev\Phi$ in our case. In general $M \geq C$, and the size of the supermultiplets are the same as the massive case without the central charge unless the equality in the above is saturated, where the size of the supermultiplets will be reduced. In the model we are considering, there will be the central charges for both topological and nontopological vortices and also for the particles in the symmetry unbroken vacuum sector. In all these cases the central charge will be equal to the mass. To see how the supermultiplets are reduced in these cases we define

$$a_1 = \frac{1}{2\sqrt{M}}(Q^1 - iQ^2), \quad a_1^\dagger = \frac{1}{2\sqrt{M}}(Q^{*1} + iQ^{2*}),$$
$$a_2 = \frac{1}{2\sqrt{M}}(Q^1 + iQ^2), \quad a_2^\dagger = \frac{1}{2\sqrt{M}}(Q^{*1} - iQ^{2*}). \tag{B14}$$

The superalgebra in Eq. (B13) will then become

$$\{a_1, a_1^\dagger\} = 0, \qquad \{a_2, a_2^\dagger\} = 1. \tag{B15}$$

Other anticommutators vanish. The operators $a_1(a_1^\dagger)$ should then be realized to be zero. Therefore we have only one pair of creation and annihilation operators and hence two states (not four) $\{|\Omega(j)\rangle$ and $a_2^\dagger|\Omega(j)\rangle$ with angular momentum j and $j - \frac{1}{2}$.

APPENDIX C: PHASE SHIFT ANALYSIS

In this appendix we will describe the phase shift analysis used for the mass correction. First consider $\kappa = 0$ case. Rewrite the Dirac equation in (3.10) as

$$L\begin{pmatrix} h_1(r) \\ h_4(r) \end{pmatrix} \equiv \begin{pmatrix} \partial_r + \frac{a+j+1/2}{r} & f \\ f & \partial_r - \frac{j-1/2}{r} \end{pmatrix} \begin{pmatrix} h_1(r) \\ h_4(r) \end{pmatrix}$$
$$= i\omega \begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix} \tag{C1}$$

and

$$-L^\dagger \begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix}$$
$$\equiv \begin{pmatrix} \partial_r - \frac{a+j-1/2}{r} & -f \\ -f & \partial_r + \frac{j+1/2}{r} \end{pmatrix} \begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix}$$
$$= i\omega \begin{pmatrix} h_1(r) \\ h_4(r) \end{pmatrix}. \tag{C2}$$

We have set $f' = \sqrt{2}evf$ and dropped the prime for simplicity. From these first-order differential equations we get the second-order equations

$$L^\dagger L \begin{pmatrix} h_1(r) \\ h_4(r) \end{pmatrix} = \omega^2 \begin{pmatrix} h_1(r) \\ h_4(r) \end{pmatrix} \tag{C3}$$

and

$$LL^\dagger \begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix} = \omega^2 \begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix}, \tag{C4}$$

where

$$L^\dagger L = \begin{pmatrix} -\partial_r^2 - \frac{\partial_r + a'}{r} + \frac{(a+j+1/2)^2}{r^2} + f^2 & 0 \\ 0 & -\partial_r^2 - \frac{\partial_r}{r} + \frac{(j-1/2)^2}{r^2} + f^2 \end{pmatrix}, \tag{C5}$$

$$LL^\dagger = \begin{pmatrix} -\partial_r^2 - \frac{\partial_r - a'}{r} + \frac{(a+j-1/2)^2}{r^2} + f^2 & 2\partial_r f \\ 2\partial_r f & -\partial_r^2 - \frac{\partial_r}{r} + \frac{(j+1/2)^2}{r^2} + f^2 \end{pmatrix}. \tag{C6}$$

Note that $LL^\dagger$ can be obtained by changing the sign of a and j in $L^\dagger L$. The equations for h_1 and h_4 are decoupled:

$$\left(-\partial_r^2 - \frac{\partial_r + a'}{r} + \frac{(a+j+1/2)^2}{r^2} + f^2\right) h_1 = \omega^2 h_1 \tag{C7}$$

and

$$\left(-\partial_r^2 - \frac{\partial_r}{r} + \frac{(j-1/2)^2}{r^2} + f^2\right) h_4 = \omega^2 h_4 . \tag{C8}$$

The functions will behave near the origin as

$$h_1 \propto r^{|j+1/2+n|}, \qquad h_4 \propto r^{|j-1/2|} . \tag{C9}$$

In the asymptotic region of $r \to \infty$, the shape of solutions are

$$h_1(r) \sim \alpha_1 J_{|j+1/2|}(\bar{\omega} r) + \beta_1 N_{|j+1/2|}(\bar{\omega} r) \tag{C10}$$

and

$$h_4(r) \sim \alpha_4 J_{|j-1/2|}(\bar{\omega} r) + \beta_4 N_{|j-1/2|}(\bar{\omega} r) \tag{C11}$$

with $\bar{\omega} = \sqrt{\omega^2 - 1}$. The coefficients of α and β will be determined to match the behavior near the origin (C9).

The influence of the background vortex will show up as a phase shift of the functions by comparing with the no-vortex background case ($a = 0$, $f = 1$). The phase shifts are given by

$$\tan\delta_j^{(1)} = \frac{\beta_1}{\alpha_1}\,, \tag{C12}$$

$$\tan\delta_j^{(4)} = \frac{\beta_4}{\alpha_4}\,. \tag{C13}$$

The other two functions h_2 and h_3 are determined by

$$\begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix} = \frac{1}{i\omega} L \begin{pmatrix} h_1(r) \\ h_4(r) \end{pmatrix}\,. \tag{C14}$$

In the limit $r \to \infty$, we have

$$\begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix} \sim \frac{1}{i\omega}\begin{pmatrix} \partial_r h_1 + h_4 \\ \partial_r h_4 + h_1 \end{pmatrix} \sim \frac{1}{i\omega}\begin{pmatrix} -\bar\omega\sin(\bar\omega r - |j+1/2|\frac{\pi}{2} - \frac{\pi}{4} - \delta_j^{(1)}) + \cos(\bar\omega r - |j-1/2|\frac{\pi}{2} - \frac{\pi}{4} - \delta_j^{(4)}) \\ -\bar\omega\sin(\bar\omega r - |j-1/2|\frac{\pi}{2} - \frac{\pi}{4} - \delta_j^{(4)}) + \cos(\bar\omega r - |j+1/2|\frac{\pi}{2} - \frac{\pi}{4} - \delta_j^{(1)}) \end{pmatrix}\,. \tag{C15}$$

First, consider the case with $j > 0$. The phase shift is given by

$$\begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix} \sim \frac{1}{i\omega}\begin{pmatrix} -\bar\omega\sin(R - \delta_j^{(1)}) + \cos(R - \delta_j^{(4)} + \frac{\pi}{2}) \\ -\bar\omega\sin(R' - \delta_j^{(4)}) + \cos(R' - \delta_j^{(1)} - \frac{\pi}{2}) \end{pmatrix} \tag{C16}$$

with

$$R = \bar\omega r - (j+1/2)\frac{\pi}{2} - \frac{\pi}{4},$$
$$R' = \bar\omega r - (j-1/2)\frac{\pi}{2} - \frac{\pi}{4} = R + \frac{\pi}{2}\,. \tag{C17}$$

Then the phase shifts become

$$\tan(\delta_j^{(2)} - \delta_j^{(1)}) = -\frac{\sin(\delta_j^{(4)} - \delta_j^{(1)})}{\bar\omega + \cos(\delta_j^{(4)} - \delta_j^{(1)})} \tag{C18}$$

and

$$\tan(\delta_j^{(3)} - \delta_j^{(4)}) = -\frac{\sin(\delta_j^{(4)} - \delta_j^{(1)})}{\bar\omega - \cos(\delta_j^{(4)} - \delta_j^{(1)})}\,. \tag{C19}$$

For the case with negative j, the phase shift will be

$$\begin{pmatrix} h_2(r) \\ h_3(r) \end{pmatrix} \sim \frac{1}{i\omega}\begin{pmatrix} -\bar\omega\sin(R'' - \delta_j^{(1)}) + \cos(R'' - \delta_j^{(4)} + \frac{\pi}{2}) \\ -\bar\omega\sin(R''' - \delta_j^{(4)}) + \cos(R''' - \delta_j^{(1)} + \frac{\pi}{2}) \end{pmatrix} \tag{C20}$$

with

$$R'' = \bar\omega r + (j+1/2)\frac{\pi}{2} - \frac{\pi}{4},$$
$$R''' = \bar\omega r + (j-1/2)\frac{\pi}{2} - \frac{\pi}{4} = R'' + \frac{\pi}{2}. \tag{C21}$$

The phase shift will then be

$$\tan(\delta_j^{(2)} - \delta_j^{(1)}) = \frac{\sin(\delta_j^{(4)} - \delta_j^{(1)})}{\bar\omega + \cos(\delta_j^{(4)} - \delta_j^{(1)})} \tag{C22}$$

and

$$\tan(\delta_j^{(3)} - \delta_j^{(4)}) = \frac{\sin(\delta_j^{(4)} - \delta_j^{(1)})}{\bar\omega + \cos(\delta_j^{(4)} - \delta_j^{(1)})}\,. \tag{C23}$$

From the above equations, not depending on the sign of j,

$$\tan(\delta_j^{(2)} - \delta_j^{(1)}) = -\tan(\delta_{-j}^{(3)} - \delta_{-j}^{(4)})\,; \tag{C24}$$

hence

$$\delta_j^{(2)} - \delta_j^{(1)} = -(\delta_{-j}^{(3)} - \delta_{-j}^{(4)})\,. \tag{C25}$$

So, for fixed ω,

$$\sum_j(\delta_j^{(2)} + \delta_j^{(3)}) - \sum_j(\delta_j^{(1)} + \delta_j^{(4)}) = \sum_j(\delta_j^{(2)} - \delta_j^{(1)}) + \sum_j(\delta_j^{(3)} - \delta_j^{(4)}) = 0\,. \tag{C26}$$

From the relation between the phase shift and the density of states [24] we get

$$n_+ = n_-. \tag{C27}$$

We turn to the case with $\kappa \neq 0$. The equations for h_2 and h_3 are

$$\left(\partial_r^2 + \frac{\partial_r + a'}{r} - \frac{(a+j-1/2)^2}{r^2} + \omega^2 - f^2 - 2\omega N - \frac{\kappa}{\omega-\kappa} f\right) h_2 + \left[\frac{1}{\omega-\kappa}\left(\partial_r + \frac{1}{r}(a+j+\tfrac{1}{2})\right) - \frac{2af}{r}\right] h_3 = 0 \tag{C28}$$

and

$$\left[\partial_r^2 + \frac{\partial_r + a'}{r} - \frac{(j+1/2)^2}{r^2} + (\omega-\kappa)^2 - \left(1-\frac{\kappa}{\omega}\right) f^2\right] h_3 + \left[\frac{-2af}{r} - \frac{\kappa}{\omega}\left(\partial_r - \frac{1}{r}(a+j-\tfrac{1}{2})\right)\right] h_2 = 0\ . \tag{C29}$$

By solving these two coupled equations, we can get h_2 and h_3. h_1 and h_4 are fixed by

$$h_1 = \frac{1}{i\omega}\left[\left(\partial_r - \frac{1}{r}(a+j-\tfrac{1}{2})\right) h_2 - f h_3\right]\ , \tag{C30}$$

$$h_4 = \frac{1}{i(\omega-\kappa)}\left[\left(\partial_r + \frac{1}{r}(j+\tfrac{1}{2})\right) h_3 - f h_2\right]\ , \tag{C31}$$

unless $\omega = 0$ and $\omega = \kappa$.

Following the steps for the case of $\kappa = 0$ we get the result

$$\delta\psi_\uparrow + \delta\chi_\downarrow = \delta\psi_\downarrow + \delta\chi_\uparrow, \tag{C32}$$

which is similar to Eq. (C27).

There may exist a solution to the Dirac equation in (3.10) for the threshold value of $\omega = \kappa$. But this gives us a discrete modes which does not contribute to the mass correction.

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