Dissipation via particle production in scalar field theories

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We study the nonequilibrium evolution of the expectation value of a scalar field in the broken and unbroken symmetry cases. We find that the particles produced by parametric amplification give rise to dissipative behavior for this mode. However, a $\Gamma\dot{\phi}$ type of term *cannot* account for the dissipational dynamics. We are able to show clearly that perturbation theory breaks down at late times, so that dissipation in field theories can only be understood nonperturbatively. When Goldstone bosons are present we find infrared divergences that require a nonperturbative resummation to describe the long-time dynamics. We use the Hartree factorization and the large N approximation to the O(N) linear σ model to numerically as well as analytically understand the long-time behavior of the zero mode as well as that of the produced particles. The O(N) model case is extremely interesting since, in the spontaneously broken case, the radial mode dissipates all of its energy into production of long-wavelength Goldstone modes. The minima of the effective action (determined by the final value of the expectation value of the scalar field) depend on the initial conditions.

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I. INTRODUCTION

It is well appreciated that the dynamics of dissipation in scalar field theories is of great importance in a variety of settings. One of the most interesting of these is that of the reheating of the Universe after an inflationary epoch has passed [1, 2]. Recall that at the end of new or chaotic inflation [3], defined by when the slow-roll conditions for the so-called inflaton field [4, 5], $\phi(t)$, fail to obtain, the inflaton begins to oscillate about its true ground state.

Since the inflaton is coupled to lighter fields (fermions or other scalars), as the scalar field oscillates around the true minimum it decays into these other particles, and eventually these particles thermalize via collisions and relax to an equilibrium state at high temperature. It is usually stated that such coupling gives rise to a term of the form $\Gamma\dot{\phi}(t)$, where Γ is the decay rate of the inflaton into the lighter fields [6, 7]. This term, which is usually put in by hand as a phenomenological result of the existence of open decay channels, acts as a friction term in the equation of motion for the inflaton, and converts the inflaton energy density into that of the lighter particles during its oscillations. These decay products are then supposed to thermalize, completing the reheating process [6, 7].

This picture bears closer scrutiny. Two questions that must be addressed are: (i) how does a time reversible theory generate dissipative dynamics, and (ii) can the phenomenological term $\Gamma\dot{\phi}(t)$ be derived from first principles?

The answer to the first question is relatively well known. To generate dissipative dynamics from a time reversible theory requires that some "coarse graining" of the field degrees of freedom be done. Essentially, one must trace out (in the functional integral) degrees of freedom other than the one that has been deemed important to the dynamics. The tracing operation turns a closed system (that of *all* the field modes) into an open one (that of the field modes of interest) where the traced out modes now become the "environment" which couples to the remaining modes. Another important necessary ingredient is nonequilibrium initial conditions.

As envisaged in the original scenarios, the process of reheating consists of two different stages. During the first one, the potential and kinetic energy of the scalar field is dissipated by the process of particle production. After that, or perhaps simultaneously with the first stage, the produced particles interact and thermalize after some time, reaching a final equilibrium temperature.

Although of definite interest for the reheating mechanism in inflationary cosmology, the issue of dissipation and thermalization is of much broader interest. In particular, similar issues must be addressed in heavy ion collisions and in general during phase transitions out of equilibrium. Our study of dissipation here is not confined to cosmological scenarios but tries to address the broader issues.

What we want to understand in this article is the dynamics of dissipation via particle production in an *a priori* fashion, i.e., how to incorporate the effect of quantum fluctuations into the dynamics of the time dependent order parameter.

Our approach is as follows. Starting from a renormalizable self-interacting scalar field theory we isolate the expectation value (which by translational invariance only depends on time) and generate its equation of motion taking quantum fluctuations into account. Since one of our direct motivations is to study reheating after an inflationary epoch, we treat the dynamics at zero temperature, since the temperature is supposed to have red-

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shifted all the way to zero during the inflationary stage.

What makes our approach different from others that have been proposed (see below) is that we are not considering an "in-out" expectation value, such as would be generated by the usual method of constructing the effective action. Rather, we use a method that generates the equations for an "in-in" expectation value of the field operator. This ensures that our equations of motion for the field expectation value are both real and causal. This has been done by others, most notably by Calzetta and Hu [8] and Paz [9].

After obtaining the renormalized, causal, and real equations of motion that determine the dynamics we provide an extensive numerical analysis of the equations.

The results from our analysis are quite interesting and we summarize them here. The equations of motion are rather complicated integro-differential equations connecting the field expectation value to the fluctuations of the nonzero momentum modes. We begin by obtaining the equations of motion in an amplitude expansion as well as in the loop expansion up to one loop. After recognizing the "dissipative terms" in these approximations, we find that these cannot be simply replaced by a term of the form $\Gamma\dot{\phi}(t)$.

In the case where there is no symmetry breaking, we generate the effective Langevin equation for the system, in the one-loop approximation, and we find a multiplicative, *non*-Markovian kernel for the dissipational term as well as a "colored" noise term. Thus this is very different from the simple friction term described above. In fact, it is easy to see that there is *no* limit in which the dissipational dynamics we find can be described by a term proportional to the time derivative of the field.

In the case in which a discrete symmetry is broken, so that there are no Goldstone modes, the one-loop equation can be linearized in the amplitude of oscillations about the nonzero expectation value of the field. Even in this linearized approximation, while a term involving the first time derivative of the field does appear, it is convolved with a nontrivial kernel, that again has no limit (at zero temperature) in which it becomes a local term.

We show that the loop expansion or the amplitude expansion for the dynamics of the field expectation value *must* break down at long times. In the loop expansion, the order $\hbar$ term has an amplitude that grows in time due to resonance phenomena induced by the existence of a two-particle threshold. Thus, it will eventually dominate over the tree amplitude, and perturbation theory will then break down. Dissipation can only be understood beyond perturbation theory. We then use the Hartree approximation in the single scalar field case and the large-N approximation in an $O(N)$ symmetric theory to try to understand the effects of quantum fluctuations on the oscillations of the field expectation value as well as particle production due to these same oscillations. Although in inflaton models of reheating there are no Goldstone bosons, the physics of dissipation via particle production is of much broader scope and we studied these processes in presence of Goldstone bosons.

The Hartree approximation reveals dissipative effects due to particle production and open channels. However, we find that the damping is not exponential, and asymptotically, the field expectation value undergoes undamped oscillations with an amplitude that depends on the coupling and the initial conditions. The same features are found in the large N limit in the $O(N)$ model in the case of unbroken symmetry. Here we find that the asymptotic oscillatory behavior can be expressed in terms of elliptic functions.

The dissipation in the system is caused by a collisionless type of process, reminiscent of Landau damping [10]. This process is a result of particle production arising from oscillations of the field expectation value via parametric amplification. These quantum fluctuations then react back on the field expectation value, but *not* in phase with it, thus damping the oscillations. This behavior is similar to that found in the damping of strong electric fields in a collisionless regime [11].

An unexpected and quite remarkable result comes from the analysis of the $O(N)$ model in large N when the symmetry is spontaneously broken. We find that the field expectation value oscillations are damped very quickly in this case, much more so than in the unbroken situation. This by itself is perhaps not so unusual, since the existence of Goldstone modes allows the field to lose energy by radiating a large number of soft Goldstone particles. What is unusual, however, is that there are some initial conditions, corresponding to "slow-roll" behavior [4], for which the field expectation value relaxes (via dissipation) to a value that is very close to the origin. This final value is a minimum of the effective action for which the expectation value of the scalar field has moved very slightly from the initial value. Most of the potential energy is converted into long-wavelength Goldstone bosons. Thus the symmetry-breaking phase transition is actually *dramatically modified* by the dissipational dynamics. These initial conditions are generically "slow-roll" conditions in which the scalar field begins very close to the top of the potential hill and the couplings are extremely weak (in our case about 10^{-7}-10^{-12}). This is one of the most striking results of this article.

We provide extensive numerical evidence for all our assertions; we can follow the evolution of the field expectation value and the quantum fluctuations, and we also compute the number of particles created via the method of Bogoliubov coefficients.

In the next section we provide an outline of the formalism we use to generate the equations for the expectation value of the scalar field that include the effects of the fluctuations of the quantum fluctuations. We then investigate the meaning of these equations within the one-loop and the amplitude expansions. In Secs. III and IV we utilize nonperturbative approximations such as the Hartree approximation and the large N approximation in the case of the $O(N)$ model to perform our numerical calculations. After this analysis, we reconcile the dissipative dynamics that we find with the concept of time reversal invariance.

Section V contains our conclusions, in which we compare our results to other results on dissipation and reheating obtained in the literature.

II. NONEQUILIBRIUM FIELD THEORY AND EQUATIONS OF MOTION

The generalization of statistical mechanics techniques to the description of nonequilibrium processes in quantum field theory has been available for a long time [12–16] and there are many clear articles in the literature using these techniques to study real time correlation functions [8, 17–22] and effective actions out of equilibrium [9, 23–25]. This formulation has already been used by some of us previously to study the dynamics of phase transitions [28].

In our analysis we will take the $\beta \to \infty$ limit (zero temperature) following the argument provided in the Introduction. We will also limit ourselves to a Minkowski space-time rather than expanding Friedmann-Robertson-Walker (FRW) space-time by considering phenomena whose time variation happens on scales much shorter than the expansion time H^{-1} of the universe, where H is the Hubble parameter.

A. Equations of motion

The method described in Ref. [28] allows the derivation of the effective equations of motion for a coarse grained field. To focus the discussion, consider the situation of a scalar field theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}\Phi^4. \tag{1}$$

The "coarse grained" field (order parameter) is defined as

$$\phi(t) = \langle \Phi(x,t) \rangle = \frac{\mathrm{Tr}\,[\Phi\rho(t)]}{\mathrm{Tr}\rho(t)}, \tag{2}$$

where we have used translational invariance. We write the field as $\Phi(x,t) = \phi(t) + \psi(x,t)$ with $\psi(x,t)$ the fluctuations, obeying $\langle \psi(x,t) \rangle = 0$ and consider $\phi(t)$ as a background field in the evolution Hamiltonian. The nonequilibrium generating functional requires

$$\mathcal{L}[\phi+\psi^+] - \mathcal{L}[\phi+\psi^-] = \Bigg\{ \frac{\delta \mathcal{L}}{\delta \phi}\psi^+ + \mathcal{L}_0[\psi^+] - \lambda\left(\frac{\phi^2(\psi^+)^2}{4} + \frac{\phi(\psi^+)^3}{6} + \frac{(\psi^+)^4}{4!}\right)\Bigg\} - \{\psi^+ \to \psi^-\} \tag{3}$$

with $\mathcal{L}_0[\psi^\pm]$ the free field Lagrangian density of a field of mass m. We can now "integrate out" the fluctuations, thus obtaining the nonequilibrium effective action for the "coarse grained" background field.

The *linear*, cubic, and quartic terms in $\psi^\pm$ are treated as perturbations, while the terms $\phi^2(t)(\psi^\pm)^2$ may either be treated as perturbations, if one wants to generate a perturbative expansion in terms of the *amplitude* of the coarse grained field, or alternatively, they may be absorbed into a time-dependent mass term for the fluctuations, if one wants to generate a *loop* expansion. We will now study both cases in detail.

B. Amplitude expansion

1. Discrete symmetry

We treat the term $\phi^2(t)(\psi^\pm)^2$ as perturbation, along with the linear, cubic, and quartic terms. The conditions

$$\langle \psi^\pm(x,t) \rangle = 0 \tag{4}$$

will give rise to the effective nonequilibrium equations of motion for the background field. This is the generalization of the "tadpole" method [30] to nonequilibrium field theory. Since the mass of the fluctuations is the bare mass, the essential ingredient for the nonequilibrium propagators introduced above are the spatial Fourier transforms of the homogeneous solutions to the free-field quadratic forms:

$$G_k^>(t_1,t_2) = \frac{i}{2\omega_k}e^{-i\omega_k(t_1-t_2)},$$
$$G_k^<(t_1,t_2) = \frac{i}{2\omega_k}e^{i\omega_k(t_1-t_2)},$$
$$\omega_k = \sqrt{k^2+m^2}.$$

To $O(\phi^3)$ the diagrams are shown in Figs. 1(a)-1(h). Since the (++) and the (−−) propagators are independent, the diagrams 1(a)–1(h) finally lead to the equation of motion

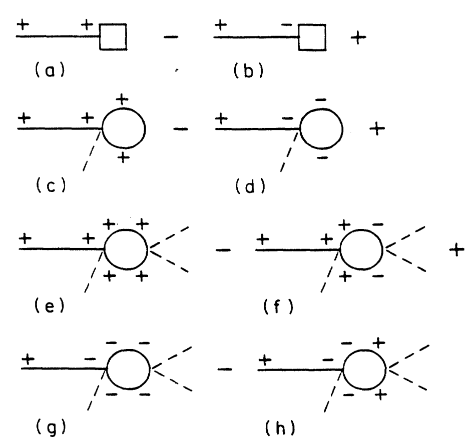


FIG. 1. Diagrams contributing to the equation of motion up to one-loop order, and in the amplitude expansion up to $O(\phi^3)$.

$$\ddot{\phi}(t)+m^2\phi(t)+\frac{\lambda}{6}\phi^3(t)+\frac{\lambda}{2}\phi(t)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}-\frac{\lambda^2}{4}\phi(t)\int_{-\infty}^{t}dt'\phi^2(t')\int\frac{d^3k}{(2\pi)^3}\frac{\sin[2\omega_k(t-t')]}{2\omega_k^2}=0. \tag{5}$$

Notice that the diagrams of Figs. 1(e) and 1(f) contain the one-loop two-particle threshold typical of the two-particle to two-particle scattering amplitude. The last term in (5) is the *real-time* expression for this one-loop contribution. We can perform an integration by parts in the time integral, discarding the contribution at $t=-\infty$ by invoking an adiabatic switching-on convergence factor to obtain

$$\ddot{\phi}(t)+m^2\phi(t)+\frac{\lambda}{6}\phi^3(t)+\frac{\lambda}{2}\phi(t)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}-\frac{\lambda^2}{4}\phi^3(t)\int\frac{d^3k}{(2\pi)^3}\frac{1}{4\omega_k^3}+\frac{\lambda^2}{4}\phi(t)$$
$$\times\int_{-\infty}^{t}dt'\phi(t')\dot{\phi}(t')\int\frac{d^3k}{(2\pi)^3}\frac{\cos[2\omega_k(t-t')]}{2\omega_k^3}=0. \tag{6}$$

The fourth term in the above equation is recognized as a mass renormalization and the fifth term as the coupling constant renormalization. Using a Fourier expansion for $\phi(t)$ and its derivative, it is straightforward to see that after integration in the time variable, the remaining integration in k is *ultraviolet finite*. There are several noteworthy features of this expression that point to a more complicated description of dissipative processes in field theory. The first such feature is an expected one. The "dissipative" contribution, that is, the last term in the above equation, has a non-Markovian (i.e., memory-retaining) kernel. Secondly, the equation is *nonlinear* in the amplitude of the coarse grained variable. These features are in striking contradiction with a simple $\Gamma\dot{\phi}$ term in the equation of motion. Originally [7] such a term was argued to arise if the scalar field in question is coupled (linearly) to other fields in the theory and truly speaking such a situation does not arise within the present context. However, the issue of a memory kernel (non-Markovian) is quite general [26], and a legitimate question to ask is: is there a Markovian (local) limit of this kernel? Such a limit would imply that

$$\mathcal{K}(t-t')\approx D\delta(t-t') \tag{7}$$

with $\mathcal{K}(t-t')$ being the nonlocal kernel present in the last term of (6). However, we find that at small $(t-t')$ the kernel has typical logarithmic divergences.

If a local approximation were valid, the "dissipative constant" D may be found by integrating the kernel in time. A straightforward calculation shows that infrared divergences give a divergent answer for such constant, clearly indicating that there is no Markovian limit for this kernel. In Ref. [24] a Markovian limit was argued to be available in the high temperature limit; since we are working at zero temperature the approximations invoked there do not apply. Thus, this lowest order calculation reveals two conclusive features that will persist in higher orders and even in nonperturbative calculations (see below): the "dissipative contribution " to the equations of motion obtained by integrating out the fluctuations are typically nonlinear and, furthermore, they do not allow for a Markovian (local) description. We will postpone a numerical analysis of these equations until we study the full one-loop equations of motion below.

2. Broken symmetry

In the case of unbroken symmetry, the above equation of motion does not admit a linearization of the dissipative contribution. However, in the case when the symmetry is spontaneously broken, such a linearization is possible [9]. In this case let $m^2=-\mu^2$ and let us look for small oscillations around one of the minima. To achieve this, write $\phi(t)=\sqrt{6\mu^2/\lambda}+\delta(t)$. The mass term for the fluctuations now becomes $2\mu^2$, and we will only keep the linear terms in δ in (3). We now follow the steps leading to the equation of motion obtained before. That is, $\langle\Psi^{\pm}\rangle=0$ is imposed to one-loop order. After integration by parts in the time integral (again with an adiabatic switching-on convergence factor), there appear several tadpole contributions. Those that are independent of δ renormalize ϕ_0 corresponding to a shift of the position of the vacuum expectation value, while those that are linear in δ renormalize the mass. We thus obtain in the linearized approximation

$$\ddot{\delta}+2\mu_R^2\delta+\frac{3\mu_R^2}{2}\lambda_R\int_{-\infty}^{t}dt'\,\dot{\delta}(t')$$
$$\times\int\frac{d^3k}{(2\pi)^3}\frac{\cos[2\tilde{\omega}_k(t-t')]}{2\tilde{\omega}_k^3}=0 \tag{8}$$

with $\tilde{\omega}_k=\sqrt{k^2+2\mu_R^2}$. Here μ_R and λ_R stand for renormalized mass and coupling constant. This expression is similar to that found by Paz [9]. In that reference the *short time* behavior was analyzed. In order to solve this equation for all times, we must specify an initial condition. We will *assume* that for $t<0$ $\delta(t<0)=\delta_i$ and $\dot{\delta}(t<0)=0$ and that Eq. (8) holds for $t>0$ [31].

Under this assumption, the linearized equation can be solved via the Laplace transform

$$\varphi(s)=\int_0^{\infty}e^{-st}\,\delta(t)\,dt. \tag{9}$$

This solution corresponds to summing Dyson's series for the propagator with the one-loop contributions depicted in Fig. 1 (for the linearized case).

We find from Eqs. (8) and (9) after some calculation,

$$\varphi(s)=\frac{\delta_i}{s}\left[1-\frac{\mu^2}{s^2+\mu^2+G\,\Sigma(s)}\right], \tag{10}$$

where $\mu^2 \equiv 2\mu_R^2$, $G \equiv 3\mu^2/(4\pi)^2\,\lambda_R$ and

$$\Sigma(s) = \sqrt{1+\left(\frac{2\mu}{s}\right)^2}\,\operatorname{arctanh}\left[\frac{1}{\sqrt{1+(\frac{2\mu}{s})^2}}\right] - 1. \tag{11}$$

The Laplace transform exhibits the two-particle cut [8, 9] in the analytic continuation of the transform variable s and one-particle poles at the renormalized masses $s = \pm iM$, where

$$\mu^2 - M^2 + G\,\Sigma(iM) = 0.$$

That is,

$$M^2 = \mu^2 - G\left(1 - \frac{\pi}{2\sqrt{3}}\right) + O(G^2). \tag{12}$$

The function $\delta(t)$ can be expressed as the inverse Laplace transform of $\varphi(s)$ using Brouwer's inversion formula. Upon deforming the integration contour, the one-particle poles are explicitly computed by the residue theorem and we find

$$\delta(t) = \delta_i\left\{\frac{\mu^2}{M\left[M - \frac{G}{2}\Sigma'(iM)\right]}\cos Mt \right.$$
$$\left. +G\mu^2\int_{2\mu}^{\infty}\frac{r(\omega)\cos(\omega t)d\omega}{\omega\left[D(\omega)^2 + (\frac{G\pi}{2})^2\,r(\omega)^2\right]}\right\}, \tag{13}$$

where

$$r(\omega) \equiv \sqrt{1-\left(\frac{2\mu}{\omega}\right)^2}$$

and

$$D(\omega) \equiv \mu^2 - \omega^2 + G\,r(\omega)\left[\operatorname{arctanh} r(\omega) - 1\right].$$

For large t the integral in Eq. (13) is dominated by the end point $\omega = 2\mu$ and tends to zero as

$$\int_{2\mu}^{\infty}\frac{r(\omega)\cos(\omega t)d\omega}{\omega\left[D(\omega)^2 + (\frac{G\pi}{2})^2\,r(\omega)^2\right]}$$
$$\overset{t\to\infty}{=} \frac{\sqrt{\pi}\,\cos(2\mu t + 3\pi/4)}{4(\mu t)^{3/2}\,(3\mu^2+G)^2} \to 0. \tag{14}$$

We see from Eqs. (13) and (14) that $\phi(t) = \sqrt{6/\lambda}\,\mu + \delta(t)$ oscillates asymptotically around a shifted minimum

$$\sqrt{6/\lambda}\,\mu - \frac{3\lambda}{2(4\pi)^2}\left(\frac{5\pi}{3\sqrt{3}} - 3\right)\delta_i + O(\lambda^2)$$

with an amplitude $\delta_i\left[1 - c\,\lambda + O(\lambda^2)\right]$ and frequency M [see (12)], where $c \equiv \frac{3}{2(4\pi)^2}\left(\frac{5\pi}{3\sqrt{3}} - 3\right) = 2.18464\ldots \times 10^{-4} > 0$. We find that the amplitude of the oscillation decreases when λ grows.

In the present ϕ^4 model the wave function renormalization is finite to one loop and takes the value

$$(Z_2)^{-1} = 1 + c\lambda + O(\lambda^2).$$

Hence, the change in the oscillation amplitude is just due to the wave function renormalization that here is $Z_2 < 1$. In other words, quantum corrections push down the ϕ amplitude.

We want to stress that Eq. (13) is more than a simple perturbative solution.

A *strict perturbative* solution would be

$$\delta(t) = \delta_i\cos[\mu t] + \lambda\delta_1(t) + \lambda^2\delta_2(t) + O(\lambda^3), \tag{15}$$

where $\delta_1(t), \delta_2(t), \ldots$ are λ-independent functions. The equation is then solved order by order in λ. The corresponding equation for $\delta_1(t)$ is now easily solved via the Laplace transform. The transform exhibits resonances at $\pm\mu$ and the large time behavior is dominated by the secular term $\delta_1(t) \approx t\sin[\mu t]$. This result also follows by expanding Eq. (13) in powers of λ.

Thus the conclusion from the perturbative series (15) is that it is valid only for *short times*; the long time behavior will not be captured by this perturbative approximation. On the contrary, the solution (13) contains in fact an infinite (but partial) resummation of perturbation theory. As we will see again below, some nonperturbative treatment is always necessary in order to properly describe dissipative behavior.

The solution (13) can only be trusted qualitatively since the higher loop contributions [$O(\lambda^2)$ and higher] are of the same order as the contributions retained there. Only through the $1/N$ expansion will we be able to control quantitatively the effects nonperturbative in λ.

The analysis provided in Refs. [8, 9, 23] in which a dissipative behavior was observed also assumes implicitly such a resummation, because their solution contains higher order terms in the coupling that are not warranted in the approximation considered. Even in this case in which there is an explicit linear velocity dependence in the "dissipative kernel," the same argument presented before applies. There is no Markovian (local) limit in which this term becomes simply $\Gamma\dot{\delta}$, despite the fact that δ has a *linear coupling* to the fluctuations.

3. Continuum symmetries and Goldstone bosons

An important issue that we want to study in detail in this article is the process of dissipation by Goldstone bosons. As discussed in the Introduction, dissipation via particle production becomes effective whenever the transferred energies become larger than multiparticle thresholds. To lowest order, as shown in the perturbative calculation of the previous section, the lowest threshold is the two-particle one, at an energy twice the mass of the particle. In the case of a spontaneously broken, continuous symmetry, there will be Goldstone bosons and the thresholds are at zero energy. In this case any amount of energy transfer may be dissipated by the Goldstone modes.

This effect may be studied in the O(2) linear σ model with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + \frac{1}{2}\mu^2(\sigma^2+\pi^2) - \frac{\lambda}{4!}(\sigma^2+\pi^2)^2. \quad (16)$$

We now write

$$\sigma(x,t) = \sqrt{\frac{6\mu^2}{\lambda}} + \delta(t) + \chi(x,t) \quad (17)$$

and use the tadpole method to impose

$$\langle\pi(x,t)\rangle = 0 \ ; \ \langle\chi(x,t)\rangle = 0. \quad (18)$$

Carrying out the same analysis in terms of Feynman diagrams to linear order in $\delta(t)$ we find the following equation of motion (after integrating by parts invoking an adiabatic switching on convergence factor and absorbing the local terms in proper renormalizations):

$$\ddot{\delta} + 2\mu_R^2 + 6\mu_R^2\lambda_R\int_{-\infty}^{t} dt'\dot{\delta}(t') \quad (19)$$

$$\times\left\{\frac{1}{4}\mathcal{K}_\chi(t-t') + \frac{1}{36}\mathcal{K}_\pi(t-t')\right\} = 0$$

with $\mathcal{K}_\chi(t-t')$ the same kernel as in Eq. (8) and

$$\mathcal{K}_\pi(t-t') = \int\frac{d^3k}{(2\pi)^3}\frac{\cos[2|k|(t-t')]}{2|k|^3} \approx \ln[M(t-t')], \quad (20)$$

where M is an infrared cutoff introduced to define the integral. This infrared divergence is the result of the fact that the threshold for Goldstone bosons is at zero energy momentum. This result clearly reflects many important features of "dissipation" via Goldstone bosons. First, as before, the dissipative kernel cannot be described in a local (Markovian) approximation even in this linearized theory in which the arguments leading to a $\Gamma\dot{\delta}$ term would be valid. Second and perhaps more important, the long time behavior is clearly *beyond perturbation theory* as the contribution from the nonlocal kernels is unbounded as a result of infrared divergences associated with Goldstone bosons.

C. One-loop equations

The full one-loop equations of motion are obtained by absorbing the terms $\phi^2(t)(\psi^\pm)^2$ in Eq. (3) as a time-dependent mass term for the fluctuating fields. In order to determine the dynamics for the background field, we will assume that $\phi(t<0)=\phi_0$ and that at $t=0$ the background field is "released" with zero velocity. Now we must find the corresponding nonequilibrium Green's functions. Consider the following homogeneous solutions of the quadratic form for the fluctuations

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2 + \frac{\lambda}{2}\phi^2(t)\right]U_k^\pm(t) = 0,$$

$$U_k^\pm(0) = 1\,;\dot{U}_k^\pm(0) = \mp i\omega_k^0, \quad (21)$$

with

$$\omega_k^0 = \left[\mathbf{k}^2 + m^2 + \frac{\lambda}{2}\phi^2(0)\right]^{\frac{1}{2}}. \quad (22)$$

The boundary conditions (21) correspond to positive U^+ and negative U^- frequency modes for $t<0$ (the Wronskian of these solutions is $2i\omega_k^0$). Notice that $U_k^-(t) = [U_k^+(t)]^*$. In terms of these mode functions we obtain

$$G^>(t_1,t_2) = \frac{i}{2\omega_k^0}U_k^+(t_1)U_k^-(t_2). \quad (23)$$

Now the equation of motion to one-loop order is obtained from the diagrams in Figs. 1(a) and 1(d) [but with the modified propagator including the contribution of $\phi(t)$]. Thus to one-loop order we find the following equations:

$$\ddot{\phi}(t) + m^2\phi(t) + \frac{\lambda}{6}\phi^3(t) + \frac{\lambda\hbar}{2}\phi(t)\int\frac{d^3k}{(2\pi)^3}\frac{|U_k^+(t)|^2}{2\omega_k^0} = 0,$$

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2 + \frac{\lambda}{2}\phi^2(t)\right]U_k^+(t) = 0, \quad (24)$$

$$U_k^+(0) = 1\,;\dot{U}_k^+(0) = -i\omega_k^0, \quad (25)$$

where we have restored the $\hbar$ to make the quantum corrections explicit. This set of equations clearly shows how the expectation value (coarse grained variable) "transfers energy" to the mode functions via a time-dependent frequency, which then in turn modify the equations of motion for the expectation value. The equation for the mode functions (24) may be solved in a perturbative expansion in terms of $\lambda\phi^2(t)$ involving the *retarded* Green's function. To first order in that expansion one recovers Eq. (5).

The renormalization aspects may be found in Refs. [28, 29], obtaining, finally the renormalized equation of motion,

$$\ddot{\phi}(t) + m_R^2\phi(t) + \frac{\lambda_R}{6}\phi^3(t) + \frac{\lambda_R\hbar}{8\pi^2}\phi(t)\int_0^\Lambda k^2dk\frac{[|U_k^+(t)|^2-1]}{\omega_k^0} + \frac{\lambda_R^2\hbar}{32\pi^2}\phi(t)[\phi^2(t)-\phi^2(0)]\ln[\Lambda/\kappa] = 0, \quad (26)$$

with Λ an ultraviolet cutoff and κ a renormalization scale. In the equations for the mode functions the mass and coupling may be replaced by the renormalized quantities to this order. One would be tempted to pursue a numerical solution of these coupled equations. However, doing so would not be consistent, since these equations were obtained only to order $\hbar$ and a naive numerical solution will produce higher powers of $\hbar$ that are not justified.

Within the spirit of the loop expansion we must be consistent and only keep terms of order $\hbar$. First we introduce dimensionless variables

$$\eta(t)=\sqrt{\frac{\lambda_R}{6m_R^2}}\phi(\tau)\,;\tau=m_Rt\,;q=\frac{k}{m_R}\,;g=\frac{\lambda_R\hbar}{8\pi^2}, \tag{27}$$

and expand the field in terms of g as

$$\eta(\tau)=\eta_{\rm cl}(\tau)+g\eta_1(\tau)+\cdots\,. \tag{28}$$

Now the equations of motion consistent up to $O(\hbar)$ become

$$\ddot{\eta}_{\rm cl}(\tau)+\eta_{\rm cl}(\tau)+\eta_{\rm cl}^3(\tau)=0, \tag{29}$$

$$\ddot{\eta}_1(\tau)+\eta_1(\tau)+3\eta_{\rm cl}^2(\tau)\eta_1(\tau)+\eta_{\rm cl}(\tau)\int_0^{\Lambda/m_R}q^2dq\frac{\left[|U_q^+(\tau)|^2-1\right]}{\left[q^2+1+3\eta_{\rm cl}^2(0)\right]^{1/2}}+\frac{3}{2}\eta_{\rm cl}(\tau)\left[\eta_{\rm cl}^2(\tau)-\eta_{\rm cl}^2(0)\right]\ln[\Lambda/m_R]=0.$$

The solution to Eq. (29) is an elliptic function. The equations for the mode functions become

$$\left[\frac{d^2}{d\tau^2}+q^2+1+3\eta_{\rm cl}^2(\tau)\right]U_q^+(\tau)=0 \tag{30}$$

with the boundary conditions as in Eq. (25) in terms of the dimensionless frequencies and where for simplicity, we have chosen the renormalization scale $\kappa=m_R$. The chosen boundary conditions $\eta(0)=\eta_0$; $\dot{\eta}(0)=0$ can be implemented as

$$\eta_{\rm cl}(0)=\eta_0;\quad \dot{\eta}_{\rm cl}(0)=0;\quad \eta_1(0)=0;\quad \dot{\eta}_1(0)=0. \tag{31}$$

In Fig. 2 we show $\eta_1(\tau)$ with the above boundary conditions with $\eta_0=1$ and $g=0.1$. A cutoff $\Lambda/m_R=100$ was chosen but no cutoff sensitivity was detected by varying the cutoff by a factor 3. Notice that the amplitude grows as a function of time. This phenomenon can be understood as follows. A first hint was obtained in the case of the linearized equations for a broken discrete symmetry. There we learned that because of resonances below the two-particle threshold, the amplitude grows at long times. This is the behavior shown in Fig. 2. An alternative and perhaps more convincing argument is the following. The mode functions $U_q^+(\tau)$ obey a Schrödinger-type equation with a potential that is a *periodic* function of time because the classical solution is periodic. Let us call the period of the classical solution T. Floquet's theorem [32] guarantees the existence of solutions that obey

$$U_q^+(\tau+T)=e^{\mu T}U_q^+(\tau). \tag{32}$$

The Floquet indices μ are functions of the parameters of the potential. The classical solution is an elliptic function, and in this case the Schrödinger equation for the modes may be shown to be a Lamé equation with $n=2$ (see Ref. [44]) whose solutions are Weierstrass functions. This yields a two-zone potential with two forbidden and two allowed bands for $q>0$. The Floquet indices μ are pure imaginary for large q, but they are real and positive for q near zero. That is, the long wavelengths belong to an unstable band.

A more intuitive understanding at a simpler level ensues if we look at small oscillations near the origin for the classical solution $\eta_{\rm cl}(\tau)\approx\eta_{\rm cl}(0)\cos(\tau)$. Then the Schrödinger equation for the modes becomes a Mathieu equation [32], for which the dependence of the Floquet indices on the parameters $(q\ ;\ \eta_{\rm cl}(0))$ is known [32]. There are unstable bands in which the Floquet index has positive real part for certain values of these parameters. Since the one-loop correction involves an integral over all wave vectors, the values of q in these bands give growing contributions to the one-loop integral. These instabilities of the one-loop equations preclude a perturbative analysis of the process of dissipation.

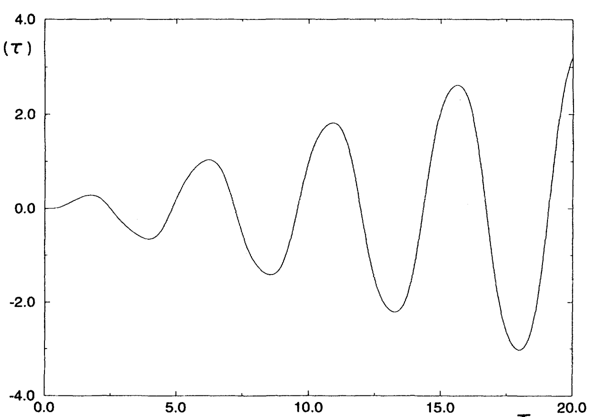


FIG. 2. First-order quantum correction for discrete symmetry case $\eta_1(\tau)$ for $\eta_1(0)=0;\dot{\eta}_1(0)=0;\eta_{\rm cl}(0)=1;\dot{\eta}_{\rm cl}(0)=0$. The cutoff is $\Lambda/m_R=100$.

1. Goldstone bosons at one loop

It proves illuminating to study the one-loop contribution to the equations of motion for the scalar field expectation value from Goldstone bosons. We proceed as in the previous case but now with the Lagrangian density of the O(2) linear σ model (16). We now write

$$\sigma(x,t)=\sigma_0(t)+\chi(x,t),$$
$$\langle\pi(x,t)\rangle=0\ ;\ \langle\chi(x,t)\rangle=0,$$

where the terms $\chi^2\sigma_0^2$; $\pi^2\sigma_0^2$ are now included with the mass terms for the respective fields. The procedure is exactly the same as in the previous case, but we now use μ_R as the mass scale and introduce the dimensionless variables of Eq. (27) in terms of this scale. Performing the proper renormalizations and a subtraction at $t=0$ we finally find the following equations for this case:

$$\frac{d^2}{d\tau^2}\eta - \eta + \eta^3 + g\eta\left\{\int_0^{\Lambda/\mu_R} \frac{q^2}{W_\chi(q)}\left[|U_q^+(\tau)|^2 - 1\right] + \frac{3}{2}\ln[\Lambda/\mu_R][\eta^2(\tau) - \eta^2(0)]\right\}$$

$$+\frac{g}{3}\eta\left\{\int_0^{\Lambda/\mu_R} \frac{q^2}{W_\pi(q)}\left[|V_q^+(\tau)|^2 - 1\right] + \frac{1}{2}\ln[\Lambda/\mu_R][\eta^2(\tau) - \eta^2(0)]\right\} = 0.$$

The mode functions satisfy the equations

$$\left[\frac{d^2}{d\tau^2} + q^2 - 1 + 3\eta^2(\tau)\right] U_q^+(\tau) = 0,$$

$$\left[\frac{d^2}{d\tau^2} + q^2 - 1 + \eta^2(\tau)\right] V_q^+(\tau) = 0,$$

where the boundary conditions are the same as in (24) but in terms of the initial frequencies $W_\chi(q)$; $W_\pi(q)$. As argued previously, for consistency we have to expand the field as in Eq. (28) and keeping only $\eta_{\rm cl}$ in the equation for the modes. Notice that whereas the minimum of the tree-level potential corresponds to $\eta_{\rm cl}^2 = 1$ and the stable region for the U modes is $\eta_{\rm cl}^2 > 1/3$, the stable region for the V modes (pion field) is for $\eta_{\rm cl}^2 > 1$. These instabilities require the choice of special initial conditions, in particular for the initial frequencies. We choose

$$W_\chi(q) = \sqrt{q^2 + 1 + 3\eta^2(0)}, \qquad (33)$$
$$W_\pi(q) = \sqrt{q^2 + 1 + \eta^2(0)}.$$

This choice corresponds to a Gaussian initial state centered at $\eta(0)$ but with *positive* frequencies at $\eta(0) = 0$. This state evolves in a broken symmetry potential from the position determined by $\eta(0)$. Alternatively, one can think of this situation as changing the sign of the mass at time $t = 0$ from positive to negative. For small oscillations of $\eta_{\rm cl}$ around the minimum at $\eta_{\rm min} = 1$, every time that the classical field oscillates to the left of the minimum the "pion" field becomes unstable and grows, and thus its contribution to the fluctuations becomes large. This behavior is depicted in Fig. 3, which shows $\eta_1(\tau)$ with the boundary conditions as in (31) and the initial frequencies (33) for the mode functions were chosen with $\eta_{\rm cl}(0) = 0.6$ and $\Lambda/\mu_R = 100$ but no cutoff sensitivity was detected by increasing the cutoff by a factor of 2. This situation clearly exhibits Floquet indices with a positive real part, and $\eta_{\rm cl}(\tau)$ is periodic for small oscillations around $\eta_{\rm cl} = 1$. The unstable wave vectors form a band when $\eta_{\rm cl}$ oscillates around this value. In each period of $\eta_{\rm cl}$ the amplitude grows because of these unstable modes.

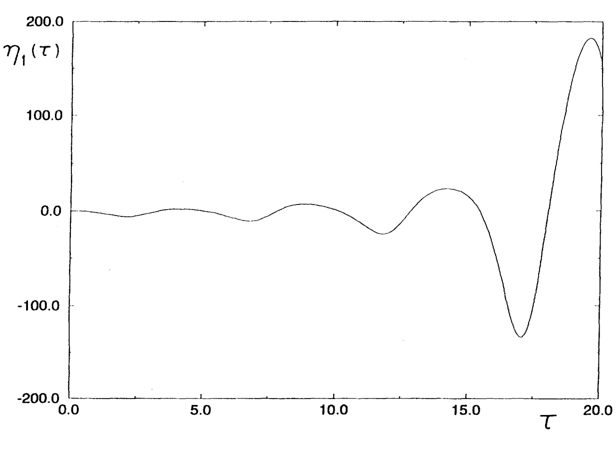


FIG. 3. First-order quantum correction for O(2) model $\eta_1(\tau)$ for $\eta_1(0) = 0; \dot\eta_1(0) = 0; \eta_{\rm cl}(0) = 0.6; \dot\eta_{\rm cl}(0) = 0$. The cutoff is $\Lambda/\mu_R = 100$.

This figure clearly shows a dramatic growth in the amplitude of the quantum correction at long times as a consequence of the instabilities associated with the Goldstone mode. As a consequence of this, perturbation theory *must* fail at long times.

2. The Langevin equation

A fundamental aspect of dissipation is that of decoherence which plays an important role in studies of quantum cosmology [25]. Furthermore, dissipation and fluctuations are related by the fluctuation-dissipation theorem. This point has been stressed by Hu and collaborators [33]. The relation between dissipative kernels and fluctuations and correlations of bath degrees of freedom is best captured in a Langevin equation.

In this section we present a derivation of the corresponding Langevin equation to one-loop order in an amplitude expansion. The first step towards deriving the Langevin equation is to determine the "system" and "bath" variables, once this is done, one integrates out the "bath" variables obtaining a (nonlocal) influence functional [25, 34, 35] for the system variables.

We separate the zero mode ($\phi^\pm$) from the field as

$$\Phi^\pm = \phi^\pm + \psi^\pm \; ; \int d^3x\; \psi^\pm(x,t) = 0 \qquad (34)$$

and consider ϕ as the "system" and ψ as the "bath" to be integrated out. Because of the condition in Eq. (34), there are no terms linear in $\psi^\pm$ in the expansion of the action in terms of these fields. Thus

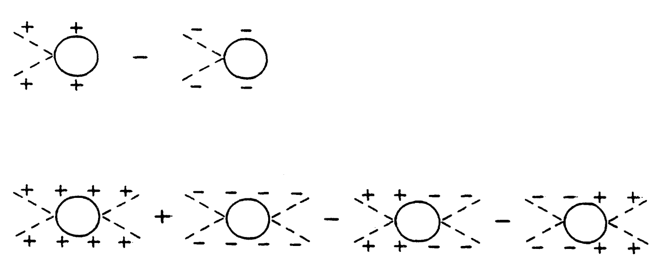


FIG. 4. One-loop diagrams contributing to the effective action up to $O((\phi^\pm)^4)$. The dashed external legs correspond to the zero mode.

$$S[\phi^+,\psi^+]-S[\phi^-,\psi^-]=\Omega\left(\mathcal{S}[\phi^+]-\mathcal{S}[\phi^-]\right)+S_0[\psi^+]-S_0[\psi^-]$$
$$-\frac{\lambda}{4}\left[[\phi^+(t)]^2\int d^3x\,[\psi^+(x,t)]^2-[\phi^-(t)]^2\int d^3x\,[\psi^-(x,t)]^2\right]+O((\psi^\pm)^3,(\psi^\pm)^4),$$

where Ω is the spatial volume, $\mathcal{S}$ the action per unit spatial volume, S_0 the free-field action, and the terms $O((\psi^\pm)^3,(\psi^\pm)^4)$ will contribute at the two-loop level and beyond. To this order, the coupling between "bath" and "system" is similar to the biquadratic coupling considered in Ref. [37] (see also [25]). Integrating out the $\psi^\pm$ fields in a consistent loop expansion gives rise to the influence functional [25, 34, 35] for the zero modes. The one-loop diagrams contributing to this functional up to $O((\phi^\pm)^4)$ are shown in Fig. 4. In order to obtain the Langevin equation it is convenient to introduce the center of mass $[\phi(t)]$ and relative $[R(t)]$ coordinates (these are the coordinates used in the Wigner transform of the coordinate density matrix) as [33, 35, 36]

$$\phi^\pm(t)=\phi(t)\pm\frac{R(t)}{2}. \tag{35}$$

Being patient with the algebra we find the effective action per unit spatial volume

$$\mathcal{S}_{\text{eff}}[\phi,R]=\int_{-\infty}^{\infty}dt\left\{\mathcal{L}[\phi+R/2]-\mathcal{L}[\phi-R/2]-\frac{\lambda}{2}R(t)\phi(t)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\right.$$
$$\left.+\frac{\lambda^2}{4}R(t)\phi(t)\int_{-\infty}^{t}dt'\phi^2(t')\int\frac{d^3k}{(2\pi)^3}\frac{\sin[2\omega_k(t-t')]}{2\omega_k^2}\right\}$$
$$+i\frac{\lambda^2}{8}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'R(t)R(t')\phi(t)\phi(t')\int\frac{d^3k}{(2\pi)^3}\frac{\cos[2\omega_k(t-t')]}{2\omega_k^2}+O(R^3). \tag{36}$$

The higher order terms $O(R^3)$ receive contributions from two and higher loops and give higher order corrections to the lowest order (one-loop) Langevin equation. The imaginary part of the effective action above (last nonlocal term) gives a contribution to the path integral that may be written in terms of a stochastic field as

$$\exp\left[-\frac{1}{2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'R(t)\mathcal{K}(t,t')R(t')\right]\propto\int\mathcal{D}\xi\mathcal{P}[\xi]\exp\left[i\int_{-\infty}^{\infty}dt\xi(t)R(t)\right],$$
$$\mathcal{P}[\xi]=\exp\left[-\frac{1}{2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\xi(t)\mathcal{K}^{-1}(t,t')\xi(t')\right]. \tag{37}$$

with

$$\mathcal{K}(t,t')=\frac{\lambda^2}{4}\phi(t)\phi(t')\int\frac{d^3k}{(2\pi)^3}\frac{\cos[2\omega_k(t-t')]}{2\omega_k^2}. \tag{38}$$

The nonequilibrium path integral now becomes (keeping track of volume factors)

$$Z\propto\int\mathcal{D}\phi\mathcal{D}R\mathcal{D}\xi\,\mathcal{P}[\xi]\exp\left\{i\Omega\left[\mathcal{S}_{\text{reff}}[\phi,R]+\int dt\xi(t)R(t)\right]\right\} \tag{39}$$

with $\mathcal{S}_{\text{reff}}[\phi,R]$ the real part of the effective action in (36) and with $\mathcal{P}[\xi]$ the Gaussian probability distribution for the stochastic noise variable given in (37).

The Langevin equation is obtained via the saddle point condition [25, 33, 35, 36]

$$\left.\frac{\delta\mathcal{S}_{\text{reff}}}{\delta R(t)}\right|_{R=0}=\xi(t) \tag{40}$$

leading to

$$\ddot{\phi}(t)+m^2\phi(t)+\frac{\lambda}{6}\phi^3(t)+\frac{\lambda}{2}\phi(t)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}-\frac{\lambda^2}{4}\phi(t)\int_{-\infty}^{t}dt'\phi^2(t')\int\frac{d^3k}{(2\pi)^3}\frac{\sin[2\omega_k(t-t')]}{2\omega_k^2}=\xi(t), \tag{41}$$

where the stochastic noise variable $\xi(t)$ has Gaussian correlations

$$\langle\langle\xi(t)\rangle\rangle=0\ ;\ \langle\langle\xi(t)\xi(t')\rangle\rangle=\mathcal{K}(t,t')=\frac{\lambda^2}{4}\phi(t)\phi(t')\int\frac{d^3k}{(2\pi)^3}\frac{\cos[2\omega_k(t-t')]}{2\omega_k^2}. \tag{42}$$

Here, the double brackets stand for averages with respect to the Gaussian probability distribution $\mathcal{P}[\xi]$. We can see that the noise is colored (not δ function correlated) and *multiplicative*. By integrating by parts the "dissipative kernel" in Eq. (41) (the last nonlocal term) in the same way as done in Eq. (6), we can clearly see that the resulting "dissipative kernel" and the noise correlation function obey a generalized fluctuation-dissipation theorem [25, 33]. In the broken symmetry case, in the *linearized approximation* (around the tree-level minimum) and *if* the k integral could be replaced by a δ function, we would obtain the usual fluctuation-dissipation relation. By taking the average over the stochastic noise of (41) with the Gaussian probability distribution (37), one obtains the equation of motion (6) by replacing the average of product of fields by the product of the averages (thus considering the field as a classical background).

The higher order terms in the effective action (influence functional) give rise to modifications to the noise correlations, making them non-Gaussian and involving more powers of ϕ in the kernels; in principle these corrections may be computed systematically in a loop expansion.

Although this Langevin equation clearly exhibits the generalized fluctuation-dissipation theorem connecting the "dissipative" kernel to the correlations of the stochastic noise, it is a hopeless tool for any evaluation of the dynamics. The long-range kernels and the multiplicative nature of the noise prevent this Langevin equation from becoming a useful tool. Its importance resides at the fundamental level in that it provides a direct link between fluctuation and dissipation including all the memory effects and multiplicative aspects of the noise correlation functions. This last correlation function is related to the decoherence functional [25]. Although the appeareance of colored noise has already been found in different problems [26, 27], our purpose in deriving the Langevin equation in this case is to point out once again that the phenomenological local damping term proportional to the time derivative of the coarse grained field is not consistent with the fluctuation dissipation theorem which is a fundamental result of nonequilibrium statistical mechanics.

3. Failure of perturbation theory to describe dissipation

This section has been devoted to a perturbative analysis of the "dissipative aspects" of the equation of motion for the scalar field. Perturbation theory has been carried out as an amplitude expansion and also up to $O(\hbar)$, both in the broken and the unbroken symmetry case. In both cases we found both analytically and numerically that the amplitude of the quantum corrections *grow as a function of time* and that the long time behavior cannot be captured in perturbation theory. This failure of perturbation theory to describe dissipation is clearly understood from a very elementary but yet illuminating example: the damped harmonic oscillator. Consider a damped harmonic oscillator

$$\ddot{q} + \Gamma\dot{q} + q = 0 \tag{43}$$

with $\Gamma = O(\lambda)$ where λ is a small perturbative coupling. One can attempt to solve Eq. (43) in a perturbative expansion in Γ. That is, set $q(t) = q_0(t) + \Gamma q_1(t) + \cdots$. The solution for $q_1(t)$ may be found by the Laplace transform

$$q_1(t) \propto \Gamma t \cos(t), \tag{44}$$

clearly exhibiting resonant behavior. This is recognized as a secular term. If Eq. (43) had been obtained as an effective equation of motion in a *perturbative expansion* in λ, this would be the consistent manner to solve this equation. However, we would be led to conclude that perturbation theory breaks down at long times. The correct solution is

$$\begin{aligned} q(t) &= e^{-\frac{\Gamma}{2}t}\cos[\omega(\Gamma^2)t] \\ &\approx \cos(t) - \frac{\Gamma}{2}t\cos(t) + O(\Gamma^2). \end{aligned} \tag{45}$$

We see that the first-order correction in Γ is correctly reproduced by perturbation theory, but in order to find appreciable damping, we must wait a time $\sim O(1/\Gamma)$ at which perturbation theory becomes unreliable.

In order to properly describe dissipation and damping one must *resum* the perturbative expansion. One could in principle keep the first order correction and exponentiate it in an ad hoc manner with the hope that this would be the correct resummation. Although ultimately this *may be* the correct procedure, it is by no means warranted in a field-theoretic perturbative expansion, since in field theory, dissipation is related to particle production and open multiparticle channels, both very subtle and nonlinear mechanisms. Another hint that points to a resummation of the perturbative series is provided by the set of equations (29) and (30). In Eq. (29), the classical solution is a periodic function of time of constant amplitude, since the classical equation has a conserved energy. As a consequence, the "potential" in the equation for the modes [Eq. (30)] is a periodic function of time with constant amplitude. Thus although the fluctuations react back on the coarse grained field, only the classically conserved part of the motion of the coarse grained field enters in the evolution equations of the mode functions. This is a result of being consistent with the loop expansion, but clearly this approximation is *not* energy conserving.

As we will point out in the next section, in an energy conserving scheme the fluctuations and amplitudes will grow up to a maximum value and then will always remain bounded at all times.

Thus, in summary for this section, we draw the conclusion that perturbation theory is not sufficient (without major ad hoc assumptions) to capture the physics of dissipation and damping in real time. A resummation scheme is needed that effectively sums up the whole (or partial) perturbative series in a consistent and/or controlled manner, and which provides a reliable estimate for the long-time behavior. The next section is devoted to the analytical and numerical study of some of these schemes.

III. NONPERTURBATIVE SCHEMES I: HARTREE APPROXIMATION

Motivated by the failure of the loop and amplitude expansions, we now proceed to consider the equations of motion in some nonperturbative schemes. First we study a single scalar model in the time-dependent Hartree approximation. After this, we study an O(N) scalar theory in the large N limit. This last case allows us to study the effect of Goldstone bosons on the time evolution of the order parameter.

In a single scalar model described by the Lagrangian density of Eq. (1), the Hartree approximation is implemented as follows. We again decompose the fields as $\Phi^{\pm} = \phi + \psi^{\pm}$ and the Lagrangian density is given by Eq. (3). The Hartree approximation is obtained by assuming the factorization (for both $\pm$ components)

$$\psi^3(\mathbf{x},t) \to 3\langle\psi^2(\mathbf{x},t)\rangle\psi(\mathbf{x},t),$$

$$\psi^4(\mathbf{x},t) \to 6\langle\psi^2(\mathbf{x},t)\rangle\psi^2(\mathbf{x},t) - 3\langle\psi^2(\mathbf{x},t)\rangle^2.$$

Translational invariance shows that $\langle(\psi^{\pm}(\mathbf{x},t))^2(\mathbf{x},t)\rangle$ can only be a function of time, and because this is an equal time correlation function, we have that

$$\langle[\psi^+(\mathbf{x},t)]^2\rangle = \langle[\psi^-(\mathbf{x},t)]^2\rangle \stackrel{\text{def}}{=} \langle\psi^2(t)\rangle. \tag{46}$$

The expectation value will be determined within a self-consistent approximation. After this factorization we find

$$\mathcal{L}[\phi+\psi^+] - \mathcal{L}[\phi-\psi^-] = \left\{\left(\frac{\delta\mathcal{L}}{\delta\phi} - \frac{1}{2}\lambda\phi\langle\psi^2(t)\rangle\right)\psi^+ + \frac{1}{2}\left(\partial_\mu\psi^+\right)^2 - \frac{1}{2}\mathcal{M}^2(t)(\psi^+)^2\right\} - \left\{(\psi^+ \to \psi^-)\right\}, \tag{47}$$

where

$$\mathcal{M}^2(t) = V''(\phi) + \frac{\lambda}{2}\langle\psi^2(t)\rangle$$
$$= m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\langle\psi^2(t)\rangle. \tag{48}$$

The resulting Hartree equations are obtained by using the tadpole method $\langle\psi^{\pm}(\mathbf{x},t)\rangle = 0$ as before. They are given by

$$\ddot{\phi} + m^2\phi + \frac{\lambda}{6}\phi^3 + \frac{\lambda}{2}\phi\langle\psi^2(t)\rangle = 0,$$

$$\langle\psi^2(t)\rangle = \int\frac{d^3k}{(2\pi)^3}\left[-iG_k^<(t,t)\right] = \int\frac{d^3k}{(2\pi)^3}\frac{\left|U_k^+(t)\right|^2}{2\omega_k(0)},$$

$$\left[\frac{d^2}{dt^2} + \omega_k^2(t)\right]U_k^+(t) = 0\,; \qquad \omega_k^2(t) = \mathbf{k}^2 + \mathcal{M}^2(t).$$

The initial conditions for the mode functions are

$$U_k^+(0) = 1\,; \qquad \dot{U}_k^+(0) = -i\omega_k(0). \tag{49}$$

It is clear that the Hartree approximation makes the Lagrangian density quadratic at the expense of a self-consistent condition. In the time-independent case, this approximation sums up all the "daisy" (or "cactus") diagrams and leads to a self-consistent gap equation.

At this stage, we must point out that the Hartree approximation is uncontrolled in this single scalar theory. This approximation does, however, become exact in the $N \to \infty$ limit of the O(N) model which we will discuss in the next section.

We refer the reader to Refs. [28, 29] for a detailed description of the renormalization procedure. With an eye towards the numerical analysis, it is more convenient to write

$$\langle\psi^2(t)\rangle_R = \left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R\right) + \langle\psi^2(0)\rangle_R \tag{50}$$

and perform a subtraction at time $t = 0$ absorbing $\langle\psi^2(0)\rangle_R$ into a further *finite* renormalization of the mass term ($m_R^2 + \langle\psi^2(0)\rangle_R = M_R^2$).

The renormalized equations that we will solve finally become

$$\ddot{\phi} + M_R^2\phi + \frac{\lambda_R}{2}\left[1 - \left(\frac{2}{3}\right)\frac{1}{1 - \frac{\lambda_R}{16\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right)}\right]\phi^3 + \frac{\lambda_R}{2}\phi\left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R\right) = 0,$$

$$\left[\frac{d^2}{dt^2} + k^2 + M_R^2 + \frac{\lambda_R}{2}\phi^2(t) + \frac{\lambda_R}{2}\left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R\right)\right]U_k^+(t) = 0,$$

and

$$\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R = \frac{1}{1 - \frac{\lambda_R}{16\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right)}\left\{\int\frac{d^3k}{(2\pi)^3}\frac{\left|U_k^+(t)\right|^2 - 1}{2\omega_k(0)} + \frac{\lambda_R}{16\pi^2}\ln\left(\frac{\Lambda}{\kappa}\right)\left(\phi^2(t) - \phi^2(0)\right)\right\} \tag{51}$$

with the initial conditions for $U_k^+(t)$

$$U_k^+(0) = 1\,; \qquad \dot{U}_k^+(0) = -i\omega_k(0)\,; \omega_k(0) = \sqrt{k^2 + M_R^2 + \frac{\lambda_R}{2}\phi^2(0)}. \tag{52}$$

It is worth noticing that there is a weak cutoff dependence on the renormalized equations of motion of the order parameter and the mode functions. This is a consequence of the well known "triviality" problem of the scalar quartic interaction in four space-time dimensions. This has the consequence that for a fixed renormalized coupling the cutoff must be kept fixed and finite. The presence of the Landau pole prevents taking the limit of the ultraviolet cutoff to infinity while keeping the renormalized coupling fixed.

This theory is sensible only as a low-energy cutoff effective theory. We then must be careful that for a fixed value of λ_R, the cutoff must be such that the theory never crosses the Landau pole. Thus from a numerical perspective there will always be a cutoff sensitivity in the theory. However, for small coupling we expect the cutoff dependence to be rather weak (this will be confirmed numerically), provided the cutoff is far away from the Landau pole.

A. Particle production

Before we engage ourselves in a numerical integration of the above equations of motion we want to address the issue of particle production, since it is of great importance for the understanding of dissipative processes. In what follows, we consider particle production due to the time varying effective mass $\mathcal{M}^2(t)$ in Eq. (48) of the quantum field ψ for the single scalar model.

The Lagrangian density for the fluctuations in the Hartree approximation is given by Eq. (47). Demanding that the order parameter satisfies its equation of motion implies that the linear term in η in Eq. (47) vanishes. The resulting Lagrangian density is

$$\mathcal{L}[\psi] = \frac{1}{2}\left(\partial_\mu \psi\right)^2 - \frac{1}{2}\mathcal{M}^2(t)\psi^2 + \frac{\lambda}{8}\langle\psi^2\rangle. \tag{53}$$

The Hartree-Fock vacuum state at $t = 0$ is chosen as the reference state. As time passes, particles (as defined with respect to this state) will be produced as a result of parametric amplification [38, 39]. We should mention that our definition differs from that of other authors [26, 39] in that we chose the state at time $t = 0$ rather than using the adiabatic modes (that diagonalize the instantaneous Hamiltonian).

We define the number density of particles as a function of time as

$$\begin{aligned}\mathcal{N}_k(t) &= \int \frac{d^3k}{(2\pi)^3}\frac{\mathrm{Tr}\left[a_k^\dagger(0)a_k(0)\rho(t)\right]}{\mathrm{Tr}\rho(0)} \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{\mathrm{Tr}\left[a_k^\dagger(t)a_k(t)\rho(0)\right]}{\mathrm{Tr}\rho(0)},\end{aligned} \tag{54}$$

where by definition

$$\begin{aligned}a_k^\dagger(t) &= U^{-1}(t,0)a_k^\dagger(0)U(t,0)\ , \\ a_k(t) &= U^{-1}(t,0)a_k(0)U(t,0)\end{aligned} \tag{55}$$

are the time-evolved operators in the Heisenberg picture. Following Refs. [29, 45], we find that the creation and annihilation operator at time t can be related to those at time $t = 0$ via a Bogoliubov transformatiom:

$$a_k(t) = \mathcal{F}_{+,k}(t)a_k(0) + \mathcal{F}_{-,k}(t)a^\dagger_{-k}(0). \tag{56}$$

The $\mathcal{F}_\pm(t)$ can be read off in terms of the mode functions $U_k^+(t)$,

$$|\mathcal{F}_{+,k}(t)|^2 = \frac{1}{4}\left|U_k^+(t)\right|^2\left[1 + \frac{\left|\dot{U}_k^+(t)\right|^2}{\omega_k^2(0)\left|U_k^+(t)\right|^2}\right] + \frac{1}{2},$$

$$|\mathcal{F}_{+,k}(t)|^2 - |\mathcal{F}_{-,k}(t)|^2 = 1.$$

After some algebra, we find

$$\mathcal{N}_k(t) = \left(2\left|\mathcal{F}_{+,k}(t)\right|^2 - 1\right)\mathcal{N}_k(0) + \left(\left|\mathcal{F}_{+,k}(t)\right|^2 - 1\right). \tag{57}$$

This result exhibits the contributions from "spontaneous" (proportional to the initial particle occupations) and "induced" (independent of it) particle production. Since we are analyzing the zero temperature case with $\mathcal{N}_k(0) = 0$, only the induced contribution results.

Before moving on to the numerical analysis, it is important to point out that the Hartree approximation is *energy conserving.* The energy density is

$$\mathcal{E} = \frac{1}{2}\dot{\phi}^2(t) + V(\phi(t)) + \frac{\mathrm{Tr}\mathcal{H}\rho(0)}{\mathrm{Tr}\rho(0)}, \tag{58}$$

with $V(\phi)$ the classical potential and $\mathcal{H}$ is the Hartree Hamiltonian with Π_ψ, ψ expanded in terms of the Hartree mode functions and creation and annihilation operators acting on the Hartree-Fock states. Using the equations of motion for $\phi(t)$ and the mode functions, and after some lengthy but straightforward algebra, one finds $\dot{\mathcal{E}} = 0$.

B. Numerical analysis

1. Unbroken symmetry case

In order to perform a numerical analysis it is necessary to introduce dimensionless quantities and it becomes convenient to choose the renormalization point $\kappa = M_R$. Thus we define

$$\eta(t) = \phi(t)\sqrt{\frac{\lambda_R}{2M_R^2}}\ ;\ q = \frac{k}{M_R}\ ;\ \tau = M_R t\ ;\ g = \frac{\lambda_R}{8\pi^2},$$

$$\Sigma(t) = \frac{4\pi^2}{M_R^2}\left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R\right), \tag{59}$$

and finally, the equations of motion become

$$\frac{d^2}{d\tau^2}\eta + \eta + \left[1 - \left(\frac{2}{3}\right)\frac{1}{1 - \frac{g}{2}\ln\left(\frac{\Lambda}{M_R}\right)}\right]\eta^3 + g\eta\Sigma(\tau) = 0,$$

$$\left[\frac{d^2}{d\tau^2} + q^2 + 1 + \eta^2(\tau) + g\Sigma(\tau)\right]U_q^+(\tau) = 0, \tag{60}$$

$$U_q^+(0) = 1\,;\ \frac{d}{d\tau}U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)}, \tag{61}$$

$$\Sigma(\tau) = \left[\frac{1}{1-\frac{g}{2}\ln\left(\frac{\Lambda}{M_R}\right)}\right] \times\left\{\int_0^{\Lambda/M_R} q^2 dq \frac{\left|U_q^+(\tau)\right|^2-1}{\sqrt{q^2+1+\eta^2(0)}} +\frac{1}{2}\ln\left(\frac{\Lambda}{M_R}\right)\left[\eta^2(\tau)-\eta^2(0)\right]\right\}. \tag{62}$$

In terms of the dimensionless quantities we obtain the number of particles within a *correlation volume* $N(\tau) = \mathcal{N}(t)/M_R^3$,

$$N(\tau) = \frac{1}{8\pi^2}\int_0^{\Lambda/M_R} q^2 dq\left\{\left|U_q^+(\tau)\right|^2 +\frac{\left|\dot{U}_q^+(\tau)\right|^2}{\sqrt{q^2+1+\eta^2(0)}}-2\right\}. \tag{63}$$

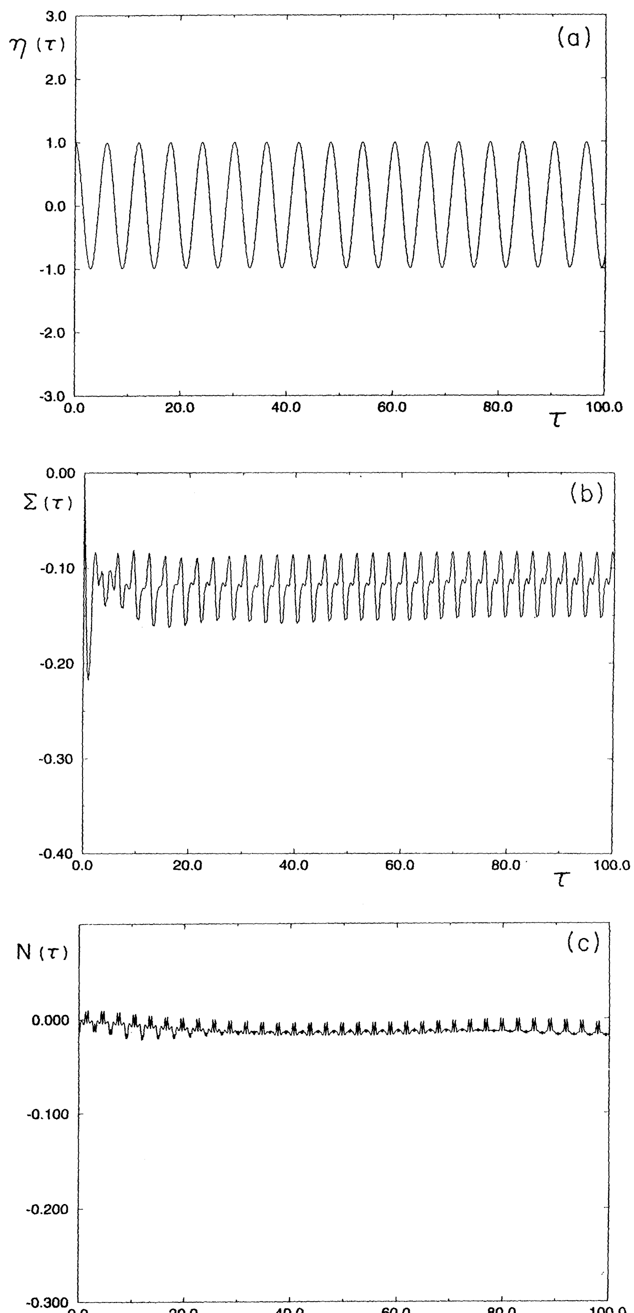


FIG. 5. (a) $\eta(\tau)$ vs τ in the Hartree approximation, unbroken-symmetry case. $g = 0.1$; $\eta(0) = 1$; $\Lambda/M_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

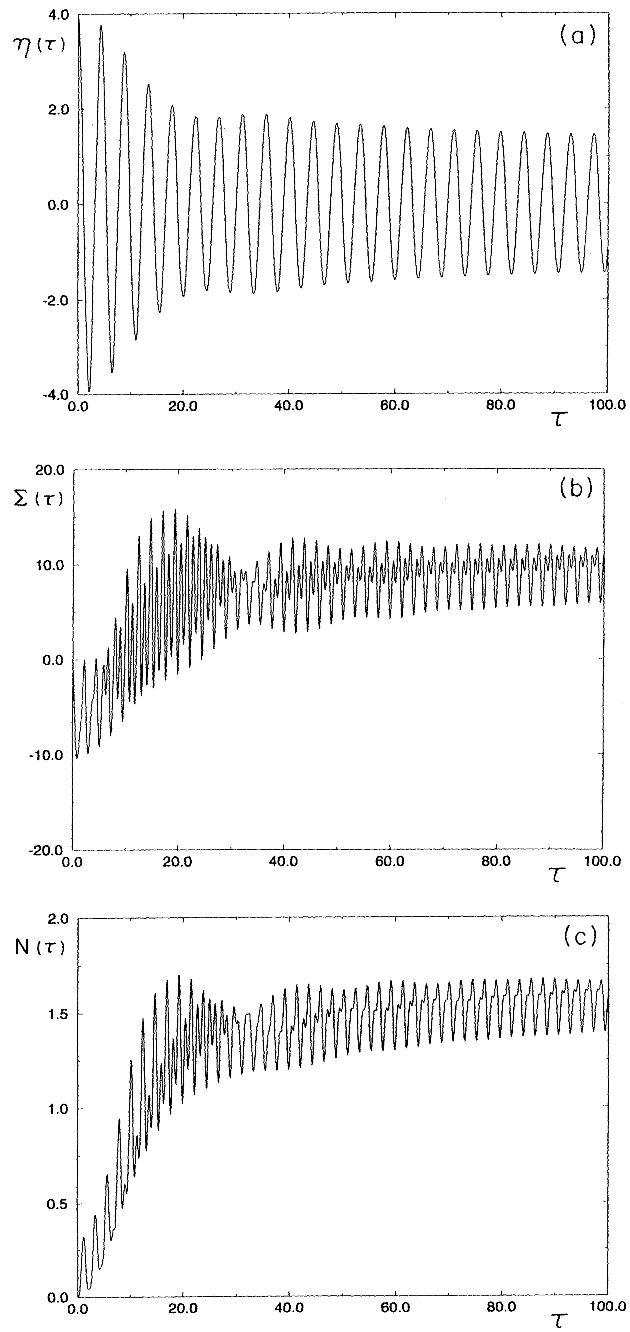


FIG. 6. (a) $\eta(\tau)$ vs τ in the Hartree approximation, unbroken-symmetry case. $g = 0.1$; $\eta(0) = 4$; $\Lambda/M_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

Figures 5(a)–5(c) show $\eta(\tau)$, $\Sigma(\tau)$, and $N(\tau)$ in the Hartree approximation, for $g = 0.1$, $\eta(0) = 1.0$, and $\Lambda/M_R = 100$; we did not detect an appreciable cutoff dependence by varying the cutoff between 50 and 200. Clearly there is no appreciable damping in $\eta(\tau)$. In fact it can be seen that the period of the oscillation is very close to 2π, which is the period of the classical solution of the linear theory. This is understood because the coefficient of the cubic term is very small and $g\Sigma(\tau)$ is negligible. Particle production is also negligible. This situation should be contrasted with that shown in Figs. 6(a)–6(c) and 7(a)–7(c) in which there is dissipation and damping in the evolution of $\eta(\tau)$ for $\eta(0) = 4, 5$, respectively, and the same values for g and the cutoff. There are several noteworthy features that can be deduced from these figures. First the fluctuations become very large, such that

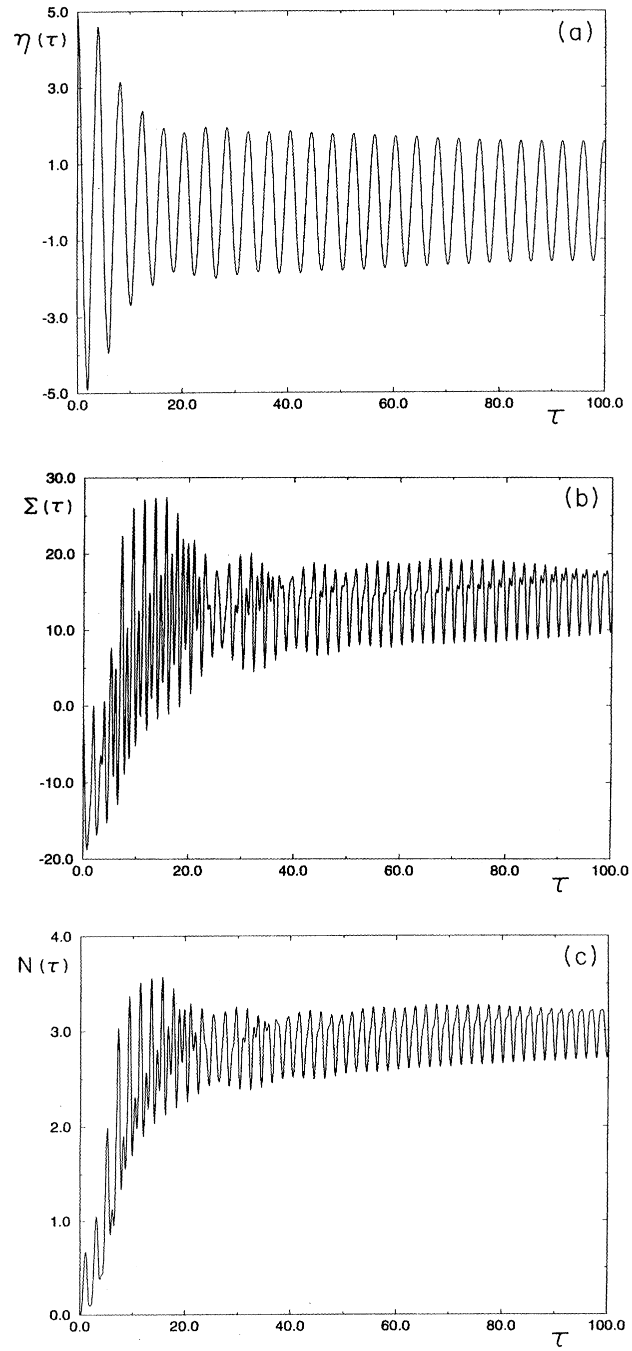


FIG. 7. (a) $\eta(\tau)$ vs τ in the Hartree approximation, unbroken-symmetry case. $g = 0.1$; $\eta(0) = 5$; $\Lambda/M_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

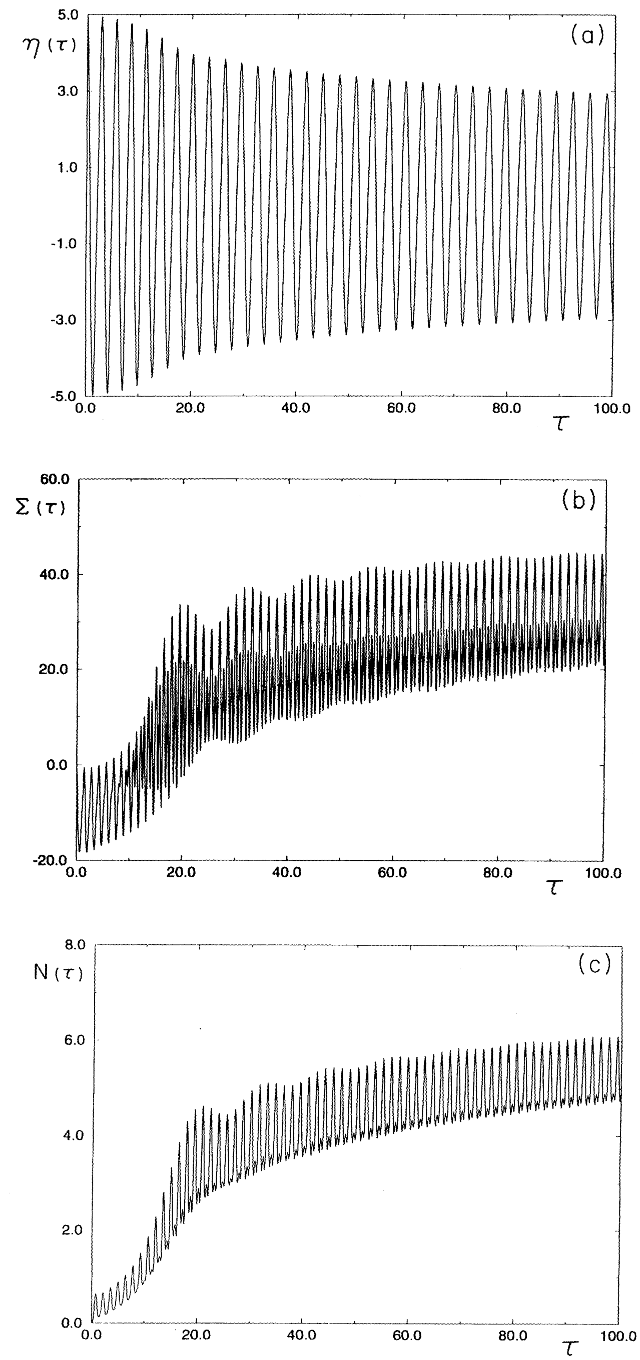


FIG. 8. (a) $\eta(\tau)$ vs τ in the Hartree approximation, unbroken-symmetry case. $g = 0.05$; $\eta(0) = 5$; $\Lambda/M_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

$g\Sigma(\tau)$ becomes ~ 1. Second, Figs. 6(a) and 7(a) clearly show that initially, channels are open and energy is transferred away from the $q = 0$ mode of the field. Eventually however, these channels shut off, and the dynamics of the expectation value settles into an oscillatory motion. The time scale for the shutting off of the dissipative behavior decreases as $\eta(0)$ increases; it is about 25 for $\eta(0) = 4$, and about 18 for $\eta(0) = 5$. This time scale is correlated with the time scale in which particles are produced by parametric amplification and the quantum fluctuations begin to plateau [Figs. 6(b), 6(c) and 7(b), 7(c)]. Clearly the dissipative mechanism which damps the motion of the expectation value is particle production. Furthermore, the long time dynamics for the expectation value *does not* correspond to exponential damping. In fact, we did not find any appreciable damping for $\tau \geq 70$ in these cases. It is illuminating to compare this situation with that of a smaller coupling depicted in Figs. 8(a)–8(c) for $\eta(0) = 5$ $g = 0.05$. Clearly the time scale for damping is much larger and there is still appreciable damping at $\tau \approx 100$. Notice also that $g\Sigma(\tau) \approx 2$ and that particles are being produced even at long times, and this again correlates with the evidence that the field expectation value shows damped motion at long times, clearly showing that the numerical analysis has not reached the asymptotic regime.

These numerical results provide some interesting observations. First we see that $\eta^2(\tau) + g\Sigma(\tau)$ has asymptotic oscillatory behavior in τ. In general this would imply unstable bands of wave vectors for the solutions of Eq. (60). However, if these unstable bands are present, the mode functions would grow and $\Sigma(\tau)$ would grow as a result since it is an integral over all wave vectors. The fact that this does not happen implies that there are no unstable bands.

We have been able to find analytically an exact oscillatory solution of Eq. (60), which is given by

$$1 + \eta^2(\tau) + g\Sigma(\tau) = -e_3 - 2\,\mathcal{P}(\tau + \omega'), \qquad (64)$$

$$\eta(\tau) = A\,\mathrm{sn}(\sqrt{e_1 - e_3}\,\tau),$$

where $\mathcal{P}$ is the Weierstrass function which is double periodic with periods ω ; ω' [43] and sn is the Jacobi sinus function. The parameters in this solution are functions of the coupling. We found that, asymptotically, the numerical solution of Eqs. (60), (61) could be fit quite precisely to the above analytic solution for precise values of the parameters. These values encode the information about the initial conditions and the coupling constant. For this potential given by (64) we can find the exact solutions for the mode functions in terms of Weierstrass σ functions [44]. In this case there are no unstable modes for real wave vector q. This is an important result because it provides an analytic understanding for the lack of forbidden bands.

Although this exact result does not illuminate the physical reason why there are no unstable bands, we conjecture that this is a consequence of the conservation of energy in the Hartree approximation. Numerically we find no evidence for exponential relaxation asymptotically; of course, we cannot rule out the possibility of power law asymptotic decay for η with an exceedingly small power.

The fundamental question to be raised at this point is, what is the origin of the damping in the evolution of the field expectation value? Clearly this is a collisionless process as collisions are not taken into account in the Hartree approximation (although the one-loop dia-

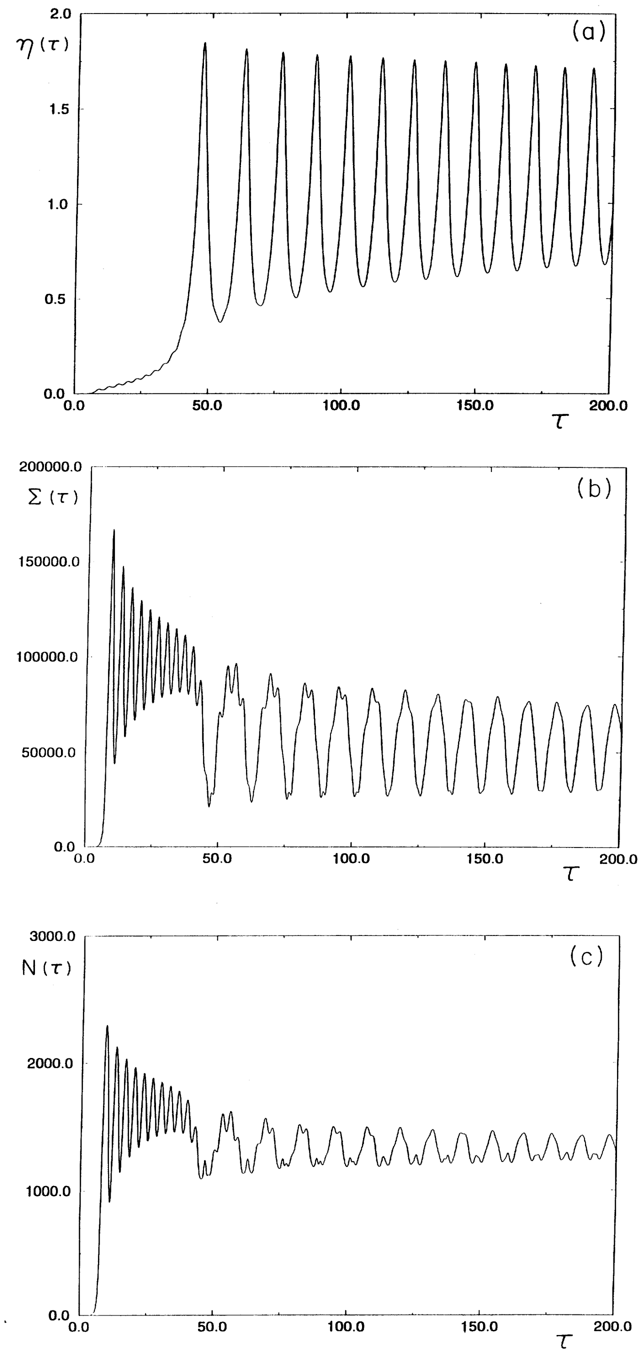


FIG. 9. (a) $\eta(\tau)$ vs τ in the Hartree approximation, broken-symmetry case. $g = 10^{-5}$; $\eta(0) = 10^{-5}$; $\Lambda/\mu_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

gram that enters in the two-particle collision amplitude with the two-particle cut is contained in the Hartree approximation and is responsible for thresholds to particle production). The physical mechanism is reminiscent of that of Landau damping in the collisionless Vlasov equation for plasmas [10] and also found in the study of strong electric fields in Ref. [11]. The difference with the usual Landau damping mechanism is that in our case damping is manifest in the zero momentum component of the scalar field, whereas in the plasma situation it occurs in a nonhomogeneous plasma, however, the main similarity is that just as in the plasma case, damping occurs in the collisionless regime.

In the case under consideration, energy is transferred from the expectation value to the quantum fluctuations, which back-react on the evolution of the field expectation value but out of phase. This phase difference between the oscillations of $\eta^2(\tau)$ and those of $\Sigma(\tau)$ can be clearly seen to be π in Figs. 6(a), 6(b), 7(a), and 7(b), since the maxima of $\eta^2(\tau)$ occur at the same times as the minima of $\Sigma(\tau)$ and vice versa.

This is an important point learned from our analysis and that is not *a priori* taken into account in the usual arguments for dissipation via collisions. The process of thermalization, however, will necessarily involve collisions and cannot be studied within the schemes addressed in this paper.

2. Broken symmetry

The broken symmetry case is obtained by writing $M_R^2 = -\mu_R^2 < 0$ and using the scale μ_R instead of M_R to define the dimensionless quantities as in Eq. (59) and the renormalization scale. The equations of motion in this case become

$$\frac{d^2}{d\tau^2}\eta - \eta + \left[1 - \frac{2}{3}\frac{1}{1-\frac{g}{2}\ln\left(\frac{\Lambda}{\mu_R}\right)}\right]\eta^3 + g\eta\Sigma(\tau) = 0,$$

$$\left[\frac{d^2}{d\tau^2} + q^2 - 1 + \eta^2(\tau) + g\Sigma(\tau)\right]U_q^+(\tau) = 0, \quad (65)$$

$$U_q^+(0) = 1\,;\, \frac{d}{d\tau}U_q^+(0) = -i\sqrt{q^2+1+\eta^2(0)}, \quad (66)$$

with $\Sigma(\tau)$ given in Eq. (62) but with M_R replaced by μ_R. The broken-symmetry case is more subtle because of the possibility of unstable modes for initial conditions in which $\eta(0) \ll 1$. We have kept the boundary conditions of Eq. (66) the same as in Eq. (61). This corresponds to preparing an initial state as a Gaussian state centered at $\eta(0)$ with real and positive covariance (width) and letting it evolve for $t > 0$ in the broken-symmetry potential [28, 45]. The number of particles produced within a correlation volume (now μ_R^{-3}) is given by Eq. (63) with M_R replaced by μ_R.

Figures 9(a)–9(c) depict the dynamics for a broken symmetry case in which $\eta(0) = 10^{-5}$, i.e., very close to the top of the potential hill. Notice that as the field expectation value rolls down the hill the unstable modes make the fluctuation grow dramatically until about $\tau \approx 50$ at which point $g\Sigma(\tau) \approx 1$. At this point, the unstable growth of fluctuations shuts off and the field begins damped oscillatory motion around a mean value of about ≈ 1.2. This point is a minimum of the effective action. The damping of these oscillations is very similar to the damping around the origin in the unbroken case. Most of the particle production and the largest quantum fluctuations occur when the field expectation value is rolling down the region for which there are unstable frequencies for the mode functions [see Eq. (65)]. This behavior is similar to that found previously [28].

IV. NONPERTURBATIVE SCHEMES II: LARGE N LIMIT IN THE O(N) MODEL

Although the Hartree approximation offers a nonperturbative resummation of select terms, it is not a consistent approximation because there is no *a priori* small parameter that defines the approximation. Furthermore, we want to study the effects of dissipation by Goldstone bosons in a nonperturbative but controlled expansion.

In this section, we consider the O(N) model in the large N limit. The large N limit has been used in studies of nonequilibrium dynamics [40–42, 45] and it provides a very powerful tool for studying nonequilibrium dynamics nonperturbatively in a consistent manner. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} - V(\sigma,\boldsymbol{\pi}),$$

$$V(\sigma,\boldsymbol{\pi}) = \frac{1}{2}m^2\boldsymbol{\phi}\cdot\boldsymbol{\phi} + \frac{\lambda}{8N}(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2, \quad (67)$$

for λ fixed in the large N limit. Here $\boldsymbol{\phi}$ is an O(N) vector, $\boldsymbol{\phi} = (\sigma, \boldsymbol{\pi})$, and $\boldsymbol{\pi}$ represents the $N-1$ pions. In what follows, we will consider two different cases of the potential $V(\sigma,\boldsymbol{\pi})$, with ($m^2 < 0$) or without ($m^2 > 0$) symmetry breaking.

Let us define the fluctuation field $\chi(\mathbf{x},t)$ as

$$\sigma(\mathbf{x},t) = \sigma_0(t) + \chi(\mathbf{x},t). \quad (68)$$

Expanding the Lagrangian density in Eq. (67) in terms of fluctuations $\chi(\mathbf{x},t)$,we obtain

$$\mathcal{L}[\sigma_0+\chi^+,\boldsymbol{\pi}^+] - \mathcal{L}[\sigma_0+\chi^-,\boldsymbol{\pi}^-] = \Bigg\{\mathcal{L}[\sigma_0,\boldsymbol{\pi}^+] + \frac{\delta\mathcal{L}}{\delta\sigma_0}\chi^+ + \frac{1}{2}\left(\partial_\mu\chi^+\right)^2 + \frac{1}{2}\left(\partial_\mu\boldsymbol{\pi}^+\right)^2$$
$$- \left(\frac{1}{2!}V''(\sigma_0,\boldsymbol{\pi}^+)\chi^{+2} + \frac{1}{3!}V^{[3]}(\sigma_0,\boldsymbol{\pi}^+)(\chi^+)^3 + \frac{1}{4!}V^{[4]}(\sigma_0,\boldsymbol{\pi}^+)(\chi^+)^4\right)\Bigg\}$$
$$- \left\{(\chi^+\to\chi^-),(\boldsymbol{\pi}^+\to\boldsymbol{\pi}^-)\right\}.$$

The tadpole condition $\langle\chi^{\pm}(\mathbf{x},t)\rangle = 0$ will lead to the equations of motion as previously discussed. We now introduce a Hartree factorization. In the presence of a nonzero expectation value the Hartree factorization is subtle in the case of continuous symmetries. A naive Hartree factorization violates the Ward identities related to Goldstone's theorem. This shortcoming is overcome if the Hartree factorization is implemented in leading order in the large N expansion [45]. We will make a series of assumptions that seem to be reasonable, but that can only be justified *a posteriori* when we recognize that with these assumptions, we obtain the equations of motion that fulfill the Ward identities. These assumptions are as follows. (i) No cross correlations between the pions and sigma field and (ii) the two-point correlation functions of the pion field are diagonal in the $O(N-1)$ space given by the remaining unbroken symmetry group. Based upon these assumptions we are led to the following Hartree factorization of the nonlinear terms in the Lagrangian density (again for both $\pm$ components):

$$\chi^4 \to 6\langle\chi^2\rangle\chi^2 + \text{const},$$
$$\chi^3 \to 3\langle\chi^2\rangle\chi,$$
$$(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 \to \left(2+\frac{4}{N-1}\right)\langle\boldsymbol{\pi}^2\rangle\boldsymbol{\pi}^2 + \text{const},$$
$$\boldsymbol{\pi}^2\chi^2 \to \langle\boldsymbol{\pi}^2\rangle\chi^2 + \boldsymbol{\pi}^2\langle\chi^2\rangle,$$
$$\boldsymbol{\pi}^2\chi \to \langle\boldsymbol{\pi}^2\rangle\chi,$$

where by const we mean the operator-independent expectation values of the composite operators which will not enter into the time evolution equation of the order parameter.

In this approximation, the resulting Lagrangian density is quadratic, with a linear term in χ :

$$\mathcal{L}[\sigma_0+\chi^+,\boldsymbol{\pi}^+] - \mathcal{L}[\sigma_0+\chi^-,\boldsymbol{\pi}^-] = \left\{\frac{1}{2}\left(\partial_\mu\chi^+\right)^2 + \frac{1}{2}\left(\partial_\mu\boldsymbol{\pi}^+\right)^2 - \chi^+V'(t) - \frac{1}{2}\mathcal{M}_\chi^2(t)(\chi^+)^2 - \frac{1}{2}\mathcal{M}_\pi^2(t)(\boldsymbol{\pi}^+)^2\right\}$$
$$-\left\{\left(\chi^+\to\chi^-\right),\left(\boldsymbol{\pi}^+\to\boldsymbol{\pi}^-\right)\right\},$$

where V' is the derivative of the Hartree potential defined below. To obtain a large N limit, we define

$$\langle\boldsymbol{\pi}^2\rangle = N\langle\psi^2\rangle\,;\sigma_0(t) = \phi(t)\sqrt{N} \tag{69}$$

with

$$\langle\psi^2\rangle \approx O(N^0)\,;\ \ \langle\chi^2\rangle \approx O(N^0)\,;\ \ \phi \approx O(N^0). \tag{70}$$

We will approximate further by neglecting the $O(\frac{1}{N})$ terms in the formal large N limit. We now obtain

$$V'(\phi(t),t) = \sqrt{N}\phi(t)\left[m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\langle\psi^2(t)\rangle\right],$$
$$\mathcal{M}_\pi^2(t) = m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\langle\psi^2(t)\rangle,$$
$$\mathcal{M}_\chi^2(t) = m^2 + \frac{3\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\langle\psi^2(t)\rangle. \tag{71}$$

Using the tadpole method, we obtain the following set of equations:

$$\ddot{\phi}(t) + \phi(t)\left[m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\langle\psi^2(t)\rangle\right] = 0,$$
$$\langle\psi^2(t)\rangle = \int\frac{d^3k}{(2\pi)^3}\frac{\left|U_k^+(t)\right|^2}{2\omega_{\pi k}^2(0)},$$
$$\left[\frac{d^2}{dt^2} + \omega_{\pi k}^2(t)\right]U_k^+(t) = 0\ ,\ \ \omega_{\pi k}^2(t) = \mathbf{k}^2 + \mathcal{M}_\pi^2(t).$$

It is clear from the above equations that the Ward identities of Goldstone's theorem are indeed fulfilled. Because $V'(\phi(t),t) = \sqrt{N}\phi(t)\mathcal{M}_\pi^2(t)$, whenever $V'(\phi(t),t)$ vanishes for $\phi \neq 0$, then $\mathcal{M}_\pi = 0$ and the "pions" are the Goldstone bosons. The initial conditions for the mode functions $U_k^+(t)$ are

$$U_k^+(0) = 1\ \ ;\qquad \dot{U}_k^+(0) = -i\omega_{\pi k}(0). \tag{72}$$

Since in this approximation, the dynamics for the $\boldsymbol{\pi}$ and χ fields decouples, and the dynamics of χ does not influence that of ϕ or the mode functions and $\langle\psi^2\rangle$, we will only concentrate on the solution for the $\boldsymbol{\pi}$ fields.

A. Renormalization

The renormalization procedure is exactly the same as that for the Hartree case in the preceding section. We carry out the same renormalization prescription and subtraction at $t=0$ as in the last section. Thus we find the equations of motion

$$\ddot{\phi} + M_R^2\phi + \frac{\lambda_R}{2}\phi^3 + \frac{\lambda_R}{2}\phi\left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R\right) = 0,$$
$$\left[\frac{d^2}{dt^2} + k^2 + M_R^2 + \frac{\lambda_R}{2}\phi^2(t) + \frac{\lambda_R}{2}\left(\langle\psi^2(t)\rangle_R - \langle\psi^2(0)\rangle_R\right)\right]U_k^+(t) = 0, \tag{73}$$

with the initial conditions given by Eq. (72) and with the subtracted expectation value given by Eq. (51).

In contrast to the Hartree equations in the preceding section, the cutoff dependence in the term proportional to ϕ^3 in Eq. (73) has disappeared. This is a consequence of the large N limit and the Ward identities, which are now obvious at the level of the renormalized equations of motion. There is still a very weak cutoff dependence in Eq. (51) because of the triviality issue which is not relieved in the large N limit, but again, this theory only makes sense as a low-energy cutoff theory.

B. Numerical Analysis

1. *Unbroken symmetry*

To solve the evolution equations in Eqs. (51) and (73) numerically, we now introduce dimensionless quantities as in Eq. (59), obtaining the following dimensionless equations:

$$\frac{d^2}{d\tau^2}\eta + \eta + \eta^3 + g\eta\Sigma(\tau) = 0,$$
$$\left[\frac{d^2}{d\tau^2} + q^2 + 1 + \eta^2(\tau) + g\Sigma(\tau)\right] U_q^+(\tau) = 0, \quad (74)$$
$$U_q^+(0) = 1\,,\ \frac{d}{d\tau}U_q^+(0) = -i\sqrt{q^2+1+\eta^2(0)},$$

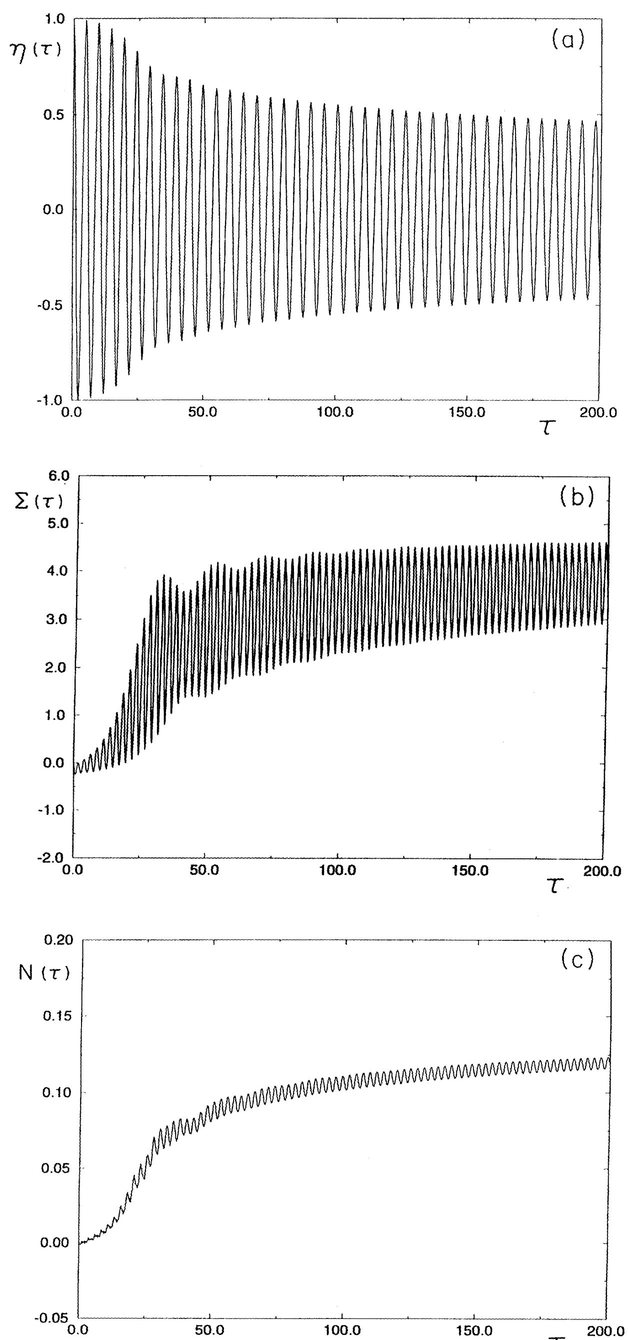


FIG. 10. (a) $\eta(\tau)$ vs τ in the large-N approximation in the O(N) model, unbroken-symmetry case. $g = 0.1$; $\eta(0) = 1$; $\Lambda/M_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

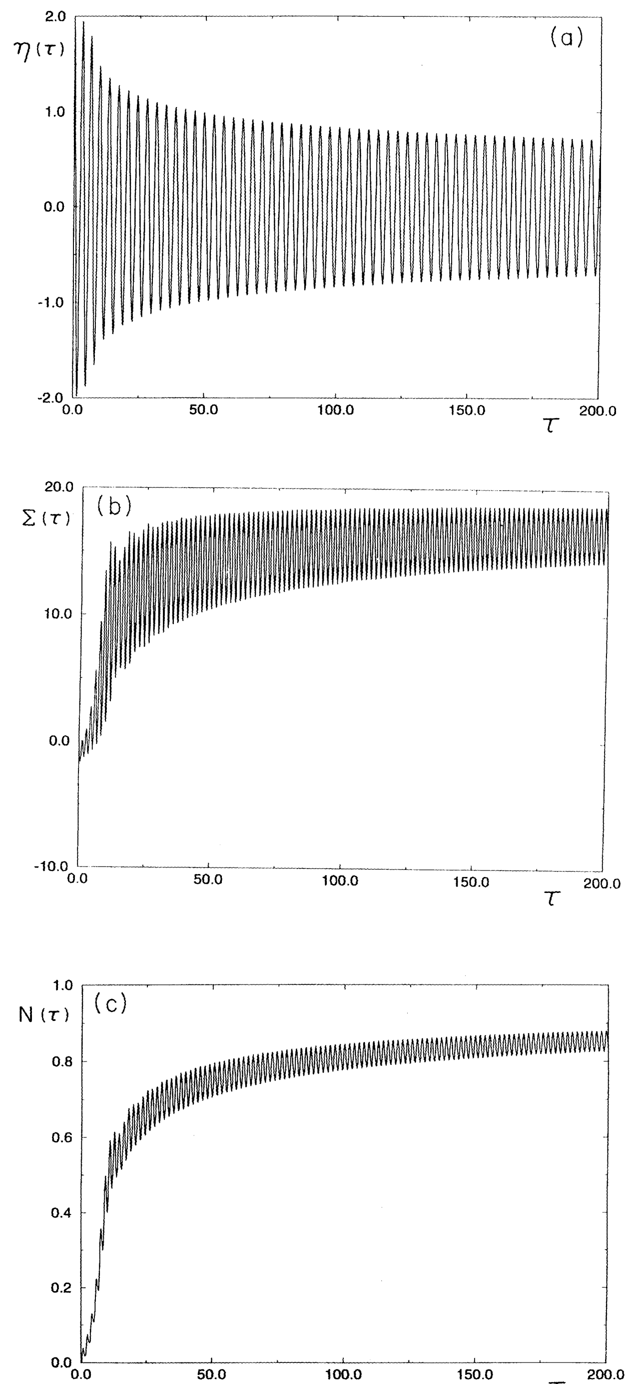


FIG. 11. (a) $\eta(\tau)$ vs τ in the large-N approximation in the O(N) model, unbroken-symmetry case. $g = 0.1$; $\eta(0) = 2$; $\Lambda/M_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

$$\Sigma(\tau) = \frac{1}{1 - \frac{g}{2}\ln\left(\frac{\Lambda}{M_R}\right)} \times \left\{ \int_0^{\Lambda/M_R} q^2 dq \frac{\left|U_q^+(\tau)\right|^2 - 1}{\sqrt{q^2 + 1 + \eta^2(0)}} + \frac{1}{2}\ln\left(\frac{\Lambda}{M_R}\right)\left[\eta^2(\tau) - \eta^2(0)\right] \right\}. \tag{75}$$

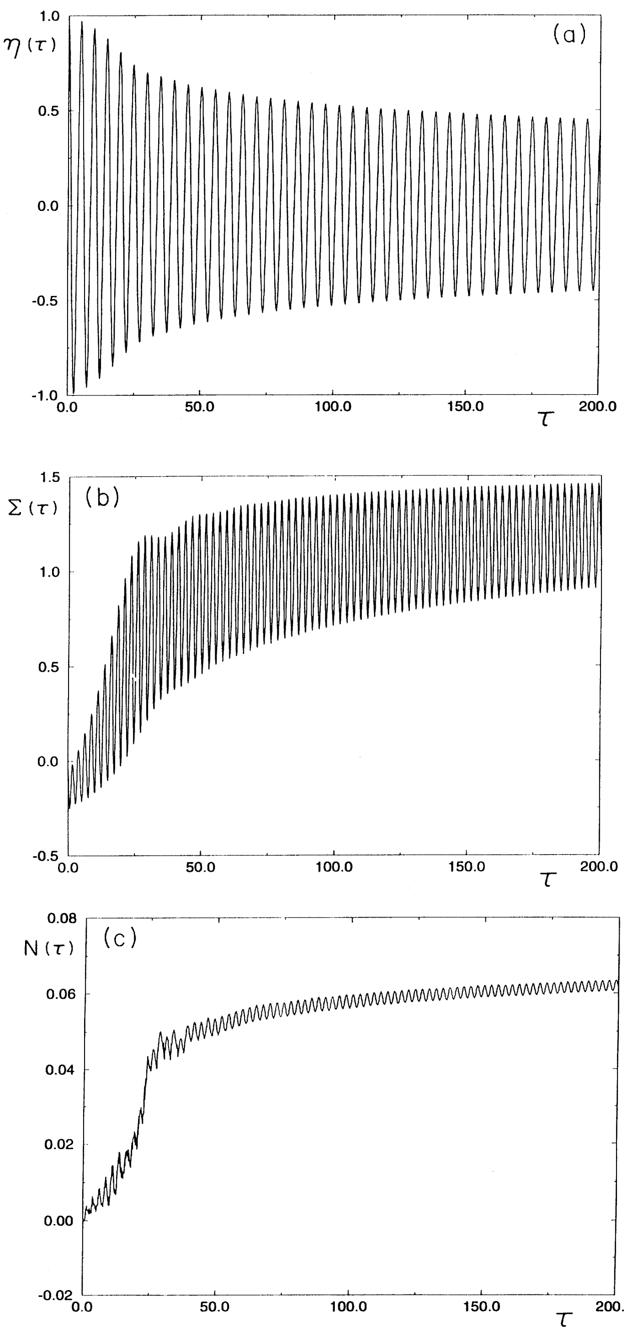


FIG. 12. (a) $\eta(\tau)$ vs τ in the large-N approximation in the O(N) model, unbroken-symmetry case. $g = 0.3$; $\eta(0) = 1$; $\Lambda/M_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

For particle production in the O(N) model, the final expression of the expectation value of the number operator for each pion field in terms of dimensionless quantities is the same as in Eq. (63), but the mode function $U_q^+(t)$ obeys the differential equation in Eq. (60) together with the self consistent condition.

Figures 10(a)–10(c), 11(a)–11(c), and 12(a)–12(c) show $\eta(\tau)$, $\Sigma(\tau)$, and $N(\tau)$ for $\eta(0) = 1, 2$, $g = 0.1, 0.3$, and $\Lambda/M_R = 100$ (although again we did not find cutoff sensitivity). The dynamics is very similar to that of the single scalar field in the Hartree approximation, which is not surprising, since the equations are very similar (save for the coefficient of the cubic term in the equation for the field expectation value). Thus the analysis presented previously for the Hartree approximation remains valid in this case.

2. Broken symmetry

The broken-symmetry case corresponds to choosing $M_R^2 = -\mu_R^2 < 0$. As in the case of the Hartree approximation, we now choose μ_R as the scale to define dimensionless variables and renormalization scale. The equations of motion for the field expectation value and the mode functions now become

$$\frac{d^2}{d\tau^2}\eta - \eta + \eta^3 + g\eta\Sigma(\tau) = 0,$$

$$\left[\frac{d^2}{d\tau^2} + q^2 - 1 + \eta^2(\tau) + g\Sigma(\tau)\right] U_q^+(\tau) = 0,$$

with $\Sigma(\tau)$ given by Eq. (75). As in the Hartree case, there is a subtlety with the boundary conditions for the mode functions because the presence of the instabilities at $\tau = 0$ for the band of wave vectors $0 \le q^2 < 1$, for $\eta^2(0) < 1$. Following the discussion in the Hartree case (broken symmetry) we chose the initial conditions for the mode functions as

$$U_q^+(0) = 1\ , \ \frac{d}{d\tau}U_q^+(0) = -i\sqrt{q^2 + 1 + \eta^2(0)}. \tag{76}$$

These boundary conditions correspond to preparing a Gaussian state centered at $\eta(0)$ at $\tau = 0$ and letting this initial state evolve in time in the "broken-symmetry potential" [28].

Figures 13(a)–13(c) show the dynamics of $\eta(\tau)$, $\Sigma(\tau)$, and $N(\tau)$ for $\eta(0) = 0.5$, $g = 0.1$, and cutoff $\Lambda/\mu_R = 100$. Strong damping behavior is evident, and the time scale of damping is correlated with the time scale for growth of $\Sigma(\tau)$ and $N(\tau)$. The asymptotic value of $\eta(\tau)$ and $\Sigma(\tau)$, $\eta(\infty)$, and $\Sigma(\infty)$, respectively, satisfy

$$-1 + \eta^2(\infty) + g\ \Sigma(\infty) = 0, \tag{77}$$

as we confirmed numerically. Thus the mode functions are "massless" describing Goldstone bosons. Notice that this value also corresponds to $V'(\phi) = 0$ in Eq. (71). An equilibrium self-consistent solution of the equations of motion for the field expectation value and the fluctuations is reached for $\tau = \infty$.

Figure 13(d) shows the number of particles produced per correlation volume as a function of (dimensionless) wave vector q at $\tau = 200$. We see that it is strongly peaked at $q = 0$, clearly showing that the particle production mechanism is most efficient for long-wavelength Goldstone bosons. Figures 13(e)–13(g) show snapshots of the number of particles as a function of wave vector for $\tau = 13, 25, 50$, respectively; notice the scale. Clearly at longer times, the contributions from $q \neq 0$ becomes smaller. Figures 14(a)–14(c) show the evolution of $\eta(\tau)$,

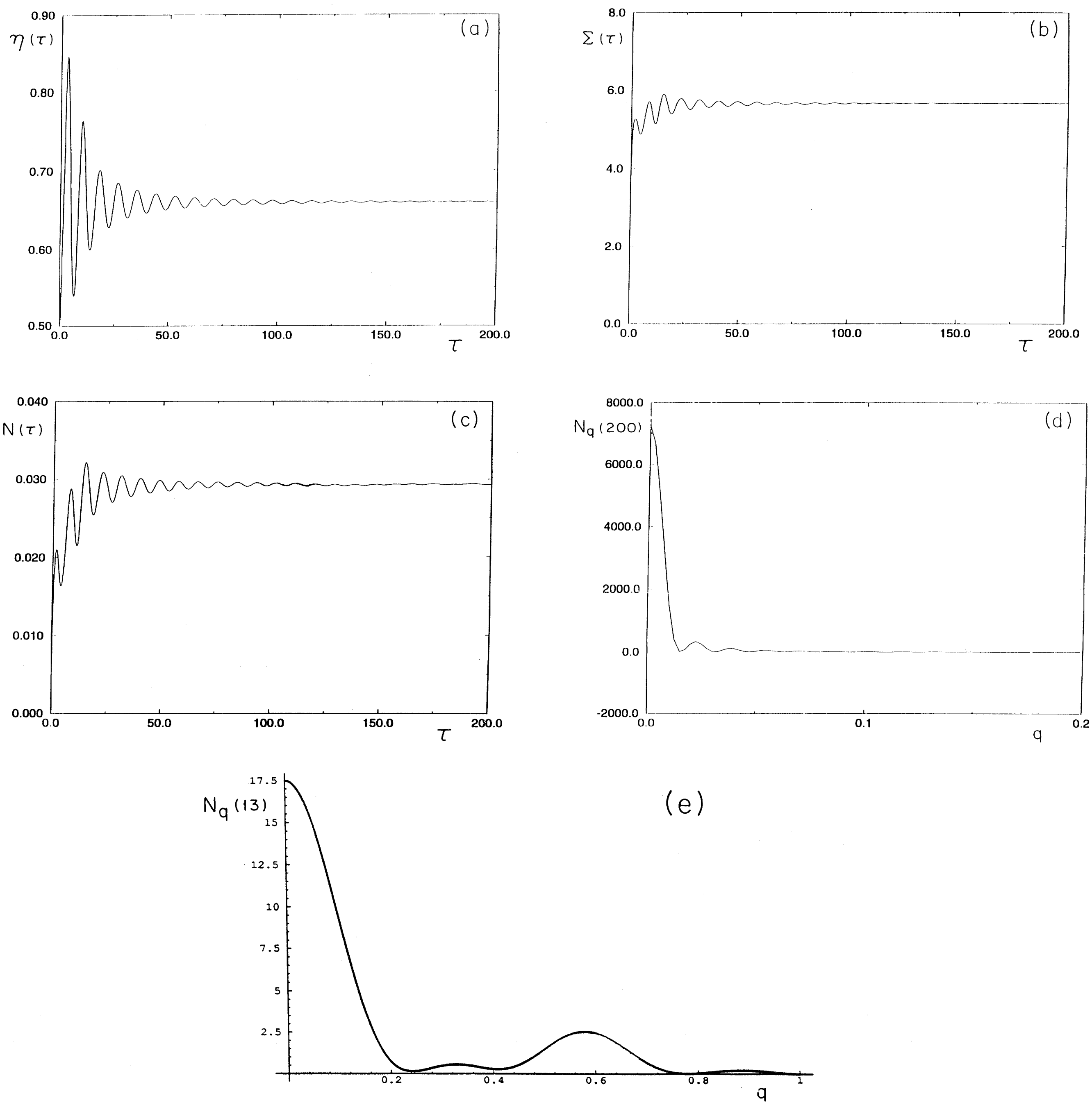
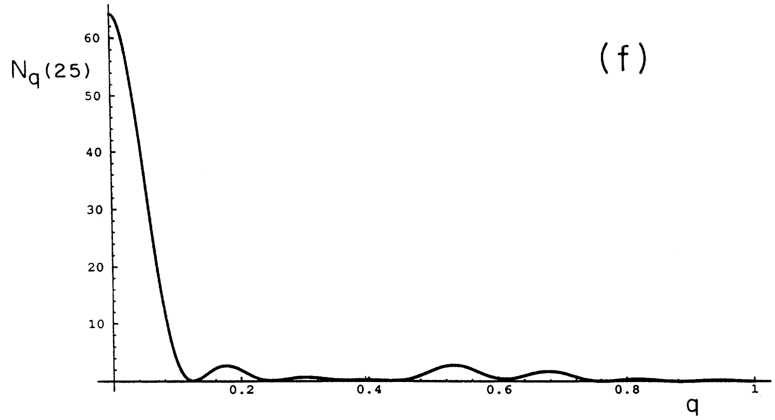


FIG. 13. (a) $\eta(\tau)$ vs τ in the large-N approximation in the O(N) model, broken-symmetry case. $g = 0.1$; $\eta(0) = 0.5$; $\Lambda/\mu_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a). (d) Number of particles in (dimensionless) wave vector q, $N_q(\tau)$ at $\tau = 200$. (e) Number of particles in (dimensionless) wave vector q, $N_q(\tau)$ at $\tau = 13$. (f) Number of particles in (dimensionless) wave vector q, $N_q(\tau)$ at $\tau = 25$. (g) Number of particles in (dimensionless) wave vector q, $N_q(\tau)$ at $\tau = 50$.



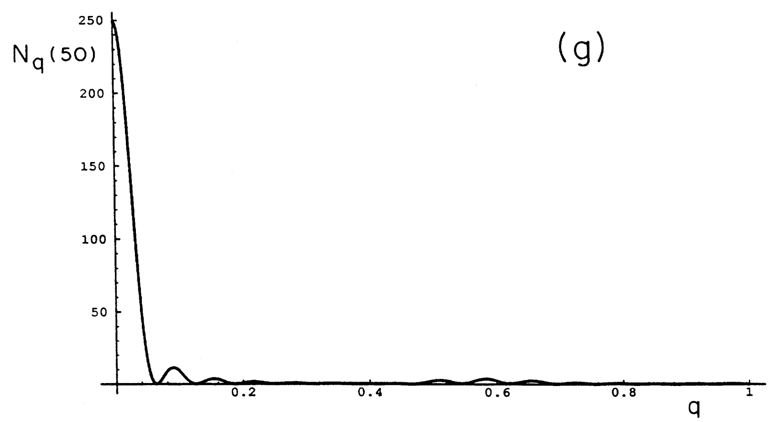


FIG. 13. (*Continued*).

$\Sigma(\tau)$, and $N(\tau)$ for $\eta(0) = 10^{-4}$, $g = 10^{-7}$. The initial value of η is very close to the "top" of the potential. Due to the small coupling and the small initial value of $\eta(0)$, the unstable modes (those for which $q^2 < 1$) grow for a long time, making the fluctuations very large. However, the fluctuation term $\Sigma(\tau)$ is multiplied by a very small coupling and it has to grow for a long time to overcome the instabilities. During this time the field expectation value rolls down the potential hill, following a trajectory very close to the classical one. The classical turning point of the trajectory beginning very near the top of the potential hill, is close to $\eta_{\rm tp} = \sqrt{2}$. Notice that $\eta(\tau)$ exhibits a turning point (maximum) at $\eta \approx 0.45$. Thus the turning point of the effective evolution equations is much closer to the origin. This phenomenon shows that the effective (nonlocal) potential is shallower than the classical potential, with the minimum moving closer to the origin as a function of time.

If the energy for the field expectation value was absolutely conserved, the expectation value of the scalar field would bounce back to the initial point and oscillate between the two classical turning points. However, because the fluctuations are growing and energy is transferred to them from the $q = 0$ mode, η is slowed down as it bounces back, and tends to settle at a value very close to the origin (asymptotically about 0.015). Figure 14(b) shows that the fluctuations grow initially and stabilize at a value for which $g\Sigma(\infty) \approx 1$. The period of explosive growth of the fluctuations is correlated with the strong oscillations at the maximum of $\eta(\tau)$. This is the time when the fluctuations begin to effectively absorb the energy transferred by the field expectation value and when the damping mechanism begins to work. Again the asymptotic solution is such that $-1 + \eta^2(\infty) + g\Sigma(\infty) = 0$, and the particles produced are indeed Goldstone bosons but the value of the scalar field in the broken-symmetry minimum is very small (classically it would be $\eta_{\rm min} = 1$, yet dynamically, the field settles at a value $\eta(\infty) \approx 0.015$!! for $g = 10^{-7}$). Figure 14(c) shows copious particle production, and the asymptotic final state is a highly excited state with a large number ($\sim 10^5$) of Goldstone bosons per correlation volume.

The conclusion that we reach from the numerical analysis is that Goldstone bosons are *extremely effective* for dissipation and damping. Most of the initial potential energy of the field is converted into particles (Goldstone bosons) and the field expectation value comes to rest at long times at a position very close to the origin.

Notice that the difference with the situation depicted in Figs. 13(a)–13(c) is in the initial conditions and the strength of the coupling. In the case of stronger coupling, the fluctuations grow only for a short time because

$g\Sigma(\tau)$ becomes ~ 1 in short time, dissipation begins to act rather rapidly and the expectation value rolls down only for a short span and comes to rest at a minimum of the effective action, having transferred all of its potential energy difference to produce Goldstone bosons.

Figures 15(a)–15(c) show a very dramatic picture and one of the most striking results of this article. For this case, we have taken $\eta(0) = 10^{-4}$, $g = 10^{-12}$. The field begins very close to the top of the potential hill. This initial condition corresponds to a "slow-roll" scenario. The fluctuations must grow for a long time before $g\Sigma(\tau)$ becomes ~ 1, during which the field expectation value evolves classically reaching the classical turning point at $\eta_{\rm tp} = \sqrt{2}$, and then bouncing back. But by the time it

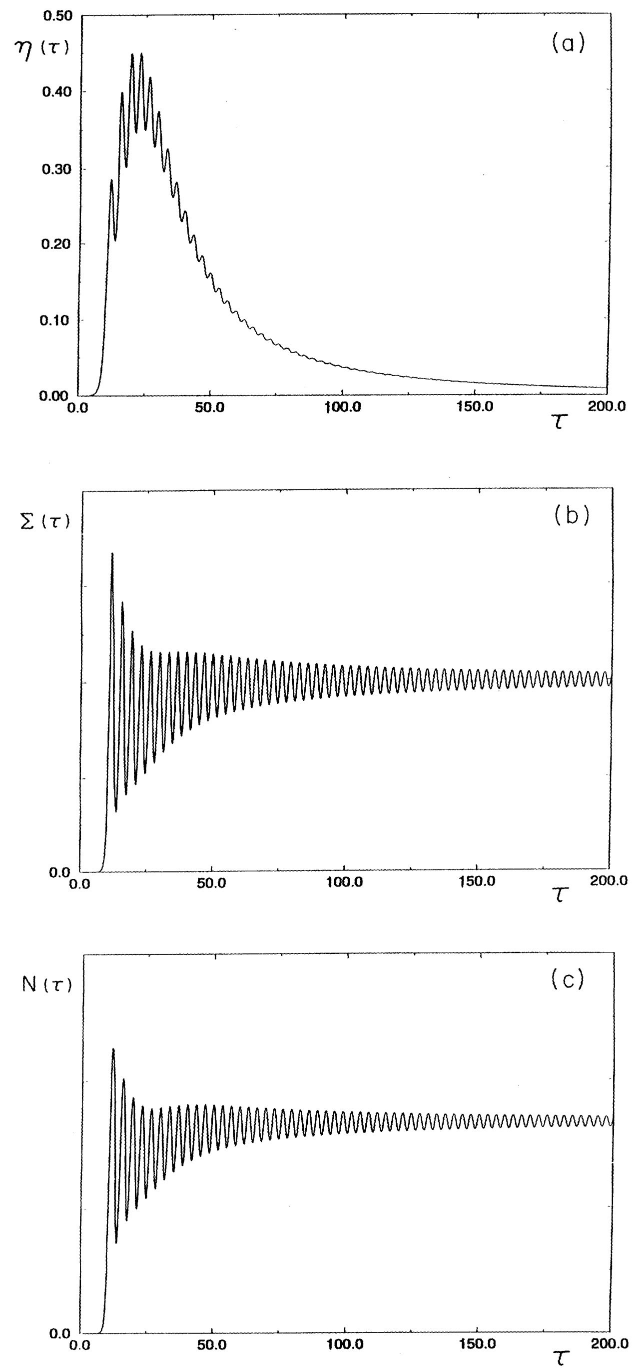


FIG. 14. (a) $\eta(\tau)$ vs τ in the large-N approximation in the O(N) model, broken-symmetry case. $g = 10^{-7}$; $\eta(0) = 10^{-5}$; $\Lambda/\mu_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

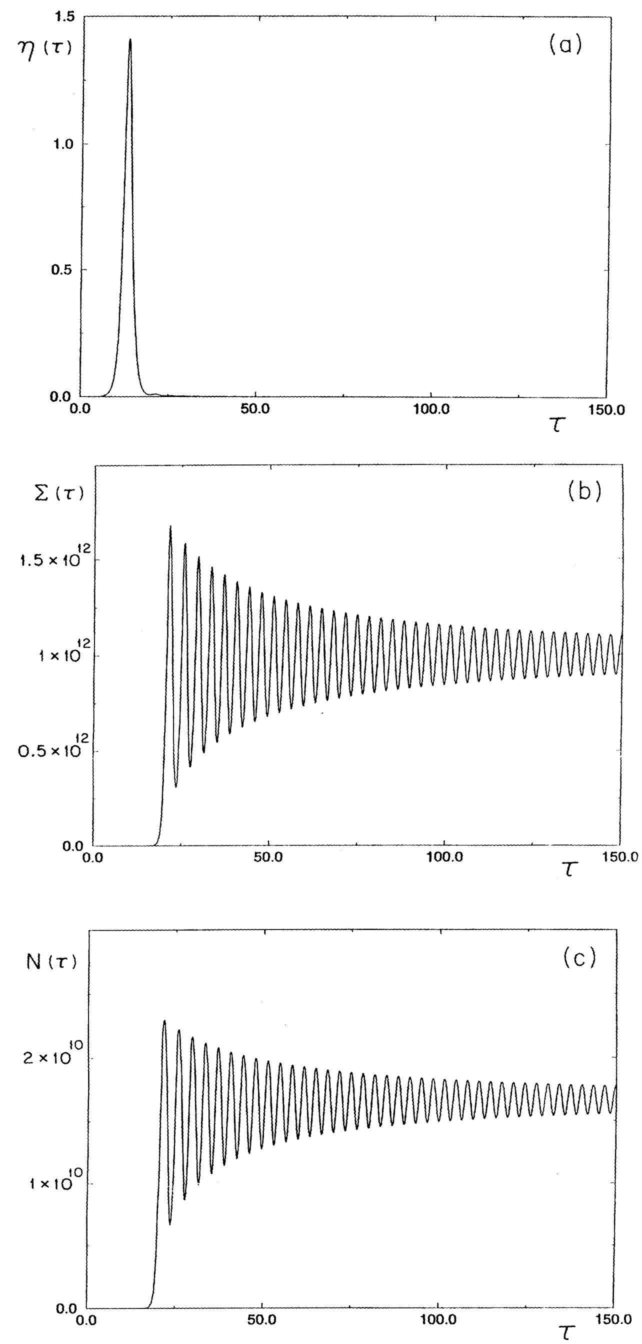


FIG. 15. (a) $\eta(\tau)$ vs τ in the large-N approximation in the O(N) model, broken-symmetry case. $g = 10^{-12}$; $\eta(0) = 10^{-5}$; $\Lambda/\mu_R = 100$. (b) $\Sigma(\tau)$ vs τ for the same case as in (a). (c) $N(\tau)$ vs τ for the same case as in (a).

gets near the origin again, the fluctuations have grown dramatically, absorbing most of the energy of the field expectation value and completely damping its motion. The condition (77) is now fulfilled with $\eta(\infty) \approx 0$. It is again clear from Figs. 15(b) and 15(c) that the time scale at which the fluctuations grow explosively is of the same order as the classical period. At this time $g\Sigma(\tau) \approx 1$ and dissipation by particle production becomes very effective. We have checked that this effect is *not* a numerical artifact. The numerical code was run changing the grid size in time and also for wave vector integration. We found the same results consistently with no appreciable variations in the numerical results. This gives us confidence that we are finding a new genuine physical result which is a consequence of dissipation by Goldstone bosons. In this case *almost all* the initial potential energy has been converted into particles. The conclusion of this analysis is that the strong dissipation by Goldstone bosons dramatically changes the dynamics of the phase transition even for very small couplings. For slow-roll initial conditions the scalar field relaxes to a final value which is very close to the origin. This is the minimum of the effective *action*, rather than of the tree-level effective potential. Thus dissipative effects by Goldstone bosons introduce a very strong dynamical correction to the effective action, leading to a very shallow effective potential (the effective action for constant field configuration). The condition for this situation to happen is that the period of the classical trajectory is of the same order of magnitude as the time scale of growth for the fluctuations.

The physical interpretation of this effect is as follows: although we are working at zero temperature, the initial potential energy stored in the scalar field is fairly large; most of this energy is effectively dissipated away in the production of low momentum Goldstone bosons. The value of the coupling constant determines the time scale of growth of the fluctuations. During this time the expectation value of the scalar field "rolls down the potential hill," when $g\Sigma(\tau) \approx 1$, dissipative effects become very efficient, and slow the rolling to a halt. If the coupling is rather large, dissipative effects become effective very early on and the field does not roll very far. If on the other hand, the coupling is very small [as in Figs. 15(a)–15(c)] the field can oscillate one (or several) periods before dissipation is effective and most of the energy is converted into Goldstone bosons. The time scales that have to be compared are the time at which $g\Sigma(\tau) \approx 1$ and the period of the (classical) trajectory (for weak coupling).

The weak coupling estimate for this dynamical nonequilibrium time scale of growth of fluctuations is $\tau_s \approx \ln(1/g)/2$, which is obtained by requiring that $g\Sigma(\tau) \approx 1$. For weak coupling the mode functions grow as $U_q^+(\tau) \approx e^\tau$ for $q^2 < 1$, and $\Sigma(\tau) \approx e^{2\tau}$. The numerical analysis confirms this time scale. For weakly coupled theories, this nonequilibrium time scale is much larger than the static correlation length (in units in which $c = 1$) and the only relevant time scale for the dynamics.

Our results pose a fundamental question: how is it possible to reconcile damping and dissipative behavior, as found in this work with time reversal invariance?

In fact we see no contradiction for the following reason: the dynamics is completely determined by the set of equations of motion for the field expectation value and the mode functions for the fluctuations described above. These equations are solved by providing non equilibrium initial conditions on the field expectation value, its derivative and the mode functions and their derivatives at the initial time $t = 0$. The problem is then evolved in time by solving the coupled *second order* differential equations. We emphasize the fact that the equations are second order in time because these are time reversal invariant. Now consider evolving this set of equations up to a positive time t_0, at which we stop the integration and find the value of the field expectation value, its derivative, the value of *all* the mode functions, and their derivatives. Because this is a system of differential equations which is second order in time, we can take these values at t_0 as initial conditions at this time and evolve *backwards* in time reaching the initial values at $t = 0$. Notice that doing this involves beginning at a time t_0 in an excited state with (generally) a large number of particles. The conditions at this particular time are such that the energy stored in this excited state is focused in the back reaction to the field expectation value that acquires this energy and whose amplitude will begin to grow. It is at this point where one recognizes the fundamental necessity of the "in-in" formalism in which the equations of motion are real and causal.

V. CONCLUSIONS AND DISCUSSION

We have focused on understanding the first stage of a reheating process, that of dissipation in the dynamics of the field expectation value of a scalar field via particle production. Starting from a scalar field theory with no apparent dissipative terms in its dynamics, we have shown how the evolution of expectation value of the scalar field is affected by the quantum fluctuations and particle production resulting in dissipative dynamics.

We started our analysis with a perturbative calculation, both in the amplitude of the expectation value of the field and to one-loop order. Analytically and numerically we find that dissipative processes cannot be described perturbatively. A systematic solution to the equation of motion reveals the presence of resonances and secular terms resulting in the growth of the corrections to the classical evolution. The perturbative study of the O(2) linear σ model reveals infrared divergences arising from the contribution to the dissipative kernels from the Goldstone modes which require nonperturbative resummation.

A Langevin equation was constructed in an amplitude expansion; it exhibits a generalized fluctuation-dissipation theorem with non-Markovian (memory) kernels and colored noise correlation functions, thus offering a more complex picture of dissipation.

Motivated by the failure of the perturbative approach, we studied the nonequilibrium dynamics in a Hartree approximation both in the symmetric as well as in the broken-symmetry case. This approximation clearly exhibits the contribution of open channels and the dissipation associated with particle production, which in this approximation is a result of parametric amplification of quantum fluctuations.

In the case of unbroken symmetry we find that asymptotically the expectation value of the scalar field oscillates around the trivial vacuum with an amplitude that depends on the coupling and *initial conditions*. An extensive numerical study of the renormalized equations of motion was performed that shows *explicitly* the dynamics of particle production during these oscillations.

Although the Hartree approximation offers a self-consistent nonperturbative resummation, it is not controlled. Thus we were led to study the large N limit in an $O(N)$ model, which also allows us to study in a nonperturbative manner the dissipative dynamics of Goldstone bosons. In the case of unbroken symmetry the results are very similar to those obtained in the Hartree approximation.

The broken-symmetry case provides several new and remarkable results. An extensive numerical study of the equations explicitly shows copious particle production while the expectation value of the field relaxes with strongly damped oscillations towards a minimum of the effective action.

It is intuitively obvious that Goldstone bosons are extremely effective for dissipation since channels are open for arbitrarily small energy transfer. What is remarkable, however, is that for "slow-roll" initial conditions, that is, when the initial expectation value of the scalar field is very close to the origin and the coupling is very weak, ($\sim 10^{-7}$ or smaller) the *final* value of the expectation value remains very close to the origin; most of the potential energy has been absorbed in the production of long-wavelength Goldstone bosons. This is confirmed by an exhaustive numerical study, including snapshots at different times of the number of particles produced for different wavelengths showing a large peak at small wave vectors at long times. This study clearly shows that particle production is extremely effective for long-wavelength Goldstone bosons. Another remarkable result is that the asymptotic value expectation value of the scalar field depends on the *initial conditions*. These asymptotic values correspond to minima of the effective action. Thus we reach the unexpected conclusion that in this approximation the effective action has a continuum manifold of minima which can be reached from different initial conditions.

We also pointed out that the basic mechanism for dissipation in this approximation is that of Landau damping through the parametric amplification of quantum fluctuations, and production of particles. These fluctuations react back in the evolution of the expectation value of the scalar field, but out of phase. This is a collisionless mechanism, similar to that found in the collisionless Boltzmann-Vlasov equation for plasmas.

Our study also reconciles dissipation in the time evolution for the coarse grained variable with time reversal invariance, as the evolution is completely specified by an *infinite set* of ordinary second order differential equations in time with proper boundary conditions.

Our formalism and techniques are sufficiently powerful to give a great deal of insight into the particulars of the dissipation process. In particular, we can see that the damping of the field expectation value ends as the particle production ends. This shows that our interpretation of the damping as being due to particle production is accurate.

It is useful to compare what we have done here with other work on this issue. We have already mentioned the work of Calzetta and Hu [8] and Paz [9]. These authors use the closed time path formalism to arrive at the effective equations of motion for the expectation value of the field. Then they solve the *perturbative* equations and find dissipative evolution at short times. In particular Paz finds the kernel that we have found for the effective equations of motion of the expectation value in the perturbative and Hartree case.

We remedy this situation by studing the nonperturbative Hartree equations, which must necesarily be solved numerically as we do.

There has been other work on the reheating problem, most notably by Abbott, Farhi, and Wise [7], Ringwald [46], and Morikawa and Sasaki [26]. In all of these works, the standard effective action is used, so that the expectation value is of the "in-out" type and hence the equations are noncausal and contain imaginary parts. In essence, they "find" dissipational behavior by adding an imaginary part to the frequency that appears in the mode equations. We see from our work (as well as that of Calzetta, Hu, and Paz) that this is not necessary; dissipation can occur even when the system is evolving unitarily. This comment deserves a definition of what we call dissipation here: it is the energy transfer from the expectation value of the $q = 0$ mode of the scalar field to the quantum fluctuations ($q \neq 0$) resulting in damped evolution for the $q = 0$ mode.

To what extent are we truly treating the reheating problem of inflationary models? As stated in the Introduction, reheating typically entails the decay of the inflaton into lighter particles during its oscillations. What we do here is understand how the quantum fluctuations and the ensuing particle production influence the dynamics of the evolution of the expectation value of the field. Thus technically speaking, this is not the reheating problem. However, we *are* able to understand where dissipation comes from in a field theory, and are able to give a quantitative description of the damping process for the expectation value of the field. We expect that the physics of dissipation when the scalar field couple to others will be very similar to the case studied in this article. Furthermore, the techniques we develop here are easily adapted to the case where the inflaton couples to fermions, a case that we are currently studying [47]. In this connection, during the writing of this paper, two related pieces of work on the reheating problem have appeared [48]. They both look at the effect of particle production from the oscillations of the inflaton field due to parametric amplification. What they do *not* do is to account for the back reaction of the produced particles on the evolution of the expectation value of the inflaton. As we have learned with our study, this back reaction will eventually shut off the particle production, so that these authors may have overestimated the amount or particle production.

We recognize that our nonperturbative treatment neglects the effect of collisions as mentioned above. Dissipation appears in a manner similar to Landau damping. In the Hartree approximation there does not seem to be a natural way to incorporate scattering processes because this is a mean-field theory. However, the large N expansion allows a consistent treatment of scattering processes for which the first contribution (two-to-two particle scattering) will appear at $O(1/N)$. This will be a necessary next step in order to fully understand the collisional thermalization which is the second stage of the reheating process. This will clearly be a fascinating and worthy endeavour that we expect to undertake soon.

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