Stress-energy tensor of quantized scalar fields in static spherically symmetric spacetimes

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A method for computing the stress-energy tensor of quantized scalar fields in static spherically symmetric spacetimes is described. The fields can be massless or massive with an arbitrary coupling ξ to the scalar curvature. They can be either in a zero temperature vacuum state or a nonzero temperature thermal state. Analytical approximations which apply to all of these cases are obtained. The method is used to numerically compute the components of the stress-energy tensor of massive and massless scalar fields in Schwarzschild and Reissner-Nordström spacetimes. The results are compared to the analytical approximations and the accuracy of the analytical approximations is discussed.

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I. INTRODUCTION

Despite decades of effort, a fully satisfactory theory of quantum gravity does not yet exist. In the absence of an adequate complete theory of quantum gravity, it behooves us to attempt to construct and study model theories which approximate the full theory. Such models can provide guidance in the quest to construct the complete theory, and can also provide insight into the sorts of physical effects which may occur in quantum gravity. Even when a successful theory of quantum gravity has been developed, model theories and approximations will still have an important role to play, as the full and correct theory is likely to be computationally intractable when applied to situations of physical interest.

One such model theory is quantum field theory in curved spacetime and the associated theory of semiclassical gravity. While the study of quantized fields in curved spaces has a long history [1], significant progress has been achieved in the last two decades due to the stimulus of Hawking's discovery [2] that black holes emit thermal radiation. During this period it has been realized that concepts such as the number of particles present are observer dependent, and that the description of the vacuum state of a quantized field should be made in terms of tensorial quantities such as the vacuum polarization $\langle\phi^2\rangle$ and the stress-energy tensor $\langle T_{\mu\nu}\rangle$. The latter quantity is also of particular interest as a source term in the Einstein equations. The semiclassical theory of gravity sets the classical Einstein tensor equal to the expectation value of the stress-energy tensor operator of the quantized matter fields present:[1]

$$G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle \ . \tag{1.1}$$

A primary computational difficulty in the theory of semiclassical gravity is that $\langle T_{\mu\nu}\rangle$ depends strongly on the metric tensor $g_{\mu\nu}$. While it is possible, with greater or lessor effort depending on the amount of symmetry present, to calculate $\langle T_{\mu\nu}\rangle$ in a specific fixed background spacetime, it is exceedingly difficult to calculate it within a general class of spacetimes, which is necessary in order to find self-consistent solutions to Eq. (1.1).

A secondary difficulty, and common criticism, of semiclassical gravity is that the effects of the quantized gravitational field are ignored. This is held to be unacceptable as the gravitons will in general perturb the classical metric by an amount of the same order as any other quantized field present. Various solutions to this objection have been proposed. A popular one is to justify ignoring the graviton contribution by working in the "large N" limit, in which the number of matter fields present is so large that the graviton contribution is negligible. An alternative approach is to study the separate effects of different sorts of quantized fields in classical space-

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[1]Throughout we use units such that $\hbar = c = G = k_B = 1$. Our sign conventions are those of Misner, Thorne, and Wheeler [3].

times. In this way one develops knowledge of the type and range of physical effects created by quantized fields. If a wide enough assortment of other kinds of quantized fields has been examined, we feel it is justifiable to assume that the graviton contributions will not be wildly different.[2] For example in homogeneous and isotropic spacetimes, gravitons can be modeled by considering the effects of minimally coupled massless scalar fields [4]. To date most computations have centered on conformally invariant quantum fields. However, the gravitational field is emphatically not conformally invariant, so it is important to investigate the effects of conformally noninvariant fields.

In this paper we describe a method of calculating $\langle T_{\mu\nu}\rangle$ for quantized scalar fields which allows both of these difficulties to be addressed. The second difficulty is addressed because the method works for scalar fields with arbitrary masses and curvature couplings. This allows quantum effects to be investigated for a large range of conformally noninvariant quantum fields. The first difficulty is addressed because the method works for arbitrary static spherically symmetric spacetimes, thus allowing the semiclassical backreaction equations to be solved in these spacetimes. Interesting examples of static spherically symmetric spacetimes include hot flat space (in a cavity), a nonrotating black hole in equilibrium with radiation in a cavity, and an extreme nonrotating black hole in empty space. Solutions to the semiclassical backreaction equations in cases such as these will provide substantial insight into the questions of how quantum effects distort the spacetime geometry near a black hole and how they affect the thermodynamic properties of black holes. An overview of our method was given in Ref. [5].

The inspiration for our method and the source of many of our ideas are the calculations by Howard and Candelas [6,7] of $\langle T_{\mu\nu}\rangle$ and $\langle\phi^2\rangle$ for the conformally invariant scalar field in Schwarzschild spacetime. A previous version was developed to compute the vacuum polarization, $\langle\phi^2\rangle$, in static spherically symmetric spacetimes. It is discussed in Ref. [8], hereafter referred to as paper I. The present version is an extension of and an improvement on that of paper I. It can be applied to computations of both $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$. Throughout this paper we will limit most of the discussion to the computation of $\langle T_{\mu\nu}\rangle$, but important results relating to the computation of $\langle\phi^2\rangle$ will be presented.

To illustrate our numerical method, we have computed the stress-energy tensor of massless and massive quantized scalar fields with arbitrary curvature couplings in the Schwarzschild and Reissner-Nordström black hole spacetimes. These are spacetimes containing uncharged and charged black holes, respectively. The fields are in the Hartle-Hawking state. These sample computations are of interest in themselves since the only numerical computations of $\langle T_{\mu\nu}\rangle$ previously completed for black hole spacetimes have been for the massless conformally coupled scale field [9,6] and for the electromagnetic field [10], both in Schwarzschild spacetime.

Numerical computations of $\langle T_{\mu\nu}\rangle$ are usually extremely computer intensive. Thus it is useful, when possible, to have analytical approximations to $\langle T_{\mu\nu}\rangle$. We present in this paper an analytical approximation which works for massless scalar fields in arbitrary static spherically symmetric spacetimes. For the special case of a conformally coupled massless field our analytical approximation is equivalent to the approximation of Frolov and Zel'nikov [11], given particular values for the arbitrary parameters in their expression. As a result, it is equivalent to Page's approximation [12] for the stress-energy tensor of a conformally coupled scalar field in any static spherically symmetric Einstein spacetime (such spacetimes satisfy $R_{\mu\nu} = \Lambda g_{\mu\nu}$). Our new approximation thus extends the previous approximation schemes of Page and Frolov and Zel'nikov to nonconformally coupled scalar fields. In addition, our derivation provides the first justification for the local approximation of Frolov and Zel'nikov from quantum field theory. The original derivation of the Frolov-Zel'nikov approximation was motivated primarily by geometric concerns rather than field theory.

For the conformally invariant field, Frolov and Zel'nikov [11] pointed out that their approximation predicts a logarithmic divergence for one component of the stress-energy tensor on the event horizon of a charged black hole. Our analytical approximation predicts the existence of such a divergence for all massless scalar fields in charged black hole spacetimes. If such a logarithmic divergence exists, then quantum effects would be important near the event horizon of any charged black hole, no matter how large the black hole or how small the charge. This would be a very surprising result. In fact our numerical computations indicate that no such divergences exist. Thus we find that the apparent logarithmic divergences are only an artifact of the approximation, and are not physical. The analytical approximation for this component is then valid near the event horizon only for Schwarzschild spacetime.

Comparisons of our analytical approximation with our numerical calculations in Reissner-Nordström spacetimes indicate that, for other components both on and away from the event horizon, the accuracy of the approximation depends on the charge to mass ratio of the black hole. The larger the charge to mass ratio, the worse the approximation.

For scalar fields with large enough masses, the DeWitt-Schwinger expansion can be used to provide approximations for both $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$. This has been done in Schwarzschild and Kerr spacetimes by Frolov [13] and Frolov and Zel'nikov [14], respectively. For the Feynman Green's function, and hence $\langle\phi^2\rangle$, the first three coefficients[3] in the DeWitt-Schwinger expansion in an ar-

[2]This assumes, of course, that one is working at scales sufficiently removed from the Planck scale so that the very notion of a classical spacetime still has some meaning.

[3]These are commonly denoted as a_0, a_1, and a_2.

bitrary spacetime have been computed by DeWitt [15]. The fourth coefficient has been computed by Gilkey [16]. The fifth coefficient has been computed by Avramidi [17] and Amsterdamski, Berkin, and O'Connor [18].

We have developed a method of deriving the De Witt-Schwinger approximation for $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ in a static spherically symmetric spacetime using the WKB approximation for the modes of the scalar field. We have used this method to compute $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ to order m^{-4} and m^{-2}, respectively, where m is the mass of the scalar field. These orders correspond to using the first four coefficients in the DeWitt-Schwinger expansion. To our knowledge, the DeWitt-Schwinger expansion for $\langle T_{\mu\nu}\rangle$ has not previously been computed to this order for a general static spherical spacetime. When applied to the Reissner-Nordström spacetime, we find that the DeWitt-Schwinger approximation provides values extremely close to the exact numerical results for values of the field mass $m \gtrsim 2M^{-1}$, where M is the black hole mass.

In Sec. II we develop an unrenormalized expression for $\langle T_{\mu\nu}\rangle$ for a scalar field with arbitrary mass and curvature coupling in a general static spherically symmetric spacetime in terms of the Euclidean Green's function. In Sec. III the resulting expression is renormalized using the method of covariant point splitting. Section IV describes our method of calculating the renormalized values of $\langle T_{\mu\nu}\rangle$ using the WKB approximation. In Sec. V we derive and discuss the analytic approximations to $\langle T_{\mu\nu}\rangle$ for massless and massive fields. In Sec. VI the stress-energy tensors of massless and massive scalar fields are numerically computed for Reissner-Nordström black hole spacetimes; the predictions of the analytic approximations are compared with the numerical results for these spacetimes. The details of some of the more tedious algebraic calculations are given in a series of Appendices.

II. AN UNRENORMALIZED EXPRESSION FOR $\langle T_{\mu\nu}\rangle$

In this section an unrenormalized expression for $\langle T_{\mu\nu}\rangle$ is derived for a scalar field in an arbitrary static spherically symmetric spacetime. It is assumed that the field is either in a thermal state at temperature T or a vacuum state defined with respect to the timelike Killing vector which always exists in a static spacetime. The calculation proceeds in a manner similar to that of the Howard-Candelas calculation of $\langle T_{\mu\nu}\rangle$ for the conformally invariant scalar field in Schwarzschild spacetime. As in their calculation, a Euclidean space approach is used. The metric for a general static spherically symmetric spacetime when continued analytically into Euclidean space is

$$ds^2 = f(r)d\tau^2 + h(r)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \ . \quad (2.1)$$

Here $\tau = it$ is the Euclidean time, and f and h are arbitrary functions of r which, if the space is asymptotically flat, become constant in the limit $r \to \infty$.

$\langle T_{\mu\nu}\rangle$ is computed using the method of point splitting [19,20]. One begins by noting that $\langle T_{\mu\nu}\rangle$ can be obtained by taking derivatives of the quantity $\langle\phi(x)\phi(x')\rangle$ and then letting $x' \to x$. The calculation is simplified by noting that in the limit $x' \to x$, the following relations hold:

$$\begin{aligned}
\langle\phi^2(x)\rangle &= \mathrm{Re}\left(\lim_{x'\to x} G_E(x,x')\right) \ , \\
\langle\phi(x)\nabla_\mu\phi(x)\rangle &= \mathrm{Re}\left(\lim_{x'\to x} \tfrac{1}{2}[G_E(x,x')_{;\mu} + g_\mu^{\alpha'} G_E(x,x')_{;\alpha'}]\right) \ , \\
\langle\nabla_\mu\phi(x)\nabla_\nu\phi(x)\rangle &= \mathrm{Re}\left(\lim_{x'\to x} \tfrac{1}{2}[g_\nu{}^{\alpha'} G_E(x,x')_{;\mu;\alpha'} + g_\mu{}^{\alpha'} G_E(x,x')_{;\alpha';\nu}]\right) \ , \\
\langle\phi(x)\nabla_\mu\nabla_\nu\phi(x)\rangle &= \mathrm{Re}\left(\lim_{x'\to x} \tfrac{1}{2}[G_E(x,x')_{;\mu;\nu} + g_\mu{}^{\alpha'} g_\nu{}^{\beta'} G_E(x,x')_{;\alpha';\beta'}]\right) \ .
\end{aligned} \quad (2.2)$$

Here G_E is the Euclidean space Green's function. It obeys the equation

$$[\Box_x - m^2 - \xi R(x)]G_E(x,x') = -g^{-1/2}(x)\delta^4(x,x') \ , \quad (2.3)$$

where m is the mass of the scalar field and ξ is its coupling to the scalar curvature R. The quantity $g_\mu^{\alpha'}$ is called the bivector of parallel transport. It parallel transports a vector at x' to one at x. It appears in Eq. (2.2) because $G_E(x,x')_{;\alpha'}$ is a vector at x' and must be parallel transported to x before the limit $x' \to x$ is taken. Similarly $G_E(x,x')_{;\alpha';\beta'}$ is a second rank tensor at x' and must be parallel transported to x before the limit $x' \to x$ is taken.

With this notation we find that the unrenormalized expectation value of the stress-energy tensor with the points split is given by the expression

$$\begin{aligned}
\langle T_{\mu\nu}\rangle_{\mathrm{unren}} = {}& \left(\frac{1}{2} - \xi\right)(g_\mu^{\alpha'} G_{E;\alpha'\nu} + g_\nu^{\alpha'} G_{E;\mu\alpha'}) + (2\xi - \tfrac{1}{2})g_{\mu\nu} g^{\sigma\alpha'} G_{E;\sigma\alpha'} - \xi(G_{E;\mu\nu} + g_\mu^{\alpha'} g_\nu^{\beta'} G_{E;\alpha'\beta'}) \\
&+ 2\xi g_{\mu\nu}(m^2 + \xi R)G_E + \xi(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R)G_E - \tfrac{1}{2}m^2 g_{\mu\nu} G_E \ .
\end{aligned} \quad (2.4)$$

Since we are interested in the values of the components of the stress-energy tensor in Lorentzian space we compute these components directly. The most straightforward way to do this is to have all of the components of all of the tensors in Eq. (2.4) be the Lorentzian space components. This implies that all of the time derivatives will be with respect to t rather than τ. An alternative approach (which we do not take in this paper) would be to perform all computations in Euclidean space and to transform the components of the renormalized stress-energy tensor back to Lorentzian space at the end of the calculation.

In paper I the form of $G_E(x,x')$ was derived for scalar fields in static spherically symmetric spacetimes when the fields are either in the Euclidean vacuum state[4] or in a thermal state at temperature T. The result is

$$G_E(x,x') = \int d\tilde{\mu}\cos[\omega(\tau-\tau')] \times \sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)C_{\omega l}p_{\omega l}(r_<)q_{\omega l}(r_>) \;, \tag{2.5}$$

where, for an arbitrary function F,

$$\int d\tilde{\mu}F(\omega) \equiv \frac{1}{4\pi^2}\int_0^{\infty} d\omega F(\omega), \;\; T=0$$
$$\equiv \frac{T}{2\pi}\sum_{n=1}^{\infty}F(\omega)+\frac{T}{4\pi}F(0), \;\; T>0 \;.$$

Here P_l is a Legendre polynomial, $\cos\gamma \equiv \cos\theta\cos\theta' + \sin\theta\,\sin\theta'\cos(\phi-\phi')$, $C_{\omega l}$ is a normalization constant and $\omega = 2\pi nT$ if $T \neq 0$. The modes $p_{\omega l}$ and $q_{\omega l}$ obey the equation

$$\frac{1}{h}\frac{d^2S}{dr^2}+\left[\frac{2}{rh}+\frac{1}{2fh}\frac{df}{dr}-\frac{1}{2h^2}\frac{dh}{dr}\right]\frac{dS}{dr} - \left[\frac{\omega^2}{f}+\frac{l(l+1)}{r^2}+m^2+\xi R\right]S=0 \;. \tag{2.6}$$

They also satisfy the Wronskian condition

$$C_{\omega l}\left[p_{\omega l}\frac{dq_{\omega l}}{dr}-q_{\omega l}\frac{dp_{\omega l}}{dr}\right] = -\frac{1}{r^2}\left(\frac{h}{f}\right)^{1/2} \;. \tag{2.7}$$

There is a WKB approximation for the modes which is very useful in both analytical and numerical calculations of $\langle T_{\mu\nu}\rangle$. The WKB approximation for the modes is obtained by the change of variables[5]

$$p_{\omega l} = \frac{1}{(2r^2W)^{1/2}}\exp\left[\int^r W\left(\frac{h}{f}\right)^{1/2}dr\right]$$
$$q_{\omega l} = \frac{1}{(2r^2W)^{1/2}}\exp\left\{-\left[\int^r W\left(\frac{h}{f}\right)^{1/2}dr\right]\right\} \;. \tag{2.8}$$

Substitution of Eq. (2.8) into Eq. (2.7) shows that the Wronskian condition is obeyed if $C_{\omega l}=1$. Substitution into the mode equation, (2.6), gives the following equation for W:

$$W^2 = \Omega^2+V_1(r)+V_2(r) +\frac{1}{2}\left[\frac{f}{hW}\frac{d^2W}{dr^2}+\left(\frac{1}{h}\frac{df}{dr}-\frac{f}{h^2}\frac{dh}{dr}\right)\frac{1}{2W}\frac{dW}{dr} -\frac{3}{2}\frac{f}{h}\left(\frac{1}{W}\frac{dW}{dr}\right)^2\right] \tag{2.9}$$

with

$$\Omega^2(r) = \omega^2+m^2f+\left(l+\frac{1}{2}\right)^2\frac{f}{r^2} \;,$$

$$V_1(r) = \frac{1}{2rh}\frac{df}{dr}-\frac{f}{2rh^2}\frac{dh}{dr}-\frac{f}{4r^2} \;,$$

$$V_2(r) = \xi Rf = -\xi f\left[\frac{1}{fh}\frac{d^2f}{dr^2}-\frac{1}{2f^2h}\left(\frac{df}{dr}\right)^2-\frac{1}{2fh^2}\frac{df}{dr}\frac{dh}{dr}+\frac{2}{rfh}\frac{df}{dr}-\frac{2}{rh^2}\frac{dh}{dr}+\frac{2}{r^2h}-\frac{2}{r^2}\right] \;. \tag{2.10}$$

Equation (2.9) can be solved iteratively. The zeroth-order solution is $W=\Omega$. The second-order solution is

$$W = \Omega+\frac{1}{2}\Omega^{-1}(V_1+V_2)+\frac{1}{4}\left[\frac{f}{h\Omega^2}\frac{d^2\Omega}{dr^2}+\left(\frac{1}{h}\frac{df}{dr}-\frac{f}{h^2}\frac{dh}{dr}\right)\frac{1}{2\Omega^2}\frac{d\Omega}{dr}-\frac{3}{2}\frac{f}{h}\frac{1}{\Omega^3}\left(\frac{d\Omega}{dr}\right)^2\right] \;. \tag{2.11}$$

[4]The Euclidean vacuum is the vacuum state that, in the Lorentzian sector, is defined with respect to the timelike Killing vector.

[5]The boundary conditions used here are, strictly speaking, correct only for an asymptotically flat spacetime with an event horizon. However, the ultraviolet divergences in $\langle T_{\mu\nu}\rangle_{\rm unren}$ are independent of the boundary conditions and the WKB approximation to $\langle T_{\mu\nu}\rangle$ will be both added and subtracted from $\langle T_{\mu\nu}\rangle_{\rm unren}$ when computing a renormalized expression for $\langle T_{\mu\nu}\rangle$. Thus there is no problem in using this version of the WKB approximation for the modes for any static spherically symmetric spacetime.

To compute an unrenormalized expression for $\langle T_{\mu\nu}\rangle$ one simply substitutes Eq. (2.5) into (2.4) and takes the derivatives. After the derivatives are taken the particular way in which the points are to be split can be decided. For numerical computations it is easiest to choose a separation in time so that $\epsilon_\tau \equiv (\tau-\tau')$, $r'=r$, $\theta'=\theta$, $\phi'=\phi$. With this point separation, the terms containing angular derivatives can be simplified substantially as shown in Appendix A. However, as was first pointed out by Candelas and Howard [7,6] for the case of Schwarzschild spacetime, the Euclidean Green's function and its derivatives have superficial divergences with this separation of points.[6] As discussed in paper I, these divergences can be removed by adding multiplies of $\delta(\tau-\tau')$ and its derivatives to $G_E(x,x')$ and its derivatives. This is permissible because, so long as the points are split, $\delta(\tau-\tau')=0$.

The superficial divergences are all divergences which occur in the sums over l when ω is held fixed. The multiples of the delta function and its derivatives that must be subtracted from the modes to remove these divergences can be most easily obtained using the WKB approximation. One first substitutes the WKB approximation for the modes into the expressions for $G_E(x,x')$ and its derivatives. The resulting expressions are then expanded in the large l limit. Terms which are divergent when the sum over l is computed with fixed ω must be subtracted from the modes.

The resulting unrenormalized expressions for $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ are

$$\langle\phi^2\rangle_{\rm unren} = G_E(x,\tau;x,\tau') = \int d\tilde{\mu}\cos[\omega(\tau-\tau')]A_1\,. \tag{2.12a}$$

$$\begin{aligned}\langle T_t{}^t\rangle_{\rm unren} = &\int d\tilde{\mu}\cos[\omega(\tau-\tau')]\Big\{\left(-\frac{1}{2}g^{tt'}+\frac{\xi}{f}+\xi f(g^{tt'})^2+\xi h(g^{tr'})^2\right)\omega^2A_1+\left(2\xi-\frac{1}{2}\right)g^{rr'}A_2\\ &+\left[\xi(g^{tr'})^2\frac{fh}{r^2}+\left(2\xi-\frac{1}{2}\right)\frac{1}{r^2}\right]A_3-\xi\frac{f'}{2fh}A_4+\left[-\xi(g^{tt'})^2\frac{ff'}{2h}-\xi(g^{tr'})^2\left(\frac{2f}{r}+\frac{f'}{2}\right)\right]A_5\\ &+\left[\xi(g^{tr'})^2fh\left(-\frac{1}{4r^2}+m^2+\xi R\right)-\left(2\xi-\frac{1}{2}\right)\frac{1}{4r^2}+\left(2\xi-\frac{1}{2}\right)(m^2+\xi R)+\xi R_t{}^t\right]A_1\Big\}\\ &+i\int d\tilde{\mu}\,\omega\sin[\omega(\tau-\tau')]\left[-\xi g^{tt'}g^{tr'}f'A_1+\left(2\xi-\frac{1}{2}\right)g^{rt'}A_4+\left(-\frac{1}{2}g^{tr'}+2\xi g^{tt'}g^{tr'}f\right)A_5\right]\,,\end{aligned} \tag{2.12b}$$

$$\begin{aligned}\langle T_r{}^r\rangle_{\rm unren} = &\int d\tilde{\mu}\cos[\omega(\tau-\tau')]\Big\{-\left[\left(2\xi-\frac{1}{2}\right)g^{tt'}+\xi(g^{rt'})^2h+\xi(g^{rr'})^2\frac{h^2}{f}+\frac{\xi}{f}\right]\omega^2A_1+\frac{1}{2}g^{rr'}A_2\\ &+\left(\frac{\xi}{r^2}[1-(g^{rr'})^2h^2]-\frac{1}{2r^2}\right)A_3+\frac{\xi}{h}\left(\frac{2}{r}+\frac{f'}{2f}\right)A_4+\left[\frac{\xi}{2}(g^{rt'})^2f'+\xi(g^{rr'})^2\left(\frac{2h}{r}+\frac{hf'}{2f}\right)\right]A_5\\ &+\left[\xi[(g^{rr'})^2h^2+1]\left(\frac{1}{4r^2}-m^2-\xi R\right)-\left(2\xi-\frac{1}{2}\right)\frac{1}{4r^2}\left(2\xi-\frac{1}{2}\right)(m^2+\xi R)+\xi R_r{}^r\right]A_1\Big\}\\ &+i\int d\tilde{\mu}\,\omega\sin[\omega(\tau-\tau')]\left\{\xi g^{rr'}g^{rt'}\frac{hf'}{f}A_1+\frac{1}{2}g^{rt'}A_4-\left[\left(2\xi-\frac{1}{2}\right)g^{tr'}+2\xi g^{rr'}g^{rt'}h\right]A_5\right\}\,,\end{aligned} \tag{2.12c}$$

$$\begin{aligned}\langle T_\theta{}^\theta\rangle_{\rm unren} = &\int d\tilde{\mu}\cos[\omega(\tau-\tau')]\Big[-\left(2\xi-\frac{1}{2}\right)g^{tt'}\omega^2A_1+\left(2\xi-\frac{1}{2}\right)g^{rr'}A_2+\frac{2\xi}{r^2}A_3-\frac{\xi}{rh}A_4-\frac{\xi}{rh}A_5\\ &+\left(-\frac{\xi}{2r^2}+\left(2\xi-\frac{1}{2}\right)(m^2+\xi R)+\xi R_\theta{}^\theta\right)A_1\Big]+i\int d\tilde{\mu}\,\omega\sin[\omega(\tau-\tau')]\left(2\xi-\frac{1}{2}\right)(g^{rt'}A_4-g^{tr'}A_5)\,,\end{aligned} \tag{2.12d}$$

where

[6] These cannot be real divergences because the Green's function and its derivatives must be finite when the points are separated.

$$A_1 = \sum_{l=0}^{\infty}\left[(2l+1)p_{\omega l}q_{\omega l} - \frac{1}{rf^{1/2}}\right] ,$$

$$A_2 = \sum_{l=0}^{\infty}\left[(2l+1)C_{\omega l}\frac{dp_{\omega l}}{dr}\frac{dq_{\omega l}}{dr} + \left(l^2+l+\frac{1}{8}\right)\frac{h}{f^{1/2}r^3} + \frac{\omega^2 h}{2f^{3/2}r} - \frac{1}{8f^{1/2}r^3} - \frac{h'}{8f^{1/2}hr^2} - \frac{5f'^2}{32f^{5/2}r} + \frac{f''}{8f^{3/2}r} - \frac{f'h'}{16f^{3/2}hr} + \frac{hm^2}{2f^{1/2}r} + \xi\frac{h}{2f^{1/2}r}R\right] ,$$

$$A_3 = \sum_{l=0}^{\infty}\left[(2l+1)\left(l+\frac{1}{2}\right)^2 C_{\omega l}p_{\omega l}q_{\omega l} - \frac{l(l+1)}{f^{1/2}r} + \frac{\omega^2 r}{2f^{3/2}} - \frac{3}{8f^{1/2}r} + \frac{1}{8f^{1/2}hr} + \frac{f'}{4f^{3/2}h} - \frac{h'}{8f^{1/2}h^2} - \frac{3f'^2 r}{32f^{5/2}h} + \frac{f''r}{8f^{3/2}h} - \frac{f'h'r}{16f^{3/2}h^2} + \frac{m^2 r}{2f^{1/2}} + \xi\frac{r}{2f^{1/2}}R\right] ,$$

$$A_4 = \sum_{l=0}^{\infty}\left[(2l+1)C_{\omega l}\frac{dp_{\omega l}}{dr}q_{\omega l} - \frac{h^{1/2}}{2f^{1/2}r^2} + \frac{f'}{4f^{3/2}r} + \frac{1}{2f^{1/2}r^2}\right] ,$$

$$A_5 = \sum_{l=0}^{\infty}\left[(2l+1)C_{\omega l}p_{\omega l}\frac{dq_{\omega l}}{dr} + \frac{h^{1/2}}{2f^{1/2}r^2} + \frac{f'}{4f^{3/2}r} + \frac{1}{2f^{1/2}r^2}\right] . \tag{2.13}$$

Note that second derivatives of the mode functions with respect to r have been eliminated from the above expressions through use of the mode equation (2.6) and time derivatives have been computed using $\partial/\partial t = i\partial/\partial\tau$.

III. A RENORMALIZED EXPRESSION FOR $\langle T_{\mu\nu}\rangle$

In this section a renormalized expression for $\langle T_{\mu\nu}\rangle$ is derived. The method of point splitting is used. In point splitting one subtracts renormalization counterterms from $\langle T_{\mu\nu}\rangle_{\rm unren}$ and takes the limit $\epsilon \to 0$. Christensen [19,20] has used the DeWitt-Schwinger expansion for the Feynman Green's function to obtain renormalization counterterms for $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ for a scalar field with arbitrary mass and curvature coupling in a general spacetime. The renormalization counterterms for $\langle\phi^2\rangle$ are given by

$$\begin{aligned}\langle\phi^2\rangle_{\rm DS} &= G_{\rm DS}(x,x') \\ &= \frac{1}{8\pi^2\sigma} + \frac{1}{8\pi^2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right] \\ &\quad \times\left[C + \frac{1}{2}\ln\left(\frac{\mu^2|\sigma|}{2}\right)\right] \\ &\quad - \frac{m^2}{16\pi^2} + \frac{1}{96\pi^2}R_{\alpha\beta}\frac{\sigma^\alpha\sigma^\beta}{\sigma} .\end{aligned} \tag{3.1}$$

Here σ is equal to one half the square of the distance between the points x and x' along the shortest geodesic connecting them, C is Euler's constant, $R_{\alpha\beta}$ is the Ricci tensor and $\sigma^\alpha \equiv \sigma^{;\alpha}$. The renormalization counterterms for the stress-energy tensor will be denoted by $\langle T_{\mu\nu}\rangle_{\rm DS}$; expressions for $\langle T_{\mu\nu}\rangle_{\rm DS}$ are displayed in Ref. [19].

The constant μ is equal to the mass m of the field for a massive scalar field. However, for a massless scalar field it is an arbitrary parameter. This constant appears in $\langle T_{\mu\nu}\rangle_{\rm DS}$ as well [19,20]. For a massless field it represents an ambiguity in the way in which the limit $m \to 0$ is computed for the renormalization counterterms. Its existence is not a problem because its coefficient is proportional to the variation of the combination of a Weyl tensor squared term and a scalar curvature squared term in the gravitational Lagrangian. Thus a particular choice of the value of μ corresponds to a finite renormalization of the coefficients of these terms in the gravitational Lagrangian. This means that the value of μ must ultimately be fixed by experiment or observation.

In Appendix B, Christensen's method [20] of expanding σ and its derivatives in powers of $\epsilon \equiv t - t'$ for static spherically symmetric spacetimes is outlined. The results are

$$\begin{aligned}\sigma^t &= \epsilon + \frac{f'^2}{24fh}\epsilon^3 - \frac{1}{120}\left(\frac{f'^4}{8f^2h^2} + \frac{3}{16}\frac{f'^3h'}{fh^3} - \frac{3}{8}\frac{f'^2f''}{fh^2}\right)\epsilon^5 \\ &\quad + O(\epsilon^7) , \\ \sigma^r &= -\frac{f'}{4h}\epsilon^2 - \frac{1}{24}\left(-\frac{f'^2h'}{8h^3} + \frac{f'f''}{4h^2}\right)\epsilon^4 + O(\epsilon^6) , \\ \sigma^\theta &= \sigma^\phi = 0 .\end{aligned} \tag{3.2}$$

A renormalized expression for $\langle T_{\mu\nu}\rangle$ can be obtained by substituting these results into $\langle T_{\mu\nu}\rangle_{\rm DS}$, subtracting from $\langle T_{\mu\nu}\rangle_{\rm unren}$ and taking the real part of the limit $\epsilon \to 0$. Schematically one has

$$\langle T_{\mu\nu}\rangle_{\rm ren} = {\rm Re}\left[\lim_{\epsilon\to 0}(\langle T_{\mu\nu}\rangle_{\rm unren} - \langle T_{\mu\nu}\rangle_{\rm DS})\right] . \tag{3.3}$$

For this to work, one must expand $\langle T_{\mu\nu}\rangle_{\rm unren}$ in powers of ϵ. The unrenormalized expressions in Eqs. (2.12a)–

(2.12d) were derived for the Euclidean sector. After the mode sums and integrals are computed, these expressions can be converted to the Lorentzian sector using the relationship $\tau = it$. The quantities $g_\mu^{\alpha'}$ in Eqs. (2.12b)–(2.12d) must also be expanded in powers of ϵ. Howard [6] has outlined a method of doing this. We use his method to obtain the necessary expansion in Appendix C. The results are

$$\begin{aligned} g^{tt'} &= -\frac{1}{f} - \frac{f'^2}{8f^2h}\epsilon^2 + \left(\frac{f'^4}{384f^3h^2} - \frac{f'^2f''}{96f^2h^2} + \frac{f'^3h'}{192f^2h^3}\right)\epsilon^4 , \\ g^{tr'} &= -g^{rt'} \\ &= -\frac{f'}{2fh}\epsilon - \left(\frac{f'^3}{96f^2h^2} + \frac{f'f''}{48fh^2} - \frac{f'^2h'}{96fh^3}\right)\epsilon^3 , \\ g^{rr'} &= \frac{1}{h} + \frac{f'^2}{8fh^2}\epsilon^2 - \left(\frac{f'^4}{384f^2h^3} - \frac{f'^2f''}{96fh^3} + \frac{f'^3h'}{192fh^4}\right)\epsilon^4 . \end{aligned} \tag{3.4}$$

If the mode equation could be solved analytically for the modes and the sums and integrals over l and ω could be computed analytically then this would be all that was required. However, for most spacetimes the mode equation must be solved numerically. In these cases additional techniques are required for practical computations of $\langle T_{\mu\nu}\rangle_{\rm ren}$. A method of computing $\langle T_{\mu\nu}\rangle_{\rm ren}$ when the mode equations must be solved numerically is discussed in the next section.

IV. USE OF THE WKB APPROXIMATION IN EVALUATING $\langle T_{\mu\nu}\rangle$

In this section we describe a method of computing the renormalized expression for $\langle T_{\mu\nu}\rangle$ given in Eq. (3.3) when the mode equation must be solved numerically. Our approach is similar to that used by Howard and Candelas [6] to compute $\langle T_{\mu\nu}\rangle$ for conformally invariant scalar fields in Schwarzschild spacetime.

A primary difficulty in evaluating the expressions in Eq. (3.3) when the mode functions are computed numerically is that a way must be found to take the limit $\epsilon \to 0$. One way to do this is to use the WKB approximation to compute an analytical approximation for $\langle T_{\mu\nu}\rangle_{\rm unren}$ which contains all of the ultraviolet divergences found in this quantity. We shall call this approximation $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$. It is to be subtracted and added to the right-hand side of Eq. (3.3) with the result that

$$\begin{aligned} \langle T_{\mu\nu}\rangle_{\rm ren} &= {\rm Re}\left\{\lim_{\epsilon\to 0}\left[(\langle T_{\mu\nu}\rangle_{\rm unren} - \langle T_{\mu\nu}\rangle_{\rm WKBdiv}) + (\langle T_{\mu\nu}\rangle_{\rm WKBdiv} - \langle T_{\mu\nu}\rangle_{\rm DS})\right]\right\} \\ &= \langle T_{\mu\nu}\rangle_{\rm numeric} + \langle T_{\mu\nu}\rangle_{\rm analytic} . \end{aligned} \tag{4.1}$$

The mode sums and integrals in $\langle T_{\mu\nu}\rangle_{\rm analytic}$ can be computed analytically. Those in $\langle T_{\mu\nu}\rangle_{\rm numeric}$ must usually be computed numerically.

The derivation of $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ proceeds as follows: First one substitutes the fourth-order WKB expansion for the modes into $\langle T_{\mu\nu}\rangle_{\rm unren}$.[7] The Plana sum formula [21] is next used to compute the sums over l in the large ω limit. The results are then expanded in inverse powers of ω and the expansion is truncated at order ω^{-1}. The details of this procedure are given in Appendix D.

The sums or integrals over ω include $\omega = 0$. Thus it is necessary to impose an infrared cutoff on those sums or integrals containing terms which are proportional to ω^{-1}. Since $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ is both added and subtracted in Eq. (4.1), it is clear that $\langle T_{\mu\nu}\rangle_{\rm ren}$ is independent of the value of this cutoff. For the zero-temperature case the infrared cutoff is made for the ω^{-1} terms by inserting a lower limit cutoff λ in the integral over ω. For the nonzero temperature case a cutoff is most easily made by not including the $n = 0$ contributions from those terms which are proportional to ω^{-1}.

The quantity $\langle T_{\mu\nu}\rangle_{\rm analytic}$ is computed by first evaluating the mode sums and integrals in $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ analytically with $\tau' \neq \tau$. Then all factors $\tau - \tau'$ are converted to powers of ϵ using the relationship $\tau - \tau' = i\epsilon$. Next the bivectors of parallel transport, $g^{\alpha\beta'}$, are expanded in powers of ϵ as are the terms in the point splitting counterterm $\langle T_{\mu\nu}\rangle_{\rm DS}$. The difference is computed and then the limit $\epsilon \to 0$ is taken. The details of this procedure are discussed in Appendix E. A similar method has been used in Schwarzschild spacetime by Candelas and Howard [7] to derive analytic contributions to $\langle\phi^2\rangle$ for massless fields, and by Howard and Candelas [6] and Jensen and Ottewill [10] to derive analytic contributions to $\langle T_{\mu\nu}\rangle_{\rm ren}$ for the conformally invariant scalar field and the electromagnetic field, respectively. Defining $\kappa = 2\pi T$, we find[8]

$$\langle\phi^2\rangle_{\rm analytic} = \frac{\kappa^2}{48\pi^2 f} + \frac{m^2}{16\pi^2} - \frac{1}{8\pi^2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\frac{1}{2}\ln\left(\frac{\mu^2 f}{4\lambda^2}\right) - \frac{f'^2}{96\pi^2 f^2 h} - \frac{f'h'}{192\pi^2 f h^2} + \frac{f''}{96\pi^2 f h} + \frac{f'}{48\pi^2 r f h} , \tag{4.2}$$

[7] A fourth-order expansion is necessary if $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ is to contain all of the divergences of $\langle T_{\mu\nu}\rangle_{\rm unren}$. For $\langle\phi^2\rangle_{\rm WKBdiv}$ a second-order WKB expansion is necessary.

[8] In paper I, a mistake was made in the evaluation of the $R_{\alpha\beta}\sigma^\alpha\sigma^\beta/\sigma$ term of Eq. (3.1). As a result the expressions in Eqs. (3.3), (3.5), and (3.8) in that paper are correct for $h = 1/f$, but incorrect for other values of h.

$$\langle T_\mu^\nu \rangle_{\rm analytic} = (T_\mu{}^\nu)_0 + \left(\xi - \frac{1}{6}\right)(T_\mu{}^\nu)_1 + \left(\xi - \frac{1}{6}\right)^2 (T_\mu{}^\nu)_2 + (T_\mu{}^\nu)_{\rm log} \,, \tag{4.3}$$

where

$$\begin{aligned}(T_t{}^t)_0 = [&-32f^4h^3 - 96\kappa^4r^4f^2h^5 + 480\kappa^2m^2r^4f^3h^5 + 32f^4h^5 + 360m^4r^4f^4h^5 - 960m^2r^3f^3h^4f' \\ &+24r^2f^2h^3f'^2 + 360m^2r^4f^2h^4f'^2 - 24r^3fh^3f'^3 + 7r^4h^3f'^4 - 64rf^4h^2h' + 80r^2f^3h^2f'h' \\ &+240m^2r^4f^3h^3f'h' - 40r^3f^2h^2f'^2h' - 2r^4fh^2f'^3h' - 56r^2f^4hh'^2 - 48r^3f^3hf'h'^2 \\ &-19r^4f^2hf'^2h'^2 + 224r^3f^4h'^3 - 56r^4f^3f'h'^3 - 480m^2r^4f^3h^4f'' + 4r^4fh^3f'^2f'' - 64r^3f^3h^2h'f'' \\ &+36r^4f^2h^2f'h'f'' + 76r^4f^3hh'^2f'' - 12r^4f^2h^3f''^2 + 32r^2f^4h^2h'' + 16r^3f^3h^2f'h'' \\ &+8r^4f^2h^2f'^2h'' - 208r^3f^4hh'h'' + 52r^4f^3hf'h'h'' - 32r^4f^3h^2f''h'' + 64r^3f^3h^3f''' - 16r^4f^2h^3f'f''' \\ &-48r^4f^3h^2h'f''' + 32r^3f^4h^2h''' - 8r^4f^3h^2f'h''' + 16r^4f^3h^3f''''] \frac{1}{46080\pi^2r^4f^4h^5} \,, \end{aligned} \tag{4.4a}$$

$$\begin{aligned}(T_t{}^t)_1 = [&32f^4h^3 - 16\kappa^2r^2f^3h^4 - 32f^4h^4 - 48m^2r^2f^4h^4 + 16\kappa^2r^2f^3h^5 + 48m^2r^2f^4h^5 \\ &-64\kappa^2r^3f^2h^4f' - 192m^2r^3f^3h^4f' - 12r^2f^2h^3f'^2 + 40\kappa^2r^4fh^4f'^2 + 12r^2f^2h^4f'^2 + 72m^2r^4f^2h^4f'^2 \\ &+88r^3fh^3f'^3 - 63r^4h^3f'^4 + 48rf^4h^2h' + 16\kappa^2r^3f^3h^3h' - 16rf^4h^3h' + 48m^2r^3f^4h^3h' \\ &-24r^2f^3h^2f'h' + 16\kappa^2r^4f^2h^3f'h' + 8r^2f^3h^3f'h' + 48m^2r^4f^3h^3f'h' + 108r^3f^2h^2f'^2h' - 66r^4fh^2f'^3h' \\ &+32r^2f^4hh'^2 + 152r^3f^3hf'h'^2 - 57r^4f^2hf'^2h'^2 - 112r^3f^4h'^3 - 56r^4f^3f'h'^3 + 16r^2f^3h^3f'' \\ &-32\kappa^2r^4f^2h^4f'' - 16r^2f^3h^4f'' - 96m^2r^4f^3h^4f'' - 144r^3f^2h^3f'f'' + 132r^4fh^3f'^2f'' - 144r^3f^3h^2h'f'' \\ &+108r^4f^2h^2f'h'f'' + 76r^4f^3hh'^2f'' - 36r^4f^2h^3f''^2 - 16r^2f^4h^2h'' - 64r^3f^3h^2f'h'' + 24r^4f^2h^2f'^2h'' \\ &+104r^3f^4hh'h'' + 52r^4f^3hf'h'h'' - 32r^4f^3h^2f''h'' + 64r^3f^3h^3f''' - 48r^4f^2h^3f'f''' - 48r^4f^3h^2h'f''' \\ &-16r^3f^4h^2h''' - 8r^4f^3h^2f'h''' + 16r^4f^3h^3f''''] \frac{1}{768\pi^2r^4f^4h^5} \,, \end{aligned} \tag{4.4b}$$

$$\begin{aligned}(T_t{}^t)_2 = [&16f^4h^2 - 32f^4h^3 + 16f^4h^4 - 8r^2f^2h^2f'^2 + 24r^2f^2h^3f'^2 - 64r^3fh^2f'^3 \\ &+21r^4h^2f'^4 - 32rf^4hh' + 32rf^4h^2h' - 80r^2f^3hf'h' + 16r^2f^3h^2f'h' - 32r^3f^2hf'^2h' + 24r^4fhf'^3h' \\ &+16r^2f^4h'^2 + 80r^3f^3f'h'^2 + 19r^4f^2f'^2h'^2 + 32r^2f^3h^2f'' - 32r^2f^3h^3f'' + 80r^3f^2h^2f'f'' \\ &-48r^4fh^2f'^2f'' - 32r^3f^3hh'f'' - 36r^4f^2hf'h'f'' + 12r^4f^2h^2f''^2 - 32r^3f^3hf'h'' \\ &-8r^4f^2hf'^2h'' + 16r^4f^2h^2f'f'''] \frac{1}{128\pi^2r^4f^4h^4} \,, \end{aligned} \tag{4.4c}$$

$$\begin{aligned}(T_r{}^r)_0 = (&32\kappa^4r^3f^2h^4 - 480\kappa^2m^2r^3f^3h^4 - 1080m^4r^3f^4h^4 + 64f^3h^2f' - 120m^2r^3f^2h^3f'^2 - 16r^2fh^2f'^3 \\ &+r^3h^2f'^4 + 16rf^3hf'h' - 8r^2f^2hf'^2h' + 2r^3fhf'^3h' - 56r^2f^3f'h'^2 + 7r^3f^2f'^2h'^2 \\ &-64rf^3h^2f'' + 64r^2f^2h^2f'f'' - 4r^3fh^2f'^2f'' + 32r^2f^3hh'f'' - 8r^3f^2hf'h'f'' - 4r^3f^2h^2f''^2 \\ &+32r^2f^3hf'h'' - 4r^3f^2hf'^2h'' - 32r^2f^3h^2f''' + 8r^3f^2h^2f'f''') \frac{1}{46080\pi^2r^3f^4h^4} \,, \end{aligned} \tag{4.4d}$$

$$\begin{aligned}(T_r{}^r)_1 = (&16\kappa^2rf^3h^3 + 48m^2rf^4h^3 - 16\kappa^2rf^3h^4 - 48m^2rf^4h^4 - 48f^3h^2f' - 16\kappa^2r^2f^2h^3f' \\ &+16f^3h^3f' - 48m^2r^2f^3h^3f' - 44rf^2h^2f'^2 - 8\kappa^2r^3fh^3f'^2 - 4rf^2h^3f'^2 - 24m^2r^3f^2h^3f'^2 \\ &+24r^2fh^2f'^3 + 9r^3h^2f'^4 - 16rf^3hf'h' + 6r^3fhf'^3h' + 28r^2f^3f'h'^2 + 7r^3f^2f'^2h'^2 \\ &+32rf^3h^2f'' - 40r^2f^2h^2f'f'' - 12r^3fh^2f'^2f'' - 16r^2f^3hh'f'' - 8r^3f^2hf'h'f'' - 4r^3f^2h^2f''^2 \\ &-16r^2f^3hf'h'' - 4r^3f^2hf'^2h'' + 16r^2f^3h^2f''' + 8r^3f^2h^2f'f''') \frac{1}{768\pi^2r^3f^4h^4} \,, \end{aligned} \tag{4.4e}$$

$$\begin{aligned}(T_r{}^r)_2 = (&16f^3hf' - 16f^3h^2f' + 20rf^2hf'^2 - 4rf^2h^2f'^2 - r^3hf'^4 - 16rf^3f'h' \\ &-8r^2f^2f'^2h' - r^3ff'^3h' + 8r^2f^2hf'f'' + 2r^3fhf'^2f'') \frac{1}{64\pi^2r^3f^4h^3} \,, \end{aligned} \tag{4.4f}$$

$$
\begin{aligned}
(T_\theta{}^\theta)_0 = {} & (32\kappa^4 r^3 f^2 h^5 - 480\kappa^2 m^2 r^3 f^3 h^5 - 1080 m^4 r^3 f^4 h^5 - 32 f^3 h^3 f' + 120 m^2 r^3 f^2 h^4 f'^2 - 16 r^2 f h^3 f'^3 \\
& + 17 r^3 h^3 f'^4 - 48 r f^3 h^2 f' h' - 24 r^2 f^2 h^2 f'^2 h' + 26 r^3 f h^2 f'^3 h' - 24 r^2 f^3 h f' h'^2 + 31 r^3 f^2 h f'^2 h'^2 \\
& + 56 r^3 f^3 f' h'^3 + 32 r f^3 h^3 f'' + 32 r^2 f^2 h^3 f' f'' - 52 r^3 f h^3 f'^2 f'' + 88 r^2 f^3 h^2 h' f'' - 64 r^3 f^2 h^2 f' h' f'' \\
& - 76 r^3 f^3 h h'^2 f'' + 28 r^3 f^2 h^3 f''^2 + 8 r^2 f^3 h^2 f' h'' - 12 r^3 f^2 h^2 f'^2 h'' - 52 r^3 f^3 h f' h' h'' + 32 r^3 f^3 h^2 f'' h'' \\
& - 48 r^2 f^3 h^3 f''' + 24 r^3 f^2 h^3 f' f''' + 48 r^3 f^3 h^2 h' f''' + 8 r^3 f^3 h^2 f' h''' - 16 r^3 f^3 h^3 f'''') \frac{1}{46080\pi^2 r^3 f^4 h^5} , \qquad (4.4g)
\end{aligned}
$$

$$
\begin{aligned}
(T_\theta{}^\theta)_1 = {} & (16 f^3 h^3 f' - 8\kappa^2 r^2 f^2 h^4 f' - 24 m^2 r^2 f^3 h^4 f' + 12 r f^2 h^3 f'^2 + 20\kappa^2 r^3 f h^4 f'^2 - 4 r f^2 h^4 f'^2 \\
& + 12 m^2 r^3 f^2 h^4 f'^2 + 48 r^2 f h^3 f'^3 - 43 r^3 h^3 f'^4 - 8\kappa^2 r^2 f^3 h^3 h' - 24 m^2 r^2 f^4 h^3 h' + 36 r f^3 h^2 f' h' \\
& + 4\kappa^2 r^3 f^2 h^3 f' h' - 4 r f^3 h^3 f' h' + 12 m^2 r^3 f^3 h^3 f' h' + 54 r^2 f^2 h^2 f'^2 h' - 45 r^3 f h^2 f'^3 h' \\
& + 30 r^2 f^3 h f' h'^2 - 38 r^3 f^2 h f'^2 h'^2 - 28 r^3 f^3 f' h'^3 - 24 r f^3 h^3 f'' - 8\kappa^2 r^3 f^2 h^4 f'' \\
& + 8 r f^3 h^4 f'' - 24 m^2 r^3 f^3 h^4 f'' - 76 r^2 f^2 h^3 f' f'' + 90 r^3 f h^3 f'^2 f'' - 48 r^2 f^3 h^2 h' f'' \\
& + 72 r^3 f^2 h^2 f' h' f'' + 38 r^3 f^3 h h'^2 f'' - 24 r^3 f^2 h^3 f''^2 - 12 r^2 f^3 h^2 f' h'' + 16 r^3 f^2 h^2 f'^2 h'' \\
& + 26 r^3 f^3 h f' h' h'' - 16 r^3 f^3 h^2 f'' h'' + 24 r^2 f^3 h^3 f''' - 32 r^3 f^2 h^3 f' f''' \\
& - 24 r^3 f^3 h^2 h' f''' - 4 r^3 f^3 h^2 f' h''' + 8 r^3 f^3 h^3 f'''') \frac{1}{768\pi^2 r^3 f^4 h^5} , \qquad (4.4h)
\end{aligned}
$$

$$
\begin{aligned}
(T_\theta{}^\theta)_2 = {} & (-24 f^3 h^2 f' + 24 f^3 h^3 f' - 12 r f^2 h^2 f'^2 + 4 r f^2 h^3 f'^2 - 22 r^2 f h^2 f'^3 + 9 r^3 h^2 f'^4 \\
& - 12 r f^3 h f' h' + 4 r f^3 h^2 f' h' - 18 r^2 f^2 h f'^2 h' + 10 r^3 f h f'^3 h' + 36 r^2 f^3 f' h'^2 + 9 r^3 f^2 f'^2 h'^2 \\
& + 8 r f^3 h^2 f'' - 8 r f^3 h^3 f'' + 28 r^2 f^2 h^2 f' f'' - 20 r^3 f h^2 f'^2 f'' - 8 r^2 f^3 h h' f'' - 16 r^3 f^2 h f' h' f'' \\
& + 4 r^3 f^2 h^2 f''^2 - 16 r^2 f^3 h f' h'' - 4 r^3 f^2 h f'^2 h'' + 8 r^3 f^2 h^2 f' f''') \frac{1}{64\pi^2 r^3 f^4 h^4} , \qquad (4.4i)
\end{aligned}
$$

$$
\begin{aligned}
(T_\mu{}^\nu)_{\log} = {} & -\frac{1}{4\pi^2}\left[\frac{1}{60}\left(R_{\rho\mu\tau}{}^\nu R^{\rho\tau}\frac{1}{4}R^{\rho\tau}R_{\rho\tau}g_\mu{}^\nu\right) - \frac{1}{180}R\left(R_\mu^\nu - \frac{1}{4}Rg_\mu^\nu\right)\right. \\
& + \frac{1}{120}(R_\mu^\nu)_{;\rho}{}^\rho - \frac{1}{360}R_{;\mu}{}^\nu - \frac{1}{720}R_{;\rho}{}^\rho g_\mu^\nu - \frac{1}{8}m^4 g_\mu^\nu + \frac{1}{2}\left(\xi - \frac{1}{6}\right)m^2\left(R_\mu^\nu - \frac{1}{2}Rg_\mu^\nu\right) \\
& \left. -\frac{1}{4}\left(\xi - \frac{1}{6}\right)^2\left(-2R\left(R_\mu^\nu - \frac{1}{4}Rg_\mu^\nu\right) + 2R_{;\mu}{}^\nu - 2R_{;\rho}{}^\rho g_\mu^\nu\right)\right]\frac{1}{2}\ln\left(\frac{\mu^2 f}{4\lambda^2}\right) . \qquad (4.4j)
\end{aligned}
$$

The zero-temperature case is obtained by setting $\kappa = 0$ in the above expressions. For the nonzero temperature case $\lambda = \kappa \exp(-C)$ if the $\omega = 0$ terms are the only ones omitted from the mode sum in $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$. C is Euler's constant.

The quantity $\langle T_{\mu\nu}\rangle_{\rm numeric}$ in Eq. (4.1) is explicitly finite in the limit $\epsilon \to 0$. Thus this limit can be taken before the mode sums are computed. We find that

$$
\begin{aligned}
\langle \phi^2\rangle_{\rm numeric} &= S_5 \\
\langle T_t{}^t\rangle_{\rm numeric} &= \left(2\xi + \frac{1}{2}\right)\frac{1}{f}S_1 + \left(2\xi - \frac{1}{2}\right)\frac{1}{h}S_2 + \left(2\xi - \frac{1}{2}\right)\frac{1}{r^2}S_3 - \frac{\xi f'}{2fh}S_4 \\
& + \left[\left(2\xi - \frac{1}{2}\right)\left(-\frac{1}{4r^2} + m^2 + \xi R\right) + \xi R_t{}^t\right]S_5 ,
\end{aligned}
$$

$$
\langle T_r{}^r\rangle_{\rm numeric} = -\frac{1}{2f}S_1 + \frac{1}{2h}S_2 - \frac{1}{2r^2}S_3 + \frac{\xi}{h}\left(\frac{2}{r} + \frac{f'}{2f}\right)S_4 + \left(\frac{1}{8r^2} - \frac{1}{2}m^2 - \frac{1}{2}\xi R + \xi R_r{}^r\right)S_5 ,
$$

$$
\langle T_\theta{}^\theta\rangle_{\rm numeric} = \left(2\xi - \frac{1}{2}\right)\frac{1}{f}S_1 + \left(2\xi - \frac{1}{2}\right)\frac{1}{h}S_2 + \frac{2\xi}{r^2}S_3 - \frac{\xi}{rh}S_4 + \left[-\frac{\xi}{2r^2} + \left(2\xi - \frac{1}{2}\right)(m^2 + \xi R) + \xi R_\theta{}^\theta\right]S_5 , \qquad (4.5)
$$

where

$$
\begin{aligned}
S_1 = \int d\tilde{\mu}\Bigg\{ & \omega^2 A_1 + \frac{1}{f}\omega^3 + \frac{1}{2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\omega \\
& -\Bigg[\frac{-49f'^4}{5760f^3h^2} + \frac{29f'^2f''}{1440f^2h^2} - \frac{f''^2}{160fh^2} - \frac{f'f'''}{120fh^2} + \frac{f''''}{360h^2} - \frac{29f'^3h'}{2880f^2h^3} + \frac{3f'f''h'}{160fh^3} - \frac{f'''h'}{120h^3} \\
& - \frac{19f'^2h'^2}{1920fh^4} + \frac{19f''h'^2}{1440h^4} - \frac{7f'h'^3}{720h^5} + \frac{f'^2h''}{240fh^3} - \frac{f''h''}{180h^3} + \frac{13f'h'h''}{1440h^4} - \frac{f'h'''}{720h^3} + \frac{f'^2m^2}{48fh} \\
& - \frac{f''m^2}{24h} + \frac{f'h'm^2}{48h^2} + \frac{fm^4}{8} + \frac{f}{360r^4} - \frac{f}{360h^2r^4} - \frac{fh'}{180h^3r^3} + \frac{f'^2}{288fh^2r^2} + \frac{f'h'}{144h^3r^2} - \frac{7fh'^2}{1440h^4r^2} \\
& + \frac{fh''}{360h^3r^2} + \frac{11f'^3}{1440f^2h^2r} - \frac{13f'f''}{720fh^2r} + \frac{f'''}{90h^2r} + \frac{f'^2h'}{120fh^3r} - \frac{13f''h'}{720h^3r} + \frac{13f'h'^2}{1440h^4r} + \frac{7fh'^3}{360h^5r} \\
& - \frac{f'h''}{240h^3r} - \frac{13fh'h''}{720h^4r} + \frac{fh'''}{360h^3r} - \frac{f'm^2}{12hr} + \left(\xi - \frac{1}{6}\right)\left(\frac{f'^2R}{48fh} - \frac{f''R}{24h} + \frac{f'h'R}{48h^2} + \frac{fm^2R}{4} - \frac{f'R}{12hr}\right. \\
& \left. - \frac{f'R'}{16h} + \frac{fh'R'}{48h^2} - \frac{fR'}{12hr} - \frac{fR''}{24h}\right) + \frac{1}{8}\left(\xi - \frac{1}{6}\right)^2 fR^2\Bigg]\frac{1}{\omega}\Bigg\}\,, \qquad (4.6a)
\end{aligned}
$$

$$
\begin{aligned}
S_2 = \int d\tilde{\mu}\Bigg\{ & A_2 - \frac{h}{3f^2}\omega^3 - \left[\frac{-5f'^2}{12f^3} + \frac{f''}{6f^2} - \frac{f'h'}{12f^2h} + \frac{hm^2}{2f} + \frac{f'}{6f^2r} + \frac{h'}{6fhr} + \frac{1}{2}\left(\xi - \frac{1}{6}\right)\frac{h}{f}R\right]\omega \\
& -\Bigg[\frac{7f'^4}{5760f^4h} - \frac{f'^2f''}{480f^3h} - \frac{f''^2}{1440f^2h} + \frac{f'f'''}{720f^2h} + \frac{f'^3h'}{960f^3h^2} - \frac{f'f''h'}{720f^2h^2} + \frac{7f'^2h'^2}{5760f^2h^3} \\
& - \frac{f'^2h''}{1440f^2h^2} + \frac{f'^2m^2}{48f^2} - \frac{f''m^2}{24f} + \frac{f'h'm^2}{48fh} + \frac{hm^4}{8} - \frac{1}{360hr^4} + \frac{h}{360r^4} + \frac{f'}{180fhr^3} - \frac{f'^2}{1440f^2hr^2} \\
& - \frac{f''}{180fhr^2} + \frac{f'h'}{720fh^2r^2} + \frac{7h'^2}{1440h^3r^2} - \frac{h''}{360h^2r^2} - \frac{f'^3}{288f^3hr} + \frac{f'f''}{120f^2hr} - \frac{f'''}{360fhr} \\
& - \frac{f'^2h'}{360f^2h^2r} + \frac{f''h'}{360fh^2r}\,\frac{7f'h'^2}{1440fh^3r} + \frac{f'h''}{360fh^2r} + \frac{h'm^2}{12hr} + \left(\xi - \frac{1}{6}\right)\left(\frac{f'^2R}{48f^2} - \frac{f''R}{24f} + \frac{f'h'R}{48fh} + \frac{hm^2R}{4}\right. \\
& \left. + \frac{h'R}{12hr} - \frac{f'R'}{48f} + \frac{h'R'}{16h} - \frac{R'}{12r} - \frac{R''}{8}\right) + \frac{1}{8}\left(\xi - \frac{1}{6}\right)^2 hR^2\Bigg]\frac{1}{\omega}\Bigg\}\,, \qquad (4.6b)
\end{aligned}
$$

$$
\begin{aligned}
S_3 = \int d\tilde{\mu}\Bigg(& A_3 - \frac{2r^2}{3f^2}\omega^3 - \left(\frac{1}{12f} - \frac{1}{3fh} + \frac{f'r}{2f^2h} + \frac{h'r}{6fh^2} - \frac{f'^2r^2}{4f^3h} + \frac{f''r^2}{6f^2h} - \frac{f'h'r^2}{12f^2h^2} + \frac{r^2}{f}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\right)\omega \\
& -\Bigg[\frac{-f'^2}{360f^2h^2} + \frac{f''}{180fh^2} - \frac{f'h'}{120fh^3} + \frac{m^2}{24} - \frac{m^2}{6h} - \frac{1}{180r^2} + \frac{1}{180h^2r^2} - \frac{f'}{180fh^2r} + \frac{h'}{180h^3r} - \frac{f'^3r}{240f^3h^2} + \frac{7f'f''r}{720f^2h^2} \\
& - \frac{f'''r}{120fh^2} - \frac{f'^2h'r}{180f^2h^3} + \frac{11f''h'r}{720fh^3} - \frac{f'h'^2r}{240fh^4} - \frac{7h'^3r}{360h^5} + \frac{f'h''r}{720fh^3} + \frac{13h'h''r}{720h^4} - \frac{h'''r}{360h^3} - \frac{f'm^2r}{12fh} \\
& + \frac{h'm^2r}{12h^2} + \frac{7f'^4r^2}{960f^4h^2} - \frac{13f'^2f''r^2}{720f^3h^2} + \frac{f''^2r^2}{144f^2h^2} + \frac{f'f'''r^2}{144f^2h^2} - \frac{f''''r^2}{360fh^2} + \frac{13f'^3h'r^2}{1440f^3h^3} - \frac{5f'f''h'r^2}{288f^2h^3} \\
& + \frac{f'''h'r^2}{120fh^3} + \frac{5f'^2h'^2r^2}{576f^2h^4} - \frac{19f''h'^2r^2}{1440fh^4} + \frac{7f'h'^3r^2}{720fh^5} - \frac{f'^2h''r^2}{288f^2h^3} + \frac{f''h''r^2}{180fh^3} - \frac{13f'h'h''r^2}{1440fh^4} + \frac{f'h'''r^2}{720fh^3} + \frac{m^4r^2}{4} \\
& + \left(\xi - \frac{1}{6}\right)\left(\frac{R}{24} - \frac{R}{6h} - \frac{f'rR}{12fh} + \frac{h'rR}{12h^2} + \frac{m^2r^2R}{2} - \frac{rR'}{3h} - \frac{f'r^2R'}{24fh} + \frac{h'r^2R'}{24h^2} - \frac{r^2R''}{12h}\right) \\
& + \frac{1}{4}\left(\xi - \frac{1}{6}\right)^2 r^2R^2\Bigg]\frac{1}{\omega}\Bigg)\,, \qquad (4.6c)
\end{aligned}
$$

$$
S_4 = \int d\tilde{\mu}\left[A_4 + A_5 - \frac{f'}{f^2}\omega + \frac{1}{2}\left(\xi - \frac{1}{6}\right)R'\frac{1}{\omega}\right]\,, \qquad (4.6d)
$$

$$
S_5 = \int d\tilde{\mu}\left\{A_1 + \frac{1}{f}\omega + \frac{1}{2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\frac{1}{\omega}\right\}\,. \qquad (4.6e)
$$

To actually compute $\langle T_{\mu\nu}\rangle_{\rm numeric}$ as it stands would require numerical solutions of the mode equations for a very large number of modes. The number of modes one needs to numerically solve for can be reduced substantially by once again adding and subtracting the WKB approximation for the modes. This time the large ω limit is not taken and the mode sums are not explicitly computed ahead of time. Schematically the result is

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{\rm numeric} &= \lim_{\epsilon\to 0}[(\langle T_{\mu\nu}\rangle_{\rm unren} - \langle T_{\mu\nu}\rangle_{\rm WKB}) \\ &\quad +(\langle T_{\mu\nu}\rangle_{\rm WKB} - \langle T_{\mu\nu}\rangle_{\rm WKBdiv})] \\ &= \langle T_{\mu\nu}\rangle_{\rm modes} + \langle T_{\mu\nu}\rangle_{\rm WKBfin}\ . \end{aligned} \tag{4.7}$$

The mode sums and integrals in $\langle T_{\mu\nu}\rangle_{\rm modes}$ can be approximately computed by truncating them for large values of l or ω. This is equivalent to using the WKB approximation for the modes at large values of l and ω. To some extent, the cutoffs in l and ω which must be used to obtain a given accuracy for the sums get smaller if larger orders in the WKB expansion are retained.[9] We have found it useful to use a sixth-order WKB expression for $\langle T_{\mu\nu}\rangle_{\rm modes}$ in the case of a massive scalar field. For a massless scalar field we have found a sixth-order expression works but the mode sums converge significantly faster for an eighth-order expression.

To compute the sums over l and/or ω more accurately in $\langle T_{\mu\nu}\rangle_{\rm modes}$, one can fit the terms by using a general linear least squares fit [22]. For the sum over l, the fit is an expansion in inverse powers of l. The fit is summed analytically from the upper limit cutoff in l to infinity. Once the sum over l has been computed, a fit for the sum or integral over ω in inverse powers of ω can be obtained. This fit is summed or integrated analytically from the upper limit cutoff in ω to infinity.

The quantity $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$ is too complicated to be computed analytically. However, for a massless field at zero temperature, the only numerical computations required are for a few integrals which do not depend on the spacetime geometry and can thus be computed once and for all. For the nonzero temperature case and/or the massive field, the numerical computations must be repeated for each value of r at which $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$ is to be computed. Efficient ways to do these computations are discussed in Appendix F.

V. ANALYTICAL APPROXIMATIONS FOR $\langle T_{\mu\nu}\rangle$

In this section we discuss two different analytical approximations for $\langle T_{\mu\nu}\rangle$ for scalar fields in static spherically symmetric spacetimes.[10] One is obtained from $\langle T_{\mu\nu}\rangle_{\rm analytic}$ and the other is the DeWitt-Schwinger approximation for a massive scalar field.

The quantity $\langle T_{\mu\nu}\rangle_{\rm analytic}$ can be used directly as an approximation for $\langle T_{\mu\nu}\rangle_{\rm ren}$ because $\langle T_{\mu\nu}\rangle_{\rm analytic}$ is conserved and, for the conformally invariant field, has a trace equal to the trace anomaly. The expression for $\langle T_{\mu\nu}\rangle_{\rm analytic}$ does not, however, yield a unique analytic approximation due to the existence of the arbitrary parameter in the log term [see Eqs. (4.3) and (4.4j)] which is due to the infrared cutoff in $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$. For a massless field this parameter can be absorbed into the definition of the arbitrary constant μ discussed in Sec. III. For a massive field $\mu = m$ so this is not possible. Instead some arbitrary value must be assigned to the parameter.

For the case of a massless conformally coupled field $\langle T_{\mu\nu}\rangle_{\rm analytic}$ is equivalent to the approximation of Frolov and Zel'nikov [11] if the arbitrary constants ${q_2}^{(0)}$ and ${q_1}^{(2)}$ in their expression for $\langle T_{\mu\nu}\rangle$ are set equal to zero. Their arbitrary constant ${q_1}^{(0)}$ is related to the arbitrary constant μ discussed in Sec. III. The original derivation of Frolov and Zel'nikov was not based on quantum field theory. Their expression for $\langle T_{\mu\nu}\rangle$ was derived by constructing the most general expression from the timelike Killing vector field, the curvature tensor, and their derivatives which was conserved and possessed a trace given by the conformal anomaly. Our demonstration that the Frolov-Zel'nikov approximation is the conformally invariant limit of our more general analytic approximation is the first justification of their approximation in terms of quantum field theory.

Since our approximation reduces to that of Frolov and Zel'nikov in the conformally invariant limit, it follows that our approximate expression for $\langle T_{\mu\nu}\rangle$ in this case duplicates Huang's results [23] for the Frolov-Zel'nikov approximation in Reissner-Nordström spacetimes. If we restrict ourselves further to the case of a conformally invariant scalar field in a static spherically symmetric Einstein ($R_{\mu\nu} = \Lambda g_{\mu\nu}$) spacetime, then our approximation for $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ is also equivalent to Page's approximation [12] for these quantities.

Unfortunately, the use of our approximation (and therefore the Frolov-Zel'nikov approximation) in black hole spacetimes is severely limited. This is due in part to logarithmic divergences which occur on the event horizon in this approximation. Examination of Eq. (4.4j) shows that for a massive field a logarithmic divergence occurs on the event horizon of any static spherically symmetric black hole for all nonzero components of $\langle T_{\mu\nu}\rangle$. Numerical computations of $\langle T_{\mu\nu}\rangle$ for massive fields in Schwarzschild and Reissner-Nordström spacetimes give no indication that such a divergence exists, as will be shown in Sec. VI. For massless fields examination of Eq. (4.4j) shows that the analytical approximation predicts the possible existence of a logarithmic divergence on the event horizon of a black hole unless $R_{\mu\nu} = 0$.[11] In

[9] This is not true to arbitrarily high order because the WKB approximation is an asymptotic expansion and thus breaks down for given values of l and ω when the order gets too high.

[10] The types of approximations discussed in this section can also be used for the quantity $\langle\phi^2\rangle$.

[11] From Eq. (4.2) it can be seen that the analytical approximation for $\langle\phi^2\rangle$ diverges on the event horizon of a black hole unless $m = 0$ and either $\xi = 1/6$ or $R = 0$.

Reissner-Nordström spacetimes one component of $\langle T_{\mu\nu}\rangle$ is predicted to have such a divergence.[12] As in the massive field case, numerical computations of $\langle T_{\mu\nu}\rangle$ in Reissner-Nordström spacetimes give no indication that this divergence exists. Thus in black hole spacetimes the approximation appears to be trustworthy near the event horizon only if $R_{\mu\nu} = 0$ and the fields are massless.[13]

Away from the event horizon our numerical computations in Schwarzschild and Reissner-Nordström spacetimes show that the approximation is not valid for intermediate or large mass fields. It is likely to be valid for very small mass fields, but we have not tested that case. For massless fields, as discussed in Sec. VI, we find that the approximation gets progressively worse as the charge to mass ratio of the black hole increases.

In contrast, the DeWitt-Schwinger approximation for a massive scalar field is valid in virtually any spacetime if the field has a large enough mass. This is because it is really an asymptotic expansion in inverse powers of the mass of the quantum field. It is also an expansion in terms of derivatives of the spacetime metric. The expansion is general in that it allows one to compute an approximate Green's function and stress-energy tensor for a massive field in an arbitrary spacetime. It forms the foundation of the renormalization method of point splitting [15,19,20].

Here we discuss an alternative derivation of the DeWitt-Schwinger approximation for $\langle T_{\mu\nu}\rangle$ for massive scalar fields in static spherically symmetric spacetimes. The derivation makes use of the WKB approximation for the modes of the quantum fields. As mentioned above the DeWitt-Schwinger expansion is an expansion in inverse powers of the mass of the quantum field and in derivatives of the metric. Therefore it is not surprising that we find a one-to-one correspondence between the order of the WKB approximation used and the resulting order of the DeWitt-Schwinger expansion in terms of powers of $1/m$. For example, a fourth-order WKB expansion results in an $O(m^0)$ DeWitt-Schwinger approximation for $\langle T_{\mu\nu}\rangle$ while a sixth-order expansion results in an $O(m^{-2})$ approximation.[14]

Christensen's point splitting counterterms [19,20] result from DeWitt-Schwinger expansions of $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ truncated at $O(m^0)$. We have reproduced these counterterms using second- and fourth-order WKB expansions, respectively. We have also derived analytical approximations for $\langle\phi^2\rangle_{\rm ren}$ and $\langle T_{\mu\nu}\rangle_{\rm ren}$ by carrying out the expansions to $O(m^{-4})$ and $O(m^{-2})$, respectively. Our results for $\langle\phi^2\rangle$ agree with those of Ref. [16]. For $\langle T_{\mu\nu}\rangle$ in Schwarzschild spacetime our results agree with those of Ref. [13,14].

Our derivation is based on the following approximation

$$\langle T_{\mu\nu}\rangle_{\rm ren} \approx \langle T_{\mu\nu}\rangle_{\rm WKBfin} + \langle T_{\mu\nu}\rangle_{\rm analytic} \,. \tag{5.1}$$

The DeWitt-Schwinger approximation is obtained to a given order by substituting the corresponding order of the WKB expansion into $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$, evaluating the mode sums and integrals in the large mass limit and substituting into Eq. (5.1). The computation of $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$ in the large mass limit is described in Appendix G. The resulting expressions for the DeWitt-Schwinger approximation for $\langle T_{\mu\nu}\rangle$ in a general static spherical spacetime are too long to be displayed here. However, explicit expressions for the Reissner-Nordström spacetime are given in the following section. Comparisons with numerical computations show that for a field with a large enough mass, the DeWitt-Schwinger expansion always provides a good approximation for $\langle T_{\mu\nu}\rangle$ in both Schwarzschild and Reissner-Nordström spacetimes.

VI. $\langle T_{\mu\nu}\rangle$ IN REISSNER-NORDSTRÖM SPACETIMES

In this section we apply the method outlined in Secs. II–IV to the computation of $\langle T_{\mu\nu}\rangle$ for massless and massive scalar fields with arbitrary curvature couplings in Schwarzschild and Reissner-Nordström spacetimes. We assume the fields are in the Hartle-Hawking state which is a thermal state at the black hole temperature $\kappa/2\pi$, where κ is the surface gravity of the black hole. We compare our numerical results with the predictions of the analytic approximations discussed in Sec. V.

For a Reissner-Nordström spacetime the metric functions f and h are

$$f = h^{-1} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \,, \tag{6.1}$$

where M is the mass of the black hole and Q is its charge. The inner and outer horizons are at

$$r_\pm = M \pm (M^2 - Q^2)^{1/2} \,. \tag{6.2}$$

We compute $\langle T_{\mu\nu}\rangle$ in the region exterior to the outer event horizon where the spacetime is static. Examination of the mode equation (2.6) shows that whenever the scalar curvature R is identically zero, the mode equation is independent of the value of the curvature coupling ξ. This is the case for Reissner-Nordström spacetimes. If $R = 0$, then $(T_{\mu\nu})_2$ in Eq. (4.3) vanishes and it can be seen from Eqs. (4.3)–(4.6) that $\langle T_{\mu\nu}\rangle_{\rm ren}$ can be written in the form

[12] In this case the component is proportional to $(\langle T_r^r\rangle - \langle T_t^t\rangle)/f$. No such divergence occurs for Schwarzschild spacetime since $R_{\mu\nu} = 0$.

[13] Numerical computations for massless fields in Reissner-Nordström spacetimes, which are discussed in Sec. VI, show that, for black holes with charge to mass ratios that are not too close to one, the analytic approximation is a reasonable approximation near the event horizon for the components $\langle T_t^t\rangle$, $\langle T_r^r\rangle$, and $\langle T_\theta^\theta\rangle$.

[14] For $\langle\phi^2\rangle$ a second-order WKB expansion results in an $O(m^0)$ DeWitt-Schwinger approximation, a fourth-order WKB expansion gives an $O(m^{-2})$ DeWitt-Schwinger approximation, and so forth.

$$\langle T_{\mu\nu}\rangle_{\rm ren} = C_{\mu\nu} + \left(\xi - \tfrac{1}{6}\right) D_{\mu\nu} \tag{6.3}$$

where $C_{\mu\nu}$ and $D_{\mu\nu}$ are tensors that are independent of ξ. This greatly simplifies the study of the effects of the curvature coupling constant ξ.

In Sec. V, we discussed an analytic approximation which is useful for the massless field and is based on $\langle T_{\mu\nu}\rangle_{\rm analytic}$. The nonzero components of this analytic approximation for a massless scalar field in a Reissner-Nordström spacetime are[15]

$$\begin{aligned}(C_t{}^t)^a = {}& (2880\pi^2 r^8 \Delta^2)^{-1}[-6\kappa^4 r^{12} + 48Q^2 r^6 + 60M^2 r^6 - 472MQ^2 r^5 - 216M^3 r^5 \\ & +275Q^4 r^4 + 1348M^2 Q^2 r^4 + 198M^4 r^4 - 1372MQ^4 r^3 - 1192M^3 Q^2 r^3 + 370Q^6 r^2 \\ & +1656M^2 Q^4 r^2 - 852MQ^6 r + 149Q^8] + \frac{Q^2\Delta}{40\pi^2 r^8}\left[C + \frac{1}{2}\ln\left(\frac{\Delta\mu^2}{4r^2}\right)\right] ,\end{aligned} \tag{6.4a}$$

$$\begin{aligned}(D_t{}^t)^a = {}& (16\pi^2 r^6 \Delta^2)^{-1}(-\kappa^2 Q^2 r^6 + 2\kappa^2 M^2 r^6 - 2\kappa^2 MQ^2 r^5 + \kappa^2 Q^4 r^4 + 4Q^2 r^4 - 4M^2 r^4 - 12MQ^2 r^3 \\ & +12M^3 r^3 + 3Q^4 r^2 + 7M^2 Q^2 r^2 - 10M^4 r^2 - 6MQ^4 r + 6M^3 Q^2 r + Q^6 - M^2 Q^4) ,\end{aligned} \tag{6.4b}$$

$$\begin{aligned}(C_r{}^r)_a = {}& (2880\pi^2 r^8 \Delta^2)^{-1}[2\kappa^4 r^{12} + 16Q^2 r^6 + 4M^2 r^6 - 112MQ^2 r^5 - 24M^3 r^5 \\ & +59Q^4 r^4 + 292M^2 Q^2 r^4 + 30M^4 r^4 - 300MQ^4 r^3 - 264M^3 Q^2 r^3 + 82Q^6 r^2 \\ & +384M^2 Q^4 r^2 - 204MQ^6 r + 37Q^8] + \frac{Q^2\Delta}{120\pi^2 r^8}\left[C + \frac{1}{2}\ln\left(\frac{\Delta\mu^2}{4r^2}\right)\right] ,\end{aligned} \tag{6.4c}$$

$$\begin{aligned}(D_r{}^r)_a = {}& (48\pi^2 r^6 \Delta^2)^{-1}(-4\kappa^2 M r^7 + 3\kappa^2 Q^2 r^6 + 6\kappa^2 M^2 r^6 - 6\kappa^2 MQ^2 r^5 + \kappa^2 Q^4 r^4 - 8Q^2 r^4 + 8M^2 r^4 \\ & +24MQ^2 r^3 - 24M^3 r^3 - 9Q^4 r^2 - 9M^2 Q^2 r^2 + 18M^4 r^2 + 14MQ^4 r - 14M^3 Q^2 r - 3Q^6 + 3M^2 Q^4) ,\end{aligned} \tag{6.4d}$$

$$\begin{aligned}(C_\theta{}^\theta)_a = {}& (2880\pi^2 r^8 \Delta^2)^{-1}[2\kappa^4 r^{12} - 32Q^2 r^6 - 8M^2 r^6 + 244MQ^2 r^5 + 24M^3 r^5 \\ & -141Q^4 r^4 - 580M^2 Q^2 r^4 - 18M^4 r^4 + 636MQ^4 r^3 + 440M^3 Q^2 r^3 - 174Q^6 r^2 \\ & -700M^2 Q^4 r^2 + 376MQ^6 r - 67Q^8] - \frac{Q^2\Delta}{60\pi^2 r^8}\left[C + \frac{1}{2}\ln\left(\frac{\Delta\mu^2}{4r^2}\right)\right] ,\end{aligned} \tag{6.4e}$$

$$\begin{aligned}(D_\theta{}^\theta)_a = {}& (48\pi^2 r^6 \Delta^2)^{-1}(2\kappa^2 M r^7 - 3\kappa^2 Q^2 r^6 + \kappa^2 Q^4 r^4 + 16Q^2 r^4 - 16M^2 r^4 - 54MQ^2 r^3 \\ & +54M^3 r^3 + 21Q^4 r^2 + 27M^2 Q^2 r^2 - 48M^4 r^2 - 40MQ^4 r + 40M^3 Q^2 r + 9Q^6 - 9M^2 Q^4) ,\end{aligned} \tag{6.4f}$$

where $\Delta \equiv r^2 f$ and C is Euler's constant.

In Sec. V our derivation of the DeWitt-Schwinger approximation for $\langle T_{\mu\nu}\rangle$ for a massive scalar field was discussed. In a Reissner-Nordström spacetime, to leading order in m^{-1}, the DeWitt-Schwinger approximation for $\langle T_{\mu\nu}\rangle$ is

$$\begin{aligned}C_t{}^t = {}& (30240\pi^2 m^2 r^{12})^{-1}(1878M^3 r^3 - 855M^2 r^4 + 810Q^2 r^4 - 1152MQ^2 r^3 \\ & -202Q^4 r^2 - 2307M^2 Q^2 r^2 + 3084MQ^4 r - 1248Q^6) ,\end{aligned} \tag{6.5a}$$

$$\begin{aligned}D_t{}^t = {}& (720\pi^2 m^2 r^{12})^{-1}(360M^2 r^4 - 792M^3 r^3 - 1008MQ^2 r^3 \\ & +728Q^4 r^2 + 2604M^2 Q^2 r^2 - 2712MQ^4 r + 819Q^6) ,\end{aligned} \tag{6.5b}$$

$$\begin{aligned}C_r{}^r = {}& (30240\pi^2 m^2 r^{12})^{-1}(315M^2 r^4 - 462M^3 r^3 + 162Q^2 r^4 - 1488MQ^2 r^3 \\ & +842Q^4 r^2 + 2127M^2 Q^2 r^2 - 1932MQ^4 r + 444Q^6) ,\end{aligned} \tag{6.5c}$$

$$\begin{aligned}D_r{}^r = {}& (720\pi^2 m^2 r^{12})^{-1}(-144M^2 r^4 + 216M^3 r^3 + 336MQ^2 r^3 - 208Q^4 r^2 \\ & -588M^2 Q^2 r^2 + 504MQ^4 r - 117Q^6) ,\end{aligned} \tag{6.5d}$$

[15]Here we have absorbed the factor of κ^2 in the log term into the definition of μ.

$$C_\theta{}^\theta = (30240\pi^2 m^2 r^{12})^{-1}(-945M^2r^4 + 2202M^3r^3 - 486Q^2r^4 + 4884MQ^2r^3 -3044Q^4r^2 - 9909M^2Q^2r^2 + 10356MQ^4r - 3066Q^6) , \quad (6.5e)$$

$$D_\theta{}^\theta = (720\pi^2 m^2 r^{12})^{-1}(432M^2r^4 - 1008M^3r^3 - 1176MQ^2r^3 + 832Q^4r^2 +3276M^2Q^2r^2 - 3408MQ^4r + 1053Q^6) . \quad (6.5f)$$

For numerical computations of $\langle T_{\mu\nu}\rangle_{\rm ren}$ in Reissner-Nordström spacetimes it is useful to scale the mass M of the black hole out of the mode equation. This can be accomplished by defining the scaled variables

$$s \equiv \frac{(r-r_+)}{M}, \quad s_\pm \equiv 1 \pm (1-q^2)^{1/2}, \quad q \equiv \frac{|Q|}{M}, \quad \tilde{\omega} \equiv \omega M, \quad \tilde{m} \equiv mM . \quad (6.6)$$

The surface gravity of the black hole is then given by

$$\kappa = \frac{s_+ - 1}{M s_+^2} . \quad (6.7)$$

In terms of these new variables the mode equation becomes

$$s(s+s_+-s_-)^2 \frac{d^2 S_{\tilde{\omega}l}}{ds^2} + (2s+s_+-s_-)(s+s_+-s_-)\frac{dS_{\tilde{\omega}l}}{ds} - \left[\frac{\tilde{\omega}^2}{s}(s+s_+)^4 + l(l+1)(s+s_+-s_-) + \tilde{m}^2(s+s_+)^2(s+s_+-s_-)\right] S_{\tilde{\omega}l} = 0 . \quad (6.8)$$

As discussed in paper I, the $p_{\omega l}$ modes are regular on the event horizon, $s = 0$, and the $q_{\omega l}$ modes are regular at $s = \infty$. Errors in the numerical integrations of the mode equations are minimized by integrating $p_{\omega l}$ from small to large values of s and integrating $q_{\omega l}$ from large to small values of s. This is because numerical errors result in effectively adding a small amount of the $q_{\omega l}$ mode to the $p_{\omega l}$ mode and vice versa. As you integrate to larger values of s, the $p_{\omega l}$ modes grow while the $q_{\omega l}$ modes are damped. The opposite happens as you integrate to smaller values of s.

After the numerical integrations the modes are normalized using Eq. (2.7). Power series solutions for $p_{\omega l}$ which can be used as starting values for the numerical integrations were given in paper I. An asymptotic series for $q_{\omega l}$ valid at large s was also given in paper I. An alternative way to obtain starting values for the $q_{\omega l}$ modes at large s is to use the WKB approximation to estimate the ratio $q'_{\omega l}/q_{\omega l}$. So long as the starting values have the correct ratio, any starting value can be used for $q_{\omega l}$ since the normalization of the modes is done after the integration of the mode equation. Typically fairly large values of s are necessary to obtain accurate starting values for the $q_{\omega l}$ modes.

Also shown in paper I was the fact that for massless fields in Reissner-Nordström spacetimes the modes $p_{0l}(s)$ and $q_{0l}(s)$ are equal to the Legendre functions $P_l(x)$ and $Q_l(x)$, respectively, where $x \equiv 2s(s_+-s_-)^{-1}+1$.[16] These modes occur in the $n = 0$ terms of $\langle T_{\mu\nu}\rangle_{\rm numeric}$. Using the identities derived by Howard [6] for the sums over l of various combinations of $P_l(x)$, $Q_l(x)$ and their derivatives, one can easily show that the $n = 0$ contributions to $\langle T_{\mu\nu}\rangle_{\rm numeric}$ are identically zero.

We have computed $\langle T_{\mu\nu}\rangle$ for both massless and massive scalar fields in spacetimes with various values of $q = |Q|/M$. Some of our results for massless fields are shown in Figs. 1–6. At large values of s the analytic approximation predicts that the magnitude of a component

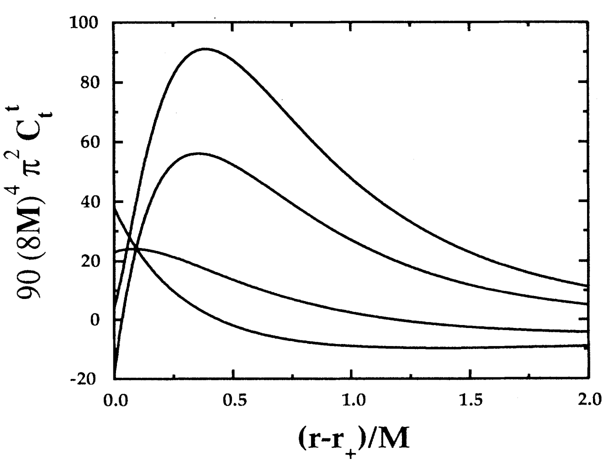

FIG. 1. The curves in this figure display the values of $C_t{}^t$ for massless scalar fields around Reissner-Nordström black holes with $|Q|/M = 0$, 0.8, 0.99, 0.95 from top to bottom at the event horizon, $r = r_+$.

[16]There is an error in the equation defining x in paper I. The correct definition is given here.

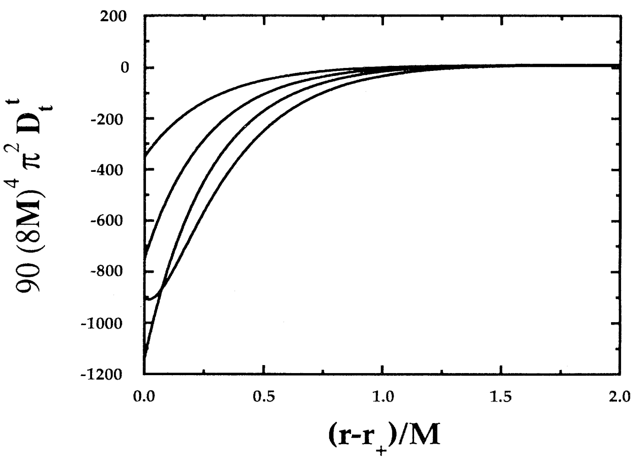

FIG. 2. The curves in this figure display the values of $D_t{}^t$ for massless scalar fields around Reissner-Nordström black holes with $|Q|/M = 0$, 0.8, 0.99, 0.95 from top to bottom at the event horizon, $r = r_+$.

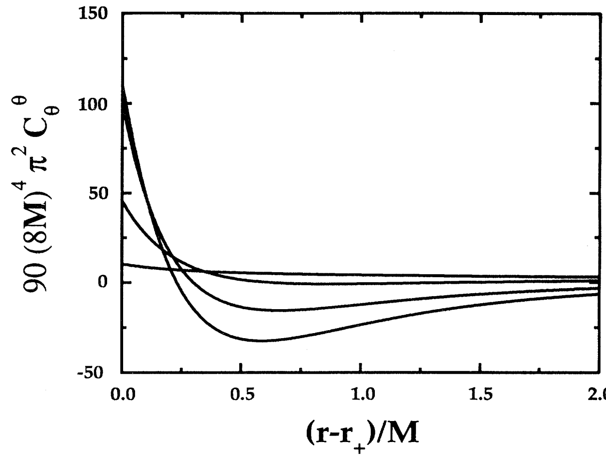

FIG. 5. The curves in this figure display the values of $C_\theta{}^\theta$ for massless scalar fields around Reissner-Nordström black holes with $|Q|/M = 0$, 0.8, 0.95, 0.99 from top to bottom at $r = r_+ + 2M$.

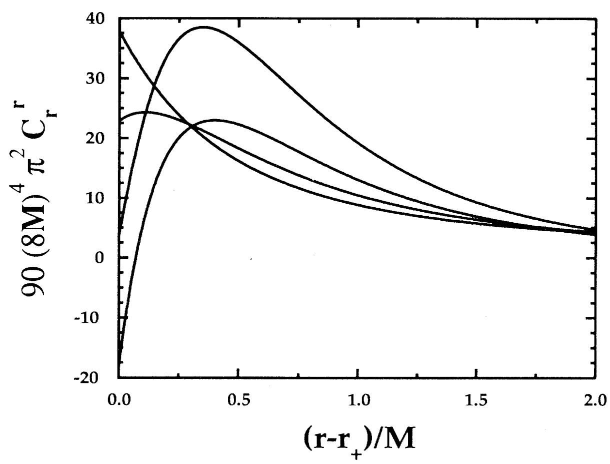

FIG. 3. The curves in this figure display the values of $C_r{}^r$ for massless scalar fields around Reissner-Nordström black holes with $|Q|/M = 0$, 0.8, 0.99, 0.95 from top to bottom at the event horizon, $r = r_+$.

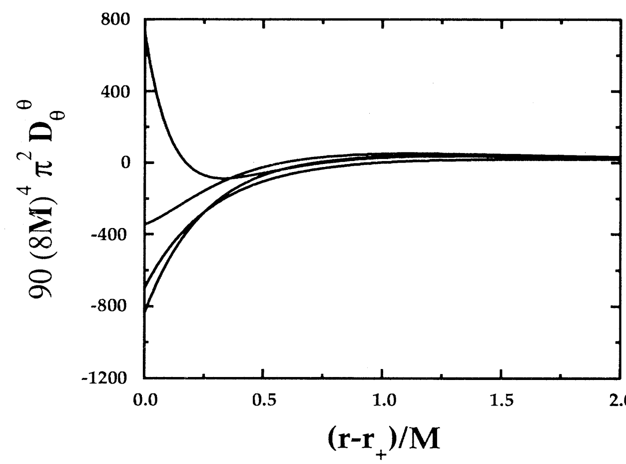

FIG. 6. The curves in this figure display the values of $D_\theta{}^\theta$ for massless scalar fields around Reissner-Nordström black holes with $|Q|/M = 0.99$, 0.95, 0, 0.8 from top to bottom at the event horizon, $r = r_+$.

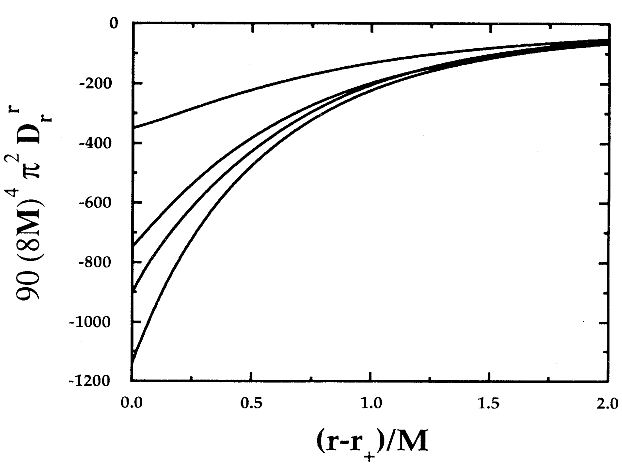

FIG. 4. The curves in this figure display the values of $D_r{}^r$ for massless scalar fields around Reissner-Nordström black holes with $|Q|/M = 0$, 0.8, 0.99, 0.95 from top to bottom at the event horizon, $r = r_+$.

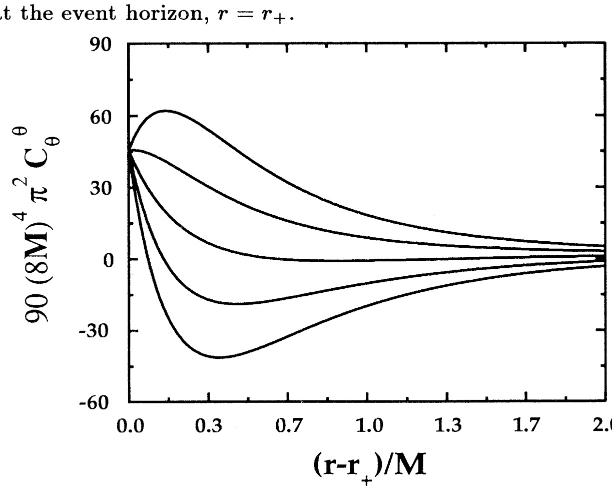

FIG. 7. The curves in this figure illustrate the dependence of $C_\theta{}^\theta$ on the arbitrary parameter μ for massless scalar fields around a Reissner-Nordström black hole with $|Q|/M = 0.8$. From top to bottom the curves are for $\mu M = 0.01$, 0.1, 0, 10, 100.

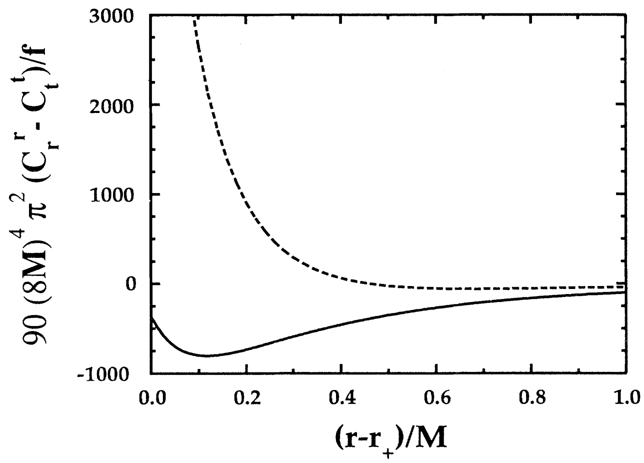

FIG. 8. The curves in this figure display the values of the quantity $(C_r{}^r - C_t{}^t)/f$ for massless scalar fields around a Reissner-Nordström black hole with $|Q|/M = 0.99$. This combination of components is related to the energy density observed by a freely falling observer. The solid curve corresponds to the numerically computed values and the dashed curve to those from the analytical approximation. The analytical approximation diverges logarithmically as $r \to r_+$, but the numerical computation shows no sign of such a divergence.

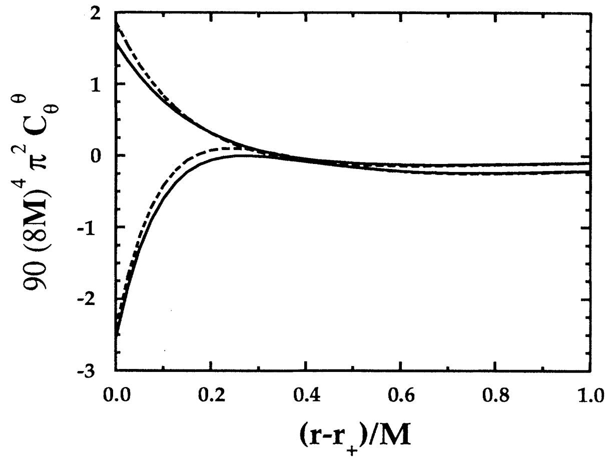

FIG. 10. The curves in this figure display the values of $C_\theta{}^\theta$ for massive scalar fields with $m = 2/M$ around Reissner-Nordström black holes with $|Q|/M = 0$, 0.95 from top to bottom at the event horizon, $r = r_+$. The solid lines correspond to the numerical computations and the dashed lines to the DeWitt-Schwinger approximation.

of the stress-energy tensor decreases with increasing q. At intermediate values of s the opposite tendency holds as can be seen in the plots. Near the event horizon many of the components are rather complicated functions of q. In this region the magnitude of a component may first increase, then decrease and sometimes increase yet again as q increases. Or, the magnitude may first decrease and later increase with increasing values of q, and so forth. The plots illustrate this behavior to some extent. However, we would need to provide two or three times as many plots to illustrate the detailed behavior of the components as functions of q.

From Eqs. (4.3) and (4.4j) it is seen that C_μ^ν is a function of the arbitrary constant μ which is discussed in

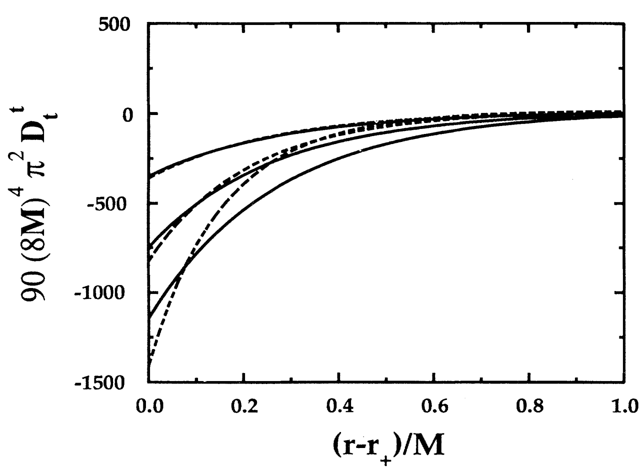

FIG. 9. The analytical approximation is compared with the numerical computation of the quantity $D_t{}^t$ for massless scalar fields around Reissner-Nordström black holes with $|Q|/M = 0$, 0.8, 0.95 from top to bottom at the event horizon $r = r_+$. The solid lines correspond to the numerical computations and the dashed lines to the analytical approximation. Note that the analytical approximation gets progressively worse as $|Q|/M$ increases.

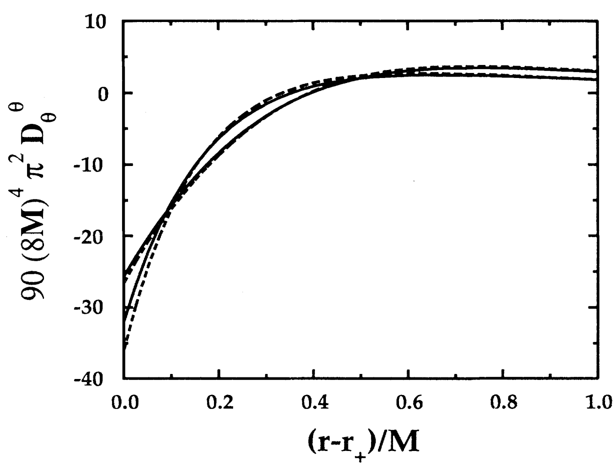

FIG. 11. The curves in this figure display the values of $D_\theta{}^\theta$ for massive scalar fields with $m = 2/M$ around Reissner-Nordström black holes with $|Q|/M = 0.95$, 0 from top to bottom at the event horizon, $r = r_+$. The solid lines correspond to the numerical computations and the dashed lines to the DeWitt-Schwinger approximation.

Sec. III. The value of this constant has a significant effect on the components of C_μ^ν for intermediate values of s. This is illustrated in Fig. 7 which shows C_θ^θ for various values of μ. In all other figures, the value $\mu = 1/M$ has been used to generate the data illustrated.

From Eqs. (6.4a) and (6.4c) it can be seen that the analytical approximation predicts that a logarithmic divergence exists at the event horizon for the combination of components $(C_r^r - C_t^t)/(-g_{tt})$, which is related to the energy density measured by a freely falling observer. This divergence exists for any nonzero value of q. If such a divergence actually existed in the physical stress-energy (as opposed to the approximation) then this would be very important because it would imply that quantum effects are large near the event horizon of any charged black hole, no matter how small its charge is. In fact when we compute this quantity numerically we find no such logarithmic divergence. An illustration is given in Fig. 8 for the case $q = 0.99$. In this plot both the analytical approximation and the numerically computed component are shown. The divergence in the analytical approximation can be clearly seen. However, no such divergence is apparent for the numerically computed component in this or any other case we have examined for $q < 1$. From this result we clearly see that the analytical approximation is not valid for this component near the event horizon if $q > 0$.

For the other nonzero components of both $C_{\mu\nu}$ and $D_{\mu\nu}$ the analytic approximation is fairly accurate both near and away from the event horizon for small values of q. It becomes progressively worse as q increases. These properties are illustrated in Fig. 9 where plots of the numerically computed values and analytic approximations for $D_t{}^t$ for the cases $q = 0$, 0.8, 0.95 are shown.[17]

Some of our results for massive fields are shown in Figs. 10 and 11. For all components we find that for $m \gtrsim 2/M$, the DeWitt-Schwinger approximation is a good approximation for small and intermediate values of s. Because, as discussed in Appendix G, the DeWitt-Schwinger approximation is independent of the state of the field, it is not a good approximation at large values of s where temperature-dependent terms dominate all nonzero components of $\langle T_{\mu\nu}\rangle$. Since the DeWitt-Schwinger approximation is constructed from local geometrical quantities, it will always be regular on the event horizon. Our numerical computations also indicate that all components of $\langle T_{\mu\nu}\rangle$ are regular on the event horizon for the case of massive fields in Reissner-Nordström spacetimes.

[17]Similar plots can be made for the nonzero components of $C_{\mu\nu}$. However in this case the value of μ in Eqs. (6.4a), (6.4c), and (6.4e) need not be the same as the value of μ used for the numerical computations. See, for example, the discussion near the beginning of Sec. V. Thus one should use the value of μ in the analytical approximation which gives the best fit to the numerically computed curves.

ACKNOWLEDGMENTS

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APPENDIX A: ANGULAR DERIVATIVES OF $G_E(x, x')$

In this appendix we compute the first and second angular derivatives of G_E which occur in Eq. (2.4). From the exact expression for G_E in Eq. (2.5) one sees that the only dependence on the angular variables occurs in the factor $P_l(\cos\gamma)$. Thus it suffices to consider derivatives of $P_l(\cos\gamma)$.

The derivatives of G_E in Eq. (2.4) must be taken assuming a general point separation. After this is done the point separation of interest may be chosen. We are using the separation $\epsilon = t - t', r = r', \theta = \theta'$, and $\phi = \phi'$. With this separation $\cos\gamma = 1$. From Eq. (2.4) it is seen that all possible combinations of partial derivatives with respect to θ and θ', ϕ and ϕ' occur to second order. In the limit $\theta' \to \theta$ and $\phi' \to \phi$ all of the first partial derivatives of $P_l(\cos\gamma)$ vanish. Thus we will only need the quantity $P_l'(1)$. It is easily obtained from the following integral representation for $P_l(z)$ [24]:

$$P_l(z) = \frac{1}{\pi}\int_0^\pi d\phi[z + (z^2-1)^{1/2}\cos\phi]^l \ . \tag{A1}$$

Taking the derivative with respect to z, expanding the integrand about the point $z = 1$, computing the integral and taking the limit $z \to 1$ yields

$$P_l'(1) = \frac{1}{2}l(l+1) \ . \tag{A2}$$

The second partial derivatives of $P_l(\cos\gamma)$ in the limit $\theta' \to \theta$ and $\phi' \to \phi$ are then found to be

$$\begin{aligned}\frac{\partial^2 P_l(\cos\gamma)}{\partial\theta^2} &= \frac{\partial^2 P_l(\cos\gamma)}{\partial\theta'^2} = -\frac{\partial^2 P_l(\cos\gamma)}{\partial\theta\partial\theta'} = -\frac{1}{2}l(l+1) \ ,\\ \frac{\partial^2 P_l(\cos\gamma)}{\partial\phi^2} &= \frac{\partial^2 P_l(\cos\gamma)}{\partial\phi'^2} = -\frac{\partial^2 P_l(\cos\gamma)}{\partial\phi\partial\phi'}\\ &= -\frac{1}{2}l(l+1)\sin^2\theta \ .\end{aligned} \tag{A3}$$

APPENDIX B: EXPANSION OF σ AND ITS DERIVATIVES IN POWERS OF ϵ

In this appendix we outline Christensen's method [20] of expanding the quantity of $\sigma(x, x')$ and its derivatives in powers of $\epsilon^\mu \equiv x^\mu - x^{\mu'}$. The procedure is to first expand $\sigma^\alpha(x, x')$ in powers of ϵ^μ and then to compute σ using the relationship $\sigma = \sigma^\alpha\sigma_\alpha/2$. This works for an arbitrary separation of points. For the separation we are interested in $\epsilon^\mu = \epsilon\delta_t{}^\mu$, with $\epsilon \equiv t - t'$. To renormalize $\langle T_{\mu\nu}\rangle$ it is necessary to compute σ to fifth order in ϵ.

$\sigma^\alpha(x, x')$ is a vector at x and a scalar at x'. This means that it can be expanded in a Taylor series about the point $x' = x$. If we use the notation

$$[\sigma^\alpha] \equiv \lim_{x' \to x} \sigma^\alpha(x, x') \ , \tag{B1}$$

then we can write

$$\sigma^\mu(x,x') = [\sigma^\mu] - [\sigma^\mu{}_{,\alpha'}]\epsilon^\alpha + \frac{1}{2!}[\sigma^\mu{}_{,\alpha'\beta'}]\epsilon^\alpha\epsilon^\beta + \cdots \ . \tag{B2}$$

The first step is to convert the partial derivatives to covariant derivatives by using the definition of the covariant derivative. For example, for a scalar function $A(x)$:

$$\begin{aligned} A_{,\alpha} &= A_{;\alpha} \ , \\ A_{,\alpha\beta} &= A_{;\alpha,\beta} = A_{;\alpha\beta} + \Gamma^\rho{}_{\alpha\beta}A_{;\rho} \ . \end{aligned} \tag{B3}$$

As pointed out by Christensen, the conversion of higher-order partial derivatives to covariant derivatives is greatly simplified for the case at hand by the fact that partial derivatives with respect to time of the connection $\Gamma^\mu{}_{\rho\sigma}$ vanish. This is because the metric is independent of time. The only derivatives of the connection that occur for our choice of point splitting are with respect to time.

After changing the partial derivatives in (B2) to covariant derivatives, one is left with a power series in ϵ which contains terms with coincidence limits of $\sigma^\alpha(x,x')$ and its covariant derivatives at x'. The next step is to use Synge's theorem [25,19,20] to convert covariant derivatives at x' to covariant derivatives at x. The theorem states that

$$\begin{aligned} [T_{\alpha_1\cdots\alpha_n\beta'_1\cdots\beta'_m;\mu'}] &= -[T_{\alpha_1\cdots\alpha_n\beta'_1\cdots\beta'_m;\mu}] \\ &\quad + [T_{\alpha_1\cdots\alpha_n\beta'_1\cdots\beta'_m}]_{;\mu} \ . \end{aligned} \tag{B4}$$

The coincidence limits of $\sigma^\alpha(x,x')$ and its covariant derivatives at x must now be evaluated. These limits have been computed by Christensen [19,20]. He finds

$$\begin{aligned} [\sigma^\mu] &= 0 \ , \\ [\sigma^\mu{}_{;\nu}] &= g^\mu{}_\nu \ , \\ [\sigma^\mu{}_{;\nu\sigma}] &= 0 \ , \\ [\sigma^\mu{}_{;\nu\sigma\tau}] &= S^\mu{}_{\nu\sigma\tau} = -\frac{1}{3}(R^\mu{}_{\sigma\nu\tau} + R^\mu{}_{\tau\nu\sigma}) \ , \\ [\sigma^\mu{}_{;\nu\sigma\tau\rho}] &= \frac{3}{4}(S^\mu{}_{\nu\sigma\tau;\rho} + S^\mu{}_{\nu\tau\rho;\sigma} + S^\mu{}_{\nu\rho\sigma;\tau}) \ . \end{aligned} \tag{B5}$$

In general a coincidence term with five covariant derivatives of σ^μ occurs in the expansion at order ϵ^5. However for the point separation we use this term is equal to zero.

After a substantial amount of algebra one finds that in a static spherically symmetric spacetime with the point separation we have chosen,

$$\begin{aligned} \sigma^\mu(x,x') &= g^\mu_t\epsilon - \frac{1}{2}\Gamma^\mu{}_{tt}\epsilon^2 + \frac{1}{6}\Gamma^\rho{}_{tt}\Gamma^\mu{}_{\rho t}\epsilon^3 \\ &\quad - \frac{1}{24}\Gamma^\rho{}_{tt}(R^\mu{}_{t\rho t} + \Gamma^\tau{}_{\rho t}\Gamma^\mu{}_{\tau t})\epsilon^4 \\ &\quad - \frac{1}{120}[-2\Gamma^\mu{}_{rt}\Gamma^r{}_{tt}R^r{}_{trt} + (\Gamma^r{}_{tt})^2R^\mu{}_{rtr} \\ &\quad - (\Gamma^r{}_{tt})^2\Gamma^t{}_{rt}\Gamma^\mu{}_{rt}]\epsilon^5 \ . \end{aligned} \tag{B6}$$

APPENDIX C: EXPANSION OF $g^\mu{}_{\nu'}$ IN POWERS OF ϵ

In this appendix we use Howard's method [6] to expand the bivectors of parallel transport $g^\mu{}_{\nu'}$ in powers of $\epsilon = t - t'$. This method exploits the fact that $g^\mu{}_{\nu'}$ parallel transports vectors from x' to x, so that

$$A^\mu = g^\mu{}_{\nu'}A^{\nu'} \ . \tag{C1}$$

Another way to parallel transport a vector from x' to x is to find its components in terms of an orthonormal tetrad at x', defined by a set of orthonormal basis vectors $\{e_a\}$ and a dual basis of orthonormal one-forms $\{\omega^a\}$. The orthonormal tetrad components at x may then be obtained by parallel transporting the tetrad from x' to x. Thus the components of the vector in terms of a tetrad with the same orientation at x are also known. From this knowledge the components of the vector in a coordinate basis at the point x can be determined. In mathematical notation the components of a vector in the orthonormal frame at x' are

$$A^a = (\omega^a)_{\nu'}A^{\nu'} \ , \tag{C2}$$

where $(\omega^a)_{\nu'}$ are the components of the basis one-form ω^a in the coordinate system at x'. The components in the coordinate frame at x are then

$$A^\mu = (e_a)^\mu A^a = \eta^{ab}(e_a)^\mu(e_b)_{\nu'}A^{\nu'} \ , \tag{C3}$$

where $(e^a)_\mu$ are the components of the orthonormal basis vector e_a in the coordinate system at x. Thus we have

$$g^{\mu\nu'} = \eta^{ab}(e_a)^\mu(e_b)^{\nu'} \ . \tag{C4}$$

In our case σ^μ is in the (r,t) plane so, following Howard, we choose two basis vectors of our tetrad along the two-sphere coordinate directions, and the timelike basis vector to lie along the tangent to the connecting geodesic in the r-t plane, i.e.,

$$\begin{aligned} e_0{}^\mu &= \frac{\sigma^\mu}{(-\sigma_\alpha\sigma^\alpha)^{1/2}} \ , \\ e_2{}^\mu &= \frac{\delta^\mu_\theta}{r} \ , \\ e_3{}^\mu &= \frac{\delta^\mu_\phi}{r\sin\theta} \ . \end{aligned} \tag{C5}$$

The fourth leg of the tetrad is determined by orthonormality; its nonzero components in the coordinate frame are

$$e_1{}^t = \left(\frac{h}{f}\right)^{1/2} \frac{\sigma^r}{(-\sigma_\alpha\sigma^\alpha)^{1/2}} \ , \tag{C6}$$

$$e_1{}^r = \left(\frac{f}{h}\right)^{1/2} \frac{\sigma^t}{(-\sigma_\alpha\sigma^\alpha)^{1/2}} \ . \tag{C7}$$

We define a similar tetrad at x' using $\sigma^{\mu'}$. In Appendix B an expression for $\sigma^\mu(x,x')$ was derived as an expansion in powers of $\epsilon = t-t'$. One can define an identical expansion for $\sigma^{\mu'}(x,x')$ in powers of $\epsilon' \equiv t'-t = -\epsilon$. Note that since $r' = r$ in our point separation scheme the coefficients of these expansions are identical. Examination of Eq. (3.2) shows that the following relationships hold:

$$\begin{aligned}\sigma^{t'}(r,\epsilon') &= -\sigma^{t}(r,\epsilon) \ ,\\ \sigma^{r'}(r,\epsilon') &= \sigma^{r}(r,\epsilon) \ ,\\ [-\sigma_{\alpha'}(r,\epsilon')\sigma^{\alpha'}(r,\epsilon')]^{1/2} &= -[-\sigma_{\alpha}(r,\epsilon)\sigma^{\alpha}(r,\epsilon)]^{1/2} \ .\end{aligned} \tag{C8}$$

Spherical symmetry and the uniqueness of the geodesic in the r-t plane connecting x and x' guarantee that the tetrad at x' is simply the tetrad at x parallel transported to x'.

Using the relationships in (C7) we find that

$$\begin{aligned} g^{tt'} &= -\left[(\sigma^t)^2 + \frac{h}{f}(\sigma^r)^2\right](-\sigma_\alpha\sigma^\alpha)^{-1} \ ,\\ g^{tr'} &= -g^{rt'} = 2\sigma^t\sigma^r(-\sigma_\alpha\sigma^\alpha)^{-1} \ ,\\ g^{rr'} &= \left[(\sigma^r)^2 + \frac{f}{h}(\sigma^t)^2\right](-\sigma_\alpha\sigma^\alpha)^{-1} \ . \end{aligned} \tag{C9}$$

Using Eq. (3.2) we can expand these expressions in powers of ϵ. The results are given in Eq. (3.4).

APPENDIX D: DERIVATION OF $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$

In this appendix we describe the derivation of the quantity $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$. To account for all of the divergences in $\langle T_{\mu\nu}\rangle_{\rm unren}$ a fourth-order WKB expansion must be used. A simple way to keep track of the order of the WKB expansion is to multiply various terms of Eq. (2.9) by a dimensionless parameter α so that the Ω^2 term is $O(\alpha^0)$, the V_1 term is $O(\alpha)$ and the other terms on the right-hand side are $O(\alpha^2)$. Then Eq. (2.9) is solved by iteration to a given order in α and the results are substituted into the quantities A_1 through A_5 in Eq. (2.13). Terms up to the given order in α are retained. At the end of the calculation α is set equal to one.

To compute $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ one first substitutes the WKB expansion for the modes into Eq. (2.13) and retains terms up to fourth order. When this has been done, the sums over l in A_1 though A_5 are all of the general form

$$L_{jk} = \sum_{l=0}^{\infty}\left(\frac{2\left(l+\frac{1}{2}\right)^{1+2j}}{\left[\omega^2 + m^2 f + \left(l+\frac{1}{2}\right)^2\frac{f}{r^2}\right]^{k/2}} - \text{subtraction terms}\right), \tag{D1}$$

where j is a nonnegative integer and k is an odd integer. There is a simple way to determine which subtraction terms go with which sum. Simply expand the function to be summed in inverse powers of l and truncate the expansion at $O(l^0)$. These are the terms which must be subtracted to make the sum finite.

The sums are next computed in the large ω limit. One way to do this is to use the Plana sum formula [21] which says that for a function $g(k)$

$$\sum_{j=k}^{\infty} g(j) = \frac{1}{2}g(k) + \int_k^\infty g(\tau)d\tau + i\int_0^\infty \frac{dt}{e^{2\pi t}-1}[g(k+it) - g(k-it)] \ . \tag{D2}$$

For sums of the form (D1) it is possible to compute the first two terms on the right-hand side in (D2) exactly. After doing so one expands them in inverse powers of ω. It is not, in general, possible to compute the last term in Eq. (D2) exactly. However, it can be computed approximately in the large ω limit by expanding the integrand in inverse powers of ω. The integrals for each term in the expansion can then be computed analytically [24].

For example, consider the sum

$$L_{10} = \sum_{l=0}^{\infty}\left[\frac{2\left(l+\frac{1}{2}\right)}{\left[\omega^2 + m^2 f + \left(l+\frac{1}{2}\right)^2\frac{f}{r^2}\right]^{1/2}} - \frac{2r}{f^{1/2}}\right] . \tag{D3}$$

Substituting (D3) into (D2), computing the first two terms on the right in (D2) analytically and expanding the third in inverse powers of ω gives

$$\begin{aligned} L_{10} &= \frac{1}{2\left(\omega^2 + m^2 f + \frac{f}{4r^2}\right)^{1/2}} - \frac{2r^2}{f}\left(\omega^2 + m^2 f + \frac{f}{4r^2}\right)^{1/2} - \frac{4}{\omega}\int_0^\infty dt\frac{t}{e^{2\pi t}-1} + \cdots \\ &= -\frac{2r^2}{f}\omega + \left(\frac{1}{12} - m^2r^2\right)\frac{1}{\omega} + O(\omega^{-3}) \ . \end{aligned} \tag{D4}$$

In this way each of the mode sums in the WKB expansions for A_1 through A_5 can be expressed as a series in inverse powers of ω. These approximations for A_1 through A_5 are then substituted into Eqs. (2.12a)–(2.12d), and only terms of order ω^{-1} or lower are retained. Terms of order ω^{-3} and higher are not ultraviolet divergent.[18]

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[18]An approximation to $\langle T_{\mu\nu}\rangle_{\rm numerical}$ can be obtained by carrying the series out to higher inverse powers of ω and substituting the result into $\langle T_{\mu\nu}\rangle_{\rm unren}$ in Eq. (4.1). The same is true for $\langle\phi^2\rangle$. This has been done by Howard and Candelas [6,7] for the conformally invariant scalar field in Schwarzschild spacetime. For that case the approximation worked well at large values of r but rather poorly near the event horizon. One problem with this procedure is that the approximation depends on an arbitrary lower limit cutoff that must be placed on the sum over n or the integral over ω. It is important to note that for the approximation to be consistent, higher orders in the WKB expansion must be used. For example to go to $O(\omega^{-3})$ a fourth-order WKB expansion for $\langle\phi^2\rangle$ and a sixth-order WKB expansion for $\langle T_{\mu\nu}\rangle$ are necessary.

APPENDIX E: COMPUTATION OF THE MODE SUMS IN $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$

To calculate $\langle T_{\mu\nu}\rangle_{\rm analytic}$ it is necessary to compute the various sums or integrals over ω in $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$ and then to expand them in inverse powers of ϵ. The results depend on whether the scalar field is in a zero- or a nonzero-temperature state. For the nonzero-temperature state Howard [6] has computed all of the relevant sums except the first one displayed below. It can be found in Ref. [24]. Here $\kappa \equiv 2\pi T$ and $\epsilon_\tau \equiv \tau - \tau'$. Note that[19] the final answer in each expression is in terms of $\epsilon \equiv t - t'$:

$$\begin{aligned}
\kappa \sum_{n=1}^{\infty} \frac{\cos(n\kappa\epsilon_\tau)}{n\kappa} &= \frac{1}{2}\ln\left(\frac{1}{2[1-\cos(\kappa\epsilon_\tau)]}\right) \\
&= -\frac{1}{2}\ln(-\kappa^2\epsilon^2) + O(\epsilon^2) \ , \\
i\kappa \sum_{n=1}^{\infty} \sin(n\kappa\epsilon_\tau) &= i\frac{\kappa}{2}\cot\left(\frac{\kappa\epsilon_\tau}{2}\right) \\
&= \frac{1}{\epsilon} + O(\epsilon) \ , \\
\kappa \sum_{n=1}^{\infty} n\kappa \cos(n\kappa\epsilon_\tau) &= \frac{d}{d\epsilon_\tau}\kappa \sum_{n=1}^{\infty} \sin(n\kappa\epsilon_\tau) \\
&= \frac{1}{\epsilon^2} - \frac{\kappa^2}{12} + O(\epsilon^2) \ , \\
i\kappa \sum_{n=1}^{\infty} n^2\kappa^2 \sin(n\kappa\epsilon_\tau) &= -i\frac{d}{d\epsilon_\tau}\kappa \sum_{n=1}^{\infty} n\kappa\cos(n\kappa\epsilon_\tau) \\
&= \frac{2}{\epsilon^3} + O(\epsilon) \ , \\
\kappa \sum_{n=1}^{\infty} n^3\kappa^3 \cos(n\kappa\epsilon_\tau) &= \frac{d}{d\epsilon_\tau}\kappa \sum_{n=1}^{\infty} n^2\kappa^2\sin(n\kappa\epsilon_\tau) \\
&= \frac{6}{\epsilon^4} + \frac{\kappa^4}{120} + O(\epsilon^2) \ . \qquad \text{(E1)}
\end{aligned}$$

As mentioned in Sec. IV, we need to impose an infrared cutoff on the terms of order $1/\omega$. This has been implicitly done in Eq. (E1) by dropping the $n = 0$ term. In the other sums in Eq. (E1) the $n = 0$ term vanishes so it makes no difference whether or not we include it.

For the zero-temperature case we impose an arbitrary lower limit cutoff λ on the integral whose integrand is proportional to $1/\omega$. The other integrals do not require such a cutoff as their integrands vanish at $\omega = 0$. For the $1/\omega$ integral we find

[19] There is an imaginary contribution from the log terms in the following expressions. This is a reflection of the fact that when the Euclidean Green's function is analytically continued back to the Lorentzian sector it is proportional to the Feynman Green's function which has both real and imaginary parts. Recall that in Eq. (2.2) we took the real part of the Euclidean Green's function and its derivatives. Thus, for the purposes of point splitting, one can simply replace the $\log(-\epsilon^2)$ terms by $\log(\epsilon^2)$ in the following equations.

$$\begin{aligned}
\int_\lambda^\infty \frac{d\omega}{\omega}\cos(\omega\epsilon_\tau) &= -ci(\lambda\epsilon_\tau) \\
&= -\frac{1}{2}\ln(-\lambda^2\epsilon^2) - C + O(\epsilon^2) \ , \qquad \text{(E2)}
\end{aligned}$$

where C is Euler's constant.

The rest of the integrals do not appear to be well defined. However they can be evaluated by taking derivatives of Eq. (E2) with respect to ϵ_τ. For example,

$$\begin{aligned}
i\int_0^\infty d\omega \sin(\omega\epsilon_\tau) &= \lim_{\lambda\to 0}\left(-i\frac{d}{d\epsilon_\tau}\int_\lambda^\infty \frac{d\omega}{\omega}\cos(\omega\epsilon_\tau)\right) \qquad \text{(E3)} \\
&= \lim_{\lambda\to 0}\left(i\frac{\cos(\epsilon_\tau\lambda)}{\epsilon_\tau}\right) = \frac{1}{\epsilon} \ .
\end{aligned}$$

The results for the other integrals are

$$\begin{aligned}
\int_0^\infty d\omega\, \omega\cos(\omega\epsilon_\tau) &= \frac{1}{\epsilon^2} \ , \\
i\int_0^\infty d\omega\, \omega^2\sin(\omega\epsilon_\tau) &= \frac{2}{\epsilon^3} \ , \qquad \text{(E4)} \\
\int_0^\infty d\omega\, \omega^3\cos(\omega\epsilon_\tau) &= \frac{6}{\epsilon^4} \ .
\end{aligned}$$

APPENDIX F: COMPUTATION OF $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$

In this appendix efficient methods to compute $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$ are described. This quantity is obtained by substituting the WKB expansion into the expressions S_1 through S_5 in Eqs. (4.6a)–(4.6e) and substituting the resulting expressions into Eq. (4.5). A WKB expansion of fourth order or higher is necessary for $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$ to be finite.

The mode sums and integrals in $\langle T_{\mu\nu}\rangle_{\rm WKBfin}$ are of the form

$$S_{ijk} = \int d\tilde{\mu}\left[\omega^{2i}\sum_{l=0}^{\infty}\left(\frac{2\left(l+\frac{1}{2}\right)^{1+2j}}{\left[\omega^2 + m^2 f + \left(l+\frac{1}{2}\right)^2 \frac{f}{r^2}\right]^{k/2}} - \text{subtraction terms}\right) - \text{subtraction terms}\right] . \qquad \text{(F1)}$$

Here $i = 0$ or 1, j is a non-negative integer and k is an odd integer.

The subtraction terms for the sum over l come from the quantities A_1 through A_5 and those for the integral over ω or sum over n come from $\langle T_{\mu\nu}\rangle_{\rm WKBdiv}$. For a given sum S_{ijk} the subtraction terms for the sum over l can be obtained by expanding the function being summed in inverse powers of l and truncating at $O(l^0)$. The subtraction terms for the integral over ω or sum over n can be obtained by using the Plana sum formula to compute the sum over l in the large ω limit in exactly the way described in Appendix D. For many values of i, j, and k there are no subtraction terms.

The computation of the quantities S_{ijk} proceeds differently depending on whether the field is at zero or nonzero temperature. For the zero-temperature case when no subtraction terms are present, the integral over ω is computed first with the result that

$$S_{ijk} = \frac{\pi^{1/2}\Gamma\left(\frac{k-2i-1}{2}\right)}{4\pi^2 2^i \Gamma\left(\frac{k}{2}\right)} \times \sum_{l=0}^{\infty} \frac{\left(l+\frac{1}{2}\right)^{1+2j}}{\left[m^2 f + \left(l+\frac{1}{2}\right)^2 \frac{f}{r^2}\right]^{(k-2i-1)/2}} . \quad \text{(F2)}$$

For a massless field this sum can be computed exactly using the relation

$$\sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right)^{-n} = (2^n - 1)\zeta(n) . \quad \text{(F3)}$$

The result is

$$S_{ijk} = \frac{\pi^{1/2}\Gamma\left(\frac{k-2i-1}{2}\right)}{4\pi^2 2^i \Gamma\left(\frac{k}{2}\right)} \left(\frac{r}{f^{1/2}}\right)^{k-2i-1} \times (2^{k-2i-2j-2} - 1)\zeta(k-2i-2j-2) . \quad \text{(F4)}$$

For a massive field one can compute the sums in (F2) numerically up to a large value $l = L$ such that $L \gg mr$. For $l > L$ the terms to be summed can be expanded in inverse powers of $(l + \frac{1}{2})$. The relation (F3) is then used to compute these terms.

If subtraction terms are present in the zero-temperature case, then the first step is to use the Plana sum formula to compute the sum over l. The first two terms on the right-hand side of (D2) are computed analytically while the third term is left as an integral over t. Next the integrals over ω are computed for each of these terms using an upper limit cutoff Λ. A lower limit cutoff λ is used for all subtraction terms proportional to ω^{-1}. Each term is then expanded in inverse powers of Λ truncating at order Λ^0.

The series for the third term in (D2) will contain some terms for which the integral over t can be computed analytically. There will also be other terms which must be computed numerically. For a massive field the latter must be computed at all the radial points of interest since the r dependence does not factor out of the integral. For a massless field the r dependence does factor out and the integrals can be computed numerically once and for all.

For the nonzero temperature case the evaluation of sums of the form (F1) must be done numerically. It is easiest in this case to break the problem up into the cases $n \le N$ and $n \ge N$ with $N^2 \gg [m^2 f + f/(4r^2)]/\kappa^2$. Recall that $\omega = n\kappa$, with $\kappa \equiv 2\pi T$ if the temperature is greater than zero. For $n \le N$ the sum over l in (F1) is computed numerically up to some large value $l = L$ such that $L^2 \gg n^2\kappa^2 r^2/f + m^2 r^2$. For $l > L$ the terms to be summed can be expanded in inverse powers of $(l + 1/2)$. The relation (F3) is then used to compute these terms.

For $n > N$ the sum over l can be computed in the large ω limit using the method described in Appendix D. However, in this case, higher-order terms of the expansion in inverse powers of ω are kept. The resulting sums over n can then be computed analytically for these terms between $N + 1$ and infinity.

APPENDIX G: COMPUTATION OF $\langle T_{\mu\nu}\rangle_{\text{WKBfin}}$ IN THE LARGE MASS LIMIT

In this appendix the computation of $\langle T_{\mu\nu}\rangle_{\text{WKBfin}}$ in the large mass limit is discussed. The computation involves evaluating sums of the form (F1) in the large mass limit. This is most easily done by first using the Plana sum formula (D2) to compute the sum over l and then expanding the results in inverse powers of the quantity $[\omega^2 + m^2 f + f/(4r^2)]^{1/2}$. The first two terms on the right-hand side of (D2) can be computed exactly. The third term must first be expanded in inverse powers of $[\omega^2 + m^2 f + f/(4r^2)]^{1/2}$, then the integral over t can be computed analytically for each term.

For the zero-temperature case, the resulting integrals over ω can easily be computed and then expanded in inverse powers of m. As an example consider S_{001}. Computing the sum over l using the Plana sum formula as described above yields

$$\begin{aligned} S_{001} = \frac{1}{4\pi^2}\int_0^\infty d\omega \Bigg[& \frac{1}{2\left(\omega^2 + m^2 f + \frac{f}{4r^2}\right)^{1/2}} - \frac{2r^2}{f}\left(\omega^2 + m^2 f + \frac{f}{4r^2}\right)^{1/2} \\ & + i\int_0^\infty \frac{dt}{e^{2\pi t} - 1}\left(\frac{(2it+1)}{\left[\omega^2 + m^2 f + \left(it + \frac{1}{2}\right)^2 \frac{f}{r^2}\right]^{1/2}} - \frac{(-2it+1)}{\left[\omega^2 + m^2 f + \left(-it + \frac{1}{2}\right)^2 \frac{f}{r^2}\right]^{1/2}}\right) \\ & + \frac{2r^2}{f}\omega - \left(\frac{1}{12} - m^2 r^2\right)\frac{1}{\omega}\Bigg] \\ = \frac{1}{4\pi^2}\int_0^\infty d\omega \Bigg[& -\frac{2r^2}{f}\left(\omega^2 + m^2 f + \frac{f}{4r^2}\right)^{1/2} + \frac{1}{3\left(\omega^2 + m^2 f + \frac{f}{4r^2}\right)^{1/2}} + \frac{f}{30r^2\left(\omega^2 + m^2 f + \frac{f}{4r^2}\right)^{3/2}} \\ & + \frac{f^2}{105r^4\left(\omega^2 + m^2 f + \frac{f}{4r^2}\right)^{5/2}} + \cdots + \frac{2r^2}{f}\omega - \left(\frac{1}{12} - m^2 r^2\right)\frac{1}{\omega}\Bigg] . \end{aligned} \quad \text{(G1)}$$

Computing the integrals over ω and taking the large mass limit yields the following asymptotic expression

$$S_{001} \approx \frac{1}{8\pi^2} m^2 r^2 \left[-1 + \ln\left(\frac{m^2 f}{4\lambda^2} \right) \right] - \frac{1}{96\pi^2} \ln\left(\frac{m^2 f}{4\lambda^2} \right) + \frac{7}{3840\pi^2 m^2 r^2} + \frac{31}{64512\pi^2 m^4 r^4} + O(m^{-6}) \ . \tag{G2}$$

For the nonzero-temperature case, the sum over l is computed as in the zero-temperature case. The resulting $n = 0$ term is expanded in inverse powers of m. The sum over n, beginning with $n = 1$, is evaluated using the Plana sum formula. As usual, the first two terms in (D2) can be evaluated exactly and then expanded in inverse powers of m. The third term must first be expanded in inverse powers of m and then the integral over t can be computed for each term. Note that for large t the third term in (D2) is exponentially damped. This means that the procedure we are following introduces errors which are exponentially damped functions of m. The result for S_{ijk} is somewhat different from that of the zero-temperature case because ω is summed over rather than integrated over. However, when the calculation of the DeWitt-Schwinger approximation for $\langle T_{\mu\nu} \rangle$ is completed, the answer turns out to be exactly the same as that found for the zero-temperature case. This is not surprising since the DeWitt-Schwinger approximation is a local approximation and should not depend on the state of the quantum field.

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