Action and entropy of extreme and nonextreme black holes

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The Hamiltonian actions for extreme and nonextreme black holes are compared and contrasted and a simple derivation of the lack of entropy of extreme black holes is given. In the nonextreme case the wave function of the black hole depends on horizon degrees of freedom which give rise to the entropy. Those additional degrees of freedom are absent in the extreme case.

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It has been recently proposed [1,2] that extreme black holes have zero entropy [3]. The purpose of this paper is to adhere to this claim by providing an economical derivation of it. The derivation also helps to set the result in perspective and to relate it to key issues in the quantum theory of gravitation, such as the Wheeler-DeWitt equation.

The argument is the application to the case of an extreme black hole [4] of an approach to black hole entropy based on the dimensional continuation of the Gauss-Bonnet theorem. The approach in question had been previously applied to nonextreme black holes only [5].

To put into evidence as clearly as possible the distinction between extreme and nonextreme holes, we first perform the analysis for the nonextreme case and then see how it is modified in the extreme case.

We will deal with gravitation theory in a spacetime of dimension D with a positive-definite signature (Euclidean formulation). To present the argument in what we believe is its most transparent form for the purpose at hand, we will start with the Hamiltonian action and will only at the end discuss the connection with the Hilbert action.

For nonextreme black holes the Euclidean spacetimes admitted in the action principle have the topology $\mathbb{R}^2 \times S^{D-2}$. It is useful to introduce a polar system of coordinates in the $\mathbb{R}^2$ factor of $\mathbb{R}^2 \times S^{D-2}$. The reason is that the black hole will have a Killing vector field, the Killing time, whose orbits are circles centered at the horizon. We will take the polar angle in $\mathbb{R}^2$ as the time variable in a Hamiltonian analysis. An initial surface of time t_1 and a final surface of time t_2 will meet at the origin. There is nothing wrong with the two surfaces intersecting. The Hamiltonian can handle that.

The canonical action

$$I_{\text{can}} = \int (\pi^{ij}\dot{g}_{ij} - N\mathcal{H} - N^i\mathcal{H}_i) \tag{1}$$

without any surface terms added can be taken as the action for the wedge between t_1 and t_2 provided the following quantities are held fixed: (i) the intrinsic geometries ${}^{(D-1)}\mathcal{G}_1$, ${}^{(D-1)}\mathcal{G}_2$ of the slices $t = t_1$, and $t = t_2$; (ii) the intrinsic geometry ${}^{(D-2)}\mathcal{G}$ of the S^{D-2} at the origin; (iii) the mass at infinity, with an appropiate asymptotic falloff for the field.

The term "mass" here refers to the conserved quantity associated with the time Killing vector at infinity. It is thus more general than the P^0 of the Poincaré group, which only exists when the spacetime is asymptotically flat. For example when there is a negative cosmological constant this mass is the value of a generator of the anti–de Sitter group.

Note that we have listed the intrinsic geometry of the S^{D-2} as a variable independent from the three-geometries of the slices $t = t_1$ and $t = t_2$. This is because in the variation of the action (1) there is a separate term in the form of an integral over S^{D-2}, which contains the variation of ${}^{(D-2)}\mathcal{G}$.

It should be observed that there will be no solution of the equations of motion satisfying the given boundary conditions if, for example, one fixes the mass at t_2 to be different from the mass at t_1. However in the quantum theory one *can* take $M_1 \neq M_2$, the path integral will then yield a factor $\delta(M_2 - M_1)$ in the amplitude. Similarly there will be no solution of the equations of motion unless the geometry of the S^{D-2} at the origin as approached from the slice $t = t_1$, coincides with the one corresponding to $t = t_2$, and unless that common value also coincides with the one taken for the geometry of the S^{D-2} at the origin. However these precautions need not be taken in the path integral, which will automatically enforce them by yielding appropiate δ functionals. This situation is the same as that arising with the action of a free particle in the momentum representation, where there is no clasical solution unless the initial and the final momenta are equal, but yet, one can (and must) compute the amplitude to go from any initial momentum to any final momentum.

To the action (1) one may add any functional of the quantities held fixed and obtain another action appropiate for the same boundary conditions. In particular one may replace (1) by

$$I = I_{\text{can}} + B[{}^{(D-2)}\mathcal{G}], \tag{2}$$

where $B[{}^{(D-2)}\mathcal{G}]$ is any functional of the $(D-2)$-geometry

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at the origin. If we only look at the wedge $t_1 \le t \le t_2$ then the demand that the action have an extremum when the equations of motion hold does not restrict B at all. However, if we look at the complete spacetime, then that same requirement fixes B uniquely. This is because when one deals with the complete spacetime the slices $t = t_1$ and $t = t_2$ are identified and neither ${}^{(D-1)}\mathcal{G}_1$ nor ${}^{(D-1)}\mathcal{G}_2$ nor ${}^{(D-2)}\mathcal{G}$ are held fixed. Now, unlike its Minkowskian signature continuation, the Euclidean black hole obeys Einstein's equations *everywhere*. Thus it should be an extremum of the action with only the asymptotic data (mass) held fixed. The demand that the action should be such as to have the black hole as an extremum with respect to variations of ${}^{(D-2)}\mathcal{G}$ fixes

$$B = 2\pi A(r_+) \quad \text{(nonextreme case complete spacetime)}\ , \tag{3}$$

where $A(r_+)$ is the area of the S^{D-2} at the origin.

Once one knows B for the complete spacetime one can infer the corresponding B for the wedge. The argument is based on demanding that the trace of the amplitude for the wedge should yield—in the semiclassical approximation—the exponential of the action for the complete spacetime. This yields [6] that B for the wedge is the *negative* of (3).

The way in which (3) arises is the following. First one writes the metric near the origin in "Schwarzschild coordinates" as

$$ds^2 = N^2(r, x^p)dt^2 + N^{-2}(r, x^p)dr^2 + \gamma_{mn}(r, x^p)dx^m dx^n, \tag{4}$$

with

$$(t_2 - t_1)N^2 = 2\Theta(x^p)(r - r_+) + O((r - r_+)^2 i) \quad \text{(nonextreme case)}. \tag{5}$$

Here r and t are coordinates in $\mathbb{R}^2$ and x^p are coordinates in S^{D-2}. The parameter Θ is the total proper angle (proper length divided by proper radius) of an arc of very small radius and coordinate angular opening $t_2 - t_1$ in the $\mathbb{R}^2$ at x^p. For this reason it is called the opening angle. When the sides of the wedge are identified $2\pi - \Theta$ becomes the deficit angle of a conical singularity in $\mathbb{R}^2$.

Next, one evaluates the variation of the canonical action (1) to obtain

$$\delta I_{\text{can}} = -\int_{S^{(D-2)}(r_+)} \Theta(x^p)\delta\gamma^{1/2}(x^p)d^{D-2}x + \beta\delta M + \int \pi^{ij}\delta g_{ij}\,\Big|_1^2 + \text{(terms vanishing on shell)}. \tag{6}$$

Here β is the Killing time separation at infinity.

Last, one observes that when the slices $t = t_1$ and $t = t_2$ are identified, the term $\int \pi^{ij}\delta g_{ij}\,|_1^2$ cancels out. Thus if M and J are kept fixed but $\gamma^{1/2}(x^p)$ is allowed to vary one must add (3) to (1) in order to obtain from the action principle that at the extremum:

$$\Theta(x^p) = 2\pi \quad \text{(complete spacetime, nonextreme case)}. \tag{7}$$

Equation (7) must hold because otherwise there would be a conical singularity at r_+ and Einstein's equations would be violated in the form of a δ-function source at the origin.

Let us now turn to the extreme case. By definition of an extreme black hole the square lapse N^2 has a double root at the origin. Thus one must replace (5) by

$$(t_2 - t_1)N^2 = O((r - r_+)^2) \quad \text{(extreme case)}. \tag{8}$$

This means that one must have

$$\Theta(x^p) = 0 \quad \text{(extreme case)}, \tag{9}$$

instead of (7). It then follows that

$$B = 0 \quad \text{(extreme case)}, \tag{10}$$

so that the canonical action (1) is appropiate *as is* for extreme black holes.

Note that Eq. (8) holds not only for the complete spacetime but also for a wedge of the extreme black hole geometry. This implies that (9) must hold also off shell (for all configurations allowed in the action principle). This is so because for the wedge there would be no way to obtain $\Theta = 0$ by extremizing the action since ${}^{(D-2)}\mathcal{G}$ is held fixed.

The difference between nonextreme and extreme cases has a topological origin. For all Θ's in the interval

$$-\infty < \Theta < +\infty, \tag{11}$$

the topology of the t, r piece of the complete spacetime is that of a disk with the boundary at infinity. When $\Theta \neq 2\pi$ the disk has a conical singularity in the curvature at the origin with deficit angle $2\pi - \Theta$. When $\Theta = 2\pi$ the singularity is absent.

However when $\Theta = 0$ the topology is different. Indeed, what would appear naively to be a source at the origin in the form of a "fully closed cone," as was misunderstood in [5], is really the signal of a spacetime with different topology. As the cone closes, its apex recedes to give rise to the infinite throat of an extreme black hole. Thus the origin is effectively removed from the manifold whose t, r piece is no longer a disk, but rather, an annulus whose inner boundary is at infinite distance.

Now, one wants to include in the action principle fields of a given topology so that one can continuously vary from one to another. Therefore for the complete spacetime of the nonextreme case all fields obeying (11) are allowed so that (7) only holds on shell. On the other hand, for the extreme case–as already remarked in the context of (8)—we must have (9) to also hold off shell. Topologically, this is so since if the origin is removed, there is no place to put a conical singularity.

We reach therefore an important conclusion: we must use a different action for extreme and nonextreme black holes. This means that these two kinds of black holes are to be regarded as drastically different physical objects, much in the same way as a particle of however small but finite mass is drastically different from one of zero mass [7]. The discontinuous jump in the action is just the way that the geometrical theory at hand has to remind us that extreme and non-extreme black holes fall into different topological classes.

The action for the complete spacetime may be rewritten as

$$I = 2\pi\chi A(r_+) + I_{\text{can}}, \tag{12}$$

and Eqs. (5) and (8) may be summarized as

$$(t_2 - t_1)N^2 = 2\chi\Theta(x^p)(r - r_+) + O((r - r_+)^2), \tag{13}$$

where χ is the Euler characteristic of the t, r factor of the complete black hole spacetime. For the nonextreme case one has $\chi = 1$ (disk), and for the extreme case $\chi = 0$ (annulus). Expression (12) had been anticipated in [5], where it emerged naturally from a study of the dimensional continuation of the Gauss-Bonnet theorem, but it was missed there that $\chi = 0$ corresponds to extreme black holes.

If one evaluates the action on the black hole solution one finds

$$I_{\text{can}}(\text{black hole}) = 0, \tag{14}$$

because the black hole is stationary ($\dot{g}_{ij} = 0$) and because the constraint equations $\mathcal{H} = \mathcal{H}_i = 0$ hold. Thus one has

$$I(\text{black hole}) = 2\pi\chi A(r_+). \tag{15}$$

Now, the action (12) is appropiate for keeping M fixed. In statistical thermodynamics this corresponds to the microcanonical ensemble. Hence, for the entropy S in the classical approximation one finds

$$S = (8\pi G\hbar)^{-1} 2\pi\chi A(r_+), \tag{16}$$

where we have restored the universal constants. Thus one sees that extreme black holes ($\chi = 0$) have zero entropy.

A word is now in place about the relation of (12) with the Hilbert action

$$I_H = \frac{1}{2}\int_M \sqrt{g} R d^D x - \int_{\partial M} \sqrt{g} K d^{D-1} x. \tag{17}$$

As was shown in [5], the action (12) for the complete spacetime (17) just differ by a boundary term at infinity, which automatically regulates the divergent functional (17). This assertion is not valid for the wedge. In that case, (2) and (17) differ not only by a boundary term at infinity but also by a boundary term at the origin. For the complete spacetime one has

$$I = I_H - B_\infty, \tag{18}$$

whereas, for the wedge,

$$I = I_H - \pi(2\chi + 1)A(r_+) - B_\infty - \pi A_\infty. \tag{19}$$

For the reasons given above we adopt (19) and not (17) as the action for the wedge.

The discontinuous change in the action between extreme and nonextreme black holes has dramatic consequences for the wave functional of the gravitational field in the presence of a black hole, which one may call for short the wave function of the black hole. Indeed, in the extreme case, the wave function has the usual arguments; namely, it may be taken to depend on the geometry of the spatial section and on the asymptotic time separation β:

$$\Psi = \Psi[{}^{(D-1)}\mathcal{G}, \beta]. \tag{20}$$

The dependence of Ψ on the three geometry is governed by the Wheeler-DeWitt equation

$$\mathcal{H}\Psi = 0, \tag{21}$$

whereas the dependence on the asymptotic time β is governed by the Schrödinger equation

$$\frac{\partial\Psi}{\partial\beta} + M\Psi = 0, \tag{22}$$

where M is the mass as defined by Arnowitt, Deser, and Misner (see, for example, [8]). On the other hand, for the nonextreme case the wave function has an extra argument which may be taken to be the opening angle Θ:

$$\Psi = \Psi[{}^{(D-1)}\mathcal{G}, \beta, \Theta], \tag{23}$$

and one has [9] the Schrödinger equation

$$\frac{\delta\Psi}{\delta\Theta(x)} - \gamma^{1/2}(x)\Psi = 0, \tag{24}$$

whose role at the horizon is analogous to that of (22) at infinity.

The additional horizon degree of freedom canonical pair $(\gamma^{1/2}, \Theta)$ may be regarded as responsible for the black hole entropy in the nonextreme case. Indeed there is no entropy in the extreme case precisely because then the origin is removed, and with it go away $(\gamma^{1/2}, \Theta)$ and the entropy.

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[1] S. W. Hawking, talk at the meeting on "Quantum Concepts in Space and Time," Durham, North Carolina, 1994 (unpublished).
[2] S. W. Hawking, G. T. Horowitz, and S. F. Ross, this issue, Phys. Rev. D **51**, 4302 (1995).
[3] For previous evidence on the lack of thermodynamic behavior of extreme black holes, see, for example, J. Preskill, P. Schwarz, A. Shapere, S. Trivedi, and F. Wilczek, Mod. Phys. Lett. A **6**, 2353 (1991); F. Wilczek, in *Quantum Mechanics of Fundamental Systems IV*, Proceedings of a Meeting held in Santiago, Chile, 1991, edited by C. Teitelboim and J. Zanelli (Plenum, New York, in press); G. W. Gibbons and R. E. Kallosh, "Topology, the Gauss Bonnet Theorem and the Entropy of Dilaton Black Holes," report (unpublished).
[4] C. Teitelboim, talk at the meeting on "Quantum Concepts in Space and Time" [1].
[5] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **72**, 957 (1994). See also C. Teitelboim, in *Proceedings of the Lanczos Centenary Conference*, edited by J. D. Brown, M. T. Chu, D. C. Ellison, and R. J. Plemmons (SIAM, Philadelphia, in press).
[6] C. Teitelboim, "Action Principle, State Counting and Statistical Thermodynamics of Black Holes," report (unpublished).
[7] G. Gibbons (private communication). This analogy is very much to the point. Indeed the number of states of a given spin, and with it the entropy, is also discontinuous ($2s+1$ states for $m \neq 0$ however small, 2 states for $m = 0$). However this jump does not make the observable properties discontinuous as was discussed by L. Bass and E. Schrödinger, Proc. R. Soc. London **232**, 1 (1955).
[8] C. Teitelboim, Phys. Rev. D **28**, 310 (1983).
[9] S. Carlip and C. Teitelboim, "The Off-Shell Black Hole," report (unpublished).