Entropy, area, and black hole pairs

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We clarify the relation between gravitational entropy and the area of horizons. We first show that the entropy of an extreme Reissner-Nordström black hole is *zero*, despite the fact that its horizon has nonzero area. Next, we consider the pair creation of extremal and nonextremal black holes. It is shown that the action which governs the rate of this pair creation is directly related to the area of the acceleration horizon and (in the nonextremal case) the area of the black hole event horizon. This provides a simple explanation of the result that the rate of pair creation of nonextreme black holes is enhanced by precisely the black hole entropy. Finally, we discuss black hole *annihilation*, and argue that Planck scale remnants are not sufficient to preserve unitarity in quantum gravity.

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I. INTRODUCTION

The discovery of black hole radiation [1] confirmed earlier indications [2] of a close link between thermodynamics and black hole physics. Various arguments were given that a black hole has an entropy which is one quarter of the area of its event horizon in Planck units. However, despite extensive discussion, a proper understanding of this entropy is still lacking. In particular, there is no direct connection between this entropy and the "number of internal states" of a black hole.

We will reexamine the connection between gravitational entropy and horizon area in two different contexts. We first consider charged black holes and show that while nonextreme configurations satisfy the usual relation $S = \mathcal{A}_{\mathrm{BH}}/4$, extreme Reissner-Nordström black holes do not. They always have zero entropy even though their event horizon has nonzero area. The entropy changes discontinuously when the extremal limit is reached. We will see that this is a result of the fact that the horizon is infinitely far away for extremal holes which results in a change in the topology of the Euclidean solution.

The second context is quantum pair creation of black holes. It has been known for some time that one can create pairs of oppositely charged monopoles arising in grand unification theories (GUT's) in a strong background magnetic field [3]. The rate for this process can be calculated in an instanton approximation and is given by e^{-I} where I is the Euclidean action of the instanton. For monopoles with mass m and charge q, in a background field B one finds (to leading order in qB) that $I = \pi m^2/qB$. It has recently been argued that charged black holes can similarly be pair created in a strong magnetic field [4–6]. An appropriate instanton has been found and its action computed. The instanton is obtained by starting with a solution to the Einstein-Maxwell equations found by Ernst [7], which describes oppositely charged black holes uniformly accelerated in a background magnetic field. This solution has a boost symmetry which becomes null on an acceleration horizon as well as the black hole event horizon, but is timelike in between. One can thus analytically continue to obtain the Euclidean instanton. It turns out that regularity of the instanton requires that the black holes are either extremal or slightly nonextremal. In the nonextremal case, the two black hole event horizons are identified to form a wormhole in space. It was shown in [6] that the action for the instanton creating extremal black holes is identical to that creating gravitating monopoles [8] (for small qB) while the action for nonextreme black holes is less by precisely the entropy of one black hole $\mathcal{A}_{\mathrm{BH}}/4$. This implies that the pair creation rate for nonextremal black holes is enhanced over that of extremal black holes by a factor of $e^{\mathcal{A}_{\mathrm{BH}}/4}$, which may be interpreted as saying that nonextreme black holes have $e^{\mathcal{A}_{\mathrm{BH}}/4}$ internal states and are produced in correlated pairs, while the extreme black holes have a unique internal state. This was not understood at the time, but is in perfect agreement with our result that the entropy of extreme black holes is zero.

To better understand the rate of pair creation, we relate the instanton action to an energy associated with

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boosts, and surface terms at the horizons. While the usual energy is unchanged in the pair creation process, the boost energy need not be. In fact, we will see that it is changed in the pair creation of nongravitating GUT monopoles. Remarkably, it turns out that it is unchanged when gravity is included. This allows us to derive a simple formula for the instanton action. For the pair creation of nonextremal black holes we find

$$I = -\frac{1}{4}(\Delta\mathcal{A} + \mathcal{A}_{\mathrm{BH}}), \qquad (1)$$

where $\Delta\mathcal{A}$ is the difference between the area of the acceleration horizon when the black holes are present and when they are absent, and $\mathcal{A}_{\mathrm{BH}}$ is the area of the black hole horizon. For the pair creation of extremal black holes (or gravitating monopoles) the second term is absent so the rate is entirely determined by the area of the acceleration horizon,

$$I = -\frac{1}{4}\Delta\mathcal{A}. \qquad (2)$$

This clearly shows the origin of the fact that nonextremal black holes are pair created at a higher rate given by the entropy of one black hole.

The calculation of each side of (2) is rather subtle. The area of the acceleration horizon is infinite in both the background magnetic field and the Ernst solution. To compute the finite change in area we first compute the area in the Ernst solution out to a large circle. We then subtract off the area in the background magnetic field solution out to a circle which is chosen to have the same proper length and the same value of $\oint A$ (where A is the vector potential). Similarly, the instanton action is finite only after we subtract off the infinite contribution coming from the background magnetic field. In [5], the calculation was done by computing the finite change in the action when the black hole charge is varied, and then integrating from zero charge to the desired q. In [6], the action was calculated inside a large sphere and the background contribution was subtracted using a coordinate matching condition. Both methods yield the same result. But given the importance of the action for the pair creation rate, one would like to have a direct calculation of it by matching the intrinsic geometry on a boundary near infinity as has been done for other black hole instantons. We will present such a derivation here and show that the result is in agreement with the earlier approaches. Combining this with our calculation of $\Delta\mathcal{A}$, we explicitly confirm the relations (1) and (2).

Perhaps the most important application of gravitational entropy is to the "black hole information puzzle." Following the discovery of black hole radiation, it was argued that information and quantum coherence can be lost in quantum gravity. This seemed to be an inevitable consequence of the semiclassical calculations which showed that black holes emit thermal radiation and slowly evaporate. However, many people find it difficult to accept the idea of nonunitary evolution. They have suggested that either the information thrown into a black hole comes out in detailed correlations not seen in the semiclassical ap-

proximation, or that the end point of the evaporation is a Planck scale remnant which stores the missing information. In the latter case, the curvature outside the horizon would be so large that semiclassical arguments would no longer be valid. However, consideration of black hole pair creation suggests another quantum gravitational process involving black holes, in which information seems to be lost yet the curvature outside the horizons always remains small.

The basic observation is that if black holes can be pair created, then it must be possible for them to annihilate. In fact, the same instanton which describes black hole pair creation can also be interpreted as describing black hole annihilation. Once one accepts the idea that black holes can annihilate (and assumes that information is not recovered in the black hole radiation), one can construct an argument for information loss as follows. Imagine pair creating two magnetically charged (nonextremal) black holes which move far apart into regions of space without a background magnetic field. One could then treat each black hole independently and throw an arbitrarily large amount of matter and information into them. The holes would then radiate and return to their original mass. One could then bring the two holes back together again and try to annihilate them. Of course, there is always the possibility that they will collide and form a black hole with no magnetic charge and about double the horizon area. This black hole could evaporate in the usual way down to Planck scale curvatures. However, there is a probability of about $e^{-\mathcal{A}_{\mathrm{BH}}/4}$ times the monopole annihilation probability that the black holes will simply annihilate, their energy being given off as electromagnetic or gravitational radiation. One can choose the magnetic field and the value of the magnetic charge in such a way that the curvature is everywhere small. Thus the semiclassical approximation should remain valid. This implies that even if small black hole remnants exist, they are not sufficient to preserve unitarity. This discussion applies to nonextremal black holes. Since extremal black holes have zero entropy, they behave differently, as we will explain.

In the next section we discuss the entropy of a single static black hole and show that an extreme Reissner-Nordström black hole has zero entropy. Section III contains a review of the Ernst instanton which describes pair creation of extremal and nonextremal black holes. In Sec. IV we discuss the boost energy and show that while it is changed for pair creation in flat space, it is unchanged for pair creation in general relativity. Section V contains a derivation of the relations (1) and (2) and the detailed calculations of the acceleration horizon area and instanton action which confirm them. Finally, Sec. VI contains further discussion of black hole annihilation and some concluding remarks.

In the appendix we consider the generalization of the Ernst instanton which includes an arbitrary coupling to a dilaton [9]. We will extend the development of the preceding sections to this case, showing that the boost energy is still unchanged in this case, and calculating the difference in area and the instanton action using appropriate boundary conditions. The result for the instanton action is in complete agreement with [6].

II. EXTREME BLACK HOLES HAVE ZERO ENTROPY

In this section we consider the entropy of a single static black hole. The reason that gravitational configurations can have nonzero entropy is that the Euclidean solutions can have nontrivial topology [10]. In other words, if we start with a static spacetime and identify imaginary time with period β, the manifold need not have topology $S^1 \times \Sigma$ where Σ is some three manifold. In fact, for nonextreme black holes, the topology is $S^2 \times R^2$. This means that the foliation one introduces to rewrite the action in Hamiltonian form must meet at a two-sphere S_h. The Euclidean Einstein-Maxwell action includes a surface term,

$$I = \frac{1}{16\pi} \int_M (-R + F^2) - \frac{1}{8\pi} \oint_{\partial M} K, \qquad (3)$$

where R is the scalar curvature, $F_{\mu\nu}$ is the Maxwell field, and K is the trace of the extrinsic curvature of the boundary. In fact, if the spacetime is noncompact, the action is defined only relative to some background solution (g_0, F_0). This background is usually taken to be flat space with zero field, but we shall consider more general asymptotic behavior. When one rewrites the action in Hamiltonian form, there is an extra contribution from the surface S_h. This arises since the surfaces of constant time meet at S_h and the resulting corner gives a delta-function contribution to K [10]. Alternatively, one can calculate the contribution from S_h as follows [11] (see [12] for another approach). The total action can be written as the sum of the action of a small tubular neighborhood of S_h and everything outside. The action for the region outside reduces to the standard Hamiltonian form,[1] which for a static configuration yields the familiar result βH. The action for the small neighborhood of S_h yields $-\mathcal{A}_{BH}/4$ where $\mathcal{A}_{BH}$ is the area of S_h. Thus the total Euclidean action is

$$I = \beta H - \frac{1}{4}\mathcal{A}_{BH}. \qquad (4)$$

The usual thermodynamic formula for the entropy is

$$S = -\left(\beta \frac{\partial}{\partial \beta} - 1\right) \ln Z, \qquad (5)$$

where the partition function Z is given (formally) by the integral of e^{-I} over all Euclidean configurations which are periodic in imaginary time with period β at infinity. The action for the solution describing a nonextremal black hole is (4) so if we approximate $\ln Z \approx -I$, we obtain the usual result

$$S = \frac{1}{4}\mathcal{A}_{BH}. \qquad (6)$$

Recall that the Reissner-Nordström metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 \qquad (7)$$
$$+ \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega.$$

For nonextreme black holes $Q^2 < M^2$, the above discussion applies. But the extreme Reissner-Nordström solution is qualitatively different. When $Q^2 = M^2$, the horizon $r = M$ is infinitely far away along spacelike directions. In the Euclidean solution, the horizon is infinitely far away along all directions. This means that the Euclidean solution can be identified with any period β. So the action must be proportional to the period $I \propto \beta$. It follows from (5) that in the usual approximation $\ln Z \approx -I$, the entropy is zero,

$$S_{\text{extreme}} = 0. \qquad (8)$$

This is consistent with the fact that gravitational entropy should be associated with nontrivial topology. The Euclidean extreme Reissner-Nordström solution (with τ periodically identified) is topologically $S^1 \times R \times S^2$. Since there is an S^1 factor, the surfaces introduced to rewrite the action in canonical form do not intersect. Thus there is no extra contribution from the horizon and the entropy is zero. Since the area of the event horizon of an extreme Reissner-Nordström black hole is nonzero, we conclude that *the entropy of a black hole is not always equal to* $\mathcal{A}_{BH}/4$; (6) holds only for nonextremal black holes.

The fact that the entropy changes discontinuously in the extremal limit implies that one should regard nonextreme and extreme black holes as qualitatively different objects. One is already used to the idea that a nonextreme black hole cannot turn into an extreme hole: the nearer the mass gets to the charge the lower the temperature is and so the lower the rate of radiation of mass is. Thus the mass will never exactly equal the charge. However, the idea that extreme and nonextreme black holes are distinct presumably also implies that extreme black holes cannot become nonextreme. At first sight this seems contrary to common sense. If one throws matter or radiation into an extreme black hole, one would expect to increase the mass and so make the hole nonextreme. However, the fact that one can identify extreme black holes with any period implies that extreme black holes can be in equilibrium with thermal radiation at any temperature. Thus they must be able to radiate at any rate, unlike nonextreme black holes, which can radiate only at the rate corresponding to their temperature. It would therefore be consistent to suppose that extreme black holes always radiate in such a way as to keep themselves extreme when matter or radiation is sent into them.

From all this it might seem that extreme and nearly extreme black holes would appear very different to outside observers. But this need not be the case. If one

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[1] The surface terms in the Hamiltonian can be obtained directly from the surface terms in the action. For a detailed discussion which includes spacetimes which are not asymptotically flat (e.g., the Ernst solution) and horizons which are not compact (e.g., acceleration horizons) see [13].

throws matter or radiation into a nearly extreme black hole, one will eventually get all the energy back in thermal radiation and the hole will return to its original state. Admittedly, it will take a very long time, but there is no canonical relationship between the advanced and retarded time coordinates in a black hole. This means that if one sends energy into an extreme black hole there is no obviously preferred time at which one might expect it back. It might therefore take as long as the radiation from nearly extreme black holes. If this were the case, a spacelike surface would intersect either the infalling matter or the outgoing radiation just outside the horizon of an extreme black hole. This would make its mass seem greater than its charge and so an outside observer would think it was nonextreme.

If extreme black holes behave just like nearly extreme ones is there any way in which we can distinguish them? A possible way would be in black hole annihilation, which will be discussed in Sec. VI.

Two-dimensional calculations [14] have indicated that the expectation value of the energy momentum tensor tends to blow up on the horizon of an extreme black hole. However, this may not be the case in a supersymmetric theory. Thus it may be possible to have extreme black holes only in supergravity theories in which the fermionic and bosonic energy momentum tensors can cancel each other. Because they have no entropy such supersymmetric black holes might be the particles of a dual theory of gravity.

There is a problem in calculating the pair creation of extreme black holes even in supergravity. As Gibbons and Kallosh [15] have pointed out, one would expect cancellation between the fermionic and bosonic energy momentum tensors only if the fermions are identified periodically. In the Ernst solution, however, the presence of the acceleration horizon means that the fermions have to be antiperiodic. Thus it may be that the pair creation of extreme black holes will be modified by strong quantum effects near the horizon.

III. THE ERNST SOLUTION

The solution describing a background magnetic field in general relativity is Melvin's magnetic universe [16],

$$ds^2 = \Lambda^2 \left(-dt^2 + dz^2 + d\rho^2 \right) + \Lambda^{-2}\rho^2 d\varphi^2, \qquad (9)$$

$$A_\varphi = \frac{\widehat{B}_M \rho^2}{2\Lambda}, \quad \Lambda = 1 + \frac{1}{4}\widehat{B}_M^2 \rho^2 . \qquad (10)$$

The Maxwell field is $F^2 = 2\widehat{B}_M^2/\Lambda^4$, which is a maximum on the axis $\rho = 0$ and decreases to zero at infinity. The parameter $\widehat{B}_M$ gives the value of the magnetic field on the axis.

The Ernst solution is given by

$$ds^2 = (x-y)^{-2}A^{-2}\Lambda^2 \left[G(y)dt^2 - G^{-1}(y)dy^2 \right. \qquad (11)$$

$$\left. + G^{-1}(x)dx^2 \right] + (x-y)^{-2}A^{-2}\Lambda^{-2}G(x)d\varphi^2,$$

$$A_\varphi = -\frac{2}{B\Lambda}\left(1 + \frac{1}{2}Bqx\right) + k, \qquad (12)$$

where the functions $\Lambda \equiv \Lambda(x,y)$, and $G(\xi)$ are

$$\Lambda = \left(1 + \frac{1}{2}Bqx\right)^2 + \frac{B^2}{4A^2(x-y)^2}G(x), \qquad (13)$$

$$G(\xi) = (1 + r_- A\xi)(1 - \xi^2 - r_+ A\xi^3), \qquad (14)$$

and $q^2 = r_+ r_-$. This solution represents two oppositely charged black holes uniformly accelerating in a background magnetic field.

It is convenient to set $\xi_1 = -1/(r_- A)$ and let $\xi_2 \leq \xi_3 < \xi_4$ be the three roots of the cubic factor in G. The function $G(\xi)$ may then be written as

$$G(\xi) = -(r_+ A)(r_- A)(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4). \qquad (15)$$

We restrict $\xi_3 \leq x \leq \xi_4$ in order for the metric to have Lorentz signature. Because of the conformal factor $(x-y)^{-2}$ in the metric, spatial infinity is reached when $x, y \to \xi_3$, while $y \to x$ for $x \neq \xi_3$ corresponds to null or timelike infinity. The range of y is therefore $-\infty < y < x$. The axis $x = \xi_3$ points towards spatial infinity, and the axis $x = \xi_4$ points towards the other black hole. The surface $y = \xi_1$ is the inner black hole horizon, $y = \xi_2$ is the black hole event horizon, and $y = \xi_3$ the acceleration horizon. We can choose $\xi_1 < \xi_2$, in which case the black holes are nonextreme, or $\xi_1 = \xi_2$, in which case the black holes are extreme.

As discussed in [9], to ensure that the metric is free of conical singularities at both poles, $x = \xi_3, \xi_4$, we must impose the condition

$$G'(\xi_3)\Lambda(\xi_4)^2 = -G'(\xi_4)\Lambda(\xi_3)^2, \qquad (16)$$

where $\Lambda(\xi_i) \equiv \Lambda(x = \xi_i)$. For later convenience, we define $L \equiv \Lambda(x = \xi_3)$. When (16) is satisfied, the spheres are regular as long as φ has period

$$\Delta\varphi = \frac{4\pi L^2}{G'(\xi_3)} . \qquad (17)$$

We choose the constant k in (12) to be $k = 2/BL^{1/2}$ so as to confine the Dirac string of the magnetic field to the axis $x = \xi_4$. We define a physical magnetic field parameter $\widehat{B}_E = BG'(\xi_3)/2L^{3/2}$, which is the value of the magnetic field on the axis at infinity. The physical charge of the black hole is defined by

$$\widehat{q} = \frac{1}{4\pi}\int F = q\frac{L^{\frac{3}{2}}(\xi_4 - \xi_3)}{G'(\xi_3)(1 + \frac{1}{2}qB\xi_4)} . \qquad (18)$$

If we also define $m = (r_+ + r_-)/2$, we can see that the solution (11,12) depends on four parameters: the physical magnetic field $\widehat{B}_E$, the physical magnetic charge $\widehat{q}$, and A and m, which may be loosely interpreted as measures of the acceleration and the mass of the black hole.

If we set the black hole parameters m and q (or equivalently, r_+, r_-) to zero in (11) we obtain

$$ds^2 = \frac{\Lambda^2}{A^2(x-y)^2}\left[(1-y^2)dt^2 - \frac{dy^2}{(1-y^2)} + \frac{dx^2}{(1-x^2)}\right] + \frac{1-x^2}{\Lambda^2(x-y)^2A^2}d\varphi^2, \qquad (19)$$

with

$$\Lambda = 1 + \frac{\widehat{B}_E^2}{4}\frac{1-x^2}{A^2(x-y)^2}. \qquad (20)$$

This is just the Melvin metric (9) expressed in accelerated coordinates, as can be seen by the coordinate transformations [6]

$$\rho^2 = \frac{1-x^2}{(x-y)^2A^2}, \quad \eta^2 = \frac{y^2-1}{(x-y)^2A^2}. \qquad (21)$$

Note that now the acceleration parameter A is no longer physical, but represents a choice of coordinates. The gauge field also reduces to the Melvin form $A_\varphi = \widehat{B}_E\rho^2/2\Lambda$. One can show [9,6] that the Ernst solution reduces to the Melvin solution at large spatial distances, that is, as $x, y \to \xi_3$.

We now turn to the consideration of the Euclidean section of the Ernst solution, which will form the instanton. We Euclideanize (11) by setting $\tau = it$. In the nonextremal case, $\xi_1 < \xi_2$, the range of y is taken to be $\xi_2 \le y \le \xi_3$ to obtain a positive definite metric (we assume $\xi_2 \ne \xi_3$). To avoid conical singularities at the acceleration and black hole horizons, we take the period of τ to be

$$\beta = \Delta\tau = \frac{4\pi}{G'(\xi_3)} \qquad (22)$$

and require

$$G'(\xi_2) = -G'(\xi_3), \qquad (23)$$

which gives

$$\left(\frac{\xi_2-\xi_1}{\xi_3-\xi_1}\right)(\xi_4-\xi_2) = (\xi_4-\xi_3). \qquad (24)$$

This condition can be simplified to

$$\xi_2 - \xi_1 = \xi_4 - \xi_3. \qquad (25)$$

The resulting instanton has topology $S^2 \times S^2 - \{pt\}$, where the point removed is $x = y = \xi_3$. This instanton is interpreted as representing the pair creation of two oppositely charged black holes connected by a wormhole.

If the black holes are extremal, $\xi_1 = \xi_2$, the black hole event horizon lies at infinite spatial distance from the acceleration horizon, and gives no restriction on the period of τ. The range of y is then $\xi_2 < y \le \xi_3$, and the period of τ is taken to be (22). The topology of this instanton is $R^2 \times S^2 - \{pt\}$, where the removed point is again $x = y = \xi_3$. This instanton is interpreted as

representing the pair creation of two extremal black holes with infinitely long throats.

IV. BOOST ENERGY

Consider the pair creation of (nongravitating) GUT monopoles in flat spacetime. In this process the usual energy is unchanged. If the background magnetic field extends to infinity, this energy will, of course, be infinite. But even if it is cut off at a large distance, the energy is conserved since in the Euclidean solution, $\nabla^\mu(T_{\mu\nu}t^\nu) = 0$, where $T_{\mu\nu}$ is the energy momentum tensor and t^ν is a time translation Killing vector. Thus the initial energy, which is the integral of $T_{\mu\nu}t^\mu t^\nu$ over a surface in the distant past, must equal the energy after the monopoles are created. However, now consider the energy associated with a boost Killing vector in the Lorentzian solution. This corresponds to a rotation ξ^μ in the Euclidean instanton. So the associated energy is

$$E_B = \int_\Sigma T_{\mu\nu}\xi^\mu d\Sigma^\nu, \qquad (26)$$

where the integral is over a surface Σ which starts at the acceleration horizon where $\xi^\mu = 0$ and extends to infinity. While the vector $T_{\mu\nu}\xi^\mu$ is still conserved, which implies that E_B is unchanged under continuous deformations of Σ that preserve the boundary conditions, this is not sufficient to prove that E_B is unchanged in the pair creation process. This is because every surface which starts at the acceleration horizon in the instanton always intersects the monopole, and cannot be deformed into a surface lying entirely in the background magnetic field. In fact, it is easy to show that E_B *is* changed. Since the analytic continuation of the boost parameter is periodic with period 2π, the Euclidean action is just $I = 2\pi E_B$. So the fact that the instanton describing the pair creation of monopoles has a different action from the uniform magnetic field means that the boost energy is different.

We now turn to the case of pair creation of gravitating monopoles, or black holes. The gravitational Hamiltonian is only defined with respect to a background spacetime, and can be expressed [13] (this form of the surface term at infinity is also discussed in [12])

$$H = \int_\Sigma N\mathcal{H} - \frac{1}{8\pi}\int_{S\infty} N(^2K - {}^2K_0), \qquad (27)$$

where N is the lapse, $\mathcal{H}$ is the Hamiltonian constraint, ^2K is the trace of the two-dimensional extrinsic curvature of the boundary near infinity, and $^2K_0$ is the analogous quantity for the background spacetime. Since the volume term is proportional to the constraint, which vanishes, the energy is just given by a surface term at infinity. The Hamiltonian for Melvin is zero since we are using it as the background in which $^2K_0$ is evaluated. We now calculate the Hamiltonian for the Ernst solution and show that it is also zero. Thus the boost energy is unchanged by pair creation in the gravitational case.

Since the spacetime is noncompact, we have to take a boundary "near infinity," and eventually take the limit

as it tends to infinity. The surface Σ in the Ernst solution is a surface of constant t in the Ernst metric (11), running from the acceleration horizon to a boundary at large distance. As a general principle, we want the boundary to obey the Killing symmetries of the metric, and in this case, we choose it to be given by $x - y = \epsilon_E$, as in [6]. The result in the limit as the boundary tends to infinity should be independent of this choice.

The first part of the surface term is computed in the Ernst metric, and the second part in the Melvin metric. We need to ensure that the boundaries that we use in computing these two contributions are identical; that is, we must require that the intrinsic geometry and the Maxwell field on the boundary are the same. Because the Ernst solution reduces to the Melvin solution at large distances, it is possible to find coordinates in which the induced metric and gauge field on the boundary agree explicitly.

The analogue of the surface Σ for the Melvin solution is a surface of constant boost time t of the Melvin metric in the accelerated form (19). We want to find a boundary lying in this surface with the same intrinsic geometry as the above. We will require that the boundary obey the Killing symmetries, but there is still a family of possible embeddings. We assume the boundary lies at $x - y = \epsilon_M$. It is not clear that the results will be independent of this assumption, but this is the simplest form the embedding in Melvin can take, so let us proceed on this basis.

If we make coordinate transformations

$$\varphi = \frac{2L^2}{G'(\xi_3)}\varphi', \quad t = \frac{2}{G'(\xi_3)}t', \tag{28}$$

and

$$x = \xi_3 + \epsilon_E \chi, \quad y = \xi_3 + \epsilon_E(\chi - 1) \tag{29}$$

in the Ernst metric (note that $\Delta\varphi' = 2\pi$, $0 \leq \chi \leq 1$, and the analytic continuation of t' has period 2π), then the metric on the boundary is

$$^{(2)}ds^2 = \frac{2L^2}{A^2\epsilon_E G'(\xi_3)}\left\{-\frac{\lambda^2 d\chi^2}{2\chi(\chi - 1)} \right. \tag{30}$$
$$\left. + \lambda^{-2}\left[2\chi + \epsilon_E \chi^2 \frac{G''(\xi_3)}{G'(\xi_3)}\right]d\varphi'^2\right\},$$

where

$$\lambda = \frac{\widehat{B}_E^2 L^2}{A^2 G'(\xi_3)\epsilon_E}\chi + \frac{\widehat{B}_E^2 L^2 G''(\xi_3)}{2A^2 G'(\xi_3)^2}\chi^2 + 1, \tag{31}$$

and everything is evaluated only up to second nontrivial order in ϵ_E, as higher-order terms will not contribute to the Hamiltonian in the limit $\epsilon_E \to 0$.

Using

$$x = -1 + \epsilon_M \chi, \quad y = -1 + \epsilon_M(\chi - 1), \tag{32}$$

the metric of the boundary in Melvin is

$$^{(2)}ds^2 = \frac{1}{\bar{A}^2\epsilon_M}\left[-\frac{\Lambda^2 d\chi^2}{2\chi(\chi - 1)} + \Lambda^{-2}\left(2\chi - \epsilon_M\chi^2\right)d\varphi^2\right], \tag{33}$$

where

$$\Lambda = \frac{\widehat{B}_M^2}{2\bar{A}^2\epsilon_M}\chi - \frac{\widehat{B}_M^2}{4\bar{A}^2}\chi^2 + 1. \tag{34}$$

Recall that $\bar{A}$ represents a choice of coordinates in the Melvin metric.

We also want to match the magnetic fields. For the Ernst solution, the electromagnetic field at the boundary is given by

$$F_{\chi\varphi'} = \frac{2A^2 G'(\xi_3)\epsilon_E}{\widehat{B}_E^3 L^2\chi^2}\left[1 - \frac{2A^2 G'(\xi_3)\epsilon_E}{\widehat{B}_E^2 L^2\chi}\right], \tag{35}$$

while for Melvin it is

$$F_{\chi\varphi} = \frac{4\bar{A}^2\epsilon_M}{\widehat{B}_M^3\chi^2}\left(1 - \frac{4\bar{A}^2\epsilon_M}{\widehat{B}_M^2\chi}\right). \tag{36}$$

If we fix the remaining coordinate freedom by choosing

$$\bar{A}^2 = -\frac{G'(\xi_3)^2}{2L^2 G''(\xi_3)}A^2, \tag{37}$$

and write ϵ_M and $\widehat{B}_M$ as

$$\epsilon_M = -\frac{G''(\xi_3)}{G'(\xi_3)}\epsilon_E(1 + \alpha\epsilon_E), \quad \widehat{B}_M = \widehat{B}_E(1 + \beta\epsilon_E), \tag{38}$$

then we can easily see that the induced metrics (30) and (33) and the gauge fields (35) and (36) of the boundary may be matched by taking $\alpha = \beta = 0$.

Note that, for the Ernst metric, the lapse (with respect to the time coordinate t') is

$$N = \left[\frac{4L^2(1 - \chi)}{A^2\epsilon_E G'(\xi_3)}\right]^{1/2}\lambda\left[1 + \epsilon_E(\chi - 1)\frac{G''(\xi_3)}{4G'(\xi_3)}\right], \tag{39}$$

where λ is given by (31). For the Melvin metric, the lapse [with respect to the boost time t appearing in (19)] is

$$N = \left[\frac{2(1 - \chi)}{\bar{A}^2\epsilon_M}\right]^{1/2}\Lambda\left[1 - \frac{1}{4}\epsilon_M(\chi - 1)\right], \tag{40}$$

where Λ is given by (34). We therefore see that the lapse functions are also matched by taking $\alpha = \beta = 0$.

We may now calculate the extrinsic curvature ^2K of the boundary embedded in the Ernst solution, which gives

$$\int_{S^\infty} N^2 K = \frac{8\pi L^2}{A^2\epsilon_E G'(\xi_3)}\left[1 - \frac{1}{4}\epsilon_E \frac{G''(\xi_3)}{G'(\xi_3)}\right]. \tag{41}$$

Calculating the extrinsic curvature $^2K_0$ of the boundary embedded in the Melvin solution gives

$$\int_{S^\infty} N^2 K_0 = \frac{4\pi}{\bar{A}^2\epsilon_M}\left(1 + \frac{1}{4}\epsilon_M\right). \tag{42}$$

Using (37) and (38) we see that these two surface terms are equal. Thus, taking the limit $\epsilon_E \to 0$, the surface

term in the Hamiltonian vanishes. Since the volume term vanishes by virtue of the equations of motion, this implies that the Hamiltonian vanishes for the Ernst solution, and thus the boost energy is unchanged.

V. ACTION AND AREA

A. The basic relations

The fact that the boost energy is unchanged in the pair creation of gravitating objects implies a simple relation between the Euclidean action I and the area of the horizons. The Euclidean action is defined only with respect to a choice of background spacetime. If both the background spacetime and original spacetime have acceleration horizons, it is shown in [13] that (4) is modified to

$$I = \beta H - \frac{1}{4}(\Delta\mathcal{A} + \mathcal{A}_{\rm BH}), \qquad (43)$$

where $\Delta\mathcal{A}$ is the difference in the area of the acceleration horizon in the physical spacetime and the background. Thus, for the case of pair creation of nonextremal black holes, we have

$$I_{\rm Ernst} = \beta H_E - \frac{1}{4}(\Delta\mathcal{A} + \mathcal{A}_{\rm BH}), \qquad (44)$$

where $\Delta\mathcal{A}$ is the difference between the area of the acceleration horizon in the Ernst metric and in the Melvin metric. In the extreme case, as shown in Sec. II, the area of the black hole horizon does not appear in the action since the horizon is infinitely far away. Therefore the action is given by

$$I_{\rm Ernst} = \beta H_E - \frac{1}{4}\Delta\mathcal{A}. \qquad (45)$$

We have shown that $H_E = 0$ in the previous section. The Ernst action is thus

$$I_{\rm Ernst} = -\frac{1}{4}(\Delta\mathcal{A} + \mathcal{A}_{\rm BH}) \qquad (46)$$

for the nonextreme case, and

$$I_{\rm Ernst} = -\frac{1}{4}\Delta\mathcal{A} \qquad (47)$$

in the extreme case. We will now show that these relations in fact hold.

The area of the black hole event horizon in the Ernst solution can be easily shown to be

$$\mathcal{A}_{\rm BH} = \int_{y=\xi_4} \sqrt{g_{xx}g_{\varphi\varphi}}dxd\varphi = \frac{\Delta\varphi_E(\xi_4 - \xi_3)}{A^2(\xi_3 - \xi_2)(\xi_4 - \xi_2)}, \qquad (48)$$

where $\Delta\varphi_E$ is given in (17). We now turn to the calculation of the other two terms in (46).

B. Change in area of the acceleration horizon

Since the acceleration horizon is noncompact, its area is infinite; to calculate the difference, we must introduce a

boundary, as we did in calculating the Hamiltonian. If we introduce a boundary in the Ernst solution at $x = \xi_3 + \epsilon_E$, the area of the region inside it is

$$\mathcal{A}_E = \int_{y=\xi_3} \sqrt{g_{xx}g_{\varphi\varphi}}dxd\varphi \qquad (49)$$

$$= \frac{\Delta\varphi_E}{A^2} \int_{x=\xi_3+\epsilon_E}^{x=\xi_4} \frac{dx}{(x - \xi_3)^2}$$

$$= -\frac{\Delta\varphi_E}{A^2(\xi_4 - \xi_3)} + \frac{\Delta\varphi_E}{A^2\epsilon_E}$$

$$= -\frac{4\pi L^2}{A^2 G'(\xi_3)(\xi_4 - \xi_3)} + \pi\rho_E^2,$$

where we have used $L = \Lambda(\xi_3)$ and (17), and defined $\rho_E^2 = 4L^2/[A^2 G'(\xi_3)\epsilon_E]$. The acceleration horizon in the Melvin solution is the surface $z = 0$, $t = 0$ in (9) (this can be seen by introducing the Rindler-type coordinates $t = \eta\sinh\hat{t}$, $z = \eta\cosh\hat{t}$). Its area inside a boundary at $\rho = \rho_M$ may similarly be calculated to be

$$\mathcal{A}_M = \int \sqrt{g_{\rho\rho}g_{\varphi\varphi}}d\rho d\varphi = 2\pi \int_{\rho=0}^{\rho=\rho_M} \rho d\rho = \pi\rho_M^2. \quad (50)$$

Note that there is no ambiguity in the choice of boundary in the Melvin solution here; $\rho = \rho_M$ is the only choice which obeys the Killing symmetry.

We must now match the intrinsic features of the boundary; we require that the proper length of the boundary and the integral of the gauge potential A_φ around the boundary be the same. For the Ernst solution, the proper length of the boundary is

$$l_E = \int \sqrt{g_{\varphi\varphi}}d\varphi \qquad (51)$$

$$= \frac{8\pi}{\widehat{B}_E^2\rho_E} \left[1 - \frac{4}{\widehat{B}_E^2\rho_E^2} - \frac{L^2 G''(\xi_3)}{G'(\xi_3)^2 A^2} \frac{1}{\rho_E^2}\right].$$

As in Sec. IV, we expand to second nontrivial order in ρ_E; higher-order terms do not affect $\Delta\mathcal{A}$ in the limit $\rho_E \to \infty$. For the Melvin solution, the proper length of the boundary is

$$l_M = \frac{8\pi}{\widehat{B}_M^2\rho_M} \left(1 - \frac{4}{\widehat{B}_M^2\rho_M^2}\right). \qquad (52)$$

The integral of the gauge potential around the boundary is, in the Ernst solution,

$$\frac{1}{2\pi} \oint A_\varphi d\varphi = \frac{2}{\widehat{B}_E} - \frac{2A^2\epsilon_E G'(\xi_3)}{\widehat{B}_E^3 L^2} = \frac{2}{\widehat{B}_E} - \frac{8}{\widehat{B}_E^3\rho_E^2}, \qquad (53)$$

while in the Melvin solution it is

$$\frac{1}{2\pi} \oint A_\varphi d\varphi = \frac{2}{\widehat{B}_M} - \frac{8}{\widehat{B}_M^3\rho_M^2}. \qquad (54)$$

If we write

$$\widehat{B}_M = \widehat{B}_E \left(1 + \frac{\beta}{\rho_E^2}\right) \quad \text{and} \quad \rho_M = \rho_E \left(1 + \frac{\alpha}{\rho_E^2}\right), \quad (55)$$

then setting the integral of the gauge fields equal gives $\beta = 0$, as before, and setting $l_E = l_M$ perturbatively gives

$$\alpha = \frac{L^2 G''(\xi_3)}{G'(\xi_3)^2 A^2}. \quad (56)$$

Substituting this into (50) gives

$$\mathcal{A}_M = \pi \rho_E^2 + 2\pi \alpha = \pi \rho_E^2 + \frac{2\pi L^2 G''(\xi_3)}{G'(\xi_3)^2 A^2}. \quad (57)$$

We can now evaluate the difference in area, letting $\rho_E \to \infty$,

$$\Delta \mathcal{A} = \mathcal{A}_E - \mathcal{A}_M = -\frac{4\pi L^2}{G'(\xi_3) A^2} \left[\frac{1}{(\xi_4 - \xi_3)} + \frac{G''(\xi_3)}{2G'(\xi_3)}\right]$$

$$= -\frac{4\pi L^2}{G'(\xi_3) A^2} \left[\frac{1}{(\xi_3 - \xi_2)} + \frac{1}{(\xi_3 - \xi_1)}\right]. \quad (58)$$

Now for the extreme case, $\xi_2 = \xi_1$, and so

$$-\frac{1}{4} \Delta \mathcal{A} = \frac{2\pi L^2}{G'(\xi_3) A^2 (\xi_3 - \xi_1)}, \quad (59)$$

which agrees with the expression for the action found in [6]. For the nonextreme case,

$$-\frac{1}{4}(\Delta \mathcal{A} + \mathcal{A}_{\text{BH}}) = \frac{\pi L^2}{G'(\xi_3) A^2} \left[\frac{2}{(\xi_3 - \xi_1)}\right.$$

$$+ \frac{(\xi_2 - \xi_1)}{(\xi_3 - \xi_2)(\xi_3 - \xi_1)}$$

$$\left. - \frac{(\xi_4 - \xi_3)}{(\xi_4 - \xi_2)(\xi_3 - \xi_2)}\right]$$

$$= \frac{2\pi L^2}{G'(\xi_3) A^2 (\xi_3 - \xi_1)}, \quad (60)$$

where we have used (48) in the first step and the no-strut condition (25) in the second. Notice that the final expression is the same as in (59). So relations (1) and (2) are confirmed provided the formula for the instanton action given in [6] is valid for both the extreme and nonextreme black holes. We now verify that this is indeed the case.

C. Direct calculation of the action

In [6] it was assumed that the divergent part of the action could simply be subtracted, using a coordinate matching condition, without affecting the correct finite contribution to the action. As we have seen above, this is not necessarily the case; we need to evaluate the action for a bounded region, impose some geometric matching conditions at the boundary to ensure that the boundaries are the same, and then let the boundary tend to infinity. Despite all this, the fact that our result above agrees with

that in [6] suggests that the answer is unchanged, as we shall see.

To evaluate the action directly, we introduce a boundary three-surface at large radius. We will take the surface to lie at $x - y = \epsilon_E$ in the Ernst solution and at $x - y = \epsilon_M$ in the accelerated coordinate system in the Melvin solution, as in Sec. IV. The volume integral of R is zero by the field equations. The volume integral of the Maxwell Lagrangian F^2 is not zero, but it can be converted to a surface term and combined with the extrinsic curvature term, as shown in [6]. Thus the action of the region of the Ernst solution inside the surface is made up of two parts: boundary contributions from the three-surface embedded in the Ernst solution, and a subtracted contribution from the three-surface embedded in the Melvin solution.

The contribution to the action from this surface in the Ernst solution is [6]

$$I_E = -\frac{1}{8\pi} \int_{x-y=\epsilon_E} d^3x \sqrt{h} e^{-\delta} \nabla_\mu (e^\delta n^\mu) \quad (61)$$

$$= \frac{\pi L^2}{A^2 G'(\xi_3)} \left[-\frac{3}{\epsilon_E} + \frac{2}{(\xi_3 - \xi_1)}\right],$$

where $e^{-\delta} = \Lambda \frac{(y-\xi_1)}{(x-\xi_1)}$, and h is the induced metric on the three-surface. The contribution from the surface in the Melvin solution may be obtained by setting $r_+ = r_- = 0$ in (61); it is

$$I_M = -\frac{\pi}{2\bar{A}^2} \frac{3}{\epsilon_M} + O(\epsilon_M). \quad (62)$$

The matching conditions on the boundary follow immediately from the conditions used to compute the Hamiltonian in Sec. IV. If we make the change of coordinates (28) and (29) in the Ernst solution, and analytically continue $\tau' = it'$, then the induced metric on the three-surface in the Ernst solution is $^{(3)}ds^2 = N^2 d\tau'^2 + {}^{(2)}ds^2$, where N is the lapse (39) and $^{(2)}ds^2$ is given by (30). Similarly, if we use (32) in the Melvin solution (19) and analytically continue $\tau = it$, the induced metric on the three-surface in the Melvin solution will be $^{(3)}ds^2 = N^2 d\tau^2 + {}^{(2)}ds^2$, where N is the lapse, given by (40), and $^{(2)}ds^2$ is given by (33). The Maxwell field on the three-surface will be the same as in Sec. IV. Therefore, we see that the intrinsic features of the three-surface may be matched by taking (37) and (38) with $\alpha = \beta = 0$. The action may now be evaluated,

$$I_{\text{Ernst}} = I_E - I_M \quad (63)$$

$$= \frac{\pi L^2}{A^2 G'(\xi_3)} \left[-\frac{3}{\epsilon_E} - \frac{3G''(\xi_3)}{G'(\xi_3)\epsilon_M} + \frac{2}{(\xi_3 - \xi_1)}\right]$$

$$= \frac{2\pi L^2}{A^2 G'(\xi_3)(\xi_3 - \xi_1)}.$$

This applies to both extremal and nonextremal instantons and agrees with the previous expressions in the literature. One can understand why the naive coordinate subtraction of divergences [6] yielded the correct answer since the boundary geometry is matched when $\alpha = 0$. Since (63) agrees with (59) and (60), we see that the relations (1) and (2) have been verified.

VI. BLACK HOLE ANNIHILATION

As discussed in the Introduction, since black holes can be pair created, it must be possible for them to annihilate. This provides a new way for black holes to disappear which does not involve Planck scale curvature. The closest analogue of the pair creation process is black hole annihilation in the presence of a background magnetic field. To reproduce the time reverse of pair creation exactly one would have to arrange that the black holes had exactly the right velocities to come to rest in a magnetic field at a critical separation. They could then tunnel quantum mechanically and annihilate each other. If the black holes came to rest too far apart their total energy would be negative and they would not be able to annihilate. If they were too near together it would still be possible for them to annihilate but now there would be energy left over which would be given off as electromagnetic or gravitational radiation. It is also possible for black holes to annihilate in the absence of a magnetic field, with all of their energy converted to radiation.

One might ask whether the generalized second law of thermodynamics is violated in this process. The answer is no. Even though the total entropy is decreased by the elimination of the black hole horizons, this is allowed since it is a rare process. The rate can be estimated as follows. Nonextremal black holes behave in pair creation as if they had e^S internal states. Since two nonextremal black holes can presumably annihilate only if they are in the same internal state, if one throws two randomly chosen black holes together the probability of direct annihilation is of order e^{-S}.

We argued in Sec. II that extreme black holes are fundamentally different from nonextreme holes since they have zero entropy. This presumably implies that two oppositely charged extreme black holes cannot form a neutral black hole. Instead, they always directly annihilate. This is consistent with the idea that extreme black holes cannot be formed in gravitational collapse, but can only arise through pair creation.

The process of black hole annihilation also seems to violate the idea that "black holes have no hair." It would appear that one could determine something about the internal state of a black hole, i.e., whether two black holes are in the same state or different states, by bringing them together and seeing if they annihilate. However, it is not clear how robust the internal state is. It is possible that simply the act of bringing the black holes together will change their state.

The fact that the pair creation of nonextremal black holes creates a wormhole in space could be taken as a geometric manifestation of their correlated state. However, we do not believe that black holes need to be connected by wormholes in order to annihilate. Imagine two pairs of black holes being created. If each pair annihilates separately, the instanton will contain two black hole loops, and one expects the action will be smaller than that of extremal black holes (or gravitating monopoles) by twice the black hole entropy. However, there should be another instanton in which the two pairs are created and then the black holes from one pair annihilate with those from the

other. This instanton will contain one black hole loop and will presumably have an action which is smaller by one factor of the black hole entropy. This can be interpreted as arising from a contribution of minus twice the black hole entropy from the pair creation of the two pairs, and a contribution of plus the black hole entropy from the annihilation of one pair. (After one pair annihilates, the other pair must be correlated, and does not contribute another factor of the black hole entropy.)

It should be pointed out that even though the nonextremal black holes are created with their horizons identified, it is still possible for them to evolve independently. In particular, their horizon areas need not remain equal. This is because the identification only requires that the interior of the two black holes be the same. On a nonstatic slice which crosses the future event horizon in Ernst, there are two separate horizons. If one throws matter into one but not the other, the areas of the two horizon components will not be equal at later times. The fact that the horizon components share a common interior region of spacetime suggests that the "internal" states of a black hole should be associated with the region near the horizon. Presumably, throwing matter into the holes will tend to decorrelate their "internal" states, but it is not clear whether just one particle is enough to decorrelate them completely, or whether that requires a large number.

APPENDIX: GENERALIZATION TO INCLUDE A DILATON

1. The dilaton Ernst solution

The above investigations of the Ernst solution can be readily extended to include a dilaton, as we now show. We consider the general action

$$I = \frac{1}{16\pi} \int d^4x [-R + 2(\nabla\phi)^2 + e^{-2a\phi}F^2] - \frac{1}{8\pi} \int K, \tag{A1}$$

which has a parameter a governing the strength of the dilaton coupling. The Melvin and Ernst solutions are extrema of (A1) with $a = 0$ and ϕ constant. The generalization of the Melvin solution to $a \neq 0$, first found by Gibbons and Maeda [17], is

$$ds^2 = \Lambda^{\frac{2}{1+a^2}} \left(-dt^2 + dz^2 + d\rho^2\right) + \Lambda^{-\frac{2}{1+a^2}} \rho^2 d\varphi^2, \tag{A2}$$

$$e^{-2a\phi} = \Lambda^{\frac{2a^2}{1+a^2}}, \quad A_\varphi = \frac{\widehat{B}_M \rho^2}{2\Lambda}, \quad \Lambda = 1 + \frac{(1+a^2)}{4} \widehat{B}_M^2 \rho^2. \tag{A3}$$

The generalization of the Ernst solution to this case is [9]

$$ds^2 = (x-y)^{-2}A^{-2}\Lambda^{\frac{2}{1+a^2}}\left\{F(x)\left[G(y)dt^2\right.\right. \tag{A4}$$

$$\left.- G^{-1}(y)dy^2\right] + F(y)G^{-1}(x)dx^2\right\}$$

$$+(x-y)^{-2}A^{-2}\Lambda^{-\frac{2}{1+a^2}}F(y)G(x)d\varphi^2,$$

$$e^{-2a\phi} = e^{-2a\phi_0}\Lambda^{\frac{2a^2}{1+a^2}}\frac{F(y)}{F(x)}, \tag{A5}$$

$$A_\varphi = -\frac{2e^{a\phi_0}}{(1+a^2)B\Lambda}\left[1 + \frac{(1+a^2)}{2}Bqx\right] + k, \tag{A6}$$

where the functions $\Lambda \equiv \Lambda(x,y)$, $F(\xi)$, and $G(\xi)$ are now given by

$$\Lambda = \left[1 + \frac{(1+a^2)}{2}Bqx\right]^2 + \frac{(1+a^2)B^2}{4A^2(x-y)^2}G(x)F(x), \tag{A7}$$

$$F(\xi) = (1 + r_-A\xi)^{\frac{2a^2}{(1+a^2)}}, \tag{A8}$$

$$G(\xi) = (1 - \xi^2 - r_+A\xi^3)(1 + r_-A\xi)^{\frac{(1-a^2)}{(1+a^2)}}, \tag{A9}$$

and $q^2 = r_+r_-/(1+a^2)$. Here it is useful to define another function,

$$H(\xi) \equiv G(\xi)F(\xi) \tag{A10}$$

$$= -(r_+A)(r_-A)(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4),$$

where $\xi_1 = -1/(r_-A)$, and ξ_2, ξ_3, ξ_4 are the roots of the cubic factor in $G(\xi)$. These roots have the same interpretation as in the Ernst solution.

We now define $L = \Lambda^{\frac{1}{1+a^2}}(\xi_3)$, and set $k = 2e^{a\phi_0}/BL^{\frac{1+a^2}{2}}(1+a^2)$. We then find that the physical magnetic field and charge are [6]

$$\widehat{B}_E = \frac{BG'(\xi_3)}{2L^{\frac{3+a^2}{2}}} \tag{A11}$$

and

$$\widehat{q} = q\frac{e^{a\phi_0}L^{\frac{3-a^2}{2}}(\xi_4 - \xi_3)}{G'(\xi_3)(1 + \frac{1+a^2}{2}qB\xi_4)}. \tag{A12}$$

We restrict x to the range $\xi_3 \le x \le \xi_4$ to get the right signature. We have to impose the condition

$$G'(\xi_3)\Lambda(\xi_4)^{\frac{2}{1+a^2}} = -G'(\xi_4)\Lambda(\xi_3)^{\frac{2}{1+a^2}} \tag{A13}$$

to ensure that the conical singularities at both poles are eliminated by choosing the period of φ to be (17). Setting the black hole parameters r_+, r_- to zero in the dilaton Ernst metric (A4) yields

$$ds^2 = \frac{\Lambda^{\frac{2}{1+a^2}}}{A^2(x-y)^2}\left[(1-y^2)dt^2 - \frac{dy^2}{(1-y^2)} + \frac{dx^2}{(1-x^2)}\right]$$

$$+\Lambda^{-\frac{2}{1+a^2}}\frac{1-x^2}{(x-y)^2A^2}d\varphi^2, \tag{A14}$$

with

$$\Lambda = 1 + \frac{(1+a^2)\widehat{B}_E^2}{4}\frac{1-x^2}{A^2(x-y)^2}, \tag{A15}$$

which is the dilaton Melvin solution (A2) written in accelerated coordinates. The dilaton Ernst solution (A4) reduces to the dilaton Melvin solution (A2) at large spatial distances, $x, y \to \xi_3$.

We obtain the Euclidean section by setting $\tau = it$. In the nonextremal case, $\xi_1 < \xi_2$, we are forced to restrict $\xi_2 \le y \le \xi_3$, and we find that to eliminate the conical singularities, we have to choose the period of τ to be (22) and impose the condition (23), which gives

$$\left(\frac{\xi_2 - \xi_1}{\xi_3 - \xi_1}\right)^{\frac{1-a^2}{1+a^2}}(\xi_4 - \xi_2) = (\xi_4 - \xi_3) \tag{A16}$$

for this metric. In the extremal case, the black hole horizon $y = \xi_2$ is at an infinite distance, so the range of y is $\xi_2 < y \le \xi_3$, and we only need to choose the period of τ to be (22) to eliminate the conical singularity. The nonextremal instanton still has topology $S^2 \times S^2 - \{pt\}$, while the extremal one has topology $R^2 \times S^2 - \{pt\}$, and they have the same interpretation as before.

2. Boost energy

We now show that the boost energy is unchanged by pair creation in this case. The Hamiltonian is still given by (27), and the volume term vanishes, so it is just given by the surface term. We choose the boundary in the Ernst solution to be given by $x - y = \epsilon_E$, and make the coordinate transformations (28) and (29). In the Melvin solution, we assume that the boundary has the form

$$x = -1 + \epsilon_M\chi[1 + \epsilon_Ef(\chi)], \tag{A17a}$$

$$y = -1 + \epsilon_M(\chi - 1)[1 + \epsilon_Eg(\chi)], \tag{A17b}$$

in the coordinates of the accelerated form (A14). In this case, we need to match the value of the dilaton on the boundary, as well as the induced metric and gauge field on the boundary. For the Ernst metric, the induced metric on the boundary is

$$^{(2)}ds^2 = \frac{2L^2F(\xi_3)}{A^2\epsilon_EG'(\xi_3)}\left\{-\frac{\lambda^{\frac{2}{1+a^2}}d\chi^2}{2\chi(\chi-1)}\right.$$

$$\times\left[1 + \epsilon_E(2\chi-1)\frac{F'(\xi_3)}{F(\xi_3)}\right]$$

$$+2\lambda^{-\frac{2}{1+a^2}}\chi\left[1 + \epsilon_E\chi\frac{H''(\xi_3)}{2H'(\xi_3)}\right.$$

$$\left.\left.-\epsilon_E\frac{F'(\xi_3)}{F(\xi_3)}\right]d\varphi'^2\right\}, \tag{A18}$$

where

$$\lambda = \frac{(1+a^2)\widehat{B}_E^2 F(\xi_3) L^2 \chi}{A^2 G'(\xi_3)\epsilon_E}\left[1 + \epsilon_E \chi \frac{H''(\xi_3)}{2H'(\xi_3)}\right] + 1.$$

(A19)

The electromagnetic field on the boundary for the Ernst solution is

$$F_{\chi\varphi'} = \frac{2L^2 F(\xi_3)\widehat{B}_E}{A^2 \epsilon_E G'(\xi_3)\lambda^2}\frac{e^{a\phi_0}}{L^{a^2}}\left[1 + \epsilon_E \chi \frac{H''(\xi_3)}{H'(\xi_3)}\right],$$

(A20)

and the dilaton at the boundary is

$$e^{-2a\phi} = e^{-2a\phi_0} L^{2a^2}\lambda^{\frac{2a^2}{1+a^2}}\left[1 - \epsilon_E \frac{F'(\xi_3)}{F(\xi_3)}\right].$$

(A21)

In the Melvin solution, the induced metric on the boundary is

$$^{(2)}ds^2 = \frac{-\Lambda^{\frac{2}{1+a^2}}}{2\chi(\chi-1)\bar{A}^2\epsilon_M}\Big\{1 - \epsilon_E(\chi-1)f(\chi) + \epsilon_E\chi g(\chi) - 2\epsilon_E\chi(\chi-1)[f'(\chi) - g'(\chi)]$$

$$- 2\epsilon_E[\chi f(\chi) - (\chi-1)g(\chi)]\Big\}\,d\chi^2 + \frac{2\Lambda^{-\frac{2}{1+a^2}}\chi}{\bar{A}^2\epsilon_M}\Big\{1 - \frac{1}{2}\epsilon_M\chi + \epsilon_E f(\chi) - 2\epsilon_E[\chi f(\chi) - (\chi-1)g(\chi)]\Big\}\,d\varphi^2,$$

(A22)

where

$$\Lambda = 1 + \frac{(1+a^2)\widehat{B}_M^2\chi}{2\bar{A}^2\epsilon_M}\left\{1 - \frac{1}{2}\epsilon_M\chi + \epsilon_E f(\chi) - 2\epsilon_E[\chi f(\chi) - (\chi-1)g(\chi)]\right\}.$$

(A23)

The gauge field on the boundary in Melvin is

$$F_{\chi\varphi} = \frac{\widehat{B}_M}{\bar{A}^2\epsilon_M\Lambda^2}\Big\{1 - \epsilon_M\chi + \epsilon_E[\chi f'(\chi) + f(\chi)] - 2\epsilon_E[\chi f(\chi) - (\chi-1)g(\chi)]$$

$$-2\epsilon_E\chi[f(\chi) + \chi f'(\chi) - g(\chi) - (\chi-1)g'(\chi)]\Big\},$$

(A24)

and the dilaton at the boundary is

$$e^{-2a\phi} = \Lambda^{\frac{2a^2}{1+a^2}}.$$

(A25)

We fix the remaining coordinate freedom by taking

$$\bar{A}^2 = -\frac{G'(\xi_3)}{2L^2 F(\xi_3)}\frac{H'(\xi_3)}{H''(\xi_3)}A^2,$$

(A26)

and write

$$e^{a\phi_0} = L^{a^2}(1 - \gamma\epsilon_E), \quad \widehat{B}_M = \widehat{B}_E(1 + \beta\epsilon_E).$$

(A27)

We then find that the intrinsic metric, gauge field, and dilaton on the boundary can all be matched by taking

$$\epsilon_M = -\frac{H''(\xi_3)}{H'(\xi_3)}\epsilon_E,$$

(A28a)

$$f(\chi) = \frac{F'(\xi_3)}{F(\xi_3)}(4\chi - 3), \quad g(\chi) = \frac{F'(\xi_3)}{F(\xi_3)}(4\chi - 1),$$

(A28b)

and

$$\beta = \gamma = \frac{1}{2}\frac{F'(\xi_3)}{F(\xi_3)}.$$

(A29)

We should note that the lapse function is also matched by these conditions. For the Ernst metric (A4), the lapse function is

$$N = \left[\frac{4L^2 F(\xi_3)(1-\chi)}{A^2\epsilon_E G'(\xi_3)}\right]^{\frac{1}{2}}\lambda^{\frac{1}{1+a^2}}$$

$$\times\left[1 + \frac{1}{4}\epsilon_E(\chi-1)\frac{H''(\xi_3)}{H'(\xi_3)} + \frac{1}{2}\epsilon_E\frac{F'(\xi_3)}{F(\xi_3)}\right],$$

(A30)

while the lapse function for the Melvin metric (A14) is

$$N = \left[\frac{2(1-\chi)}{\bar{A}^2\epsilon_M}\right]^{\frac{1}{2}}\Lambda^{\frac{1}{1+a^2}}\left\{1 - \frac{1}{4}\epsilon_M(\chi-1)\right.$$

(A31)

$$\left. + \frac{1}{2}\epsilon_E g(\chi) - \epsilon_E[\chi f(\chi) - (\chi-1)g(\chi)]\right\}.$$

We see that (A28) and (A29) make (A30) and (A31) equal as well.

The extrinsic curvature of the boundary embedded in the Ernst solution is

$$^2K = \frac{A\epsilon_E^{1/2}G'(\xi_3)^{1/2}}{LF(\xi_3)^{1/2}\lambda^{\frac{1}{1+a^2}}}\left[1 + \frac{1}{4}\epsilon_E\frac{H''(\xi_3)}{H'(\xi_3)}(4\chi - 3)\right.$$

(A32)

$$\left. - \frac{1}{2}\epsilon_E\frac{F'(\xi_3)}{F(\xi_3)}(4\chi - 3)\right],$$

while the extrinsic curvature of the boundary embedded in the Melvin solution is

$$^2K_0 = \frac{\bar{A}\epsilon_M^{1/2}\sqrt{2}}{\Lambda^{\frac{1}{1+a^2}}}\left[1 - \frac{1}{4}\epsilon_M(4\chi - 3)\right. \qquad \text{(A33)}$$
$$\left. - \frac{1}{2}\epsilon_E\frac{F'(\xi_3)}{F(\xi_3)}(24\chi - 13)\right].$$

Using the matching conditions (A26) and (A28), we may now evaluate

$$^2K - {}^2K_0 = \frac{5A\epsilon_E^{3/2}G'(\xi_3)^{1/2}}{LF(\xi_3)^{1/2}\lambda^{\frac{1}{1+a^2}}}\frac{F'(\xi_3)}{F(\xi_3)}(2\chi - 1). \quad \text{(A34)}$$

Therefore, taking the limit $\epsilon_E \to 0$, the Hamiltonian is

$$H_E = -\frac{1}{4}\int_0^1 d\chi N\sqrt{h}(^2K - {}^2K_0) \qquad \text{(A35)}$$
$$= -\frac{5L^2F'(\xi_3)}{A^2G'(\xi_3)}\int_0^1 d\chi(2\chi - 1) = 0,$$

where h is the determinant of the metric [(A18) or (A22)]. Thus, (1) and (2) still hold, which we will now confirm by direct calculation.

3. Horizon area and instanton action

We begin by calculating the difference in area. The area of the black hole is now given by

$$\mathcal{A}_{\text{BH}} = \frac{F(\xi_2)\Delta\varphi_E(\xi_4 - \xi_3)}{A^2(\xi_3 - \xi_2)(\xi_4 - \xi_2)} \qquad \text{(A36)}$$
$$= \frac{4\pi F(\xi_2)L^2}{A^2G'(\xi_3)}\frac{(\xi_4 - \xi_3)}{(\xi_3 - \xi_2)(\xi_4 - \xi_2)},$$

and the area of the acceleration horizon in the Ernst solution, inside a boundary at $x = \xi_3 + \epsilon_E$, is

$$\mathcal{A}_E = \frac{F(\xi_3)\Delta\varphi_E}{A^2}\int_{x=\xi_3+\epsilon_E}^{x=\xi_4}\frac{dx}{(x - \xi_3)^2} \qquad \text{(A37)}$$
$$= -\frac{4\pi F(\xi_3)L^2}{A^2G'(\xi_3)(\xi_4 - \xi_3)} + \pi\rho_E^2,$$

where $\rho_E^2 = 4F(\xi_3)L^2/[G'(\xi_3)A^2\epsilon_E]$ now. Also, $\mathcal{A}_M$ is still given by (50). The boundary conditions in this case are that the proper length of the boundary, the integral of the gauge potential around the boundary, and the value of the dilaton at the boundary are the same.

The proper length of the boundary in the dilaton Ernst solution is

$$l_E = \frac{4\pi 2^{\frac{1-a^2}{1+a^2}}\rho_E^{\frac{a^2-1}{1+a^2}}}{[(1+a^2)\widehat{B}_E^2]^{\frac{1}{1+a^2}}}\left\{1 + \frac{F(\xi_3)L^2}{G'(\xi_3)A^2}\frac{1}{\rho_E^2}\right. \qquad \text{(A38)}$$
$$\left. \times \left[\frac{a^2-1}{1+a^2}\frac{H''(\xi_3)}{H'(\xi_3)} - \frac{2F'(\xi_3)}{F(\xi_3)}\right] - \frac{4}{\widehat{B}_E^2\rho_E^2(1+a^2)^2}\right\},$$

and the proper length of the boundary in the dilaton Melvin solution is

$$l_M = \frac{4\pi 2^{\frac{1-a^2}{1+a^2}}\rho_M^{\frac{a^2-1}{1+a^2}}}{[(1+a^2)\widehat{B}_M^2]^{\frac{1}{1+a^2}}}\left[1 - \frac{4}{\widehat{B}_M^2\rho_M^2(1+a^2)^2}\right]. \quad \text{(A39)}$$

It is interesting to note that the proper length behaves quite differently for $a^2 < 1$ and $a^2 > 1$. The integral of the gauge potential around the boundary curve is, in the dilaton Ernst solution

$$\oint A_\varphi d\varphi = \frac{4\pi e^{a\phi_0}}{(1+a^2)L^2a^2\widehat{B}_E}\left[1 - \frac{4}{\widehat{B}_E^2\rho_E^2(1+a^2)}\right], \qquad \text{(A40)}$$

while for the dilaton Melvin solution it is

$$\oint A_\varphi d\varphi = \frac{4\pi}{(1+a^2)\widehat{B}_M}\left[1 - \frac{4}{\widehat{B}_M^2\rho_M^2(1+a^2)}\right]. \quad \text{(A41)}$$

Finally, the dilaton field at the boundary is, for the dilaton Ernst solution

$$e^{-2a\phi} = e^{-2a\phi_0}\Lambda^{\frac{2a^2}{1+a^2}}\frac{F(\xi_3)}{F(\xi_3 + \epsilon_E)} \qquad \text{(A42)}$$
$$= e^{-2a\phi_0}L^{2a^2}\left[\frac{(1+a^2)\widehat{B}_E^2\rho_E^2}{4}\right]^{\frac{2a^2}{1+a^2}}$$
$$\times \left[1 + \frac{4a^2}{1+a^2}\frac{F(\xi_3)L^2H''(\xi_3)}{G'(\xi_3)A^2H'(\xi_3)}\frac{1}{\rho_E^2}\right.$$
$$\left. + \frac{8a^2}{(1+a^2)^2}\frac{1}{\widehat{B}_E^2\rho_E^2} - \frac{4F'(\xi_3)L^2}{G'(\xi_3)A^2}\frac{1}{\rho_E^2}\right],$$

while for dilaton Melvin it is

$$e^{-2a\phi} = \left[\frac{(1+a^2)\widehat{B}_M^2\rho_M^2}{4}\right]^{\frac{2a^2}{1+a^2}}\left[1 + \frac{8a^2}{(1+a^2)^2}\frac{1}{\widehat{B}_M^2\rho_M^2}\right]. \qquad \text{(A43)}$$

We now write

$$\widehat{B}_M = \widehat{B}_E\left(1 + \frac{\beta}{\rho_E^2}\right), \quad \rho_M = \rho_E\left(1 + \frac{\alpha}{\rho_E^2}\right), \quad \text{(A44)}$$

and

$$e^{a\phi_0} = L^{a^2}\left(1 - \frac{\gamma}{\rho_E^2}\right). \qquad \text{(A45)}$$

We may solve for $\alpha, \beta,$ and γ by setting the various quantities equal perturbatively. This gives

$$\alpha = \frac{F(\xi_3)L^2}{G'(\xi_3)A^2}\left[\frac{H''(\xi_3)}{H'(\xi_3)} - \frac{2F'(\xi_3)}{F(\xi_3)}\right], \qquad \text{(A46)}$$

$$\beta = \gamma = \frac{2F'(\xi_3)L^2}{G'(\xi_3)A^2}. \qquad \text{(A47)}$$

We may now calculate the difference in area:

$$\Delta\mathcal{A} = -\frac{4\pi L^2 F(\xi_3)}{A^2 G'(\xi_3)(\xi_4 - \xi_3)} - 2\pi\alpha \qquad (A48)$$

$$= -\frac{4\pi L^2 F(\xi_3)}{A^2 G'(\xi_3)}\left[\frac{1}{(\xi_4 - \xi_3)} + \frac{H''(\xi_3)}{2H'(\xi_3)} - \frac{F'(\xi_3)}{F(\xi_3)}\right]$$

$$= -\frac{4\pi L^2 F(\xi_3)}{A^2 G'(\xi_3)}\left[\frac{(\xi_2 - \xi_1)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} + \frac{F'(\xi_3)}{a^2 F(\xi_3)}\right].$$

For the extreme case, $\xi_2 = \xi_1$, and so,

$$-\frac{1}{4}\Delta\mathcal{A} = \frac{\pi L^2 F'(\xi_3)}{a^2 A^2 G'(\xi_3)}, \qquad (A49)$$

which agrees with the expression for the instanton action in [6]. For the nonextreme case,

$$-\frac{1}{4}(\Delta\mathcal{A} + \mathcal{A}_{BH}) = \frac{\pi L^2}{A^2 G'(\xi_3)}\left[\frac{F'(\xi_3)}{a^2} \qquad (A50)\right.$$

$$+\frac{F(\xi_3)(\xi_2 - \xi_1)}{(\xi_3 - \xi_2)(\xi_3 - \xi_1)}$$

$$\left.-\frac{F(\xi_2)(\xi_4 - \xi_3)}{(\xi_4 - \xi_2)(\xi_3 - \xi_2)}\right]$$

$$= \frac{\pi L^2 F'(\xi_3)}{a^2 A^2 G'(\xi_3)},$$

where we have used (A16) to cancel the last two terms.

Now we turn to the direct calculation of the action. The contribution to the action from a boundary at $x - y = \epsilon_E$ embedded in the Ernst solution is [6]

$$I_E = -\frac{1}{8\pi}\int_{x-y=\epsilon_E} d^3 x\sqrt{h}e^{-\phi/a}\nabla_\mu(e^{\phi/a}n^\mu) \qquad (A51)$$

$$= \frac{\pi L^2}{A^2 G'(\xi_3)}\left[-\frac{3F(\xi_3)}{\epsilon_E} + \frac{F'(\xi_3)}{a^2}\right].$$

As the solution is independent of τ, the metric on this boundary is just $^{(3)}ds^2 = {}^{(2)}ds^2 + N^2 d\tau^2$, where $^{(2)}ds^2$ is given by (A18), and the gauge field and dilaton on the boundary are (A20) and (A21). Thus, if we assume the boundary in the Melvin solution has the form (A17), then we may see that (A28) and (A29) will match the metric, gauge field, and dilaton on the boundary. The contribution to the action from the boundary embedded in the Melvin solution is then

$$I_M = -\frac{1}{8\pi}\int_{bdry} d^3 x\sqrt{h}e^{-\phi/a}\nabla_\mu(e^{\phi/a}n^\mu) \qquad (A52)$$

$$= \frac{\pi}{8\bar{A}^2\epsilon_M}\int_0^1 d\chi\left[-12 + 5\epsilon_M(2\chi - 1)\right.$$

$$\left.+\frac{103F'(\xi_3)}{2F(\xi_3)}\epsilon_E(2\chi - 1)\right]$$

$$= -\frac{3\pi}{2\bar{A}^2\epsilon_M}.$$

Thus, using (A26), we may evaluate the action,

$$I_{Ernst} = I_E - I_M = \frac{\pi L^2 F'(\xi_3)}{A^2 G'(\xi_3)a^2}, \qquad (A53)$$

which is in perfect agreement with [6]. As (A53) agrees with (A49) and (A50), we have explicitly shown that (1) and (2) hold for general a.

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