Universal scaling and echoing in the gravitational collapse of a complex scalar field

Eric W. Hirschmann*

Department of Physics, University of California, Santa Barbara, California 93106-9530

Douglas M. Eardley[†]

Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030

This paper studies the gravitational collapse of a complex scalar field at the threshold for black hole formation, assuming that the collapse is spherically symmetric and continuously self-similar. A new solution of the coupled Einstein-scalar field equations is derived, after a small amount of numerical work with ordinary differential equations. The universal scaling and echoing behavior discovered by Choptuik in spherically symmetrical gravitational collapse appears in a somewhat different form. The properties of the end state of the collapse are derived: The collapse leaves behind an irregular outgoing pulse of scalar radiation, with an approximately flat spacetime within it.

PACS number(s): 04.20.Jb

I. INTRODUCTION

Recently there has been a lot of new interest in gravitational collapse just at the threshold for the formation of black holes, inspired by the striking numerical results of Choptuik [1] on the spherically symmetric gravitational collapse of scalar field configurations. Further numerical results for vacuum relativity in axial symmetry by Abrahams and Evans [2] suggest that the phenomena discovered by Choptuik is not just restricted to spherical symmetry, and a number of extensions have been proposed [3–8].

Gravitational collapse has two kinds of possible end states, according to current views. The first end state consists of a black hole, plus some outgoing matter and outgoing gravitational radiation. The second end state consists of a stationary remnant star, plus some outgoing matter and outgoing gravitational radiation, but no black hole. The choice between these end states depends on initial conditions; gravitational collapse ends in a black hole only if the gravitational field becomes strong enough, in the sense of the singularity theorems [9].

A third possibility, the naked singularity, is thought not to occur (cosmic censorship conjecture [10]) but is not rigorously ruled out.

The thought experiment employed by Choptuik, in the context of numerical relativity, is to "tune" across the critical threshold in the space of initial conditions that separates the non-black-hole end state from the black hole end state, and to carefully study the critical behavior of various quantities at this threshold [1,11]. Among the infinite number of parameters that characterize the initial state, he chooses to tune a single convenient parameter p that influences the strength of gravity in the initial state. One could imagine that the threshold behavior would be intricate, and depend strongly on exactly how one tunes across the threshold. However, he finds impressive evidence that the threshold behavior is, in important respects, universal.

For gravitational collapse of fermionic matter from suitably regular initial conditions, the Chandrasekhar mass sets the scale of minimum mass for any black hole, as long as the initial conditions are sufficiently regular. Some condition on regularity of the initial conditions is clearly necessary; otherwise, one could create black holes of arbitrarily small mass with focused beams of ultrarelativistic fermions, evading the threshold set by the Chandrasekhar limit. Barring irregular initial conditions, the black hole mass $M_{\rm BH}(p)$ for fermionic configurations therefore behaves in a simple and rather uninteresting way at threshold: It is a step function of p.

For gravitational collapse of bosonic matter from regular initial conditions, one might have thought that $M_{\rm BH}(p)$ would likewise be a step function of p, with the role of the absent Chandrasekhar mass being played by some mass scale of the initial data. However, this is not the case. In the spherically symmetric collapse of a massless real scalar field ϕ coupled to gravity, Choptuik finds the power-law behavior

$$M_{\rm BH}(p) \propto (p - p^*)^\gamma, \quad \gamma \approx 0.37, \tag{1}$$

at threshold, and conjectures that γ might be universal. For a massless scalar field with no self-coupling, no bound star can exist, so that the end state consists of outgoing scalar radiation, plus a possible black hole.

Furthermore, exactly at the threshold p^*, he finds a unique field configuration acting as an attractor for all nearby initial conditions on threshold — and so as an "intermediate attractor" for all initial conditions. This field configuration, which we will call a "Choptuon," has a discrete self-similarity, by virtue of which it exhibits a

---
*Electronic address: ehirsch@dolphin.physics.ucsb.edu
[†]Electronic address: doug@itp.ucsb.edu

striking, recurrent "echoing" behavior: It repeats itself at ever-decreasing time and length scales,

$$t' = e^{-n\Delta} t \ , \tag{2a}$$

$$r' = e^{-n\Delta} r \ , \tag{2b}$$

$$ds'^2 = e^{-2n\Delta} ds^2 \ , \tag{2c}$$

$$\phi(t', r') = \phi(t, r) \tag{2d}$$

$$(n = 1, 2, 3, \dots) \ , \tag{2e}$$

where (t, r) are spherical coordinates and $\Delta \approx \ln 30$ is a constant belonging to the Choptuon, determined numerically [1]. The solution is thus invariant under a discrete family of scale transformations. This Choptuon is itself regular, and acts as an intermediate attractor for a very wide class of regular initial data in spherical symmetry.

Why study the field configuration exactly at threshold? After all, we understand what happens just below threshold (outgoing radiation) and just above (tiny black hole plus outgoing radiation). Beyond the obvious fascination of the Choptuons, there are at least two important motivations. First, Choptuons amount to a new kind of counterexample to some formulations of the cosmic censorship conjecture, a counterexample that cannot be blamed on bad choice of matter fields. Technically they are a counterexample because regions of arbitrarily strong curvature are visible to observers at future null infinity. One can always bypass this new counterexample by reformulating the conjecture — for instance, to specify that *generically* such behavior does not occur — but the more serious point is that Choptuons threaten to obstruct any proof of the cosmic censorship conjecture using the global theory of nonlinear partial differential equations. Therefore it will be necessary to confront and understand them.

Second, and even more importantly, Choptuons represent in principle a means by which effects of extremely strong field gravity, even quantum gravity, can be observable in the present Universe. An experimenter, tuning across the black hole threshold in a succession of gravitational collapses, should be able to observe events in which some outgoing radiation appears from regions where the spacetime curvature is as strong as the Planck value, and where the quantum behavior of general relativity or string theory may be studied. Therefore, the quantum corrections to the classical Choptuon, or, indeed, the quantum gravitational and stringy generalizations of the Choptuon, demand study.

Striking though they are, the numerical computations involving partial differential equations (PDE's) do not give complete information about the Choptuons. For instance, output is restricted to the domain covered by the numerical coordinate system adopted, which may not cover the whole domain of dependence of the initial data. This makes it hard to study the end state, the spacetime singularity to which the Choptuon collapses, and the burst of radiation that it emits. Therefore it is valuable to find further Choptuons that can be studied by analytic techniques or mostly analytic techniques, including numerical solution of ordinary differential equations (ODE's). Evans and Coleman [12] have proposed the idea

of continuously self-similar Choptuons, a special case of discretely self-similar Choptuons. In particular, they obtained continuously self-similar Choptuons for the spherically symmetric collapse of hot gas. The Evans-Coleman configurations show the threshold behavior of M_{BH} and are intermediate attractors in spherical symmetry, but do not show echoing. Continuously self-similar solutions are convenient because they are governed by ODE's, not PDE's. The greater numerical accuracy obtainable with ODE's is desirable, not for its own sake, but because of the light it may shed on the deeper questions of universality.

The subject of this paper is the behavior at threshold in gravitational collapse of a complex scalar field ϕ, under spherical symmetry. The use of a complex scalar field, rather than a real one, will allow "echoing" to occur in the form of phase oscillations: ϕ changes phase, but not amplitude, under a scale transformation. We thereby construct and study a continuously self-similar, complex Choptuon as solutions of the coupled Einstein-scalar-field equations. In spherical coordinates (t, r) our scalar field takes the form

$$\phi(t, r) = (-t)^{i\omega} f(-r/t), \tag{3}$$

where ω is a constant to be fixed by a nonlinear eigenvalue problem arising from the field equations. This field is continuously self-similar: Under a scale transformation by an arbitrary parameter λ, it transforms solely by a phase factor:

$$t' = e^{-\lambda} t \ , \tag{4a}$$

$$r' = e^{-\lambda} r \ , \tag{4b}$$

$$ds'^2 = e^{-2\lambda} ds^2 \ , \tag{4c}$$

$$\phi(t', r') = e^{-i\omega\lambda} \phi(t, r) \tag{4d}$$

$$(0 < \lambda < \infty). \tag{4e}$$

This solution thus exhibits "linear" phase oscillations, a form of the "echoing," superimposed on an exact continuous scale symmetry. It is thus similar to, but "less nonlinear" than, the solution of Choptuik [1].

We are able to construct a Choptuon as a nonextendible spacetime, singular only at the single point $(t, r) = (0, 0)$, which evolves from regular initial data. A burst of outgoing scalar radiation, irregular but of finite energy, emerges just prior to the retarded time at which the singular point forms at $(0, 0)$, and explodes along the future light cone of that point. The most surprising aspect of this Choptuon is that spacetime appears numerically to be flat throughout the interior of the future light cone of the singular point. Thus, in this example anyway, a Choptuon leaves behind flat space, with no numerically measured radiation after the irregular outgoing burst.

Since a real ϕ is a special case of a complex ϕ, all of Choptuik's numerical results immediately apply as special cases to a complex field; in addition, we find that there exist intrinsically complex Choptuons which do not reduce to the real case. Which of these Choptuons is the strongest intermediate attractor is an important question not addressed in this paper; we will return to it in

future papers. Preliminary numerical evidence by Choptuik [13] seems to show that the real Choptuon is at least not rendered unstable when complex degrees of freedom are added.

II. FIELD EQUATIONS

We begin with the spacetime metric in the form used by Choptuik [1]:

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2 , \qquad (5)$$

where $\alpha(t,r)$ and $a(t,r)$ are functions of time and radius. This is an example of a "radial gauge" because the area of spheres (given by the coefficient of $d\Omega^2$) defines the radial coordinate. The radial gauge breaks down at an apparent horizon, and so another coordinate system will need to be used if an apparent horizon appears. The time coordinate is chosen so that gravitational collapse on the axis of spherical symmetry first occurs at $t = 0$ and the metric is regular for $t < 0$. This metric remains invariant in form under transformations

$$t = t(t') \qquad (6)$$

of the time coordinate, and such a transformation can be used to set $\alpha(t,0) = 1$ for $t < 0$ on the axis. By regularity (no cone singularity), $a(t,0) = 1$ on the axis for $t < 0$ as well.

Matter consists of a free, massless, complex scalar field ϕ that obeys the wave equation

$$\Box \phi = 0 ,$$

while the Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$
$$= 8\pi \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi \right) .$$

These equations amount to

$$\frac{1}{r^2} \partial_r \left(\frac{\alpha}{a} r^2 \partial_r \phi \right) = \partial_t \left(\frac{a}{\alpha} \partial_t \phi \right) , \qquad (7a)$$

$$\frac{1}{\alpha} \partial_r \alpha - \frac{1}{a} \partial_r a = \frac{a^2 - 1}{r} , \qquad (7b)$$

$$\frac{2}{a^2 r} \left(\frac{1}{\alpha} \partial_r \alpha + \frac{1}{a} \partial_r a \right) = 8\pi \left(\frac{1}{a^2} \partial_r \phi^* \partial_r \phi + \frac{1}{\alpha^2} \partial_t \phi \partial_t \phi^* \right) , \qquad (7c)$$

$$\frac{2}{ar} \partial_t a = 4\pi \left(\partial_r \phi^* \partial_t \phi + \partial_r \phi \partial_t \phi^* \right) . \qquad (7d)$$

These equations admit global U(1) symmetries for a constant Λ,

$$\phi' = e^{i\Lambda} \phi, \quad -\infty < \Lambda < \infty , \qquad (8)$$

leaving the metric invariant.

III. CONTINUOUS SELF-SIMILARITY

We now derive the form of the scaling transformations, Eqs. (4), for the fields. We assume spacetime is spheri-

cally symmetric, and admits a homothetic Killing vector field [14,12] ξ obeying

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2g_{\mu\nu}, \qquad (9)$$

which generates a continuous one-parameter family of homothetic motions (self-similarities) on spacetime, Eqs. (4). Here, $\mathcal{L}$ denotes the Lie derivative. In a coordinate system, Eq. (5), ξ may always be taken in the form

$$\xi^\mu \partial_\mu = t\partial_t + r\partial_r, \qquad (10)$$

possibly after a further coordinate transformation, Eq. (6). Equations (9) and (10) are implemented by Eqs. (4a)–(4c). The metric functions are then of the special form

$$\alpha(t,r) = \alpha(-r/t), \quad a(t,r) = a(-r/t). \qquad (11)$$

The minus sign is chosen so that $-r/t > 0$ where $t < 0$ to the past of the singularity. Dimensional analysis (in classical general relativity, not quantum field theory) suggests that $\phi(t,r)$ should have dimensions $(\text{length})^0$. Therefore, should we assume that ϕ is invariant under ξ, $\mathcal{L}_\xi \phi = 0$ or $\phi(t,r) = \phi(r/t)$? No, the global U(1) symmetry must be accommodated too. Quite generally, one must allow for spacetime symmetries to get mixed up with internal symmetries; for instance, the Maxwell equations for the potential A_μ are conformally invariant only in the sense that a gauge transformation be allowed to accompany each conformal transformation. In this case, we must allow some U(1) transformation, Eq. (8), to accompany each scale transformation:

$$\mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi = i\omega \phi , \qquad (12)$$

under an infinitesimal scale transformation, or

$$\phi(t',r') = \exp(-i\omega\lambda)\phi(t,r) ,$$

under the finite scale transformation, Eqs. (4), with ω a constant of the solution.

This transformation law for ϕ can be conveniently implemented by adopting the form

$$\phi(t,r) = (-t)^{i\omega} f(z). \qquad (13)$$

The time coordinate t has now been redefined, Eq. (6), so that the first singularity of the collapse is at $t = 0$; and a new independent variable z has been introduced by

$$z = -r/t; \qquad (14)$$

z is invariant under scale transformations, Eqs. (4). Also, from Eq. (11),

$$\alpha = \alpha(z), \quad a = a(z). \qquad (15)$$

With the scale symmetry implemented by Eqs. (13), (14), and (15), the next step is to solve the field equations. We can give initial conditions in the hypersurface $t = -1$ and then evolve forward in time. It follows from a theorem of Berger [15] that the Einstein equations are

compatible with homothetic symmetry, in the sense that scale-invariant initial data will always evolve to a scale-invariant spacetime; therefore, the field equations ought to have a solution.

Remarks should be made as to how suitable the assumption of self-similarity is. Self-similar spacetimes can never be asymptotically flat [because the Arnowitt-Deser-Misner (ADM) or Bondi mass would define a length scale, breaking self-similarity]; furthermore, there are some reasons to believe that self-similar spacetimes can never be spatially compact [16]. Spatial compactness is irrelevant here, but should not gravitational collapse be modeled by an asymptotically flat spacetime? However, the self-similar solution should be interpreted as a model of the gravitationally collapsing region out to some radius R, where it can match smoothly onto a non-self-similar, asymptotically flat region. To anticipate, we

will find that our solutions always have a certain horizon called the past similarity horizon; as long as the matching radius R is outside the past similarity horizon, the gravitational collapse that we study here will remain entirely within the domain of dependence of the self-similar region of the initial data. For instance, for the complex Choptuon discussed below, a value $R > 5.004$ suffices.

Transform the metric variables α, a to a new set b, u given by

$$b(z) = \alpha(z)/a(z),$$

$$u(z) = a^2(z) - 1.$$

Under the similarity hypothesis, $b = b(z)$ and $u = u(z)$ are functions of $z \equiv -r/t$ alone. Under this transformation, the metric, Eq. (5), becomes

$$ds^2 = e^{2\tau} \left\{ (1+u) \left[-(b^2 - z^2)d\tau^2 + 2z d\tau dz + dz^2 \right] + z^2 d\Omega^2 \right\} , \tag{16}$$

where $\tau \equiv \ln t$. Following Choptuik's notation, represent ϕ in terms of complex functions $(\Phi(t,r), \Pi(t,r))$ where

$$\Phi = \partial_r \phi = (-t)^{i\omega - 1} q(z) ,$$
$$\Pi = \frac{a}{\alpha} \partial_t \phi = -(-t)^{i\omega - 1} p(z) ,$$

where the functions $q(z)$, $p(z)$ are

$$q(z) = \frac{df}{dz} ,$$
$$p(z) = \frac{a}{\alpha} \left(i\omega f - z \frac{df}{dz} \right) .$$

The field equations now take the form

$$\frac{d}{dz} \begin{pmatrix} q \\ p \end{pmatrix} = -\frac{1}{z} \begin{pmatrix} u+2 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \frac{1}{\Delta} \begin{pmatrix} -z & b \\ b & -z \end{pmatrix} \begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} , \tag{17a}$$

$$\frac{db}{dz} = \frac{bu}{z} , \tag{17b}$$

$$\frac{du}{dz} = (u+1) \left[4\pi z(|q|^2 + |p|^2) - \frac{u}{z} \right] , \tag{17c}$$

$$\frac{du}{dz} = -8\pi(u+1)b\,\mathrm{Re}(q^*p), \tag{17d}$$

where

$$\beta_\pm = i\omega + u \pm 1,$$
$$\Delta = b^2 - z^2.$$

The boundary conditions are as follows. At $z = 0$, regularity of solutions on the axis of spherical symmetry demands

$$b(0) = 1,$$
$$u(0) = 0,$$
$$q(0) = 0.$$

The boundary value $P = p(0)$ of p is a free boundary condition, and will henceforth be taken real and non-

negative by a global phase transformation of ϕ, without loss of generality. Integration of Eqs. (17) from $z = 0$ to $z = +\infty$ is tantamount to solving the initial value problem in the spacelike hypersurface $t = -1$, along with the further constraint equations that enforce self-similarity on the initial data [15]. Evolution in t is then fixed by self-similarity.

The wave equation (17a) has a singular point when Δ vanishes, i.e., when $b(z) = z$. This value of z will be called z_2, and the point will be called a *similarity horizon*. It represents the null hypersurface in spacetime where the homothetic Killing vector, timelike near the axis, becomes null. The value of u there,

$$\kappa \equiv 1 - u(z_2),$$

is free, while $b(z_2) = z_2$ is of course determined by z_2.

The flat space wave equation provides a simple introduction; see Fig. 1. Here the metric is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \qquad (18)$$

and the general spherically symmetric solution to the massless wave equation is

$$\phi(t,r) = \frac{f(t-r) + g(t+r)}{r} ,$$

where f and g are arbitrary functions. The homothetic Killing vector is $\xi = t\partial_t + r\partial_r$ and all self-similar solutions, Eq. (3), are linear combinations of

$$\phi^{\pm}(t,r) = \frac{(t \mp r)^{i\omega+1}}{r} = (-t)^{i\omega} \frac{(1 \pm z)^{i\omega+1}}{z} , \qquad (19)$$

where

$$z = -r/t.$$

Here ϕ^+ is an outgoing wave, is regular at the past similarity horizon $z = +1$, and is irregular at the future similarity horizon $z = -1$. In contrast, ϕ^- is an ingoing wave, is irregular at the past similarity horizon $z = +1$, and is regular at the future similarity horizon $z = -1$. All solutions are regular at the hypersurface $t = 0$ or $z = \infty$, except at the single spacetime point $(t,r) = (0,0)$. Only the linear combination $\phi^+ - \phi^-$ is regular at the origin $r = 0$ of spherical coordinates (for all $t \neq 0$). We can transform from the coordinates (t,r) to the coordinates (τ,z) where $\tau = \ln t$. Then the metric becomes

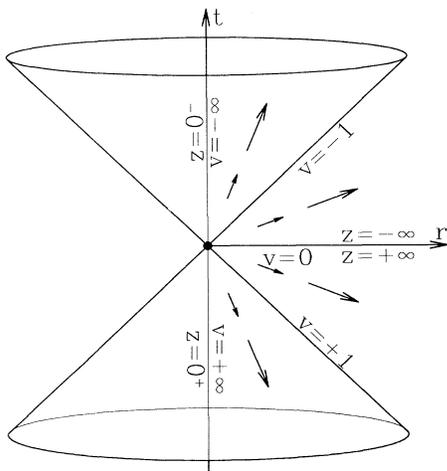$$ds^2 = e^{2\tau} \left[-(1-z^2)d\tau^2 + 2z d\tau dz + dz^2 + z^2 d\Omega^2 \right] . \qquad (20)$$

FIG. 1. Coordinate systems in Minkowski spacetime. Arrows show the homothetic Killing vector field $\xi \equiv t\partial_t + r\partial_r$. Shown also are the past and future light cones of the origin; in the terminology of this paper, these light cones are past and future similarity horizons, where ξ becomes null. For coordinates z and v, see Sec. III.

The coordinates (τ, z) have a singularity on the spacelike hyperplane $t = 0$. Yet another coordinate system for flat spacetime, regular at $t = 0$, is (ρ, v), where $\rho = \ln(r)$ and $v = 1/z = -t/r$; the metric is

$$ds^2 = e^{2\rho} \left[-dv^2 - 2v dv d\rho + (1-v^2)d\rho^2 + d\Omega^2 \right] . \qquad (21)$$

Returning now to the curved spacetime metric, Eq. (5), analysis of the wave equation near $z = z_2$ shows that $q(z)$ and $p(z)$ likewise have a regular solution and an irregular solution at the similarity horizon. The regular solution behaves like

$$f \sim \text{const} , \qquad (22a)$$

$$(q,p) \sim \text{const} , \qquad (22b)$$

and the irregular solution behaves like

$$f \sim (z - z_2)^{(i\omega+1)/\kappa} , \qquad (23a)$$

$$(q,p) \sim (z - z_2)^{(i\omega+1)/\kappa-1} . \qquad (23b)$$

This behavior is identical to the case of flat spacetime except for one aspect: the presence of the quantity

$$\kappa \equiv 1 - \frac{db}{dz}\bigg|_{z=z_2}$$

in the exponent. For flat spacetime $\kappa = 1$. In a self-similar spacetime, κ plays a role parallel to that of the surface gravity of the event horizon of a black hole in a stationary spacetime, and in this paper we will simply call it the "surface gravity."

The regular solution again represents outgoing radiation crossing the similarity horizon, and the irregular solution represents ingoing radiation propagating along the horizon. Since the subject of study is gravitational collapse from regular initial conditions, and since the similarity horizon $z = z_2$ is in the Cauchy development of the initial data — it is to the past of the earliest singular point $(t,r) = (0,0)$ — regularity of ϕ will be demanded on the similarity horizon. This means that a linear combination of q and p must vanish at $z = z_2$,

$$\beta_+ q(z_2) - \beta_- p(z_2) = 0, \qquad (24)$$

so that, for instance, we can choose $p(z_2)$ freely as a complex-valued boundary condition at the similarity horizon, and $q(z_2)$ is then fixed.

To sum up, the free data for the system will be taken as

$$\omega, \qquad (25a)$$

$$P \equiv p(0) \quad (\text{real}), \qquad (25b)$$

$$z_2, \qquad (25c)$$

$$\kappa = 1 - u(z_2), \qquad (25d)$$

$$p(z_2) \quad (\text{complex}), \qquad (25e)$$

amounting to six real constants. Solutions to the system Eq. (17) must be found by searching in the six-dimensional data space.

IV. CONSTRUCTION AND PROPERTIES OF THE COMPLEX CHOPTUON

The numerical methods we used followed Chap. 16 of Ref. [17]: We handled the system as a two-point boundary value problem with one fixed boundary (at $z = 0$) and one free boundary (at $z = z_2$). We "shot" with an adaptive step ODE solver from each boundary point, to meet at a point z_1 in the middle. The six free data values were adjusted in an outer loop with a Newton's-method solver for nonlinear equations, the six nonlinear equations being the matching conditions for the six ODE's (17) at z_1. Convergence of Newton's method then identified a solution on the domain $0 \leq z \leq z_2$. Initial work was done with a Runge-Kutta integrator, while a Bulirsch-Stoer integrator was subsequently used for higher precision. The solution was then continued to larger z without the need for further boundary conditions.

To enforce numerically the boundary condition on the horizon is difficult, because the irregular part of ϕ, which we want to annul, vanishes anyway if $\kappa < 1$, and the smaller κ, the faster it vanishes; see Eq. (23). Said differently, the difficulty is that we want to start a purely regular solution at the horizon and continue it, but the unwanted irregular solution grows rapidly as it is continued away from the solution; this is a numerically unstable situation. We handled this by computing analytically the second derivative of the regular solution at $z = z_2$, and using a second-order Taylor expansion to start the solution there. Even so we encountered numerical problems for $\kappa \lesssim 1/3$, as discussed in the Appendix.

A single new solution was found on a domain $0 \leq z < z_2$, obtained with the values

FIG. 2. Behavior of the complex Choptuon, the solution for $\omega = 1.915\,444\,6$, over the domain $0 \leq z \leq 10$. The past similarity horizon is located at $z_2 = 5.003\,538\,0$, and ω was determined by demanding regularity there. (a) The complex functions $q(z)$ and $p(z)$ which represent the scalar field ϕ. (b) The metric functions $b(z)$ and $u(z)$.

$$\omega = 1.915\,444\,6 \pm 0.000\,000\,1,$$

$$P \equiv p(0) = 0.672\,172\,63 \pm 0.000\,000\,4,$$

$$z_2 = 5.003\,538\,0 \pm 0.000\,000\,2,$$

$$u(z_2) \equiv 1 - \kappa = 0.397\,072\,05 \pm 0.000\,000\,03,$$

$$p(z_2) = (0.020\,305\,344 \pm 0.000\,000\,003) + (0.007\,153\,158 \pm 0.000\,000\,002)i \ .$$

This solution will be called the "complex Choptuon." Its complex conjugate solution, with the sign of ω changed, also of course exists. Figure 2 displays the functions $q(z)$, $p(z)$, $b(z)$, $u(z)$ for the complex Choptuon.

We also found a solution at $\omega = 0$, corresponding to a real scalar field ϕ; this, however, is nothing but the flat ($k = 0$) Robertson-Walker cosmological model with a real scalar field as matter, reversed in time. Further details of the numerical results are reported in the Appendix.

The complex Choptuon solution was continued smoothly past $z = z_2$ to large z. The nature of the point $z = \infty$ on the z axis must now be clarified. Examination shows this to be a singular point of Eqs. (17). However, on physical grounds there should be no spacetime singularity here, since $z = \infty$ corresponds to the spacelike hypersurface $t = 0$, which should be regular except at the axis, since it lies in the Cauchy development of the initial data. Any apparent singularity there must fall under
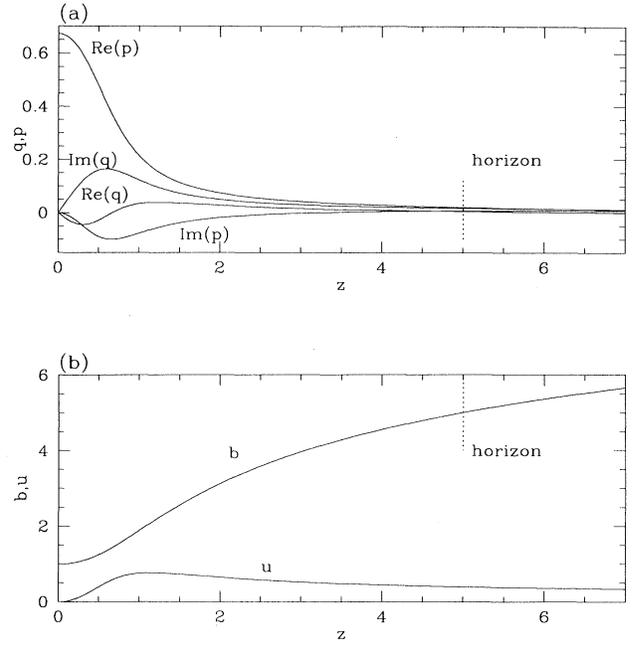
strong suspicion as a coordinate singularity caused by a bad choice of time coordinate, Eq. (6), near $t = 0$. Indeed, the system of equations can be rendered regular at $z = \infty$ by the change of variables

$$dw = b(z)\frac{dz}{z^2}, \quad w = 0 \text{ at } z = \infty, \tag{26a}$$

$$\begin{pmatrix} Q(w) \\ P(w) \end{pmatrix} = z^{1-i\omega} \begin{pmatrix} q(z) \\ p(z) \end{pmatrix}, \tag{26b}$$

$$v(w) = \frac{b(z)}{z}, \tag{26c}$$

$$u(w) = u(z). \tag{26d}$$

The spacetime metric, Eqs. (5) and (16), becomes

$$ds^2 = e^{2\rho} \left\{ (1+u) \left[-dw^2 + 2v\, dw\, d\rho + (1-v^2) d\rho^2 \right] + d\Omega^2 \right\}, \tag{27}$$

where $\rho \equiv \ln r$. Moreover, the scalar field written as Eq. (3) appears irregular at $z = \infty$, but can be written in a regular form as

$$\phi(t,r) = r^{i\omega} F(w), \tag{28}$$

and $(Q(w), P(w))$ are derived in a regular way from $F(w)$. In terms of the new independent variable w, the equations of motion become

$$\frac{d}{dw}\begin{pmatrix} Q \\ P \end{pmatrix} = \frac{1}{1-v^2}\begin{pmatrix} v & -1 \\ -1 & v \end{pmatrix}\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}, \tag{29a}$$

$$\frac{dv}{dw} = u - 1, \tag{29b}$$

$$\frac{du}{dw} = \frac{u+1}{v}\left[4\pi(|Q|^2 + |P|^2) - u \right], \tag{29c}$$

$$\frac{du}{dw} = -8\pi(u+1)\mathrm{Re}(Q^*P). \tag{29d}$$

From the last two equations, $u(w)$ can be expressed as

$$u(w) = 4\pi[|Q|^2 + |P|^2 + 2v\mathrm{Re}(Q^*P)]. \tag{30}$$

The point $z = \infty$ is now marked by $v(w) = 0$, and the system is clearly regular at this point. [The apparent pole at $v = 0$ in Eq. (29c) is canceled by a zero in the numerator, from Eq. (30).] The singular points $v = \pm 1$ are, in contrast, true singular points of the system, and are, respectively, the past similarity horizon at $z = z_2$ and a new future similarity horizon.

Integration of Eqs. (29) is tantamount to integration of the field equations in a timelike hypersurface $r = 1$. Because of self-similarity, this in turn is tantamount to evolution in a timelike direction of a whole spatial hypersurface $0 \leq r < \infty$, in the region outside of any similarity horizons.

Numerically we proceed as follows. Having found a solution for $0 \leq z \leq z_2$, we integrate it a little further in z and then transform by Eqs. (26) to the w variables. At this point v is a little less than 1 and decreasing. Then we integrate in w (noting that the solution, if it approaches $w = 0$, always continues smoothly through) until one of two things happen.

The first alternative is that $u \to \infty$ at some point; we interpret such a point as an apparent horizon in spacetime, where radial gauge coordinates [which we use throughout; see Eq. (5)] break down. Such an apparent horizon probably means that the solution represents, not a Choptuon, but a black hole that is growing by self-similar accretion of scalar field. We believe that such a solution, if continued further in a different coordinate system, would always encounter a spacetime singularity within the black hole, but we will not pursue this issue here.

The second alternative is that $v \to -1$ at some finite w. We interpret this as a second, future similarity horizon. The complex Choptuon behaves in this way. What

are the proper boundary conditions at the future similarity horizon? Above, at the past similarity horizon, we argued that ϕ should remain regular because it arose from regular initial conditions, and was not influenced by the spacetime singularity to its future. That argument does not hold here, because the future similarity horizon is outside the domain of dependence of the initial data; in particular, it is the Cauchy horizon of that domain. Observers on the future similarity horizon will see data coming from the singularity, and indeed it is a very interesting question to ask what they will see. Therefore no boundary condition at all is enforced on ϕ at the future similarity horizon; (ϕ, Q, P) may be an arbitrary combination of regular and irregular solutions. This is good, because we have already used up all our boundary conditions at the axis and the past similarity horizon, and we would be embarrassed to have to obey further boundary conditions. However, we must still decide how to continue the solution across the horizon, because the horizon represents a singular point of Eqs. (29) and furthermore is a Cauchy horizon in spacetime. Evolution across a Cauchy horizon in classical general relativity is never unique. Quantum considerations, which might fix the evolution in some way, are beyond the scope of this paper, though later papers will treat it. Therefore we will continue with a conservative assumption about how to evolve.

Drop self-similarity for a moment, and look at the general boundary conditions for a wave equation in curved spacetime. A test wave function ϕ is allowed to have discontinuity, or a discontinuity in some higher derivative, across the characteristic surfaces of the wave equation, which are the null hypersurfaces of spacetime. However, if the wave equation is coupled to gravity, a discontinuity in value of ϕ will cause an infinite-mass singularity, and is therefore forbidden: $T_{\mu\nu} \sim (\nabla\phi)^2 \sim \delta(\ )^2$. However, a discontinuity in the first or higher derivative of ϕ is allowed. In terms of the characteristic initial value problem, the initial data for the wave equation on a characteristic hypersurface are the value of ϕ itself, but do not include any derivatives of ϕ.

We now assume that spacetime continues to be self-similar to the future of the future similarity horizon. So turn now back to our self-similar equations for ϕ. The matching rules for ϕ at the future similarity horizon are as follows. The wave function ϕ must be continuous across the horizon, to forbid an infinite-mass singularity at the horizon. The regular part of ϕ represents incoming radiation crossing the horizon from the regular region exterior to it. In contrast, the irregular part of ϕ consists of outgoing radiation originating very close to the spacetime singularity, streaming along the similarity horizon. In w coordinates, the wave variables (Q, P) of the irregular part of the solution oscillate an infinite number of times while approaching the future similarity horizon, and also die as a power law if $\kappa > 0$. However, if $\kappa = 1$, the amplitude of the oscillations remains constant. The irregular part of the solution is allowed to be discontinuous; that

is, the amplitude of the irregular part of (Q, P) can be different on the two sides of the horizon. This causes at worst an (allowable) jump in the stress energy.

In general we expect to find that (Q, P) contains both regular and irregular parts at the future similarity horizon. Then the regular part must be continuous, but the irregular part is allowed to jump arbitrarily. The solution can still be fixed uniquely, if we are willing to postulate that spacetime is smooth along the future time axis $t > 0$, $r = 0$. In that case, ϕ and the spacetime geometry must obey boundary conditions at the future time axis, which are essentially the same as the boundary conditions we have already imposed at the past time axis. Just counting degrees of freedom in the boundary conditions, we expect this boundary condition to fix a unique solution (or perhaps a discrete set of solutions).

However, at this point a numerical "miracle" happens at the future similarity horizon. We find that for the complex Choptuon, ϕ is purely irregular there, so that the regular piece of ϕ vanishes, which entails that $\kappa = 1$ (to an accuracy of 10^{-6}), so that the future similarity horizon carries initial data for flat spacetime with constant ϕ. In particular the mass aspect vanishes on this null hypersurface.

Figure 3 displays the functions $Q(w)$, $P(w)$, $v(w)$, $u(w)$. Note that, at the future similarity horizon marked by $v = -1$, $u = 0$, which means the mass aspect vanishes there; this is the numerical miracle discussed above. As indicated in Fig. 4, the scalar field is irregular at the future similarity horizon, and oscillates an infinite number of times. A distant observer, measuring outgoing radia-
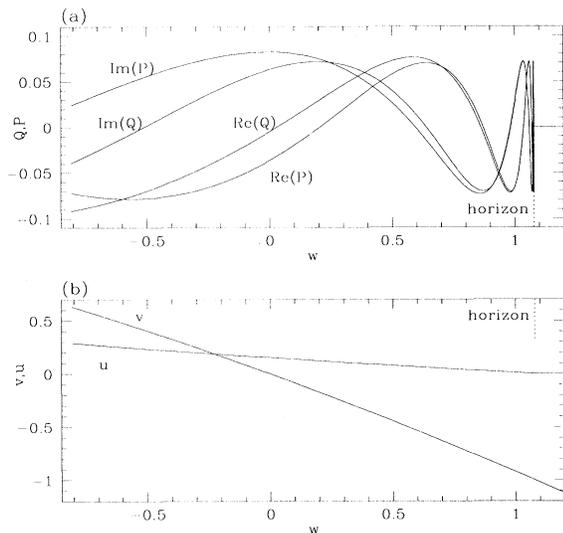

FIG. 4. Same as Fig. 3, except that the horizontal axis is now logarithmic, and shows $\ln(1 + v)$. (a) The complex functions $Q(w)$, $P(w)$ which represent the scalar field ϕ. They oscillate infinitely many times in approaching the future similarity horizon. (b) The metric functions $v(z)$ and $u(z)$, plotted as $\ln(1 + v)$ and $\ln(u)$, both of which approach 0 on the future similarity horizon.

FIG. 3. Behavior of the complex Choptuon, the solution for $\omega = 1.915\,444\,6$, over the domain $-0.8 \leq w \leq 10$. The future similarity horizon is located at the value z_4 of z where $v(z) = -1$. (a) The complex functions $Q(w)$, $P(w)$ which represent the scalar field ϕ. They oscillate infinitely many times in approaching the future similarity horizon. (b) The metric functions $v(z)$ and $u(z)$. Note that $u(z_4) = 0$ to numerical accuracy, showing that the solution matches onto flat spacetime at the future similarity horizon.
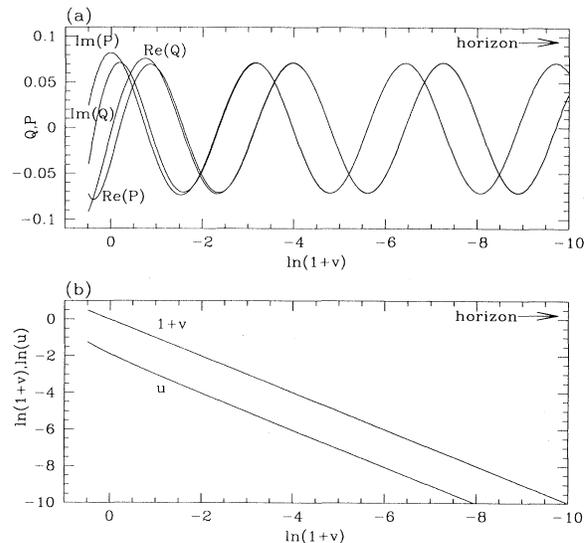
tion in the scalar field, would see such a signal coming from the threshold gravitational collapse.

Consider again the evolution of the solution to the future of the future similarity horizon. In view of the boundary conditions, we are allowed to choose an arbitrary irregular part just to the future of this horizon. The choice of zero irregular part gives flat spacetime to the future of this horizon. Any other choice will give a negative mass in this region, and will create a negative mass naked singularity along the future time axis $t > 0$, $r = 0$. Because we are attempting to evolve across a Cauchy horizon, the choice is not obligatory. However, the most sensible choice is clearly the one that gives flat spacetime. We therefore conclude that the complex Choptuon leaves behind a flat spacetime to the future of the singular point at $(t, r) = (0, 0)$, to an accuracy of 1 part in 10^6.

V. DISCUSSION AND OUTLOOK

In Fig. 5, we show the interpretation arrived at for the complex Choptuon. In region I, there is a collapsing ball of a scalar field, bound by its own self-gravity; it acts as a near zone for scalar radiation. The ball collapses toward the spacetime singularity at the origin, shown as a single point. As the ball collapses, the scalar field is partially trapped by spacetime curvature, but also continually leaks out of the gravitationally bound region, across, the past similarity horizon at $v = +1$, and radiates outwards through regions IIa and IIb. The boundary between regions IIa and IIb is the surface $t = 0$ in Choptuik coordinates or $z = \infty$. However, this is merely a

coordinate singularity, and there is no physical boundary between the two regions; they should be considered as a single region II, which acts as a transition zone between the near and far field for scalar radiation. Spacetime is curved in region II, and in general some backscatter of outgoing radiation to ingoing radiation is to be expected. However, the numerical miracle discussed in Sec. IV indicates that the backscattered ingoing radiation vanishes exactly at the future similarity horizon $v = -1$, for reasons that we do not understand. In any case, an irregular pulse of outgoing radiation propagates outwards, with an infinite number of wave fronts piling up at the future similarity horizon. This is the nature of the pulse that a distant observer sees.

This irregular pulse originates in the "echoing" behavior of the collapse in region I. Choptuik found the "echoing" scale factor

$$e^\Delta \approx 30$$

in his real Choptuon, Eqs. (2). In the complex Choptuon the echoing behavior is somewhat different, but the comparable scale for the solution to recur is measured by 2π rad in the phase oscillation of ϕ, so that

$$e^\Delta = e^{2\pi/\omega} = 26.583\,086 \pm 0.000\,005,$$

a value which clearly differs from Choptuik's. Therefore, this critical exponent Δ is not universal between the two solutions.
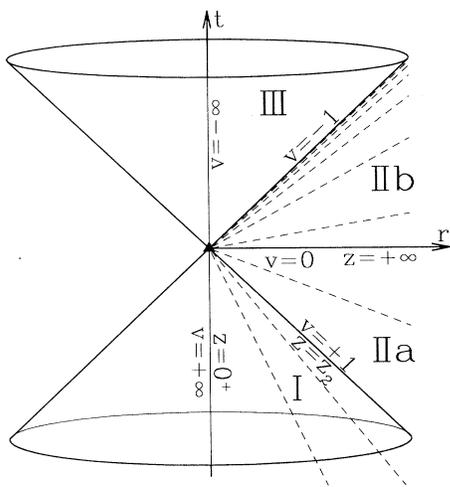
There is an exact self-similar solution for a collaps-

FIG. 5. Interpretation of the complex Choptuon. The similarity horizons lie at $v = +1$ (past) and $v = -1$ (future). Dotted lines show peaks and valleys of one function representing the scalar field [q for $v < -1$, Q for $v > -1$], to indicate its oscillations; however, the U(1) phase rotation [$(-t)^{i\omega}$ for $v < -1$, $r^{i\omega}$ for $v > -1$] is not shown. Region I is a collapsing sphere of gravitationally bound scalar field; in regions IIa and IIb this blends smoothly into an outgoing scalar wave. The outgoing scalar wave oscillates infinitely many times, approaching the future similarity horizon at $v = -1$. Region III is flat.

ing pulse of real scalar field ϕ in spherically symmetric general relativity [18,20,3,4,7], found independently by at least four groups. It would not be surprising to learn of further independent discoveries. This solution, which we shall call the Maithreyan-Christodolou-Brady-Oshiro-Nakamura-Tomimatsu (MCBONT) solution, exhibits the critical behavior of the black hole mass at threshold [3,4,7]. However, the MCBONT solution appears different from the Choptuons in important respects. It does possess a past similarity horizon; however, the scalar field ϕ is not regular but irregular there, having a step in its derivative. A closely related issue is that the MCBONT solution does not evolve from regular initial conditions for gravitational collapse. It remains to be seen whether or not the MCBONT solution is an intermediate attractor. We shall return to this solution in a future paper.

In this paper we have worked exactly at the threshold for black hole formation, and have not addressed the critical behavior of the black hole mass, Eq. (1). This critical behavior can be determined by first-order perturbation theory around the complex Choptuon, just as it can be addressed for the Evans-Coleman hot-gas Choptuons [12,19]. Perturbation theory can also determine whether or not the solution is an attractor. We will return to these issues in a future paper.

Region III, to the future of the singular point, is flat in our solution, to our numerical resolution ($\sim 10^{-6}$). Since the future similarity horizon is a Cauchy horizon, evolution cannot be unique; according to classical relativity, anything could come out of the singular point. However, it is remarkable that the solution admits an evolution into singularity-free and approximately flat spacetime, and we have argued that this is the preferred evolution.

Thus our solution provides one possible answer in classical gravity to the following question: What is the end state of gravitational collapse at the threshold for black hole production? The end state is an outgoing, irregular pulse of radiation from the singular point in the wave zone, together with a most conservative end state in the near zone: approximately flat spacetime.

Further physical insight about this question must ultimately come from quantum gravity or string theory, since strong curvatures comparable to the Planck value occur near the singular point.

Note added. After this paper was submitted, Horne [21] carried out new and more precise numerical work on this problem. His early results show that spacetime is not precisely flat in region III; rather, $u \approx 8.49 \times 10^{-7}$ on the future similarity horizon, and ϕ has a small regular part ($\sim 10^{-4}$) there. Horne also finds that the resultant weak fields within region III can be constructed under the assumption that spacetime is smooth along the future time axis, as discussed in Sec. IV above. Physically this means that backscatter into region III is very weak, but does not vanish exactly.

ACKNOWLEDGMENTS

PHY90-08502, and parts were carried out at the Aspen Center for Physics. We are especially grateful to Matt Choptuik for providing some numerical results by private communication, and to Jim Horne for enlightening communications about this and related problems. We are grateful to J. Traschen and D. Kastor for helpful conversations at ITP, and to A. Rendall for comments on the manuscript. D.M.E. is grateful to C. Evans, R. Price, C. Gundlach, D. Garfinkle, J. Pullin, and B. Schmidt for helpful conversations at Aspen.

APPENDIX

In Table I we present all numerical solutions returned by our algorithm. As explained in Sec. IV, accuracy is lost near the past similarity horizon $z = z_2$, because of decay of the irregular part of ϕ there, Eqs. (23). Physically we would like the solution ϕ to be C^∞ at $z = z_2$. However, our numerical algorithm cannot distinguish between the degree of smoothness, C^3 and C^∞, because Taylor expansions of the regular solution around z_2 were carried out only through second order. (If we had carried them to nth order, the algorithm could distinguish C^n from C^∞, but could not distinguish C^{n+1}.) In turn, the degree of smoothness of the irregular solution depends on κ, Eqs. (23), in such a way that our algorithm is reliable for $\kappa \lesssim 1$, but loses reliability for $\kappa \approx 1/3$. The solutions marked "Reliable? No" in Table I all have $\kappa \approx 1/3$ and therefore we do not know whether ϕ is irreg-

TABLE I. Numerical solutions found. The solution is started at $z = 0$ with values ω and P. The past similarity horizon appears at $z = z_2$. All solutions marked "Reliable? No" have $\kappa \approx 1/3$ there, and hence may have lost accuracy.

ω	P	z_2	Reliable?	Interpretation
0.0	0.16286	1.10668	Yes	Robertson-Walker solution
0.0	0.21412	1.24793	No	Black Hole?
0.15265	0.23468	1.35664	No	Black Hole?
0.29825	0.24502	1.40583	No	Black Hole?
0.43076	0.25919	1.47793	No	Black Hole?
0.55509	0.27670	1.57717	No	Black Hole?
0.68307	0.29923	1.72316	No	Black Hole?
0.81757	0.32998	1.97453	No	Black Hole?
0.94825	0.37269	2.49240	No	Choptuon?
1.04980	0.44740	4.56352	No	Choptuon?
1.91544	0.67217	5.00354	Yes	Complex Choptuon; see Sec. IV

ular and C^3 or regular and C^∞, at $z = z_2$ for them. The "Interpretation" depends on the behavior of the solution outside the past similarity horizon at $z = z_2$. Solutions marked "Black Hole" encounter a numerical singularity that appears to be an apparent horizon. Such an apparent horizon probably means that the solution represents, not a Choptuon, but a black hole that is growing by self-similar accretion of scalar fields. Solutions marked "Choptuon" encounter no such apparent horizon, but do encounter a future similarity horizon.

[1] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[2] A.M. Abrahams and C.R. Evans, Phys. Rev. Lett. **70**, 2980 (1993).
[3] P.R. Brady, Class. Quantum Grav. **11**, 1255 (1994).
[4] Y. Oshiro, K. Nakamura, and A. Tomimatsu, Prog. Theor. Phys. **91**, 1265 (1994).
[5] A. Strominger and L. Thorlacius, Phys. Rev. Lett. **72**, 1584 (1994).
[6] J. Traschen, Phys. Rev. D **50**, 7144 (1994).
[7] P.R. Brady, Phys. Rev. D **51**, 4168 (1995).
[8] D. Garfinkle, "Choptuik scaling in null coordinates," Report No. gr-qc/9412008, 1994 (unpublished).
[9] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[10] R. Penrose, in *General Relativity: An Einstein Centennary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), pp. 581–638.
[11] See also D. Christodoulou, Commun. Math. Phys. **105**, 337 (1986); **106**, 587 (1986); **109**, 591 (1987); **109**, 613 (1987).
[12] C.R. Evans and J.S. Coleman, Phys. Rev. Lett. **72**, 1782 (1994).
[13] M.W. Choptuik (private communication).
[14] D.M. Eardley, Commun. Math. Phys. **37**, 287 (1974).
[15] B.K. Berger, J. Math. Phys. **17**, 1268 (1976).
[16] D. Eardley, J. Isenberg, J. Marsden, and V. Moncrief, Commun. Math. Phys. **106**, 137 (1986).
[17] W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, *Numerical Recipes* (Cambridge University Press, Cambridge, England, 1986).
[18] T. Maithreyan, Ph.D. thesis, Boston University, 1985.
[19] C.R. Evans (private communication).
[20] D. Christodoulou, Commun. Pure Appl. Math. **44**, 339 (1991); **46**, 1131 (1993).
[21] J.H. Horne (private communication).